USPC
Université Sorbonne Paris Cité

université PARIS DIDEROT PARIS 7

Thèse de doctorat
de l'Université Sorbonne Paris Cité
Préparée à l'Université Paris Diderot

École doctorale de sciences mathématiques de Paris centre

Discipline : Mathématiques

présentée par

Nicolina Istrati

Conformal structures on compact complex manifolds

---

dirigée par Andrei Moroianu

Soutenue le 15 juin 2018 devant le jury composé de :

M. Paul Gauduchon	Directeur de recherche	École Polytechnique	Président du jury
M. Paul Laurain	Maitre de conférences	Université Paris Diderot	Examinateur
M. Andrei Moroianu	Directeur de recherche	Université Paris - Sud	Directeur
Mme Mihaela Pilca	Professeur	Universität Regensburg	Examinatrice
M. Uwe Semmelmann	Professeur	Universität Stuttgart	Examinateur

Au vu des rapports de :

M. Andrew Swann	Professeur	Aarhus University
M. Vestislav Apostolov	Professeur	Université du Québec à Montréal

Institut de mathématiques de Jussieu-Paris Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Université Sorbonne Paris Cité.
Campus Paris Diderot.
École doctorale de sciences mathématiques de Paris centre.
Case courrier 7012
8 Place Aurélie Nemours
75 205 Paris cedex 13

Remerciements

Je dois commencer par remercier mon directeur de thèse, Andrei Moroianu, qui a accepté dès mon M2 de surveiller et guider mes débuts dans la recherche. De m'avoir consacré énormément de temps et de patience en m'écoutant, en lisant mes nombreux essais de preuves et en m'indiquant des pistes de recherche. D'avoir corrigé beaucoup de mes erreurs et m'avoir constamment rappelé l'importance de l'esprit critique pour faire des mathématiques. Je remercie aussi Simone Diverio, qui nous a accompagnés pendant la première partie de cette aventure.

Ensuite je remercie Vestislav Apostolov et Andrew Swann d'avoir accepté d'être rapporteurs de ce manuscrit, ainsi que pour leurs remarques avisées. Merci à Paul Gauduchon, Paul Laurain, Mihaela Pilca et Uwe Semmelmann de faire partie du jury, leur présence me fait grand honneur.

Mes remerciements vont aussi à Liviu Ornea, qui a été mon mentor à Bucarest, et sur le soutien duquel j'ai toujours pu compter. Merci encore à Victor Vuletescu et à Sergiu Moroianu pour leurs conseils et pour avoir veillé à mon parcours. Finalement, merci à Aurel Bârsan, qui a été le premier professeur à me faire prendre les mathématiques au sérieux.

Merci à mon amie mathématique, Alexandra. C'est une grande chance qu'on puisse partager nos dilemmes pour en faire des articles. Merci aussi à Cristina, qu'on a accompagnée vers la topologie, mais à la fin seule elle y a survécu.

Je suis reconnaissante envers mes camarades du couloir qui ont fait que l'expérience de thésard à Jussieu soit aussi sympathique. Premièrement au petit groupe Macarena, Louis, Léo et Adrien B, avec lesquels au début je me sentais bien sans trop comprendre ce qu'ils disaient, mais avec les bières, les belotes et les balades espagnoles, ça a fini par arriver. À eux se rajoutent les autres collègues de génération Jesua, Johannes, Adrien S, Arthur, Amiel, Cyrus. Ensuite à ceux qui nous ont si bien accueillis et appris — Liana, Maylis, Juliette, François, Olivier, Thibaut, Amine, Anne, John, Malick, Miguel, Ramanujan, Lucas, Valentin, Arthur-César, Joaquin, Hsueh-Yung. Et finalement à ceux qui sont venus après et prennent fièrement le relais — Hugo, Thomas, Justin, Xavier, Mathieu, Eckhard, Vincent, Michou, Malo, Peiyi, Linyuan, Benoit.

Aussi importants ont été mes colocataires le long de ces quatre années. Merci à Apple qui m'a accueillie avec autant de chaleur et de générosité. Ensuite merci à la troupe d'Ivry, où il faut encore nommer Hanane et Fede, pour leur bonne humeur et leurs repas délicieux.

Les savoir là, surtout quand je revenais pour une bouffée d'air roumain, a aussi contribué à mener à bonne fin cette aventure. Merci à Tatiana et Buga Chan, à Irina et Marian, à Anca et

Sergiu. Mais surtout merci à Ioana, sa compréhension et son encouragement ont été essentiels pour que je continue.

Je suis fortunée d'avoir rencontré Louis au milieu de tous ces mathématiques, partager ces dernières années ensemble a été une quête au moins aussi intéressante que le travail pour cette thèse. Je lui suis reconnaissante pour le savoir à mes côtés.

Finalement, je veux remercier ma famille. À mon frère, dont les propres recherches ont toujours été une inspiration pour moi. Et à mes parents, qui m'ont offert leur soutien et la liberté de marcher dans mes propres pas. Merci.

Résumé

Dans cette thèse on s'intéresse à deux types de structures conformes non-dégénérées sur une variété complexe compacte donnée. La première c'est une forme holomorphe symplectique twistée (THS), i.e. une deux-forme holomorphe non-dégénérée à valeurs dans un fibré en droites. Dans le deuxième contexte, il s'agit des métriques localement conformément kähleriennes (LCK).

Dans la première partie, on se place sur un variété de type Kähler. Les formes THS généralisent les formes holomorphes symplectiques, dont l'existence équivaut à ce que la variété admet une structure hyperkählerienne, par un théorème de Beauville. On montre un résultat similaire dans le cas twisté, plus précisément: une variété compacte de type kählerien qui admet une structure THS est un quotient fini cyclique d'une variété hyperkählerienne. De plus, on étudie sous quelles conditions une variété localement hyperkählerienne admet une structure THS.

Dans la deuxième partie, les variétés sont supposées de type non-kählerien. Nous présentons quelques critères pour l'existence ou non-existence de métriques LCK spéciales, en terme du groupe de biholomorphismes de la variété. En outre, on étudie le problème d'irréductibilité analytique des variétés LCK, ainsi que l'irréductibilité de la connexion de Weyl associée. Dans un troisième temps, nous étudions les variétés LCK toriques, qui peuvent être définies en analogie avec les variétés de Kähler toriques. Nous montrons qu'une variété LCK torique compacte admet une métrique de Vaisman torique, ce qui mène à une classification de ces variétés par le travail de Lerman.

Dans la dernière partie, on s'intéresse aux propriétés cohomologiques des variétés d'Oeljeklaus-Toma (OT). Plus précisément, nous calculons leur cohomologie de de Rham et celle twistée. De plus, on démontre qu'il existe au plus une classe de de Rham qui représente la forme de Lee d'une métrique LCK sur un variété OT. Finalement, on détermine toutes les classes de cohomologie twistée des métriques LCK sur ces variétés.

Mots-clés

Forme holomorphe symplectique, variété hyperkählerienne, métrique localement conformément kählerienne, métrique de Vaisman, géométrie torique, variété d'Oeljeklaus-Toma, cohomologie twistée.

Abstract

In this thesis, we are concerned with two types of non-degenerate conformal structures on a given compact complex manifold. The first structure we are interested in is a twisted holomorphic symplectic (THS) form, i.e. a holomorphic non-degenerate two-form valued in a line bundle. In the second context, we study locally conformally Kähler (LCK) metrics.

In the first part, we deal with manifolds of Kähler type. THS forms generalise the well-known holomorphic symplectic forms, the existence of which is equivalent to the manifold admitting a hyperkähler structure, by a theorem of Beauville. We show a similar result in the twisted case, namely: a compact manifold of Kähler type admitting a THS structure is a finite cyclic quotient of a hyperkähler manifold. Moreover, we study under which conditions a locally hyperkähler manifold admits a THS structure.

In the second part, manifolds are supposed to be of non-Kähler type. We present a few criteria for the existence or non-existence for special LCK metrics, in terms of the group of biholomorphisms of the manifold. Moreover, we investigate the analytic irreducibility issue for LCK manifolds, as well as the irreducibility of the associated Weyl connection. Thirdly, we study toric LCK manifolds, which can be defined in analogy with toric Kähler manifolds. We show that a compact toric LCK manifold always admits a toric Vaisman metric, which leads to a classification of such manifolds by the work of Lerman.

In the last part, we study the cohomological properties of Oeljeklaus-Toma (OT) manifolds. Namely, we compute their de Rham and twisted cohomology. Moreover, we prove that there exists at most one de Rham class which represents the Lee form of an LCK metric on an OT manifold. Finally, we determine all the twisted cohomology classes of LCK metrics on these manifolds.

Keywords

Holomorphic symplectic form, hyperkähler manifold, locally conformally Kähler metric, Vaisman metric, toric geometry, Oeljeklaus-Toma manifold, twisted cohomology.

Contents

Introduction **vii**

Notation and conventions xvi

1 Twisted Holomorphic Symplectic Forms **1**

1.1 Introduction 1

1.2 Holomorphic symplectic manifolds 2

1.3 Twisted holomorphic symplectic manifolds 3

1.4 A characterization 7

1.5 Examples 12

2 Locally Conformally Kähler Geometry **15**

2.1 Introduction 15

2.2 Basic definitions and properties 16

2.3 Connections 19

2.4 Special LCK metrics 22

2.5 Infinitesimal automorphisms of LCK manifolds 25

2.6 Examples 28

2.6.1 Diagonal Hopf manifolds 28

2.6.2 Non-diagonal Hopf surfaces 31

2.6.3 LCK manifolds obtained from ample vector bundles 32

2.6.4 LCK metrics on blow-ups 34

2.6.5 Complex surfaces 35

3 Existence Criteria for LCK Metrics **37**

3.1 Introduction 37

3.2 The Lee vector field 38

3.3 Existence of LCK metrics with potential 42

3.4 Existence of Vaisman metrics 44

3.5 Torus principal bundles 47

3.6 Analytic irreducibility of complex manifolds of LCK type 48

3.7 Weyl reducible manifolds 51

4 Toric LCK Manifolds **55**
4.1 Introduction 55
4.2 Twisted Hamiltonian Vector Fields 56
4.3 Torus actions on LCS manifolds 58
4.4 Proof of the Main Theorem 59
4.5 Examples 63
4.6 Final remarks and questions 65

5 Cohomological properties of OT manifolds **67**
5.1 Introduction 67
5.2 Oeljeklaus-Toma manifolds 68
5.2.1 The construction 68
5.2.2 Metric properties 69
5.3 Technical Preliminaries 71
5.3.1 Leray-Serre spectral sequence of a locally trivial fibration 71
5.3.2 Twisted cohomology 73
5.4 The de Rham cohomology 76
5.5 The Leray-Serre spectral sequence of OT manifolds 79
5.6 Twisted cohomology of OT manifolds 81
5.7 Applications and Examples 83

Bibliography **89**

Introduction

In the present dissertation, we are interested in certain non-degenerate conformal structures on a given compact complex manifold. The conformal nature can be encoded in a line bundle over the manifold. As such, given a complex manifold (M, J) and a line bundle L over M, we want to study non-degenerate two-forms:

$$\omega \in \Gamma(\bigwedge^2 T^*M \otimes L).$$

There are two different settings which we investigate. In the first one, we suppose that L is a holomorphic line bundle and ω is an L-valued holomorphic two-form. This kind of structure will be called *twisted holomorphic symplectic* (THS). In the second context, we suppose that L is an oriented real line bundle endowed with a flat connection ∇, and ω is a positive $(1,1)$-form with values in L which is d_∇-closed. Such a structure will be called a *locally conformally Kähler form* (LCK).

Both these structures are natural generalisations of the well-known non-twisted ones: the first one coincides with a holomorphic symplectic form when L is holomorphically trivial, while the second one is simply a Kähler metric when $(L, \nabla) = (M \times \mathbb{R}, d)$. We wish to understand what kind of restrictions the existence of such structures imposes on the manifold, and to what extent the properties of the corresponding non-twisted structures generalise to our setting.

As it turns out, if one assumes that (M, J) is of Kähler type, then one reduces quite easily to the non-twisted situation. The first chapter proves and explains this reduction in the context of THS structures. On the other hand, we make no assumption of Kählerness in the second situation, and although LCK structures are just conformal generalisations of Kähler structures, they behave quite differently from the latter. LCK structures are studied throughout chapters 2 to 4. The last chapter presents a certain family of non-Kähler complex manifolds, called Oeljeklaus-Toma manifolds, focusing on their topological properties. This part is related to the rest of the discussion by the fact that some of these manifolds admit LCK forms.

Twisted holomorphic symplectic forms

Let (M, J) be a compact complex manifold of Kähler type, of complex dimension $2m$. It is well known, by a theorem of Beauville [Bea83b] based on Yau's proof of the Calabi conjecture, that the existence of a holomorphic symplectic form ω on M is equivalent to the manifold admitting a hyperkähler structure, i.e. a metric compatible with the complex structure, whose holonomy group of the Levi-Civita connection sits in $Sp(m)$. One might hope to obtain less rigid structures if one assumes that the symplectic form takes values in a holomorphic line bundle instead.

This expectation is quite natural if we take a look at the analogous symmetric situation: suppose that L is a holomorphic line bundle over (M, J) and that there exists a non-degenerate holomorphic section $g \in H^0(M, S^2T^*M \otimes L)$, called a *holomorphic conformal structure*. Still under the Kählerness assumption, Inoue, Kobayashi and Ochiai [IKO80] proved that if L is holomorphically trivial then (M, J) is a finite quotient of a complex torus. On the other hand, if L is not trivial, then new examples appear, and in fact classifications are known only for the compact surfaces ([KO82]), and for projective threefolds ([JR05]). Let us note that the standard example of such a manifold is given by the hyperquadric:

$$\mathbb{Q}_n = \{[z_0 : \cdots : z_{n+1}] | -2z_0z_{n+1} + \sum_{k=1}^{n} z_k^2 = 0\} \subset \mathbb{P}^n$$

with the structure $g = -2dz_0dz_{n+1} + dz_1dz_1 + \ldots dz_ndz_n \in H^0(\mathbb{Q}_n, S^2T^*\mathbb{Q}_n \otimes O(2))$.

It turns out that things are different in the symplectic setting, as we show that the conformal case is quite similar to the standard one:

Theorem A (Theorem 1.3.5, Theorem 1.4.1)**.** *Let (M^{2m}, J, L, ω), $m > 1$, be a compact THS manifold of Kähler type, and let $\alpha \in H^2(M, \mathbb{R})$ be a Kähler class. Then L is unitarily flat, and there exists a unique Kähler metric g with respect to J representing α so that a finite cyclic cover of (M, g, J) has holonomy in $Sp(m)$. Moreover, the form ω is parallel with respect to the natural connection induced by g on $\bigwedge^2 T^*M \otimes L$.*

The main point of the proof is to show that the line bundle L has torsion first Chern class, as everything else follows similarly to the non-twisted case, via Yau's theorem and the Weitzenböck formula. This is true, because we manage to construct a holomorphic connection in L, naturally induced by ω via the Lefschetz operator

$$\mathrm{Lef}_\omega : \Omega^\bullet \to \Omega^{\bullet+2} \otimes L, \quad \eta \mapsto \eta \wedge \omega.$$

By the Kähler assumption, this will imply then that $c_1(L) = 0 \in H^2(M, \mathbb{R})$.

Let us note that the hypothesis $m > 1$ is natural, as any complex surface admits a THS form: simply take $L = \bigwedge^2 TM$, and note that $\Omega^2_M \otimes L$ is holomorphically trivial.

An equivalent definition for a hyperkähler structure on M is a Riemannian metric g together with three integrable complex structures I, J, K compatible with g, parallel for the Levi-Civita connection of g and verifying the quaternionic relations $IJ = K = -JI$. Moreover, we say that (M, g) is locally hyperkähler if the universal Riemann cover $(\tilde{M}, \tilde{g})$ is hyperkähler, so that the structures I, J and K are defined only locally on M. Note that the above theorem gives the existence of a local hyperkähler structure on M which is particular: it is formed of a global complex structure J which we have fixed, together with two more local complex structures. We call such a structure *Kähler locally hyperkähler* (KLH).

In the last part of Chapter 1, we investigate under what conditions a KLH manifold admits a twisted holomorphic symplectic form. The presence of a THS structure forces the fundamental group of the manifold to have a certain structure, which we describe. This depends mainly on the (local) de Rham reducibility of the manifold. We first note that, although a product of two hyperkähler manifolds is again hyperkähler, THS manifolds are de Rham irreducible (Corollary 1.4.2). At the same time, we show that for locally irreducible manifolds, the existence of a THS structure is equivalent to the manifold being KLH (Corollary 1.4.3). For the intermediate case of irreducible, locally reducible manifolds, we need to do a discussion depending on the finiteness of the fundamental group. The results of this part are obtained by an analysis of the structure of an isometry of certain Riemannian products, and the main tool we use is the holomorphic Lefschetz fixed-point formula.

Locally conformally Kähler metrics

A Kähler metric g on a complex manifold (M, J) is a Hermitian metric whose fundamental form $\Omega := g(J\cdot, \cdot)$ is closed. There are many well-known obstructions to the existence of such metrics on a compact manifold, the basic one being that b_1 needs to be even. One way to generalize such metrics is to consider metrics that are conformal to them. It can be easily seen that if Ω is Kähler and $f \in \mathcal{C}^\infty(M, \mathbb{R})$, then $\mathrm{e}^f\Omega$ will not be Kähler unless f is constant. More generally, in order to get rid of some of the topological obstructions, one can consider metrics that are only locally conformal to Kähler metrics: these are the LCK metrics. More precisely, Ω is LCK if every point of M has a neighbourhood U on which there exists a Kähler metric Ω_U which is conformal to Ω, i.e.

$$\Omega|_U = \mathrm{e}^{f_U}\Omega_U, \quad f_U \in \mathcal{C}^\infty(U, \mathbb{R}). \tag{0.0.1}$$

In the above equation, although the function f_U is local, $\theta = df_U$ is a global real closed one-form on M, called the *Lee form* of Ω, provided that $\dim_{\mathbb{C}} M > 1$. This allows us to give an equivalent definition: g is LCK if Ω verifies $d\Omega = \theta \wedge \Omega$, with θ a closed one-form on M. On the minimal cover $\hat{M}$ of M on which θ becomes exact, the local Kähler metric in (0.0.1) pulls back to a global Kähler metric Ω_K, and $(\hat{M}, J, \Omega_K)$ is called *the minimal Kähler cover.* The most basic manifold of non-Kähler type admits an LCK metric: this is the (standard) Hopf surface

$$H = \mathbb{C}^2 - \{0\}/_{(z_1,z_2)\sim(\alpha z_1, \alpha z_2)}, \quad 0 < \alpha < 1,$$

with the Boothby metric $\Omega = |z|^{-2}dd^c|z|^2$. This is also the first LCK example appearing in the literature, but although the metric was constructed by Boothby in [Bo54], it was noticed only after twenty years that it is LCK by Vaisman, who was the one to start a systematic study of these structures.

Any LCK metric on a manifold of Kähler type is globally conformal to a Kähler metric ([Va80]). For this reason, we will always assume tacitly that our manifolds are not of Kähler type, in order to study only strict LCK metrics. In this setting, a first obstruction appears for manifolds of LCK type, namely: $0 < b_1 < 2h^{0,1}$, where $h^{0,1} = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M)$. As a matter of fact, this is the only cohomological obstruction known for a general LCK manifold. Vaisman had conjectured that such a manifold should always have b_{2k+1} odd for some $k \in \mathbb{N}$, however this was disproved by the OT manifolds [OT05].

There are a few special LCK metrics which are better understood. The most important one is a *Vaisman metric* (which used to be called generalized Hopf), defined by the condition $\nabla^g\theta = 0$, where ∇^g is the Levi-Civita connection determined by g. Their study started with Vaisman ([Va82]) and is still ongoing. Vaisman manifolds admit a transversal Kähler foliation, which allows one to study some of their properties by means of the better known ones from Kähler geometry. For instance, this was done in order to determine their cohomological properties by Vaisman [Va82] and Tsukada [Ts94], and it turns out that the Frölicher spectral sequence degenerates at the first page for manifolds of Vaisman type (although they don't admit a Hodge decomposition, as already noticed).

Moreover, a normalised Vaisman metric (Ω, θ) on (M^n, J) has the form

$$\Omega = -dJ\theta + \theta \wedge J\theta, \tag{0.0.2}$$

(see Corollary 2.4.8 in Chapter 2), and the corresponding Kähler metric on $\hat{M}$ can be written as $\Omega_K = dd^c e^{-\varphi}$ where $\varphi \in \mathcal{C}^\infty(\hat{M}, \mathbb{R})$ is a function satisfying $\theta = d\varphi$ on $\hat{M}$. Thus Ω_K has a positive potential. This was first noted by Verbitsky [Ve04], and as a consequence Ornea-Verbitsky [OV10] introduced and started the study of the more general notion of a *LCK metric with (positive) potential.* These are LCK metrics whose Kähler metric writes

$$\Omega_K = dd^c(f e^{-\varphi}), \quad f \in \mathcal{C}^\infty(M, \mathbb{R}). \tag{0.0.3}$$

This class of metrics has the advantage of being closed under small deformations ([OV10], [Go14]), while the Vaisman manifolds are not (see [Bel00]). Even more general than this is the notion of an *exact LCK metric*, which is an LCK metric whose Kähler metric has the form:

$$\Omega_K = d(e^{-\varphi}\eta), \quad \eta \in \mathcal{E}^1(M, \mathbb{R}). \tag{0.0.4}$$

A closed one-form θ on a manifold M induces a twisted differential operator

$$d_\theta : \mathcal{E}^k \to \mathcal{E}^{k+1}, \quad \alpha \mapsto d\alpha - \theta \wedge \alpha$$

which verifies $d_\theta^2 = 0$. This defines the twisted cohomology $H_\theta(M) = \operatorname{Ker} d_\theta / \operatorname{Im} d_\theta$, which plays an important role in LCK geometry. Note that an LCK structure (Ω, θ) induces a cohomology a class $[\Omega] \in H^2_\theta(M)$, which is zero precisely for exact LCK metrics. Moreover, by [LLMP03], one has $H^\bullet_\theta(M) = 0$ if θ is the Lee form of a Vaisman metric.

Finally, let us say a few words on an interesting problem in LCK geometry, which we also tackle in the special case of OT manifolds. It concerns determining the set, or at least the geometry of the set:

$$\mathcal{L}(M, J) = \{a \in H^1(M, \mathbb{R}) | \text{ there exists an LCK structure } (\Omega, \theta) \text{ with } \theta \in a\}$$

where (M, J) is a compact complex manifold. It was first studied by Tsukada in [Ts94] in the case of a Vaisman manifold (M, J). He showed that any element of $\mathcal{L}(M, J)$ is the Lee class of a Vaisman metric, and that for a given element $a_0 \in \mathcal{L}(M, J)$, one has $\mathcal{L}(M, J) = \{ta_0 + b | t > 0, b \in \mathcal{H} \subset H^1(M, \mathbb{R})\}$. Here, $\mathcal{H} \subset H^1(M, \mathbb{R})$ is formed by all the de Rham classes of forms whose $(1, 0)$-part are holomorphic d-closed one forms. Tsukada showed that on a Vaisman manifold, $\mathcal{H}$ is a hyperplane in $H^1(M, \mathbb{R})$. More recently, the set $\mathcal{L}(M, J)$ was studied by Apostolov and Dloussky [AD16a], [AD16b] and by Otiman [O16] in the case of compact complex surfaces, and it has been completely determined for a number of cases. By the above cited articles, together with our result for OT manifolds, it turns out that for all the known examples of LCK manifolds, $\mathcal{L}(M, J)$ is either a point or an open subset of $H^1(M, \mathbb{R})$.

Existence of LCK metrics

The first question one probably asks is how often do LCK metrics arise? As noted, simply connected compact complex manifolds of non-Kähler type, such as Calabi-Eckmann manifolds for instance, cannot admit such metrics, so the class is strict. The existence of LCK metrics on compact complex surfaces is fairly well undestood ([Tr82], [LeB91], [GO98], [Bel00], [FP10], [Bru11]), and the only ones known to not admit LCK metrics are a certain class of Inoue-Bombieri surfaces. The surfaces of this class are in fact small deformations of some other Inoue-Bombieri surfaces which admit LCK metrics, so in particular the category of LCK manifolds is not closed under small deformations. Moreover, one encounters all the special LCK metrics defined above, or the lack of such metrics, already in the surface case.

Moving to higher dimension, there are natural ways of constructing Vaisman manifolds: starting from any ample holomorphic vector bundle over a Kähler manifold, one has a naturally associated Vaisman manifold ([Va76], [Va80], [Ts97], [Ts99]). We extend this construction in Section 2.6.3 to obtain manifolds with LCK metrics with positive potential. Moreover, any complex submanifold of dimension bigger than 1 of a Vaisman manifold is again Vaisman ([Va82], [Ts97]). However, it is more difficult to construct examples of manifolds of LCK type, not admitting LCK metrics with potential (or exact, for that matter). In fact, all known manifolds of higher dimension of this kind are either Oeljeklaus-Toma manifolds, or blow-ups of LCK manifolds. We should note at this point that indeed, the blow-up of an LCK manifold along a submanifold of Kähler type admits an LCK metric ([Tr82], [Vu09], [OVV13]), but never an exact one.

It should also be noted that the product metric of two LCK manifolds cannot be LCK [Va80], however it is still unknown whether such a product manifold can admit some other LCK metric. Some particular cases are known, such as: if M_1 and M_2 are of Vaisman type, then their product admits no LCK metric [Ts99], and if M_1 is not a curve and verifies the $\partial\bar{\partial}$-lemma, then again $M_1 \times M_2$ admits no LCK metric [OPV14]. We moreover prove:

Theorem B (Theorem 3.6.3, Proposition 3.6.4)**.** *Suppose that M_1 and M_2 are two compact complex manifolds. Then $M_1 \times M_2$ cannot admit a Vaisman metric. Moreover, if M_1 is of Vaisman type, then $M_1 \times M_2$ admits no LCK metric at all.*

Theorem C (Proposition 3.6.8)**.** *Let M_1 be a compact complex curve, let M_2 be a complex manifold and suppose that $M := M_1 \times M_2$ admits an LCK metric. Then M_2 admits an LCK metric with positive potential.*

A related problem concerns the reducibility of the natural connections associated to an LCK metric. Madani, Moroianu, and Pilca showed in [MMP16] that the holonomy group of the Levi-Civita connection of an LCK metric is irreducible and generic, unless the metric is Vaisman, in which case it equals $\mathrm{SO}(2n-1)$, n being the complex dimension of the manifold. Another natural connection associated to an LCK metric (Ω, θ) on (M, J) is the standard Weyl connection $\mathcal{D}$, defined as being the unique torsion free connection on M which satisfies:

$$\mathcal{D}J = 0, \quad \mathcal{D}\Omega = \theta \otimes \Omega.$$

As this connection coincides with the Levi-Civita (or also the Chern) connection of the local Kähler metrics, it encodes in some sense more interesting properties. For instance, the Weyl connection of an LCK metric can have reducible holonomy in a non-trivial way: this is the case for the Oeljeklaus-Toma manifolds. Kourganoff [Kou15] gave a structure theorem for a more general class of manifolds with Weyl-reducible connection, which we adapt to the LCK context, in order to show:

Theorem D (Theorem 3.7.7)**.** *Any exact LCK metric on a compact complex manifold is Weyl-irreducible, unless it is the standard LCK metric on the standard Hopf manifold.*

The fact that we know very few examples of non-exact LCK metrics parallels with the lack of criteria for the existence or non-existence of a general LCK metric. Again, things look better if we turn to the case of metrics with positive potential or of Vaisman metrics. Ornea-Verbitsky ([OV12], [OV17]) gave an existence criterion for LCK metrics with positive potential, whose proof we revise:

Theorem E ([OV12], [OV17], Theorem 3.3.1)**.** *Let* (M, J, Ω, θ) *be a compact LCK manifold admitting a holomorphic action of* $\mathbb{S}^1$ *which, on the minimal cover* $\hat{M}$*, lifts to an effective* $\mathbb{R}$*-action. Then there exists an LCK metric with positive potential whose Lee form is cohomologous to* θ*.*

Kamishima-Ornea [KO05] gave a criterion for a given LCK conformal class $[g]$ on a compact complex manifold (M, J) to admit a Vaisman metric, namely they show that this is equivalent to the automorphism group $\mathrm{Aut}(M, J, [g])$ containing a one-dimensional complex Lie group which does not act isometrically on the corresponding Kähler metric. We generalise their criterion in a way that does not involve a given fixed conformal class:

Theorem F (Theorem 3.4.3)**.** *A connected compact complex manifold* (M, J) *of LCK type admits a Vaisman metric if and only if* $\mathrm{Aut}(M, J)$ *contains a torus* $\mathbb{T}$ *whose Lie algebra* $\mathfrak{t}$ *verifies* $\dim_{\mathbb{C}}(\mathfrak{t} \cap i\mathfrak{t}) > 0$*.*

As a consequence, we obtain a criterion of non-existence of LCK metrics:

Corollary G (Corollary 3.4.5)**.** *Let* (M, J) *be a compact complex manifold, and suppose that the group of biholomorphisms* $\mathrm{Aut}(M, J)$ *contains a compact torus whose Lie algebra* $\mathfrak{t}$ *verifies* $\dim_{\mathbb{C}}(\mathfrak{t} \cap i\mathfrak{t}) > 1$*. Then* (M, J) *admits no LCK metric.*

An immediate application of this criterion is the classification of manifolds of LCK type among all the torus principal bundles (Proposition 3.5.1), in analogy with a theorem of Blanchard [Bl54].
Let us recall at this point that an LCK metric (Ω, θ) induces naturally two vector fields B and $A = JB$, called the *Lee* and *Reeb* vector fields, via:

$$\iota_A\Omega = -\theta, \quad \iota_B\Omega = J\theta.$$

In the case of a Vaisman metric, these vector fields are part of the Lie algebra $\mathfrak{aut}(M, J, \Omega)$, and are the ones to generate a non-real torus as in the above criterion. Moreover, the condition of B being Killing easily implies that the metric is Vaisman. One could then ask what happens if we impose B to be holomorphic. This problem is studied in the recent paper [MMO17], where A. Moroianu, S. Moroianu and L. Ornea show that if, moreover, the LCK metric is Gauduchon, or if B has constant norm, then the metric is Vaisman. At the same time, they construct an example of a non-Vaisman LCK metric with holomorphic Lee field, showing that one needs more hypothesis then just the holomorphicity of B. In Proposition 3.2.2 we show that if Ω is an LCK metric with constant potential, i.e. of the form (0.0.2), and B is holomorphic, then again Ω is Vaisman. Moreover, we make the remark that the example of [MMO17] can be chosen with positive potential (Lemma 3.2.7), so our hypothesis cannot be relaxed.

Toric LCK manifolds

Recall that a symplectic manifold (M^{2n}, ω) of real dimension $2n$ is called a toric manifold if the compact torus $\mathbb{T}^n$ acts effectively in a Hamiltonian way on (M, ω). If $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}^n$, this means that there exists a $\mathbb{T}^n$-invariant map $\mu : M \to \mathfrak{t}^*$, called the moment map, verifying that for each vector field $X \in \mathfrak{t} \subset \mathfrak{aut}(M)$, letting $\mu_X =< \mu, X >\in \mathcal{C}^\infty(M)$, one has $d\mu_X = \iota_X\omega$. As is well-known, by the Delzant construction, toric manifolds (M, ω, μ) are completely determined by the image of their moment map, which is a Delzant polytope. Moreover, all their analytic and geometric properties can be read off their moment polytope, and in particular they admit a compatible complex structure with respect to which ω is Kähler.

If one forgets the complex structure in the LCK context, then one deals with *locally conformally symplectic* (LCS) forms, namely non-degenerate forms Ω verifying $d\Omega = \theta \wedge \Omega$ for some closed real one-form θ. For these structures, there exist analogous notions of Hamiltonians and moment maps, introduced by Vaisman in [Va85]. A vector field X on M is *twisted Hamiltonian* with respect to (Ω, θ) if there exists a function $f_X \in \mathcal{C}^\infty(M)$ so that

$$\iota_X\Omega = d_\theta f_X. \tag{0.0.5}$$

If we consider the associated minimal symplectic cover $(\hat{M}, \Omega_K)$ as in the LCK context, then this definition is equivalent to asking for the lift of X to $\hat{M}$ to be Hamiltonian for Ω_K. If θ is not exact and M is compact, then d_θ is injective on $\mathcal{C}^\infty(M)$, which implies that if a function as in (0.0.5) exists, then it is unique. Finally, let us note that this definition only depends on the conformal class $[\Omega]$, and not on Ω itself: X is twisted Hamiltonian for Ω if and only if it is so for $\mathrm{e}^f\Omega$, $f \in \mathcal{C}^\infty(M, \mathbb{R})$.

A *toric LCS manifold* is an LCS manifold $(M^{2n}, [\Omega])$ together with the effective action of a compact torus $\mathbb{T}^n$ so that every induced vector field $X \in \mathfrak{Lie}(\mathbb{T}^n) \subset \mathfrak{aut}(M)$ is twisted Hamiltonian. If, moreover, there exists a complex structure on M so that Ω is LCK, and the torus acts by biholomorphisms, then we have a *toric LCK manifold*. If μ is the moment map of a toric LCS form (Ω, θ), then the minimal symplectic cover is a toric symplectic manifold with corresponding moment map $\hat{\mu} = \mathrm{e}^{-\varphi}\mu$, where $\theta = d\varphi$.

Although introduced early in the history of LCS/LCK geometry, general twisted Hamiltonian group actions have not been extensively studied. The reduction procedure from symplectic geometry has been adapted to this context by Haller and Rybicki in [HR01] for LCS manifolds, and by Gini, Ornea and Parton in [GOP05] for LCK manifolds. Moreover, twisted Hamiltonian actions were studied by Otiman in [O15] for the purpose of constructing LCS bundles.

On the other hand, recently there have been some new advances concerning toric LCK manifolds. In order to explain them, let us first recall that Vaisman metrics are closely related to Sasaki structures: their minimal Kähler covers are Kähler cones over Sasaki manifolds, cf. [GOP06]. Pilca showed in [Pi16] that a compact Vaisman manifold is toric if and only if the associated Sasaki manifold is toric, and one action naturally induces the other. Moreover, Madani, Moroianu and Pilca showed in [MMP17] that the first Betti number of a toric Vaisman manifold is $b_1 = 1$, implying that the associated toric Sasaki manifold is compact.

In the same paper, the authors gave a classification of compact toric LCK surfaces, and it turns out that they all admit toric Vaisman metrics. Hence the question was raised of whether this is always the case, regardless of dimension. The main result of Chapter 4 is an affirmative answer to it:

Theorem H (Theorem 4.4.1)**.** *Let $(M, J, [\Omega])$ be a compact toric LCK manifold. Then there* *exists a Vaisman metric Ω', possibly nonconformal to Ω, with respect to which the same action is still twisted Hamiltonian.*

Let us note that compact toric Sasaki manifolds can also be understood via the image of the moment map, as a result of the paper [Ler03] of Lerman, in which he completes the classification of toric compact contact manifolds. Indeed, the image of their moment map to which one adds $\{0\}$ is a cone over a convex polytope with certain combinatorial properties which makes it *a good cone*. Moreover, to each good cone one can associate in a unique way a compact contact toric manifold (N, α). On the symplectic manifold naturally associated to this contact manifold, there always exists a compatible complex structure, inducing a toric Sasaki structure on N. Moreover, just like in the compact symplectic case, all such complex structures can be described only in terms of certain functions defined on the moment cone, as shown by Martelli, Sparks and Yau [MSY06], see also Abreu [Ab10].

With this in mind, and as a corollary of our result, we can thus describe also toric LCK manifolds in terms of combinatorial data coming from certain moment cones. However, all information is not preserved: from the good cones we can recover only some of the toric LCS structures, namely the ones giving Vaisman metrics.

We end this part by a remark on the differences between the symplectic case and the LCS case. Recall that a compact toric symplectic manifold admits a compatible integrable complex structure with respect to which the manifold is toric Kähler. On the contrary, Example 4.5.5 shows that on a general toric LCS manifold, there does not always exist a compatible complex structure, making it into a toric LCK manifold.

Oeljeklaus-Toma manifolds

OT manifolds were introduced by Oeljeklaus and Toma in [OT05] as higher dimensional analogues of a class of Inoue-Bombieri surfaces. They are compact complex manifolds of non-Kähler type, obtained as quotients of $\mathbb{H}^s \times \mathbb{C}^t$ by discrete groups of affine transformations arising from a number field K and a particular choice of a subgroup of units U of K. Usually, such a manifold is said to be of type (s, t), and is denoted by $X(K, U)$.

More specifically, start with a number field K which admits exactly s real embeddings in $\mathbb{C}$, $\sigma_1, \ldots, \sigma_s$, and $2t$ complex conjugate ones $\sigma_{s+1} = \overline{\sigma_{s+t+1}}, \ldots, \sigma_{s+t} = \overline{\sigma_{s+2t}}$. Then there exists a choice of a subgroup of units U of the ring of integers of K, O_K, so that the semi-direct product $\Gamma := U \rtimes O_K$ acts freely and properly discontinuously on $\mathbb{H}^s \times \mathbb{C}^t$, and the quotient $X = X(K, U) := \mathbb{H}^s \times \mathbb{C}^t/\Gamma$ is compact. Both groups O_K and U act diagonally on $\mathbb{H}^s \times \mathbb{C}^t$ via the first $s + t$ embeddings: O_K acts by translations, while U acts by dilatations.

OT manifolds of type $(s, 1)$ are known to admit LCK metrics. But as they carry no holomorphic vector fields, they admit no Vaisman metrics. In fact, along with the blown-ups of LCK manifolds, these are the only known examples of LCK manifolds in higher dimension which admit no exact LCK metric, by a result of Otiman [O16]. Thus they are a good testing ground for conjectures concerning cohomological properties of LCK manifolds. Indeed, when introduced, they disproved a long standing conjecture of Vaisman, according to which the odd index Betti numbers of an LCK manifold should be odd.

So far, significant advances have been made in the study of OT manifolds. Many of their properties are closely related to the arithmetical properties of (K, U), as can be seen particularly in the papers of M. Parton and V. Vuletescu [PV12] and of O. Braunling [Bra17]. OT manifolds were shown to carry the structure of a solvmanifold by H. Kasuya [Kas13a], and those of type $(s,1)$ to contain no non-trivial complex submanifolds by L. Ornea and M. Verbitsky [OV11]. A delicate issue seems to be the existence of LCK metrics on OT manifolds which are not of type $(s,1)$. Some progress in this direction has been made by V. Vuletescu [Vu14] and A. Dubickas [Du14], but the question remains open in general.

In the present text, we are interested in the cohomological properties of OT manifolds. Their first Betti number and the second one for a certain subclass of manifolds, called *of simple type*, were computed in [OT05]. More recently, H. Kasuya computed in [Kas13b] the de Rham cohomology of OT manifolds of type $(s,1)$, using their solvmanifold structure. We will compute the de Rham cohomology algebra (Theorem 5.4.1) and the twisted cohomology (Theorem 5.6.1) of any OT manifold. This is done in terms of numerical invariants coming from $U \subset K$.

We do this by two different approaches. In order to explain them, let us first note the differentiable fiber bundle structures appearing in the construction of an OT manifold. $\hat{X} := \mathbb{H}^s \times \mathbb{C}^t / O_K$ has the structure of a trivial principal $\mathbb{T}^n$-bundle over $\mathbb{R}^s$, where $n = s + 2t$. This structure descends to X to a flat $\mathbb{T}^n$-fiber bundle structure over $\mathbb{T}^s$. Note that $\mathbb{T}^n$ acts on $\hat{X}$, but not on X. Our first approach consists in reducing to the study of the cohomology of $\mathbb{T}^n$-invariant differential forms. In the second one, we study the Leray-Serre spectral sequence associated to the fiber bundle structure of X, which turns out to degenerate at the second page.

In the rest of the chapter, we present a few applications, focusing on the OT manifolds of LCK type. First of all, we show:

Theorem I (Proposition 5.2.2)**.** *Let X be an OT manifold of LCK type. Then X admits only one Lee class.*

Next, we identify all the possible classes of LCK forms in the twisted cohomology group $H^2_\theta(X, \mathbb{R})$ on an OT manifold of LCK type (Corollary 5.7.8). As a consequence of this, we obtain that an LCK form (Ω, θ) on an OT manifold induces a non-degenerate Lefschetz map in cohomology, in the sense that $\mathrm{Lef}_\Omega : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$ is injective for $k \leq \dim_{\mathbb{C}} X$ and surjective for $k \geq \dim_{\mathbb{C}} X$.

Notation and conventions

- M, N will generally denote smooth manifolds.
- $\tilde{M}$ will always denote the universal cover of a manifold M.
- $\pi_1(M)$ will be the fundamental group of M, and Γ will usually denote some normal subgroup of $\pi_1(M)$ (or even $\pi_1(M)$ itself). These groups will be automatically identified with the deck groups of the associated coverings of M.
- We will denote by capital letters the compact Lie groups, such as G, H etc, and by lowercase Gothic letters their corresponding Lie algebras, i.e. $\mathfrak{g}$, $\mathfrak{h}$ etc.
- g will denote a Riemannian metric on a given manifold.
- Connections will be denoted by D, ∇, $\mathcal{D}$, and the curvature corresponding to a Chern connection, by Θ.
- I, J, K will denote complex structures on a given manifold. If we fix a complex structure J on a smooth manifold M, then we will sometimes use the notation M also for the complex manifold (M, J), when there is no ambiguity.
- K_M will denote the canonical bundle of a given complex manifold (M, J). $\mathcal{O}_M$ will denote the sheaf of holomorphic functions of M.
- Ω^k_M will denote the sheaf of holomorphic k-forms on a complex manifold (M, J), and $\mathcal{E}^{p,q}_M$ the sheaf of smooth (p, q)-forms.
- $\mathcal{E}^k_M$ will denote the sheaf of real-valued smooth k-forms on M, and $\mathcal{E}^k_M \otimes \mathbb{C}$ the sheaf of smooth $\mathbb{C}$-valued k-forms.
- Given a holomorphic vector bundle E over a complex manifold (M, J), $H^0(M, E)$ will denote the holomorphic sections of E. Its corresponding smooth sections will be denoted by $\mathcal{C}^\infty(M, E)$ or by $\Gamma(M, E)$. Also, by some abuse of notation, we will denote by $\mathcal{E}^{p,q}_M \otimes E$ or by $\mathcal{E}^{p,q}_M(E)$ the sheaf of (p, q)-forms on M valued in E.
- Let (L, h) be a Hermitian line bundle over (M, J), and let Θ_h denote the curvature of the induced Chern connection of L. We use the convention that $c_1(L)$, the first Chern class of L, is the de Rham cohomology class of $\frac{i}{2\pi}\Theta_h$. We will either view it in $H^2(M, \mathbb{R})$ or in $H^2(M, \mathbb{Z})$.
- $\mathbb{T}^n$ denotes the n-dimensional compact torus, seen as a real Lie group. We will denote by $\mathbf{T}$ a complex compact torus.
- For $X \in \Gamma(TM)$ a smooth vector field on a manifold M, ι_X denotes the contraction with X, while $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

In the context of Locally Conformally Kähler geometry:

- Ω will denote the LCK form. $\theta \in \mathcal{E}^1(M)$ will denote the Lee form corresponding to Ω, verifying $d\Omega = \theta \wedge \Omega$.

- $[\Omega]$ will denote the conformal class of Ω, that is the set $\{e^f\Omega | f \in \mathcal{C}^\infty(M)\}$, where M is the ambient manifold. Similarly, $[g]$ will denote the conformal class of a Riemannian metric g.

- B will denote the Lee vector field corresponding to Ω, defined by $\iota_B\Omega = J\theta$, and A the Reeb field $A = JB$, also defined by $\iota_A\Omega = -\theta$. Equivalently, if there exists a compatible complex structure on the given manifold and $g = \Omega(\cdot, J\cdot)$ is the corresponding Riemannian metric, then B and A are the duals of θ and $J\theta$ with respect to g.

- $\hat{M}$ will denote the minimal cover of (M, θ) on which θ becomes exact. φ will be a function on $\hat{M}$ or on $\tilde{M}$ satisfying $d\varphi = \theta$, and Ω_K will denote the symplectic form on $\hat{M}$ or on $\tilde{M}$ corresponding to Ω, defined by $\Omega_K = e^{-\varphi}\Omega$.

Chapter 1

Twisted Holomorphic Symplectic Forms

1.1 Introduction

This chapter is basically the content of [Is16], in which we are concerned with compact complex manifolds which admit a particular kind of structure: holomorphic non-degenerate 2-forms valued in a line bundle. Manifolds admitting such a structure will be called twisted holomorphic symplectic (THS). The problem has different analogues that have been intensively studied. On the one hand, there is the non-twisted problem concerning holomorphic symplectic forms. On the other hand, its symmetric avatar consists in the study of holomorphic (conformal) metrics.

In the compact setting, the class of complex manifolds of Kähler type admitting holomorphic symplectic forms coincides with the class of hyperkähler manifolds, as shown in [Bea83b]. There is a rich literature concerning this subject, and its study is ongoing. Turning to the symmetric counterpart, the situation is somewhat different. Although the class of compact Kähler manifolds admitting a holomorphic metric is rather small – they are all finitely covered by complex tori, as shown in [IKO80], as soon as one allows the structure to be twisted – thus studying holomorphic conformal structures – one enters a very rich class of manifolds. A complete classification of these has been reached only in dimension 2 and 3, in [KO82] and [JR05].

Even though one could expect that the class of THS manifolds is also wide, it turns out that the situation is not much different from the non-twisted case. More precisely, we show in Theorem 1.3.5 that compact THS manifolds of Kähler type are locally hyperkähler. In particular, the presence of such a structure ensures the existence of a Ricci-flat Kähler metric, and with respect to the connection induced by this metric the form is parallel.

Roughly speaking, the proof goes as follows: we first notice that the THS form induces local Lefschetz-type operators acting on the sheaves of holomorphic forms Ω^*, which then determine a local splitting of Ω^3 into Ω^1 and some other summand. This, in turn, allows us to find local holomorphic 1-forms which behave like connection forms on the line bundle where the twisted form takes its values. Finally, this means that the bundle admits a holomorphic connection, thus also a flat one, and that the manifold is Ricci-flat locally holomorphic symplectic, thus locally hyperkähler.

In the next section, we give a more precise description of THS manifolds. In Theorem 1.4.1

we show that they are finite cyclic quotients of hyperkähler manifolds. Then we investigate under which conditions a locally hyperkähler manifold admits a THS form. The two classes do not coincide, and this is essentially because locally hyperkähler manifolds behave well on products, while THS manifolds never do, as shown in Corollary 1.4.2. Still, for locally irreducible manifolds, the two classes coincide by Corollary 1.4.3. Finally, for the intermediate case of irreducible, locally reducible manifolds, a discussion depending on the compactness of the universal cover is done in the remaining part of Section 1.4. As a consequence, we also obtain that strict THS manifolds with finite fundamental group are necessarily projective.

1.2 Holomorphic symplectic manifolds

We start by discussing the complex symplectic case. For this, let us first define the objects we will be interested in:

Definition 1.2.1: A Riemannian manifold (M, g) is called *hyperkähler* if it admits three complex structures I, J and K which:

1. are compatible with the metric, i.e.

$$g(\cdot,\cdot) = g(I\cdot, I\cdot) = g(J\cdot, J\cdot) = g(K\cdot, K\cdot)$$

2. verify the quaternionic relations:

$$IJ = -JI = K$$

3. are parallel with respect to the Levi-Civita connection given by g.

In particular, a hyperkähler manifold is Kähler with respect to its fixed metric and any complex structure $aI + bJ + cK$, with a, b and c real constants verifying $a^2 + b^2 + c^2 = 1$.

Equivalently, we could say that a $4n$-dimensional Riemannian manifold (M, g) is hyperkähler iff its holonomy group is a subgroup of $\mathrm{Sp}(n)$.

Definition 1.2.2: A holomorphic 2-form on a complex manifold M, $\omega \in H^0(M, \Omega^2_M)$, is called a *holomorphic symplectic form* if it is nondegenerate in the following sense:

$$\iota_v \omega_x = 0 \Rightarrow v = 0, \quad \forall x \in M, \forall v \in T^{1,0}_x M,$$

where ι_v is the contraction with v.

We call a manifold admitting such a form a *holomorphic symplectic manifold.*

In particular, a holomorphic symplectic manifold (M, ω) has even complex dimension $2m$ and ω^m is a nowhere vanishing holomorphic section of the canonical bundle K_M=det Ω^1_M. Thus, K_M is holomorphically trivial and $c_1(M) = 0$.

It can be easily seen that, once we fix a complex structure on a hyperkähler manifold M, say I, there exists a holomorphic symplectic form ω on (M, I) defined by:

$$\omega(\cdot,\cdot) = g(J\cdot,\cdot) + ig(K\cdot,\cdot)$$

Thus, a hyperkähler manifold is a holomorphic symplectic manifold (but not in a canonical way). In the compact case, the converse is also true:

Theorem 1.2.3: *(Beauville, [Bea83b]) Let (M, I) be a compact complex manifold of Kähler type admitting a holomorphic symplectic form. Then, for any Kähler class $\alpha \in H^2(M, \mathbb{R})$, there exists a unique metric g on M which is Kähler with respect to I, representing α, so that (M, g) is hyperkähler.*

Moreover, the manifold (M, I) admits a metric with holonomy exactly $\mathrm{Sp}(m)$ if and only if it is simply connected and admits a unique holomorphic symplectic form up to multiplication by a scalar.

Remark 1.2.4: The existence and uniqueness of the Kähler metric representing the given Kähler class comes from Yau's theorem: it is exactly the unique representative in the class that has vanishing Ricci curvature. Consequently, the holomorphic symplectic form in the theorem is parallel with respect to the Levi-Civita connection given by this Ricci-flat metric.

1.3 Twisted holomorphic symplectic manifolds

We will now concentrate on the twisted case, and see that the situation is similar to the non-twisted one. Specifically, we will show that a Kähler manifold admitting a non-degenerate twisted holomorphic form admits a locally hyperkähler metric which is moreover Kähler for the given complex structure. With respect to the connection induced by this metric, the form will be parallel.

Definition 1.3.1: A Riemannian manifold (M^{4m}, g) is called *locally hyperkähler* if its universal cover with the pullback metric is hyperkähler or, equivalently, if the restricted holonomy group $\mathrm{Hol}^0(g)$ is a subgroup of $\mathrm{Sp}(m)$. If, moreover, the manifold admits a global complex structure I which is parallel with respect to the Levi-Civita connection induced by g, we will call it *Kähler locally hyperkähler*, or KLH for short.

Hence, a locally hyperkähler manifold is one which admits locally three orthogonal complex structures parallel for the Levi-Civita connection and verifying the quaternionic relations. It can be shown that in the case of a KLH manifold (M, g, I), one of these complex structures can be taken to be I, so that an equivalent definition for KLH is a Kähler manifold which admits two local parallel complex structures preserved by g which verify the quaternionic relations together with I.

Definition 1.3.2: Let (M, I) be a compact complex manifold, let L be a holomorphic line bundle over M. A non-degenerate L-valued holomorphic form

$$\omega \in H^0(M, \Omega^2_M \otimes L)$$

is called a *twisted holomorphic symplectic* form, or THS, and also the manifold with the endowed structure (M, I, L, ω) is called a THS manifold.

Remark 1.3.3: Like in the symplectic setting, the existence of a THS form implies that M is of even complex dimension $2m$. Moreover, ω^m is a nowhere vanishing holomorphic section of the line bundle $K_M \otimes L^m$. Thus, we have a holomorphic isomorphism $L^m \cong K_M^*$. In particular, any metric on M naturally induces one on L, and we also have

$$c_1(M) = mc_1(L).$$

Remark 1.3.4: Any complex surface M, of Kähler type or not, is THS in a tautological way. Simply take L to be K_M^*, so that $\Omega_M^2 \otimes L = K_M \otimes K_M^*$ is holomorphically trivial. Thus, any non-zero section of this bundle is a twisted-symplectic form, which in local holomorphic coordinates (z, w) on M is of the form

$$\omega = \lambda dz \wedge dw \otimes \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial w}, \quad \lambda \in \mathbb{C}.$$

Therefore, the class of THS manifolds is interesting only starting from complex dimension 4.

Our main result in this section is the following:

Theorem 1.3.5: *Let (M^{2m}, I, L, ω), $m > 1$, be a compact THS manifold of Kähler type, and let $\alpha \in H^2(M, \mathbb{R})$ be a Kähler class. Then there exists a unique Kähler metric g with respect to I representing α so that (M, g, I) is KLH. Moreover, L is unitary flat and ω is parallel with respect to the natural connection induced by g on L.*

Proof. Let $\{U_i\}_i$ be a trivializing open cover for the line bundle L and for each i, let $\sigma_i \in H^0(U_i, L)$ be a holomorphic frame, so that the holomorphic transition functions $\{g_{ij}\}_{ij}$ are given by $\sigma_i = g_{ij}\sigma_j$. Then, if we write over U_i

$$\omega = \omega_i \otimes \sigma_i$$

we get local holomorphic symplectic forms ω_i that verify, on $U_i \cap U_j$, $\omega_i = g_{ji}\omega_j$.

The ω_i's, being holomorphic, induce the morphisms of sheaves of $\mathcal{O}_{U_i}$-modules over U_i:

$$L_k : \Omega_{U_i}^k \to \Omega_{U_i}^{k+2}$$

$$L_k\alpha = \omega_i \wedge \alpha.$$

Lemma 1.3.6: *For $m > 1$ we have an isomorphism of sheaves of $\mathcal{O}_{U_i}$-modules:*

$$\Omega_{U_i}^3 \cong \Omega_{U_i}^1 \oplus \Omega_{0,U_i}^3$$

where Ω_{0,U_i}^3 is the sheaf $\operatorname{Ker}(L_3^{m-2} : \Omega_{U_i}^3 \to \Omega_{U_i}^{n-1})$ *and $n = 2m$.*

Proof. We claim that $L_1^{m-1} : \Omega_{U_i}^1 \to \Omega_{U_i}^{n-1}$ is an isomorphism of sheaves over U_i. We inspect this at the germ level, so we fix $z \in U_i$. Since the corresponding free $\mathcal{O}_z$-modules have the same rank, it suffices to prove the injectivity of $L_{1,z}^{m-1}$. But this becomes a trivial linear algebra problem, noting that we can always find a basis over $\mathbb{C}$ in $T^{1,0}M_z^*$ $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ so that

$$\omega_i(z) = \sum_{s=1}^{m} e_s \wedge f_s.$$

Next, since $L_1^{m-1} = L_3^{m-2} \circ L_1$ we get that L_1 is injective and L_3^{m-2} is surjective. Hence, we have an exact sequence of sheaves:

$$0 \longrightarrow \Omega_{0,U_i}^3 \longrightarrow \Omega_{U_i}^3 \xrightarrow{T} \Omega_{U_i}^1 \longrightarrow 0$$

where $T := (L_1^{m-1})^{-1} \circ L_3^{m-2}$. But T admits as a section $L_1 : \Omega_{U_i}^1 \to \Omega_{U_i}^3$, as $TL_1 = \mathsf{id}$. Thus, the sequence splits and we get the desired isomorphism. This ends the proof of the lemma. ∎

Now, we have $d\omega_i \in \Omega^3_M(U_i)$, so we can write:

$$d\omega_i = \omega_i \wedge \theta_i + \xi_i \tag{1.3.1}$$

with $\theta_i \in \Omega^1_M(U_i)$ and $\xi_i \in \Omega^3_{0,M}(U_i)$ holomorphic sections uniquely determined by the previous lemma. Since $\omega_i = g_{ji}\omega_j$, we get:

$$dg_{ji} \wedge \omega_j + g_{ji} d\omega_j = g_{ji}\omega_j \wedge \theta_i + \xi_i$$

whence

$$\omega_j \wedge \theta_j + \xi_j = d\omega_j = \omega_j \wedge \theta_i + \frac{1}{g_{ji}}\xi_i - \frac{dg_{ji}}{g_{ji}} \wedge \omega_j$$

Thus, applying again the previous lemma, we obtain that the θ_i's change by the rule:

$$\theta_i = \theta_j + d\log g_{ji}. \tag{1.3.2}$$

Hence, the differential operator $D : C^\infty(M, L) \to C^\infty(M, T^*M \otimes L)$ given over U_i by

$$D(f \otimes \sigma_i) = (df - \theta_i) \otimes \sigma_i$$

is a well defined connection on L. On the other hand, given some Hermitian metric h on L, its Chern connection D^h must differ from D by a linear operator:

$$D^h = D + A, \quad A \in C^\infty(T^*M \otimes \mathrm{End}L).$$

Moreover, since $D^{0,1} = (D^h)^{0,1} = \bar\partial_L$, A must be a global $(1,0)$-form on M.

Now $\Theta(D^h) = \Theta(D) + dA$, and since $\Theta(D)_{U_i} = -d\theta_i$ is of type (2,0) and $i\Theta(D^h)$ is a real (1,1)-form, we have that $i\Theta(D^h) = i\bar\partial A$ is exact in $H^{1,1}(M, \mathbb{R})$. But on a compact Kähler manifold $H^{1,1}(M, \mathbb{R}) \subset H^2_{dR}(M, \mathbb{R})$, so $2\pi c_1(L) = [i\Theta(D^h)] = 0 \in H^2_{dR}(M, \mathbb{R})$.

Thus we also get $c_1(M) = mc_1(L) = 0$. So, by Yau's theorem, there exists a unique Ricci-flat Kähler metric g whose fundamental form ω_g represents the given class α.

Now, on the sections of $\mathcal{E}^{2,0}_M \otimes L$ we have the Weitzenböck formula (see for instance [M]):

$$2\bar\partial^*\bar\partial = \nabla^*\nabla + \mathcal{R}$$

where ∇ is the naturally induced connection by g on $\mathcal{E}^{2,0}_M \otimes L$ and $\mathcal{R}$ is a curvature operator which on decomposable sections is given by:

$$\mathcal{R}(\beta \otimes s) = i\rho_g\beta \otimes s + \beta \otimes \mathrm{Tr}_{\omega_g}(i\Theta(L))s$$

with $\rho_g : \mathcal{E}^{2,0}_M \to \mathcal{E}^{2,0}_M$ the induced action of the Ricci form on $\mathcal{E}^{2,0}_M$. Now, since g is Ricci-flat, $\rho_g \equiv 0$. Also, if we consider the curvatures induced by g, we have:

$$0 = -i\rho = \Theta(K^*_M) = \Theta(L^m)$$

so the induced connection on L is flat and $\mathcal{R}$ vanishes.

Hence, applying the Weitzenböck formula to ω, we get $0 = \nabla^*\nabla\omega$ or also, after integrating over M, $\|\nabla\omega\|^2_{L^2} = 0$. Thus $\nabla\omega = 0$.

Finally, if we let $\pi : (\tilde{M}, \tilde{g}, \tilde{I}) \to (M, g, I)$ be the universal cover with the pullback metric and complex structure, we have that π^*L is holomorphically trivial and $\tilde{\omega} = \pi^*\omega \in H^0(\tilde{M}, \Omega^2_{\tilde{M}})$ is a holomorphic symplectic form. By the Cheeger-Gromoll theorem, $\tilde{M} \cong \mathbb{C}^l \times M_0$, where M_0 is compact, simply connected, Kähler, Ricci-flat, and $\mathbb{C}^l$ has the standard Kähler metric. Moreover, by the theorems of de Rham and Berger, the holonomy of M_0 is a product of groups of type $\mathrm{Sp}(k)$ and $\mathrm{SU}(k)$. We have that $\tilde{\omega}$ is a parallel section of

$$\bigwedge^2 T^*\tilde{M} = \bigwedge^2 \mathrm{pr}_1^*T^*\mathbb{C}^l \oplus (\mathrm{pr}_1^*T^*\mathbb{C}^l \otimes \mathrm{pr}_2^*T^*M_0) \oplus \bigwedge^2 \mathrm{pr}_2^*T^*M_0$$

But $\mathrm{pr}_1^*T^*\mathbb{C}^l \otimes \mathrm{pr}_2^*T^*M_0 \cong (T^*M_0)^{\oplus l}$ has no parallel sections by the holonomy principle, so $\tilde{\omega}$ is of the form $\omega_c + \omega_0$, with ω_c, ω_0 holomorphic symplectic forms on $\mathbb{C}^l$, M_0 respectively. Thus, l is even, so $\mathbb{C}^l$ is hyperkähler, and also, by Theorem 1.3, M_0 is hyperkähler. It follows that (M, g, I) is KLH.

This concludes the proof of the theorem. ■

Remark 1.3.7: Note that $D^{1,0}$ is actually a holomorphic connection on L, so this gives another reason of why L must be unitary flat.

Remark 1.3.8: The flat connection induced by g on L does not depend on the Kähler class α. It is uniquely determined by ω and is equal to the connection D given in the above proof. To see this, let D^g be the Chern connection on L induced by g and write $D^g\sigma_i = \tau_i \otimes \sigma_i$. Then we have:

$$0 = \nabla\omega = \nabla\omega_i \otimes \sigma_i + \omega_i \otimes \tau_i \otimes \sigma_i.$$

So, denoting by $a : \mathcal{E}^{2,0}_M \otimes (T^*M \otimes \mathbb{C}) \otimes L \to (\mathcal{E}^{3,0}_M \oplus \mathcal{E}^{2,1}_M) \otimes L$ the antisymmetrization map, we get:

$$d\omega_i = a(\nabla\omega_i) = -\omega_i \wedge \tau_i.$$

Thus, by (1.3.1) we deduce that $\xi_i = 0$ and $\tau_i = -\theta_i$, i.e. $D^g = D$.

Remark 1.3.9: If we only suppose that ω is a non degenerate $(2,0)$ twisted form, not necessarily holomorphic, then ω still induces a connection on L in the same manner. This time, we have the morphisms of sheaves of $\mathcal{E}_{U_i}$-modules $L_k : \mathcal{E}^{k,0}_{U_i} \to \mathcal{E}^{k+2,0}_{U_i}$ which induce isomorphisms $\mathcal{E}^{3,0}_M(U_i) \cong \mathcal{E}^{1,0}_M(U_i) \oplus \mathcal{E}^{3,0}_{0,M}(U_i)$. Writing

$$\mathcal{E}^{3,0}_M(U_i) \ni \partial\omega_i = \omega_i \wedge \theta_i + \xi_i,$$

we get the $(1,0)$-forms θ_i which define a connection D just as before. It is only at this point that the holomorphicity of ω becomes essential in order to have that D defines a holomorphic connection on L.

Actually, the complex manifolds which admit a non degenerate (2,0)-form valued in a complex line bundle are exactly those which have a topological $\mathrm{Sp}(m)\mathrm{U}(1)$ structure. As expected, these are not necessarily locally hyperkähler: a counterexample is given by the quadric $\mathbb{Q}_6 = \mathrm{SO}(7)/\mathrm{U}(3) \subset \mathbb{P}^7\mathbb{C}$, which is a Kähler manifold with topological $\mathrm{Sp}(3)\mathrm{U}(1)$ structure, see [MPS13], but is not KLH, since it has positive first Chern class.

Remark 1.3.10: Note that the Kähler hypothesis was heavily used during the proof. So one could ask two questions in the non-Kähler setting:

(1) Does it follow that a compact complex THS manifold has holomorphic torsion canonical bundle, so that ω determines a holomorphic symplectic form on some finite unramified cover of M?

(2) Which are the compact complex manifolds admitting a holomorphic symplectic form?

For the first question, the problem comes from cohomology. For a general compact complex manifold one can define many cohomologies (de Rham, Dolbeault, Bott-Chern, Aeppli) which are not necessarily comparable. In particular, one does not always have a map from $H^{1,1}_{\bar{\partial}}(M, \mathbb{C})$ to $H^2_{dR}(M, \mathbb{C})$. Since what we actually show is that $c_1(K_M)_{\bar{\partial}} = 0$, we cannot conclude that this Chern class vanishes in all other cohomologies (except for Aeppli). Moreover, even if it was the case, this would still not imply that K_M is holomorphically torsion, see [To15] for a detailed discussion and for examples showing the nonequivalence of the notions.

For Fujiki's class $\mathcal{C}$ manifolds, the answer is yes though. Since these manifolds satisfy the $\partial\bar{\partial}$-lemma, we can conclude that the first Chern class of the manifold vanishes in all cohomologies. We then use the result of [To15] stating that a Fujiki's class $\mathcal{C}$ manifold M with Bott-Chern class $c_1(M)_{BC} = 0$ has holomorphic torsion canonical bundle.

Examples of THS manifolds which do not verify the $\partial\bar{\partial}$-lemma can be given as follows: let S be a primary Kodaira surface. It admits a closed holomorphic symplectic form, thus also S^m does. Moreover, if $\Gamma = <\gamma> \subset \mathrm{Aut}(S)$ is a finite cyclic group so that $S/_\Gamma$ is a secondary Kodaira surface, then $S^m/_{<\gamma,\dots,\gamma>}$ is THS by Theorem 1.4.1 in the next section. Note that this manifold still has holomorphic torsion canonical bundle.

Regarding the second question, what we can say for sure is that the holomorphic symplectic class strictly contains the hyperkähler manifolds. Compact non-Kähler manifolds with holomorphic symplectic forms were constructed by Guan and Bogomolov (see [Gu94] and [Bo96]). A non-Kähler example with non-closed non-degenerate holomorphic form is given as follows: start with a global complex contact manifold, the Iwasawa 3-fold for instance, that is a complex manifold M^{2m+1} admitting a global holomorphic form $\eta \in H^0(M, \Omega^1_M)$ such that $\eta \wedge d\eta^m$ is nowhere zero. Let $\mathbf{T}$ be a 1-dimensional complex torus, and take on $X = M \times \mathbf{T}$ the form $\omega = d\eta + \theta \wedge \eta$, where θ is a generator of $H^0(\mathbf{T}, \Omega^1_{\mathbf{T}})$. Then ω is holomorphic symplectic and verifies $0 \neq d\omega = \theta \wedge \omega$. More examples can be constructed as complex *mapping tori* over M: let $f \in \mathrm{Aut}(M, \eta)$ be a contactomorphism, i.e. $f^*\eta = \eta$. Write $\mathbf{T} = \mathbb{C}/_\Lambda$, $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, and let Λ act on M by $1.x = x$ and $\tau.x = f(x)$. Then ω descends to $M_f := M \times_\Lambda \mathbb{C}$, which is again holomorphic symplectic. These examples are the holomorphic version of what is usually called locally conformally symplectic manifolds.

1.4 A characterization

In this section, we want to investigate the converse problem. It is not true that all KLH manifolds are twisted holomorphic symplectic. Already we will see that a product of strictly THS manifolds is never THS, but it turns out that being reducible is not the only obstruction. In what follows, we will give some description of THS manifolds and their fundamental groups.

By a strictly twisted holomorphic symplectic manifold we always mean a THS manifold (M, I, L, ω) such that the line bundle L is not holomorphically trivial.

Theorem 1.4.1: *A compact Kähler manifold M of complex dimension > 2 is THS if and only if there exists a holomorphic symplectic form ω_0 on its universal cover $\tilde{M}$ so that the* *action of* $\Gamma = \pi_1(M)$ *on* $H^0(\tilde{M}, \Omega^2_{\tilde{M}})$ *preserves* $\mathbb{C}\omega_0$*. In particular, any THS manifold is a finite cyclic quotient of a hyperkähler manifold.*

Proof. Suppose first that M admits a twisted-symplectic form

$$\omega \in H^0(M, \Omega^2_M \otimes L).$$

Then, by Theorem 1.3.5, L is unitary flat, and thus given by a unitary representation $\rho : \Gamma \to \mathrm{U}(1)$, i.e. if we see $\pi : \tilde{M} \to M$ as a Γ-principal bundle over M, we have $L = \tilde{M} \times_\rho \mathbb{C}$. Let $s_i : U_i \to \tilde{M}$ be local sections of $\pi : \tilde{M} \to M$ over a trivializing cover $\{U_i\}$. We then have $s_i = \gamma_{ij} s_j$ on $U_i \cap U_j$, where $\gamma_{ij} : U_i \cap U_j \to \Gamma$ are the transition functions for $\tilde{M}$. Then, $\sigma_i := [s_i, 1]$ are local frames for L, where $[\cdot, \cdot]$ denotes the orbit of an element of $\tilde{M} \times \mathbb{C}$ under the left action of Γ. The locally constant functions $g_{ij} := \rho(\gamma_{ij}^{-1})$ are the transition functions for L verifying

$$\sigma_i = [\gamma_{ij} s_j, 1] = [s_j, \rho(\gamma_{ij}^{-1})] = g_{ij}\sigma_j.$$

Since $\pi^* L$ is trivial, there exist $f_i \in \mathcal{O}^*_{\tilde{M}}(\pi^{-1}U_i)$ such that $\pi^* g_{ij} = \frac{f_i}{f_j}$ on $\pi^{-1}U_i \cap \pi^{-1}U_j$. Also, the sections $\frac{\pi^*\sigma_i}{f_i} \in H^0(\pi^{-1}U_i, \pi^* L)$ all coincide on intersections and are non vanishing, thus giving a global frame for $\pi^* L$ which we can suppose equal to 1, so that $\pi^*\sigma_i = f_i$. Thus, if we write $\omega = \omega_i \otimes \sigma_i$ and define $\omega_0 := \pi^*\omega$, we get:

$$\omega_0|_{\pi^{-1}U_i} = \pi^*\omega_i f_i$$

and, for any $\gamma \in \Gamma$:

$$\gamma^*\omega_0|_{\pi^{-1}U_i} = \pi^*\omega_i \gamma^* f_i = \omega_0 \frac{\gamma^* f_i}{f_i}$$

Moreover, for any γ, we have on $\pi^{-1}U_i \cap \pi^{-1}U_j$:

$$\frac{f_j}{f_i} = \frac{f_j \circ \gamma}{f_i \circ \gamma} \Leftrightarrow g_{ij} \circ \pi = g_{ij} \circ \pi \circ \gamma$$

hence the constant function $\frac{f_i \circ \gamma}{f_i}$ does not depend on i.
On the other hand, we have:

$$\frac{\gamma^* f_i}{f_i} = \frac{[s_i \circ \pi \circ \gamma, 1]}{[s_i \circ \pi, 1]} = \frac{[s_i \circ \pi, \rho(\gamma^{-1})]}{[s_i \circ \pi, 1]} = \frac{1}{\rho(\gamma)} \tag{1.4.1}$$

Hence Γ preserves the subspace $\mathbb{C}\omega_0 \subset H^0(\tilde{M}, \Omega^2_{\tilde{M}})$ and ρ is determined by the action of Γ on the holomorphic symplectic form ω_0 by:

$$\frac{1}{\rho(\gamma)} \cdot \omega_0 = \gamma^*\omega_0.$$

Conversely, suppose a holomorphic symplectic form ω_0 is an eigenvector for Γ acting on $H^0(\tilde{M}, \Omega^2_{\tilde{M}})$. Define

$$\rho : \Gamma \to \mathbb{C}^*$$
$$\gamma \mapsto \frac{\omega_0}{\gamma^*\omega_0}$$

Let $L := \tilde{M} \times_\rho \mathbb{C}$ and, with the same data for L as before, define $\omega \in H^0(M, \Omega^2_M \otimes L)$ by $\omega|_{U_i} = \omega_i \otimes \sigma_i$, where $\omega_i = s_i^* \frac{\omega_0}{f_i}$. Then ω is twisted holomorphic symplectic and, seeing $s_i\pi$ as an element of Γ, we have, by (3.1), on $\pi^{-1}(U_i)$:

$$\pi^*\omega = \frac{\pi^* s_i^* \omega_0}{\pi^* s_i^* f_i} f_i = \frac{1}{\rho(s_i\pi)} \omega_0 \frac{f_i}{(s_i\pi)^* f_i} = \omega_0.$$

To prove the last part, suppose M is THS and let, as in Theorem 1.3.5, $\tilde{M} = \mathbb{C}^{2l} \times M_1 \times \ldots \times M_k$, with M_i irreducible hyperkähler manifolds. The manifold M has a finite unramified cover $M' = \mathbf{T} \times M_1 \times \ldots \times M_k$, where $\mathbf{T}$ is a $2l$-dimensional compact complex torus, so that $M = M'/\Gamma'$ and $\Gamma \cong \mathbb{Z}^{4l} \ltimes \Gamma'$. The symplectic form ω_0 is preserved under the action of $\mathbb{Z}^{4l}$, so it descends to a holomorphic symplectic form on M', which we will also denote by ω_0. The group Γ' preserves $\mathbb{C}\omega_0$.

Let $\rho' : \Gamma' \to$U(1) be the representation induced by ρ. Denote by N' its kernel, and by $N := \mathbb{Z}^{4l} \ltimes N'$. Then N is normal inside Γ, so there exists a Galois covering $M_N \to M$ with $\pi_1(M_N) = N$. Moreover, since $\pi_1(M') = \mathbb{Z}^{4l}$ is normal in N, also $M' \to M_N$ is a covering whose deck transformation group is $N/\mathbb{Z}^{4l} \cong N'$.

We thus have that $M_N \cong M'/N'$ and N' preserves ω_0, so ω_0 descends to M_N. Since M_N is compact holomorphic symplectic, it is hyperkähler.

Finally, $\rho(\Gamma) = \rho'(\Gamma')$ is a finite subgroup of U(1), so cyclic, and $\Gamma/N \cong \Gamma'/N' \cong \rho(\Gamma)$, so M_N is a finite cyclic covering of M.

This concludes the proof of the theorem. ■

Corollary 1.4.2: *A compact strictly THS manifold of dimension > 2 is de Rham irreducible.*

Proof. Suppose $M \cong M_1 \times M_2$ is strictly twisted holomorphic symplectic. Let $M' \cong M_1' \times M_2'$ be a finite unramified cover of M with holomorphic symplectic form $\omega_0 = \omega_1 + \omega_2$ preserved up to constants by $\Gamma' \cong \Gamma_1' \times \Gamma_2'$, where $\pi_1(M_i) = \mathbb{Z}^{2l_i} \ltimes \Gamma_i'$, $i = 1, 2$.

Then we should have that $\rho(\Gamma') = \rho(\Gamma_1') \times \rho(\Gamma_2')$ is a non-trivial cyclic group of the same order as $\rho(\Gamma_1')$, $\rho(\Gamma_2')$, which is impossible. ■

Corollary 1.4.3: *A compact locally irreducible Kähler manifold of dimension > 2 is KLH if and only if it is THS. In this case, the twisted-symplectic form is valued in the canonical bundle.*

Proof. Let M be a locally irreducible KLH manifold and $\tilde{M}$ its universal cover endowed with a holomorphic symplectic form ω_0. Since $\tilde{M}$ is irreducible, it is compact and $H^0(\tilde{M}, \Omega^2_{\tilde{M}}) = \mathbb{C}\omega_0$. Hence $\Gamma = \pi_1(M)$ preserves $\mathbb{C}\omega_0$ in a trivial way and M is twisted holomorphic symplectic by the previous theorem.

In particular, this implies that Γ is cyclic. Let d be its order. Then $d|m+1$, where dim$M = 2m$. To see this, let $\gamma \in \Gamma$ be a generator, so that $\gamma^*\omega_0 = \xi \cdot \omega_0$, with ξ a primitive d-root of unity. Since γ has no fixed points, by the holomorphic Lefschetz fixed-point formula we must have that its Lefschetz number, which by definition is:

$$L(\gamma) = \sum_q (-1)^q \mathrm{tr}\gamma^*|_{H^q(\tilde{M}, \mathcal{O})}$$

must vanish. On the other hand, we have

$$\overline{H}^{\bullet}(\tilde{M},\mathcal{O}_{\tilde{M}})\cong H^0(\tilde{M},\Omega^{\bullet}_{\tilde{M}})\cong\frac{\mathbb{C}[\omega_0]}{(\omega_0^{m+1})}$$

so $L(\gamma)=1+\xi+\ldots+\xi^m$. Thus, $L(\gamma)=0$ implies $d|m+1$.

Let $\rho:\Gamma\to\mathrm{U}(1)$ be given by the action of Γ on ω_0 and $L:=\tilde{M}\times_{\bar{\rho}}\mathbb{C}$, so that the twisted holomorphic symplectic form is L-valued. Since the action of Γ on $K_{\tilde{M}}$ is given by ρ^m, we also have that $K_M=\tilde{M}\times_{\rho^m}\mathbb{C}$. Now, $\rho^{m+1}=1$ implies $\bar{\rho}^m\cdot\bar{\rho}=1$, or also $K_M^*\otimes L=\underline{\mathbb{C}}_M$, i.e. we have a holomorphic isomorphism $L\cong K_M$. ■

Remark 1.4.4: For a THS manifold (M,I,L,ω), we always have, by Remark 1.3.3, that L is a root of K_M^*. In the particular case when M is locally irreducible, we obtain, moreover, that L is precisely (up to isomorphism) K_M.

It is difficult to give a nice criterion for being THS in the case of de Rham irreducible, locally reducible KLH manifolds. We can, though, give a somewhat more precise description of fundamental groups of THS manifolds. For this, we first give some lemmas concerning isometries of Riemannian products.

Lemma 1.4.5: *Let $k>0$ and for each $1\leq i\leq k$ let (M_i,g_i) be a complete locally irreducible Riemannian manifold of dimension bigger than 1. Let $M_0=M_1\times\ldots\times M_k$ be endowed with the product metric. Let γ be an isometry of M_0 and let $\gamma_i:=p_i\gamma$, where $p_i:M_0\to M_i$ are the canonical projections. Then γ_i is of the form $\gamma_i=\tilde{\gamma}_i p_{\sigma(i)}$, where $\tilde{\gamma}_i:M_{\sigma(i)}\to M_i$ is an isometry and σ a permutation of $\{1,\ldots,k\}$.*

Proof. We have that $\tilde{g}_i:=\gamma_i^*g_i$ is a parallel section of $S^2(T^*M_0)$. On the other hand,

$$S^2(T^*M_0)\cong\sum S^2(T^*M_i)\oplus\sum_{i<j}(T^*M_i\otimes T^*M_j).$$

Now, $T^*M_i\otimes T^*M_j$ admits no parallel section for $i<j$, while the space of parallel sections of $S^2(T^*M_i)$ is exactly $\mathbb{R}g_i$. Indeed, by the holonomy principle, this is equivalent to saying that $G_i\times G_j$ has no fixed points when acting on $T_x^*M_i\otimes T_y^*M_j$, while the only G_i-invariant elements of $S^2(T_x^*M_i)$ are the multiples of $(g_i)_x$, where $x\in M_i$, $y\in M_j$ are any points and G_s is the restricted holonomy group of M_s, $s=1,...,k$. The first assertion follows from the dimension hypothesis and the more general fact that if U is a G-irreducible space and V a H-irreducible space, then $U\otimes V$ is a $G\times H$-irreducible space. The second assertion is equivalent to Schur's lemma if we identify $S^2(T_x^*M_i)$ with the symmetric endomorphisms of $T_x^*M_i$ via g_i.

Next, we want to show that for every i, there is exactly one $j=j(i)$ so that $a_{ij}\neq 0$. Thus, if we let $A(i)=\{j|a_{ij}\neq 0\}$, we need to show that $A(i)\neq\emptyset$ for each i and $A(i)\cap A(j)=\emptyset$ for all $i\neq j$. Now, since g_i is definite and $d\gamma_i$ is surjective, we have that $\ker\tilde{g}_i:=\{X\in TM_0|\tilde{g}_i(X,\cdot)=0\}=\ker d\gamma_i$. Hence, since $\ker d\gamma_i\neq TM$, the first assertion follows.

For the second assertion, first note that $\ker\tilde{g}_i\cap TM_k\neq 0$ if and only if $a_{ik}=0$, in which case $TM_k\subset\ker\tilde{g}_i$. Therefore, $(\ker d\gamma_i)^\perp=\sum_{j\in A(i)}TM_j$. Hence, for $i\neq j$, $A(i)\cap A(j)=\emptyset$ is equivalent to $\{0\}=(\ker d\gamma_i)^\perp\cap(\ker d\gamma_j)^\perp=(\ker d\gamma_i+\ker d\gamma_j)^\perp$. But we have

$$\ker d\gamma_i+\ker d\gamma_j=d\gamma^{-1}(\ker dp_i+\ker dp_j)=d\gamma^{-1}(\sum_{s\neq i}TM_s+\sum_{s\neq j}TM_s)=TM.$$

It follows that there exists a permutation σ of $\{1,\dots,k\}$ so that $A(i)=\{\sigma(i)\}$ for each i. Hence, since for any j, $\sum_i a_{ij}=1$, we have that $a_{i\sigma(i)}=1$ and $\gamma_i=\tilde{\gamma}_i p_{\sigma(i)}$ with $\tilde{\gamma}_i : M_{\sigma(i)} \to M_i$ an isometry. ■

In what follows, we will omit writing the projections and identify γ_i with $\tilde{\gamma}_i$.

Lemma 1.4.6: *Let $k>0$ and for $1\leq i\leq k$ let M_i be an irreducible compact hyperkähler manifold. Let $M_0=M_1\times\ldots\times M_k$ be endowed with the product metric and a holomorphic symplectic form ω_0. Then any isometry of M_0 preserving ω_0 has fixed points.*

Proof. By Theorem 1.2.3, each manifold M_i is simply connected and admits an unique holomorphic symplectic form ω_i up to a scalar, so we have:

$$H^0(M_0,\Omega^2_{M_0})=\mathbb{C}\omega_1\oplus\ldots\oplus\mathbb{C}\omega_k.$$

Hence we can suppose, after rescaling the ω_i's, that $\omega_0=\omega_1+\ldots+\omega_k$. Let γ be an isometry of M_0 with $\gamma^*\omega_0=\omega_0$.

Consider first the case where all the manifolds M_i are isometric, so that $M_0\cong M_1^k$. Let σ be the permutation determined by γ as in the previous lemma and let l be the order of σ. If we define, for $i=1,...,k$:

$$\gamma_i'=\gamma_i\gamma_{\sigma(i)}\cdots\gamma_{\sigma^{l-1}(i)}$$

then $\gamma^l(x_1,\dots,x_k)=(\gamma_1'(x_1),\dots,\gamma_k'(x_k))$ for any $(x_1,\dots,x_k)\in M_0$. If γ acts freely, then also γ^l acts freely. Otherwise, suppose $\gamma^l(y_1,\dots,y_k)=(y_1,\dots,y_k)$ for some $(y_1,\dots,y_k)\in M_0$. Let $i_1,\dots,i_t\in\{1,\dots,k\}$ represent the orbits of the group spanned by σ, of corresponding cardinals $l_1,\dots,l_t$, and define $(x_1,\dots,x_k)\in M_0$ by

$$x_{i_\alpha}:=y_{i_\alpha}\quad\text{and}\quad x_{\sigma^j(i_\alpha)}:=\gamma_{\sigma^j(i_\alpha)}\cdots\gamma_{\sigma^{l_\alpha-1}(i_\alpha)}(y_{i_\alpha})$$

for $\alpha=1,\dots,t$ and $j=1,\dots,l_\alpha-1$. The fact that

$$\gamma_{i_\alpha}\gamma_{\sigma(i_\alpha)}\cdots\gamma_{\sigma^{l_\alpha-1}(i_\alpha)}(y_{i_\alpha})=y_{i_\alpha}$$

implies that $(x_1,\dots,x_k)$ is a fixed point for γ, which is a contradiction.

Now, $\gamma^*\omega_0=\omega_0$ implies $(\gamma^l)^*\omega_0=\sum_i(\gamma_i')^*\omega_1=\omega_0$, or also $(\gamma_i')^*\omega_1=\omega_1$ for any $i=1,...,k$. On the other hand, the fact that γ^l acts freely implies that some γ_{i_0}' acts freely on M_1. By the holomorphic Lefschetz fixed-point formula, its Lefschetz number must then vanish. But $L(\gamma_{i_0}')=m+1$, where $\dim M_1=2m$, contradiction.

In the general situation, write $M_0=(M_1)^{k_1}\times\ldots\times(M_s)^{k_s}$, with M_i irreducible and $M_i\not\cong M_j$ for all $i\neq j$. By the previous lemma, $\gamma=(\gamma_1,\dots,\gamma_s)$, with γ_i an isometry of $(M_i)^{k_i}$. Again, $\gamma^*\omega_0=\omega_0$ implies $\gamma_i^*\tilde{\omega}_i=\tilde{\omega}_i$, where the $\tilde{\omega}_i$'s are the induced symplectic forms on $(M_i)^{k_i}$, $i=1,\dots,s$. Also, if γ acts freely on M_0, then some γ_i acts freely on $(M_i)^{k_i}$ and we already showed that this is impossible. ■

Remark 1.4.7: We can now say slightly more about THS manifolds M with compact universal cover $\tilde{M}$. In this case, with the notations of Theorem 1.3.5, $\Gamma=\Gamma'$, the representation ρ is faithful by the previous lemma, so $\Gamma=\rho(\Gamma)$ is cyclic. Thus, if γ is a generator of Γ of order d and $\gamma^*\omega_0=\xi\omega_0$, then ξ is necessarily a primitive d-root of unity. Moreover, if we write $\gamma=(\gamma_1,\dots,\gamma_k)$ just as in Lemma 1.4.5, then all γ_i's must have the same order d. To see this, let $d_i = \mathrm{ord}\gamma_i$. Then $d_i | d = \mathrm{lcm}(d_i)_i$. Since $\gamma^*\omega_0 = \sum_i \gamma_i^*\omega_i = \xi \sum_i p_i^*\omega_i$, we have, for all i, $\gamma_i^*\omega_i = \xi\omega_{\sigma(i)}$, hence $\xi^{d_i} = 1$. But ξ was primitive, so $d_i = d$. We can conclude:

Corollary 1.4.8: *If the fundamental group of a compact THS manifold is finite, then it is cyclic and of the form $\Gamma =< \gamma = (\gamma_1, \dots, \gamma_k) >$, with γ_i isometries of the irreducible components of the universal cover, all of the same order.*

Remark 1.4.9: When M is THS but $\tilde{M}$ is not compact, it is not necessarily the case that Γ' is cyclic, i.e. $\rho' : \Gamma' \to \mathrm{U}(1)$ need not be faithful. By the same type of arguments as in Lemma 1.4.5 and with the notations of Theorem 1.3.5, it can be seen that an element of Γ' is of the form $\gamma = (\gamma_T, \gamma_0)$, with $\gamma_T \in \mathrm{Aut}(\mathbf{T})$ and $\gamma_0 \in \mathrm{Aut}(M_0)$. There exist fixed point free complex symplectomorphisms of $\mathbf{T}$ of finite order (for instance translation by a torsion element $a \in \mathbf{T}$). So, if γ_T is one such symplectomorphism and γ_0 is a symplectomorphism of M_0 of the same order as γ_T, (γ_T, γ_0) is an element in the kernel of ρ'.

Corollary 1.4.10: *A compact strictly twisted holomorphic symplectic manifold M of dimension > 2 with finite fundamental group is projective.*

Proof. Let $\pi : \tilde{M} \to M$ be the compact universal covering, where, by Theorem 1.4.1, $\tilde{M} = M_1 \times \ldots \times M_k$ with M_i irreducible hyperkähler manifolds. Then, by Lemma 1.4.6, each M_i admits an automorphism which is not symplectic. By a result of A. Beauville [Bea83a, Proposition 6], such manifolds are necessarily projective, hence so is $\tilde{M}$. But it is a well known fact that a compact Kähler manifold is projective if and only if some finite unramified covering is, thus the conclusion follows. ∎

1.5 Examples

Concerning examples of Kähler type, finding locally irreducible KLH manifolds is equivalent to finding a fixed point free automorphism γ of an irreducible symplectic manifold, so that all powers of γ also act freely.

In complex dimension 2, by Remark 1.3.4 all manifolds are twisted holomorphic symplectic. On the other hand, the only finite cyclic quotients of hyperkähler surfaces are the Enriques surfaces and some bielliptic surfaces. The first ones are quotients $K/_{<\iota>}$, with K a K3 surface admitting a fixed point free involution ι, and they are locally irreducible. The bielliptic surfaces are quotients of products of two elliptic curves by some groups of order 2, 3, 4 or 6. This shows that Theorem 1.4.1 does not hold in complex dimension 2.

Next, one can easily construct locally reducible THS manifolds of any dimension by iterating the KLH examples from above. For instance, if (K, ι) is a K3 surface as before, then $K^m/_{<\iota,\dots,\iota>}$ is THS. In the same way, let $\mathbf{T} = \mathbb{C}/_{(\mathbb{Z}\oplus i\mathbb{Z})}$ and let P be a 4-torsion point on $\mathbf{T}$. Denote by $(x_1, \dots, x_m, y_1, \dots, y_m)$ the local holomorphic coordinates on $\mathbf{T}^{2m}$ coming from the standard coordinates on $\mathbb{C}^{2m}$ and define $f \in \mathrm{Aut}(\mathbf{T}^{2m})$ by:

$$f([x_1, \dots, y_m]) := ([ix_1, \dots, ix_m], [y_1, \dots, y_m] + (P, \dots, P)).$$

Then f spans a group of order 4 acting freely on $\mathbf{T}^{2m}$ and $\mathbf{T}^{2m}/_{<f>}$ is a THS manifold, with twisted symplectic form given locally by $\sum_{j=1}^m dx_j \wedge dy_j$. More generally, if M_1 and M_2 are hyperkähler and $\gamma_i \in \mathrm{Aut}(M_i)$, $i = 1, 2$, have the same order, with $< \gamma_1 >$ acting freely but $< \gamma_2 >$ possibly not, then $(M_1 \times M_2)/_{<\gamma_1,\gamma_2>}$ is again a smooth THS manifold.

Finding locally irreducible THS manifolds of higher dimension is more difficult. For the Hilbert schemes of points on K3 surfaces, see [Bea83b] for the construction, all known automorphisms of order bigger thatn 2 have fixed points, so we have no hope of constructing examples out of them. On the other hand, there is hope with the generalized Kummer varieties K_r, see again [Bea83b] for the definition. In [BNS11] and [OS11] the authors find fixed point free cyclic groups of automorphisms Γ of order 3 for the manifolds K_2 and K_5, and of order 4 for K_3. The corresponding quotients give the desired examples of dimension 4, 10 and 6, respectively.

Chapter 2

Locally Conformally Kähler Geometry

2.1 Introduction

This chapter puts together the definitions and the basic results in locally conformally Kähler (LCK) geometry which are relevant for the thesis. Most of the chapter consists in already known facts, although sometimes presented in a different way, except perhaps the last part of example 2.6.3.

We start by giving all the equivalent definitions of an LCK structure in Section 2.2. Although we do not discuss the locally conformally symplectic (LCS) structures, the obvious adaptation of all the given definitions are valid in this context as well. Next, in Section 2.3 we present the relevant linear connections appearing in the context of LCK geometry, as well as the relation between all of them.

In Section 2.4 we move on to introducing particular classes of LCK metrics, namely Vaisman metrics, LCK metrics with (positive) potential and exact LCK metrics. We discuss the relation between these classes, as well as their important or elementary properties. This section is important for the rest of the text, as some of our main results concern special LCK metrics.

Also essential for the discussion that will follow, namely in Chapter 4, is Section 2.5 concerning the properties of the automorphism group of an LCK/LCS manifold. In particular, in this section we give a criterion for lifting the action of a compact Lie group on a compact LCS manifold to its minimal symplectic cover (Proposition 2.5.4).

In Section 2.6 we present the main examples and known methods of constructing LCK manifolds. We start by giving the metric construction of diagonal Hopf manifolds and non-diagonal Hopf surfaces. This is well known and follows the ideas of Gauduchon-Ornea [GO98] and Belgun [Bel00]. However, we chose to give all the details, in a way that clearly generalizes to the next example we present, namely LCK manifolds obtained form ample vector bundles over Kähler manifolds. This last construction generalizes the one given by Tsukada [Ts97]: given any linear Hopf manifold or non-diagonal Hopf surface H, one chooses suitable line bundles over a projective manifold in order to form an LCK manifold fibering in H over the projective manifold. Lastly, we discuss shortly the LCK metric structure of complex surfaces, as well as of blown-up manifolds.

2.2 Basic definitions and properties

We start by introducing the equivalent definitions of a locally conformally Kähler structure. Let (M, J) be a complex n-dimensional manifold and let $\Omega \in \mathcal{E}_M^{1,1}(M)$ be a non-degenerate $(1,1)$-form on M which is positive, meaning that $g(\cdot,\cdot) := \Omega(\cdot, J\cdot)$ is a J-invariant Riemannian metric.

Definition 2.2.1: The form Ω is called *locally conformally Kähler*, abbreviated *LCK*, if there exists a covering of M with open sets $\{U_\alpha\}_{\alpha\in I}$ such that Ω restricted to each of them is conformal to a Kähler form:

$$\Omega|_{U_\alpha} = \mathrm{e}^{\varphi_\alpha}\Omega_\alpha \text{ with } d\Omega_\alpha = 0 \tag{2.2.1}$$

where $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R})$ and $\Omega_\alpha \in \mathcal{E}_M^{1,1}(U_\alpha)$, for any $\alpha \in I$.
Let Ω be an LCK form. By differentiating the relation (2.2.1), we obtain:

$$d\Omega = d\varphi_\alpha \wedge \Omega \quad \text{on } U_\alpha\ \forall \alpha \in I.$$

Suppose for the moment that $n > 1$. In this case, the morphism $\mathcal{E}_M^1(M) \to \mathcal{E}_M^3(M)$, $\eta \mapsto \eta \wedge \Omega$ is injective as Ω is non-degenerate, hence the above implies that on the intersections $U_\alpha \cap U_\beta$ we have $d\varphi_\alpha = d\varphi_\beta$. Thus the collection $\{d\varphi_\alpha\}_{\alpha\in I}$ glues up to give a real closed 1-form θ globally defined on M which verifies $d\Omega = \theta \wedge \Omega$.
For $n = 1$, the above argument does not work, and in fact any two $(1,1)$-forms, locally or globally defined on M, are conformal. Nonetheless, for any real 1-form θ on M (and there always exist some), the relation $d\Omega = \theta \wedge \Omega$ is trivially satisfied, since both terms are 0 because of their degree.
Conversely, suppose that θ is a closed 1-form on M satisfying $d\Omega = \theta \wedge \Omega$. By the Poincaré lemma, M is covered by open sets $\{U_\alpha\}_{\alpha\in I}$ on which θ becomes exact: $\theta = d\varphi_\alpha$, $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R})$. Setting then $\Omega_\alpha := \mathrm{e}^{-\varphi_\alpha}\Omega$ on U_α, one verifies easily that this is a closed form on U_α conformal to Ω, and so Ω is LCK. Thus we also have:

Definition 2.2.2: The form Ω is called LCK if there exists a real closed 1-form θ on M, called the *Lee form*, satisfying:

$$d\Omega = \theta \wedge \Omega. \tag{2.2.2}$$

Remark 2.2.3: It's worth mentioning that, for $n > 2$, the morphism $\mathcal{E}^2(M) \to \mathcal{E}^4(M)$ given by wedging with Ω is injective, hence differentiating the relation (2.2.2) gives us $d\theta = 0$ automatically.

In order to get to other equivalent definitions, we need to relate the Lee form to a connection in a real line bundle over M. To do this, let us first note that, by the Universal Coefficient Theorem and the fact that $H_1(M, \mathbb{Z})$ is the abelianisation of $\pi_1(M)$, we have the following group isomorphisms:

$$H^1(M, \mathbb{R}) \cong \mathrm{Hom}(H_1(M, \mathbb{R}), \mathbb{R}) \cong \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{R}) \cong \mathrm{Hom}(\pi_1(M), \mathbb{R}).$$

The isomorphism between the first and the last group is given by taking the de Rham class of a closed 1-form η to the homomorphism $\gamma \in \pi_1(M) \mapsto \int_\gamma \eta$, which of course does not depend on the chosen representative η. Moreover, if we identify $\pi_1(M)$ with the deck group of $\pi : \tilde{M} \to M$ acting on the universal cover $\tilde{M}$, this homomorphism is the same as:

$$\begin{aligned} \tau : \pi_1(M) &\to (\mathbb{R}, +) \\ \gamma &\mapsto \gamma^*\varphi - \varphi. \end{aligned} \tag{2.2.3}$$

where $\pi^*\eta = d\varphi$. Again, this homomorphism depends neither on η, nor on the chosen primitive φ.

At the same time, the group $\mathrm{Hom}(\pi_1(M), \mathbb{R})$ is in one to one correspondence to the set of the isomorphism classes of oriented real line bundles with flat connection over M, given in the following way. Let $\tau : \pi_1(M) \to (\mathbb{R}, +)$ be a homomorphism and let $\rho = \exp\tau : \pi_1(M) \to \mathbb{R}_{>0}$. Then we define the real line bundle over M:

$$L_\tau := \tilde{M} \times_\rho \mathbb{R} = \tilde{M} \times \mathbb{R}/\pi_1(M)$$

where $\gamma \in \pi_1(M)$ acts on $(\tilde{x}, v) \in \tilde{M} \times \mathbb{R}$ by $(\gamma(\tilde{x}), \rho(\gamma)v)$. As before, let us represent τ by a closed form η, and on $\tilde{M}$ we choose a primitive $\varphi \in \mathcal{C}^\infty(\tilde{M}, \mathbb{R})$ for $\pi^*\eta$. Then $\tilde{\sigma}(\tilde{x}) := (\tilde{x}, \mathrm{e}^{\varphi(\tilde{x})})$ descends to a trivialising section of $L_\tau \to M$, denoted by σ, and we define a flat connection ∇ in L_τ by setting $\nabla\sigma = \eta \otimes \sigma$. Adding an exact form to η or a locally constant function to φ does not change the isomorphism class of (L_τ, ∇).

Suppose now (Ω, θ) is an LCK structure on M, and let us pull back all structures from M to $\tilde{M}$ and denote them respectively with $\tilde{\cdot}$. Clearly $\tilde{\Omega}$ is still an LCK form on $(\tilde{M}, \tilde{J})$ verifying $d\tilde{\Omega} = \tilde{\theta} \wedge \tilde{\Omega}$. If we take $\varphi \in \mathcal{C}^\infty(\tilde{M}, \mathbb{R})$ such that $\tilde{\theta} = d\varphi$, then $\Omega_K := \mathrm{e}^{-\varphi}\tilde{\Omega}$ is Kähler on $\tilde{M}$. Let us denote, as before, by τ the morphism (2.2.3), by $\rho = \mathrm{e}^\tau$ and by (L_τ, ∇) the associated flat line bundle. The group $\pi_1(M)$ acts on Ω_K by homotheties:

$$\gamma^*\Omega_K = \rho(\gamma)^{-1}\Omega_K.$$

This equivariance tells us that Ω_K can be identified with a smooth positive section $\omega = \Omega \otimes \sigma^{-1}$ of $\mathcal{E}_M^{1,1} \otimes L_\tau^*$ which is d_∇-closed. Here, d_∇ is the differential operator acting on $\mathcal{E}_M^\bullet \otimes L_\tau^*$ induced by the flat connection $\nabla^{L_\tau^*}$, and the positivity of ω means that in any positively oriented trivialisation of L_τ^*, ω gives a positive $(1,1)$-form.

Definition 2.2.4: An LCK structure on (M, J) is given by a Kähler form Ω_K on $(\tilde{M}, \tilde{J})$ on which $\pi_1(M)$ acts by homotheties.

Definition 2.2.5: An LCK structure on (M, J) is given by an oriented flat real line bundle (L, ∇) over M and a positive section $\omega \in \mathcal{E}_M^{1,1}(M, L)$ which is d_∇-closed.

Remark 2.2.6: If we forget about the complex structure of M, and so also about the condition of positivity of Ω, then the above definitions introduce the notion of a *locally conformally symplectic (LCS)* structure. We will not be interested in these structures in the present text, and for the sake of continuity we will not make any remarks or adaptations concerning them, but indeed some of the facts that we state for LCK structures apply, more generally, to LCS structures.

Given an LCK structure (Ω, θ), there are two distinguished smooth vector fields on M, B and $A = JB$ defined by:

$$\iota_B\Omega = J\theta, \quad \iota_A\Omega = -\theta. \tag{2.2.4}$$

Usually, B is called *the Lee vector field*, as it is the metric dual of the Lee form, and A is called *the Reeb vector field*, and is the metric dual of $J\theta$. Moreover, because $d\theta = 0$, A is always an infinitesimal symplectomorphism:

$$\mathcal{L}_A\Omega = -d\theta + \iota_A(\theta \wedge \Omega) = \theta \wedge \theta \wedge \Omega = 0. \tag{2.2.5}$$

Remark 2.2.7: Let us note that, because $d\theta = 0$, the form $dJ\theta$ is a real $(1,1)$-form. Indeed, let $\theta = \theta^{1,0} + \theta^{0,1}$ be the decomposition of θ in its $(1,0)$ and $(0,1)$ parts, where $\theta^{1,0} = \frac{1}{2}(\theta + iJ\theta)$. As $d\theta = 0$, it follows that $0 = (d\theta)^{2,0} = \partial\theta^{1,0}$, hence $d\theta^{1,0} = \bar{\partial}\theta^{1,0} \in \mathcal{E}_M^{1,1}(M)$. Thus $d\theta = 0$ implies that $dJ\theta = \frac{2}{i}d\theta^{1,0} \in \mathcal{E}_M^{1,1}(M)$. In fact, as we will see soon, this form plays a special role in LCK geometry.

If (Ω, θ) is an LCK structure on M such that the form θ is exact on M: $\theta = d\varphi$, $\varphi \in \mathcal{C}^\infty(M, \mathbb{R})$, then $\Omega_\varphi := \mathrm{e}^{-\varphi}\Omega$ is closed. Hence Ω is globally conformal to a Kähler metric (also abbreviated *GCK*). Conversely, suppose that Ω is GCK. If $n > 1$, then one easily sees that the form θ is exact by the same argument as before, but for $n = 1$, this is clearly false. From the above discussion, we see that the GCK condition for Ω is equivalent also to the corresponding flat line bundle (L, ∇) being isomorphic to $(M \times \mathbb{R}, d)$ for $n > 1$.

We will sometimes call an LCK structure which is not GCK *strict*, but later we will only consider this type of structure, and so omit the word strict. In fact, an early result in LCK geometry, due to I. Vaisman [Va80], states that for an LCK structure on a compact complex manifold which is not a Riemann surface, being strict is equivalent to the manifold being non-Kählerian.

Theorem 2.2.8: *(Vaisman, [Va80]) Let (M, J, Ω) be a compact complex manifold of complex dimension $n > 1$ endowed with an LCK structure. If M verifies the $\partial\bar{\partial}$-lemma for real $(1,1)$-forms, then Ω is GCK.*

Proof. The proof consists in two assertions:

<u>Fact 1</u>: If M verifies the $\partial\bar{\partial}$-lemma for real $(1,1)$-forms, then there exists an LCK form Ω' conformal to Ω whose corresponding Lee form θ' verifies $dJ\theta' = 0$.

We have $dd^c = 2i\partial\bar{\partial}$, and the $\partial\bar{\partial}$-lemma applied to $dJ\theta$, which is a real $(1,1)$-form, gives us the existence a smooth real function φ on M verifying $dJ\theta = dd^c\varphi$. Define the LCK form $\Omega' := \mathrm{e}^{-\varphi}\Omega$, with corresponding Lee form $\theta' = \theta - d\varphi$. We then have:

$$dJ\theta' = dJ\theta - dd^c\varphi = 0.$$

<u>Fact 2</u>: If an LCK metric (Ω, θ) on a compact manifold verifies $dJ\theta = 0$, then $\theta = 0$.

Suppose that $\theta \neq 0$. Since the form $\theta \wedge J\theta$ is always semi-positive, so is $\theta \wedge J\theta \wedge \Omega^{n-1}$, so the integral $\int_M \theta \wedge J\theta \wedge \Omega^{n-1}$ is a strictly positive number. On the other hand, we have:

$$0 < \int_M \theta \wedge J\theta \wedge \Omega^{n-1} = -\int_M J\theta \wedge (\frac{1}{n-1}d(\Omega^{n-1})) = -\frac{1}{n-1}\int_M dJ\theta \wedge \Omega^{n-1} = 0 \tag{2.2.6}$$

where we used Stokes' formula and the last equality follows by hypothesis, and so we arrive at a contradiction. ■

Let (M, J, Ω, θ) be a compact LCK manifold, and consider the associated morphism $\tau : \pi_1(M) \to \mathbb{R}$. Let Γ_0 be its kernel, which is a normal subgroup of $\pi_1(M)$, hence there exists a

Galois covering $\hat{M} \to M$ with deck group $\Gamma = \pi_1(M)/\Gamma_0$. Note that $\Gamma \cong \operatorname{Im}\tau \subset \mathbb{R}$, hence it is a free abelian group, isomorphic to $\mathbb{Z}^k$ for some $k \in \mathbb{N}$. The rank k is sometimes called *the LCK rank* of (Ω, θ). By construction, $\hat{M}$ is the minimal covering space on which θ becomes exact, and hence on which Ω becomes GCK. Alternatively, Γ_0 is the maximal subgroup of $\pi_1(M)$ acting trivially on the corresponding Kähler form Ω_K, which then descends to $\hat{M}$ and that we also denote by Ω_K. We will call $(\hat{M}, \Omega_K)$ the minimal Kähler cover of (M, Ω). Note that, if Ω is strict LCK, neither the minimal, nor the universal covering spaces are compact. Moreover, it is shown in [BM16] that the corresponding Kähler metric is never complete, and the metric completion of $(\hat{M}, \Omega_K)$ adds exactly one point to $\hat{M}$.

Any closed 1-form θ on M induces an integrable differential operator:

$$\begin{aligned} d_\theta : \mathcal{E}^k(M) &\to \mathcal{E}^{k+1}(M) \\ d_\theta \eta &= d\eta - \theta \wedge \eta. \end{aligned} \tag{2.2.7}$$

Indeed, $d\theta = 0$ implies $d_\theta^2 = 0$. However one should note that d_θ does not satisfy Leibniz' rule, in the sense that $d_\theta(\alpha \wedge \beta) \neq d_\theta \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_\theta \beta$ for $\alpha, \beta \in \mathcal{E}^\bullet(M)$. If Ω is an LCK form with Lee form θ, then one has $d_\theta \Omega = 0$. A simple, but very useful fact in LCK geometry, is the following lemma. We give here the proof of [MMP17], but one can also look for a slightly different one in [Va85].

Lemma 2.2.9: *Let M be a connected differential manifold and θ a real-valued closed 1-form on M. Then $\ker d_\theta \subset \mathcal{C}^\infty(M)$ is trivial if and only if θ is not exact.*

Proof. Clearly, if $\theta = d\varphi$, with $\varphi \in \mathcal{C}^\infty(M)$, then e^φ is in the kernel of d_θ. Conversely, suppose that there exists $f \in \mathcal{C}^\infty(M)$ so that

$$df = f\theta \tag{2.2.8}$$

on M. If f has some zero at $x \in M$, then as (2.2.8) is a linear first order differential system, the initial condition $f(x) = 0$ determines its unique solution $f = 0$. Otherwise, f never vanishes, but in this case (2.2.8) implies $\theta = d \ln |f|$, so we are done. ∎

Let us note at this point that the notion of LCK structure is a conformal one. More precisely, if Ω is an LCK form, then clearly also $\Omega_f := e^f \Omega$ is. Moreover, if θ is the Lee form of Ω, then $\theta_f := \theta + df$ is the Lee form of Ω_f. In particular, the de Rham class of the Lee form is invariant under conformal changes of the metric, and so are the minimal and universal coverings with the Kähler metrics, as well as the isomorphism class of (L, ∇) corresponding to Ω. For this reason, we will sometimes call an LCK structure the conformal class of an LCK form Ω, denoted by $[\Omega]$, instead of the form itself.

2.3 Connections

On an LCK manifold (M, J, g, Ω) there are three distinguished affine connections which, unlike in the Kähler case, do not coincide: the Levi-Civita connection, the standard Weyl connection and the Chern connection. Recall that the first one is the unique torsion free affine connection preserving the metric g, denoted by ∇^g. The second one is the unique torsion free affine connection preserving the conformal class $[g]$ as well as J, denoted by $\nabla^{[g]}$. Finally, the Chern connection is the unique $\mathbb{C}$-linear connection on the holomorphic vector bundle TM whose

$(0,1)$-part coincides with $\bar{\partial}$ and which preserve the Hermitian structure $h := g - i\Omega$, and we will denote it by D.

On the other hand, M is covered by open sets $\{U_\alpha\}_{\alpha\in I}$ so that $g = \mathrm{e}^{\varphi_\alpha} g_\alpha$ and g_α is Kähler on each (U_α, J). This time, the Levi-Civita connection of (U_α, J, g_α), denoted by ∇^α, coincides with the Chern connection, as well as with the standard Weyl connection preserving J, both corresponding to (U_α, J, g_α). Thus, the relation between the three connections of the LCK structure will be given by comparing them to the connection ∇^α corresponding to g_α.

Recall first that the Levi-Civita connections of two conformal metrics h and $\tilde{h} = \mathrm{e}^f h$ relate via the formula:

$$\nabla^{\tilde{h}}_X Y = \nabla^h_X Y + \frac{1}{2}(df(X)Y + df(Y)X - h(X,Y)\mathrm{grad}_h f).$$

If we apply this to our context and remark that $d\varphi_\alpha = \theta$ is globally defined on M, as well as $\mathrm{grad}_g \varphi_\alpha = B$, we obtain that the connections $\{\nabla^\alpha\}_{\alpha\in I}$ glue up to a globally defined torsion free connection $\mathcal{D}$ on M which preserves J, given by the relation:

$$\mathcal{D}_X Y = \nabla^g_X Y - \frac{1}{2}(\theta(X)Y + \theta(Y)X - g(X,Y)B). \tag{2.3.1}$$

Moreover, on any U_α we have:

$$\mathcal{D}g = \mathcal{D}(\mathrm{e}^{\varphi_\alpha} g_\alpha) = d\varphi_\alpha \otimes g + \mathrm{e}^{\varphi_\alpha}\mathcal{D}g_\alpha = \theta\otimes g \tag{2.3.2}$$

meaning that $\mathcal{D}$ preserves also the conformal class $[g]$. Therefore, $\mathcal{D}$ is in fact the standard Weyl connection $\nabla^{[g]}$. Finally, as $\mathcal{D}$ preserves J, we obtain the following formula for $\nabla^g J$:

$$\begin{aligned}\nabla^g_X J &= \frac{1}{2}(\theta(X)J - J\theta\otimes X - g(X, J\cdot)B - \theta(X)J - \theta\otimes JX + g(X,\cdot)A)\\ &= \frac{1}{2}(-J\theta\otimes X - \theta\otimes JX + g(X,\cdot)A + \Omega(X,\cdot)B)\end{aligned} \tag{2.3.3}$$

from which it follows that $\nabla^g_A J = 0$ and $\nabla^g_B J = 0$. Equivalently, this formula writes for Ω:

$$\nabla^g_X\Omega = g(X,\cdot)\wedge J\theta + \iota_X\Omega\wedge\theta.$$

Let us now compare the standard Weyl and the Chern connections of g. In this case, we think of TM as a holomorphic vector bundle over M, and up to shrinking them, we can suppose that the open sets U_α trivialise holomorphically TM. Let $h := g - i\Omega$ be the induced hermitian structure on TM, fix $\alpha\in I$ and let also $h_\alpha = \mathrm{e}^{-\varphi_\alpha}h$ be the hermitian structure induced by g_α on TU_α. Let $\sigma = \{\sigma_1,\ldots,\sigma_n\}$ be a holomorphic frame of TM over U_α. Recall that if we denote by H the $n\times n$ matrix of functions $(h(\sigma_j,\sigma_k))_{j,k=1,n}$, then the Chern connection of h with respect to σ is given by $D_\sigma = \overline{H}^{-1}\partial\overline{H}$. Thus, if we also denote by H_α the matrix $(h_\alpha(\sigma_j,\sigma_k))_{j,k=1,n}$, we have:

$$\begin{aligned}D_\sigma &= (\mathrm{e}^{-\varphi_\alpha})\overline{H}_\alpha^{-1}\partial(\mathrm{e}^{\varphi_\alpha}\overline{H}_\alpha)\\ &= \theta^{1,0}\otimes\mathsf{id} + \nabla^\alpha_\sigma.\end{aligned}$$

In particular, the difference $D_\sigma - \nabla^\alpha_\sigma$ does not depend on the trivialisation, and so we have:

$$D = \theta^{1,0}\otimes\mathsf{id}_{TM} + \mathcal{D}. \tag{2.3.4}$$

Finally, let us recall that for an LCK manifold (M, J, Ω, θ) we have a naturally associated line bundle with flat connection (L, ∇_L) corresponding to $\theta \in \mathcal{E}^1_M(M)$, so that Ω is L^*-valued. Now take the complex line bundle $L^c = L \otimes \mathbb{C}$ with the connection ∇_L extended by $\mathbb{C}$-linearity, and define the differential operator $\bar{\partial}_L : \mathcal{C}^\infty(M, L^c) \to \mathcal{E}^{0,1}_M(M,\ L^c)$ acting on smooth sections of L^c by:

$$\bar{\partial}_L := \nabla^{0,1}_L = \bar{\partial} + \theta^{0,1} \wedge \cdot.$$

Note that, as $d\theta = 0$, also $\bar{\partial}\theta^{0,1} = 0$, hence $\bar{\partial}_L \circ \bar{\partial}_L = 0$, which implies that $\bar{\partial}_L$ induces a holomorphic structure on L^c. We will denote by $\mathcal{L}$ the associated holomorphic line bundle. Next, we also have a natural hermitian product h on $\mathcal{L}$. Indeed, recall that σ, defined on $\tilde{M}$ by $\sigma(x) = (x, \mathrm{e}^{\varphi(x)})$, where $d\varphi = \theta$, is a smooth trivializing section of L. Thus, we can define h so that σ is orthonormal, i.e. for a smooth section $s = f\sigma$ of $\mathcal{L}$, with $f \in \mathcal{C}^\infty(M, \mathbb{C})$, we put $h(s, s) := |f|^2$. The Chern curvature of the associated Chern connection of $(\mathcal{L}, h)$ can easily be computed. Before giving its formula, let us recall the definition of the $(1, 1)$-Bott-Chern cohomology groups of (M, J):

$$H^{1,1}_{BC}(M, \mathbb{C}) := \frac{\{\alpha \in \mathcal{E}^{1,1}_M(M, \mathbb{C}) | d\alpha = 0\}}{i\partial\bar{\partial}(\mathcal{C}^\infty(M, \mathbb{C}))} \quad H^{1,1}_{BC}(M, \mathbb{R}) := \frac{\{\alpha \in \mathcal{E}^{1,1}_M(M, \mathbb{R}) | d\alpha = 0\}}{i\partial\bar{\partial}(\mathcal{C}^\infty(M, \mathbb{R}))}$$

and we have $H^{1,1}_{BC}(M, \mathbb{C}) \cong H^{1,1}_{BC}(M, \mathbb{R}) \otimes \mathbb{C}$.

Lemma 2.3.1: *The Chern connection $D_{\mathcal{L}}$ corresponding to $(\mathcal{L}, h)$ is given by $D_{\mathcal{L}} = \nabla_{L^c} - 2\theta^{1,0}$, and its Chern curvature is $\Theta(D_{\mathcal{L}}) = -idJ\theta$. In particular, if (M, J, Ω) is strict LCK, then the Bott-Chern class $c_1(\mathcal{L})_{BC} = [\frac{1}{2\pi} dJ\theta]_{i\partial\bar{\partial}}$ of $\mathcal{L}$ does not vanish.*

Proof. Recall that, with respect to a local holomorphic frame s of $\mathcal{L}$, $D_{\mathcal{L}}$ is given by $D_{\mathcal{L}} s = \alpha \otimes s$, with $\alpha = \partial \ln h(s, s)$, and the Chern curvature by $\Theta(D_{\mathcal{L}}) = \bar{\partial}\alpha = d\alpha$, which is well-defined globally on M. Now the local section $s = \mathrm{e}^{-\varphi}\sigma$ verifies $\bar{\partial}_L s = 0$, and we have thus $\alpha = -2\partial\varphi = -2\theta^{1,0}$. On the other hand, s is parallel with respect to ∇_{L^c}:

$$\nabla_{L^c} s = -\mathrm{e}^{-\varphi} d\varphi \otimes \sigma + \mathrm{e}^{-\varphi}\theta \otimes \sigma = 0$$

so $D_{\mathcal{L}} = \nabla_{L^c} - 2\theta^{1,0}$. Also, we have:

$$\Theta(D_{\mathcal{L}}) = -2d\theta^{1,0} = -d(\theta + iJ\theta) = -idJ\theta. \tag{2.3.5}$$

Finally, we obtain

$$c_1(\mathcal{L})_{BC} = [\frac{i}{2\pi}\Theta]_{i\partial\bar{\partial}} \in H^{1,1}_{BC}(M, \mathbb{R})$$

and the proof of Theorem 2.2.8 implies that the vanishing of this class is equivalent to Ω being GCK. ∎

Remark 2.3.2: Note that there exists a natural map $F : H^{1,1}_{BC}(M, \mathbb{R}) \to H^{1,1}(M, \mathbb{R})$, where

$$H^{1,1}(M, \mathbb{R}) := \frac{\{\alpha \in \mathcal{E}^{1,1}_M(M, \mathbb{R}) | d\alpha = 0\}}{d(\mathcal{C}^\infty(M, \mathbb{R}))}.$$

Theorem 2.2.8 implies that $0 \neq c_1(\mathcal{L})_{BC} \in \ker F$. On the other hand, we have a natural injection $H^1(M, \mathbb{R}) \to H^1(M, \mathcal{O})$ and $\ker F \cong \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{R})}$, cf. [Ga76]. Thus, for a compact complex manifold of (strict) LCK type, we have a strict inequality

$$b_1 < 2h^{0,1}$$

where $h^{0,1} := \dim_{\mathbb{C}} H^1(M, \mathcal{O})$, which is also equivalent to the fact that the Picard variety

$$\mathrm{Pic}^0(M) := \ker(c_1 : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z})) \cong \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})}$$

is non-compact. For more details on this discussion, see [Ga76] and [Ga84].

Remark 2.3.3: The above lemma shows that we have a natural morphism

$$G : H^1(M, \mathbb{R}) \to H^{1,1}_{BC}(M, \mathbb{R}), \quad [\theta]_d \mapsto [dJ\theta]_{i\partial\bar{\partial}}$$

which is the composition of $[\theta] \mapsto \mathcal{L}_{[\theta]} \mapsto 2\pi c_1(\mathcal{L}_{[\theta]})_{BC}$. If $[\theta]$ is the Lee form of an LCK metric, $G([\theta]) \neq 0$.

2.4 Special LCK metrics

Not much is known about general LCK manifolds, or about constraints on the existence of such metrics. Moreover, this class is not necessarily well behaved under natural operations, such us deforming the complex structure. However, there are some special classes of LCK metrics which are quite well understood, which we present in this section.

First of all, in any conformal class, there exists a special representative which is sometimes very useful to work with. Let us fix a Hermitian manifold (M, J, g, Ω) of complex dimension $n > 1$, where $g = \Omega(\cdot, J\cdot)$. The metric g induces a L^2 inner product on $\mathcal{E}^\bullet(M)$. We denote by d^* the adjoint of d with respect to this inner product, and by $\Delta := dd^* + d^*d$ the corresponding Laplacian. We recall that on a complex manifold we have the formula: $d^* = - \star d \star$, where $\star$ is the Hodge star operator with respect to g.

On the other hand, Ω induces a Lefschetz map $\mathrm{Lef}_\Omega := \Omega \wedge \cdot$ acting on $\mathcal{E}^\bullet(M)$, so that $\mathrm{Lef}_\Omega : \mathcal{E}^1(M) \to \mathcal{E}^3(M)$ is injective, as $n > 1$, and we have an isomorphism $L^{n-1}_\Omega : \mathcal{E}^1_M(M) \to \mathcal{E}^{2n-1}_M(M)$. Define

$$\theta := \frac{1}{n-1}(\mathrm{Lef}^{n-1}_\Omega)^{-1} d(\Omega^{n-1}).$$

This means that we have a decomposition:

$$d\Omega = \theta \wedge \Omega + \xi$$

with $\xi \wedge \Omega^{n-2} = 0$.

Remark 2.4.1: If we extend the action of J to 1-forms by $(J\alpha)(X) := -\alpha(JX)$, and then to k-forms as a 0-degree derivation via the rule: $J\alpha \wedge \beta := J\alpha \wedge \beta + \alpha \wedge J\beta$, then θ can also be defined directly by the formula:

$$\theta = \frac{1}{n-1} J d^* \Omega. \tag{2.4.1}$$

Indeed, we have:

$$d^*\Omega = - \star d \star \Omega = - \star d \frac{1}{(n-1)!}\Omega^{n-1}$$
$$= -\frac{1}{(n-2)!} \star (\theta \wedge \Omega^{n-1}).$$

Now, by working in a local orthogonal frame of $T^*M \otimes \mathbb{C}$ diagonalising Ω, one can easily prove the identity:

$$\star\theta \wedge \Omega^{n-1} = (n-1)! J\theta$$

from which (2.4.1) follows.

Definition 2.4.2: The Hermitian metric g is called a *Gauduchon metric* if θ is d^*-closed. For an LCK metric (g, Ω, θ), this is equivalent to saying that θ is Δ-harmonic.

The Gauduchon condition for the metric also translates into an equation for the corresponding fundamental form, namely $d^*\theta = 0$ is equivalent to

$$dd^c\Omega^{n-1} = 0. \tag{2.4.2}$$

Indeed, this follows by (2.4.1). As J commutes with $\star$ and $\star^2$ is a constant multiple of the identity, we have the following equivalences:

$$\begin{aligned} d^*\theta = 0 &\Leftrightarrow d \star (Jd^*\Omega) = 0 \\ &\Leftrightarrow dJd \star \Omega = 0 \\ &\Leftrightarrow dJd\Omega^{n-1} = 0. \end{aligned}$$

Finally, on any complex manifold, one has the commutation relation $[J, d] = d^c$, which implies that $dJd = dd^c$ and the conclusion follows.

By a result of P. Gauduchon [Ga77], on a compact complex connected manifold, in any conformal class of a Hermitian metric there exists a Gauduchon metric, and it is unique up to multiplication by a positive constant provided that the complex dimension of the manifold is greater than 1. Hence, this class of metrics is to be viewed as a useful tool in non-Kähler geometry.

In the LCK context, there are some other special metrics, whose existence imposes restrictions on the manifold:

Definition 2.4.3: Let (M, J, g, Ω) be an LCK manifold and let $\nabla := \nabla^g$ denote the Levi-Civita connection of (M, g). If we have $\nabla\theta = 0$, then g (or sometimes Ω) is called a *Vaisman structure* or *metric*. Equivalently, denoting by B the Lee vector field, then g is Vaisman if $\nabla B = 0$.

Vaisman manifolds are closely related to Sasaki manifolds: the universal cover of a Vaisman manifold with its Kähler metric is isometric to the Kähler cone over a Sasaki manifold. We recall that a Sasaki manifold $(S, g_S, \tilde{J})$ is a Riemannian manifold (S, g_S) together with a complex structure $\tilde{J}$ on the Riemann cone $(C(S) = S \times \mathbb{R}, g_K := \mathrm{e}^{-2t}(g_S + dt^2))$ so that g_K is Kähler.

Remark 2.4.4: A Vaisman metric is Gauduchon. Indeed, writing d^* as $d^* = -\sum_{j=1}^{2n} \iota_{e_j}\nabla_{e_j}$ for a local orthonormal real basis of TM, we see that $\nabla\theta = 0$ implies $d^*\theta = 0$. In particular, two conformal Vaisman metrics must also differ by a multiplicative constant.

For Vaisman metrics, the Lee vector field is particularily important. First of all, let us note that $\nabla B = 0$ implies that B is of constant norm, and that B has no zeroes. In particular, the Euler characteristic of M is 0. Moreover, we have the following well known properties, cf. [Va82]:

Proposition 2.4.5: *Let $(M, J, g, \Omega, \theta)$ be an LCK manifold with corresponding Lee and Reeb* *vector fields B and A. If g is Vaisman, then A and B are real-holomorphic vector fields preserving both g and Ω. Conversely, if B is Killing, then g is Vaisman.*

Proof. Let us first see that B is holomorphic, which will then imply that also $A = JB$ is. Using the fact that $\nabla_B J = 0$ and $\nabla B = 0$, we obtain, for any $X \in \Gamma(TM)$:

$$(\mathcal{L}_B J)X = [B, JX] - J[B, X] = \nabla_B JX - J\nabla_B X = (\nabla_B J)X = 0.$$

In particular, as A always preserves the form Ω by 2.2.5, it follows that A is Killing. Now let us see that B is Killing. For this, take any two vector fields $X, Y \in \Gamma(TM)$ and write:

$$\begin{aligned}(\mathcal{L}_B g)(X, Y) &= B(g(X, Y)) - g([B, X], Y) - g(X, [B, Y]) \\ &= g(\nabla_B X, Y) + g(X, \nabla_B Y) - g(\nabla_B X, Y) - g(X, \nabla_B Y) = 0.\end{aligned}$$

Thus, as B preserves J and g, it also preserves Ω.

Conversely, if B is Killing then $\nabla\theta$ is antisymmetric. But $d\theta$ is the antisymmetrisation of $\nabla\theta$, hence $0 = d\theta = 2\nabla\theta$, i.e. g is Vaisman. ■

In particular, on a Vaisman manifold one has that $[A, B] = 0$, hence A and B span an integrable distribution giving rise to a complex analytic foliation $\mathcal{F}$, called *the canonical foliation.* This foliation has many interesting properties, such as being Riemannian (and in fact transversally Kählerian). Its leaves are parallelizable one-dimensional complex analytic manifolds, embedded as totally geodesic submanifolds of (M, J, Ω). For the details of this, see [Va82]. Moreover, we have the following important result:

Theorem 2.4.6: *([Va82, Thm 5.1], [Ts97, Thm 3.2], [Ve04, Prop 6.5]) Let (M, J, g) be a compact Vaisman manifold and $j : (N, J') \to (M, J)$ be an immersed complex submanifold. Then N is foliated, in the sense that for any $x \in N$, the leaf of $\mathcal{F}$ passing through $j(x)$ is contained in $j(N)$. In particular, if N is compact and verifies $\dim_{\mathbb{C}} N \geq 2$, then j^*g is a Vaisman metric on (N, J').*

Remark 2.4.7: The above result implies that any complex curve C of a compact complex manifold of Vaisman type (M, J) is foliated with respect to $\mathcal{F}$, thus complex paralellizable. In particular, if C is compact, then C is an elliptic curve. As a consequence, (M, J) cannot contain rational curves, and so cannot be the blow-up of some other manifold.

As another consequence of Proposition 2.4.5, we obtain that for a Vaisman metric, Ω has a very special form, completely determined by the Lee form:

Corollary 2.4.8: *If (Ω, θ) is a Vaisman structure on the complex manifold (M, J) normalised so that $\|\theta\|^2 = 1$, then one has:*

$$\Omega = -dJ\theta + \theta \wedge J\theta. \tag{2.4.3}$$

Equivalently, on the minimal Kähler cover $(\hat{M}, \Omega_K)$, if $\theta = d\varphi$, then:

$$\Omega_K = dd^c \mathrm{e}^{-\varphi}. \tag{2.4.4}$$

Proof. Clearly the two formulas are equivalent, and (2.4.3) is equivalent to $\mathcal{L}_B\Omega = 0$. Indeed, by the Cartan formula:

$$\mathcal{L}_B\Omega = dJ\theta + \iota_B(\theta \wedge \Omega) = dJ\theta + \|\theta\|^2\Omega - \theta \wedge J\theta. \tag{2.4.5}$$

■

However, formula (2.4.3) does not characterize Vaisman structures. In fact, we have a name for structures having this form, which were introduced by L. Ornea and M. Verbitsky [OV10]:

Definition 2.4.9: An LCK structure (Ω, θ) on (M, J) is called an *LCK metric with potential* if there exists $f \in \mathcal{C}^\infty(M, \mathbb{R})$ so that

$$\Omega = d_\theta d^c_\theta f.$$

This is equivalent to saying that there exists a Γ-equivariant smooth function $\hat{f}$ on the minimal cover $\pi : \hat{M} \to M$ so that $\Omega_K = dd^c\hat{f}$.

Moreover, the LCK structure is called *with positive potential* if f can be chosen strictly positive.

Remark 2.4.10: The Γ-equivariant potential in the above definition is given by the formula:

$$\hat{f} = \mathrm{e}^{-\varphi} f$$

where $\pi^*\theta = d\varphi$.

Remark 2.4.11: The above definitions are conformally invariant. Indeed, if $\Omega = d_\theta d^c_\theta f$ and $h \in \mathcal{C}^\infty(M)$, then:

$$\Omega_h = \mathrm{e}^h\Omega = d_{\theta_h} d^c_{\theta_h}(\mathrm{e}^h f), \quad \theta_h = \theta + dh.$$

In particular, if $\Omega = d_\theta d^c_\theta f$ is an LCK form with positive potential, then there exists $\Omega' \in [\Omega]$ verifying (2.4.3):

$$\Omega' = f^{-1}\Omega = d_\theta d^c_\theta 1.$$

Finally, even this notion of an LCK metric can be generalized:

Definition 2.4.12: An LCK structure (Ω, θ) is called *exact* if Ω is d_θ exact, i.e. $\Omega = d_\theta\eta$, for some form $\eta \in \mathcal{E}^1_M(M)$. In this case, for any other LCK form in the same conformal class $\Omega_f = \mathrm{e}^f\Omega$ with Lee form $\theta_f = \theta + df$ we have $\Omega_f = d_{\theta_f}(\mathrm{e}^f\eta)$. Hence we call an LCK manifold $(M, J, [\Omega])$ *LCK exact* if some, and hence any representative Ω is exact.

Remark 2.4.13: We should note that no two notions of LCK structures defined above coincide. As we will see in the examples that will follow, the class of manifolds admitting LCK structures with positive potential is strictly larger than the one admitting Vaisman structures, and moreover, not all LCK manifolds admit exact LCK structures. Also, given an LCK metric with positive potential, in some cases one can add to the potential a convenient pluriharmonic function to obtain an LCK metric with non-positive potential, as pointed out in the introduction of [OV17]. Finally, Goto showed in [Go14] that there exist exact LCK metrics on the standard Hopf manifolds (see 2.6.1) which do not admit any potential.

2.5 Infinitesimal automorphisms of LCK manifolds

In this section we will take a closer look at the Lie algebra of infinitesimal automorphisms of LCK manifolds, and will distinguish a special subalgebra that will play a particular role. From now on, even if not specified, we only work with LCK structures on complex manifolds of complex dimension greater that 1.

For an LCK manifold $(M, J, [\Omega])$, the automorphism group $\mathrm{Aut}(M, J, [\Omega])$ is formed by all the conformal biholomorphisms $\Phi : M \to M$, $\Phi^*\Omega \in [\Omega]$. Denote by $\mathfrak{aut}(M, J, [\Omega])$ the corresponding Lie algebra of infinitesimal automorphisms. We have the first well-known property of this Lie group:

Proposition 2.5.1: *Let $(M, J, [\Omega])$ be a compact LCK manifold, and let $\Omega_0 \in [\Omega]$ be a Gauduchon metric. Then $\mathrm{Aut}(M, J, [\Omega]) = \mathrm{Aut}(M, J, \Omega_0)$. In particular, the automorphism group of a compact LCK manifold is compact.*

Proof. Let $\Phi \in \mathrm{Aut}(M, J, [\Omega])$. As Ω_0 is Gauduchon, we have $dd^c\Omega_0^{n-1} = 0$, an as Φ is a biholomorphism, Φ^* commutes with d and with d^c. Hence we also have $dd^c\Phi^*\Omega_0^{n-1} = 0$, i.e. $\Phi^*\Omega_0$ is also a Gauduchon metric. But we have $\Phi^*\Omega_0 \in [\Omega_0]$, so by the uniqueness up to scalars of such metrics in a given conformal class, we must have $\Phi^*\Omega_0 = \lambda\Omega_0$ with $\lambda \in \mathbb{R}_{>0}$. Finally, as M is compact and Φ is bijective, we have $\int_M \Omega^n = \int_M \Phi^*\Omega^n = \lambda^n \int_M \Omega^n > 0$, implying that $\lambda = 1$.

Thus, the group $\mathrm{Aut}(M, J, [\Omega_0])$ is a closed subgroup of the compact Lie group of isometries $\mathrm{Aut}(M, \Omega_0(\cdot, J\cdot))$, hence also compact Lie group. ■

Similarly, given an LCS manifold $(M, [\Omega])$, the group of automorphisms $\mathrm{Aut}(M, [\Omega])$ is formed by all the conformal diffeomorphisms $\Phi : M \to M$, $\Phi^*\Omega \in [\Omega]$. Next, we want to investigate the algebraic structure of the corresponding Lie algebra $\mathfrak{aut}(M, [\Omega])$. First of all, note that $X \in \mathfrak{aut}(M, [\Omega])$ means $\mathcal{L}_X\Omega = f_X\Omega$. This implies $(f_X - \theta(X))\Omega = d_\theta(\iota_X\Omega)$. Hence $d_\theta((f_X - \theta(X))\Omega) = 0$, or also $(df_X - d(\theta(X))) \wedge \Omega = 0$ and since we are working under the supposition that $\dim_{\mathbb{C}} M > 1$, it follows that $\theta(X) - f_X = c_X \in \mathbb{R}$. By straightforward computations it can be seen that the constants c_X are conformally invariant. Hence we have a linear map:

$$\begin{aligned} l : \mathfrak{aut}(M, [\Omega]) &\to \mathbb{R} \\ X &\mapsto c_X = \theta(X) - f_X. \end{aligned} \tag{2.5.1}$$

One can check that l is in fact a Lie algebra morphism, or can consult [Va85] for the details. I. Vaisman studied the restriction of l to $\mathfrak{aut}(M, \Omega)$ and the LCS manifolds for which this restriction is not identically zero, which he named *LCS manifolds of the first kind.* As noted in [Va85], being of the first kind is not a conformally invariant notion. However, we have the following result in the conformal setting:

Lemma 2.5.2: *The map l is surjective if and only if $(M, [\Omega])$ is LCS exact.*

Proof. First of all fix $\Omega \in [\Omega]$ an LCS form. Suppose l is not identically zero and choose $C \in \mathfrak{aut}(M, [\Omega])$ such that $l(C) = 1$. Then we have:

$$\theta(C)\Omega - \Omega = \mathcal{L}_C\Omega = d\iota_C\Omega + \theta(C)\Omega - \theta \wedge \iota_C\Omega$$

hence:

$$\Omega = d_\theta(-\iota_C\Omega).$$

Conversely, suppose $\Omega = d\eta - \theta \wedge \eta$. Define $C \in \Gamma(TM)$ by: $\iota_C\Omega = -\eta$. We compute:

$$\begin{aligned} \mathcal{L}_C\Omega =& d\iota_C\Omega + \iota_C(\theta \wedge \Omega) \\ =& -d\eta + \theta(C)\Omega + \theta \wedge \eta \\ =& (\theta(C) - 1)\Omega. \end{aligned}$$

Hence $C \in \mathfrak{aut}(M, [\Omega])$ and $l(C) = 1$. ■
Consider the kernel of l, which is also conformally invariant:

$$\mathfrak{aut}'(M, [\Omega]) := \{X \in \Gamma(TM) | \mathcal{L}_X \Omega = \theta(X)\Omega\}. \tag{2.5.2}$$

We will call elements of this subalgebra *horizontal* or *special conformal vector fields.*
Let now $\pi : \hat{M} \to M$ be the minimal cover corresponding to $[\Omega]$, with deck group Γ and corresponding symplectic form denoted by Ω_K. For a Lie algebra $\mathfrak{a}$ of vector fields on $\hat{M}$ fixed by the action of Γ, we denote by $\mathfrak{a}^\Gamma$ the corresponding subalgebra of Γ-invariant elements of $\mathfrak{a}$.

Lemma 2.5.3: *We have a natural isomorphism between* $\mathfrak{aut}(M, [\Omega])$ *and* $\mathfrak{aut}(\hat{M}, [\Omega_K])^\Gamma$ *given by* π^*. *In particular, under this isomorphism,* $\mathfrak{aut}'(M, [\Omega])$ *is in bijection with* $\mathfrak{aut}(\hat{M}, \Omega_K)^\Gamma$, *the Lie algebra of* Γ*-invariant infinitesimal symplectomorphisms of* Ω_K.

Proof. In general, for $X \in \Gamma(TM)$ and $\hat{X} := \pi^* X \in \Gamma(T\hat{M})$ its lift to $\hat{M}$, we have the following formula:

$$\mathcal{L}_{\hat{X}} \Omega_K = \mathrm{e}^{-\varphi} \pi^* (\mathcal{L}_X \Omega - \theta(X)\Omega) \tag{2.5.3}$$

where $\pi^*\theta = d\varphi$. In particular, if $X \in \mathfrak{aut}(M, [\Omega])$ then $\mathcal{L}_{\hat{X}} \Omega_K = -l(X)\Omega_K$. Conversely, if $\hat{X} = \pi^* X \in \mathfrak{aut}(\hat{M}, [\Omega_K])^\Gamma$, then X is in fact a homothety:

$$\mathcal{L}_X \Omega_K = f\Omega_K \Rightarrow 0 = d(\mathcal{L}_X \Omega_K) = df \wedge \Omega_K \Rightarrow df = 0.$$

Moreover, by (2.5.3), $\mathcal{L}_X \Omega_K = f\Omega_K$ implies then

$$\mathcal{L}_X \Omega = (f + \theta(X))\Omega$$

so $X \in \mathfrak{aut}(M, [\Omega])$.
Finally, for $X \in \mathfrak{aut}(M, [\Omega])$ and $\hat{X} = \pi^* X$ we have:

$$X \in \mathfrak{aut}'(M, [\Omega]) \Leftrightarrow l(X) = 0 \Leftrightarrow \mathcal{L}_{\hat{X}} \Omega_K = 0 \Leftrightarrow \hat{X} \in \mathfrak{aut}(\hat{M}, \Omega_K)^\Gamma.$$

■

In LCS and LCK geometry, often one needs to switch between the compact LCS manifold M and the non-compact symplectic covering $\hat{M}$. We would like to know what happens to Lie group actions in the process. In general, an action of a group G on $\hat{M}$ descends to an action of G on M if and only if G commutes with Γ, where Γ is the deck group of $\hat{M} \to M$. Conversely, if one has an action of a Lie group G on M, then one can always lift it to an action of $\tilde{G}$ on $\hat{M}$, $\tilde{G}$ being the universal cover of G. The question is then when does the lifted action of $\tilde{G}$ factors through an action of G itself on $\hat{M}$. Next we give an answer to this in the case G is a compact torus, which is an analogue of [MMP17, Proposition 4.4] for abelian groups in the LCS setting. The above result can be shown to hold for any compact Lie group, following the arguments of [MMP17] and using the structure theorem of compact Lie groups.

Proposition 2.5.4: *Let* $(M, [\Omega], [\theta]_{dR})$ *be an LCS manifold and* $\mathbb{T}$ *be a compact torus acting on* M *by conformal automorphisms. Then the action of* $\mathbb{T}$ *lifts to the minimal cover* $\hat{M}$ *if and only if* $\mathfrak{Lie}(\mathbb{T}) = \mathfrak{t} \subset \mathfrak{aut}'(M, [\Omega])$.

Proof. We can suppose that $\mathbb{T} = \mathbb{S}^1$, for otherwise we make use of the same argument for each generator of the $\mathbb{T}$-action. Fix an LCS form Ω. Denote by X the generator of the infinitesimal action of $\mathbb{S}^1$ on M, by $\hat{X}$ its lift to $\hat{M}$ and by Φ_t and $\hat{\Phi}_t$ their corresponding flows, so that $\Phi_0 = \Phi_1 = \mathsf{id}_M$. Then the action of $\mathbb{T}$ lifts to an action of $\mathbb{T}$ on $\hat{M}$ if and only if $\hat{\Phi}_t$ is periodic in t.

Suppose first that $\mathfrak{t} \subset \mathfrak{aut}'(M, [\Omega])$. Equation (2.5.3) implies then that $\hat{X} \in \mathfrak{aut}(\hat{M}, \Omega_K)$, hence $\{\hat{\Phi}_t\}_t$ are symplectomorphisms. On the other hand, $\hat{\Phi}_1$ is an element of Γ since it covers the identity of M. Thus $\hat{\Phi}_1 \in \operatorname{Ker} \tau = \{\mathsf{id}\}$ by the definition of the minimal cover, where $\tau : \Gamma \mapsto \mathbb{R}$ is the homomorphism corresponding to $[\theta]_{dR}$.

Conversely, suppose $\hat{\Phi}_t$ is periodic in t. As we have already seen, $\hat{X}$ acts by homotheties on Ω_K, hence the exists a periodic $\mathcal{C}^\infty$ function $c : \mathbb{R} \to \mathbb{R}$ such that $\hat{\Phi}_t^* \Omega_K = c(t)\Omega_K$. Moreover, we have, for any $t_1, t_2 \in \mathbb{R}$:

$$c(t_1 + t_2)\Omega_K = \hat{\Phi}_{t_1}^*(\hat{\Phi}_{t_2}^* \Omega_K) = \hat{\Phi}_{t_1}^*(c(t_2))\hat{\Phi}_{t_1}^* \Omega_K = c(t_2)c(t_1)\Omega_K.$$

Hence, for any $t \in \mathbb{R}$:

$$\dot{c}(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h} = \lim_{h \to 0} \frac{c(t)c(h) - c(t)c(0)}{h} = c(t)\dot{c}(0).$$

On the other hand, since c is periodic, it must have some critical point, implying that $\dot{c}(0) = 0$. Therefore $\mathcal{L}_{\hat{X}} \Omega_K = \dot{c}(0)\Omega_K = 0$, or also, by (2.5.3), $X \in \mathfrak{aut}'(M, [\Omega])$. ∎

Remark 2.5.5: In the LCK setting, we can reformulate the above criterion. Suppose that $\mathbb{S}^1$ acts holomorphically on a complex manifold (M, J) of LCK type. Take any LCK structure on M and average it over $\mathbb{S}^1$ in order to get an $\mathbb{S}^1$-invariant structure (Ω, θ). In particular, if X is a generator of the $\mathbb{S}^1$-action, $\theta(X)$ is constant. By the above, the action lifts to an $\mathbb{S}^1$-action on the minimal cover $\hat{M}$ corresponding to $[\theta]_{dR}$ if and only if $\theta(X) \neq 0$.

2.6 Examples

2.6.1 Diagonal Hopf manifolds

The standard Hopf manifold is the most basic example of a compact complex manifold not admittig a Kähler metric, and it is also the first example of an LCK manifold appearing in the literature (I. Vaisman, [Va76]).

By definition, a Hopf manifold is an n-dimensional compact complex manifold whose universal cover is biholomorphic to $W := \mathbb{C}^n - \{0\}$, and whose fundamental group is infinite cyclic, generated by a contraction $c \in \operatorname{Aut}(\mathbb{C}^n)$ fixing $0 \in \mathbb{C}^n$. We recall that c is called a contraction if the eigenvalues of $d_0 c$ are all of module smaller than 1. If c is diagonal, meaning that it is of the form $c(z_1, \ldots, z_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$, then $H_\lambda := W/ < c >$ is called a *diagonal* or *linear* Hopf manifold. Moreover, when $\lambda_1 = \ldots = \lambda_n$, H_λ is usually called the *standard* Hopf manifold.

All diagonal Hopf manifolds are known to admit LCK metrics. On the standard ones, these are easy to construct and were given by I. Vaisman. More generally, consider a diagonal Hopf manifold H_λ given by $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $|\lambda_1| = \ldots = |\lambda_n|$. In this case, one defines on

$W = \mathbb{C}^n - \{0\}$:

$$f(z) = |z|^2 = \sum_{k=1}^{n} |z_k|^2, \quad \varphi = -\ln f, \quad \Omega_K = dd^c f, \quad \theta = d\varphi, \tag{2.6.1}$$

where we denote by $z = (z_1, \ldots, z_n)$ the holomorphic coordinates on W. Then Ω_K is the standard Kähler metric on W induced from $\mathbb{C}^n$, and so $\Omega := f^{-1}\Omega_K$ descends to an LCK metric on H_λ, with corresponding Lee form θ.

P. Gauduchon and L. Ornea constructed in [GO98] LCK metrics on any diagonal Hopf surface, and then the same metrics were constructed by a different approach by F. Belgun in [Bel00]. These metrics are Vaisman, but the potential is given only implicitly, by a geometrical construction, which we shall next describe, following [GO98]. Their construction has a straightforward generalization to any n-dimensional diagonal Hopf manifold, and we will directly describe this case.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ with $0 < |\lambda_1| \leq \ldots \leq |\lambda_n| < 1$, and let $c(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$. After choosing arguments for all λ_k so that $\lambda_k = a_k \mathrm{e}^{iu_k}$, $k \in \{1, \ldots, n\}$, we can always extend the action of $\Gamma := < c >$ on W to a holomorphic action of $\mathbb{R}$ on W by:

$$\Phi_\lambda^t(z) = (\lambda_1^t z_1, \ldots, \lambda_n^t z_n) = (a_1^t \mathrm{e}^{iu_1 t} z_1, \ldots, a_n^t \mathrm{e}^{iu_n t} z_n), \quad t \in \mathbb{R} \tag{2.6.2}$$

so that $\Phi_\lambda^1 = c$. Denote the real holomorphic vector field generating this action by C_λ, i.e.

$$C_\lambda = \sum_{k=1}^{n} 2\,\mathrm{Re}((\ln a_k + iu_k) z_k \frac{\partial}{\partial z_k}).$$

Consider the sphere $S = \mathbb{S}^{2n-1}$ embedded in W as $S = \varphi^{-1}(0)$, where φ was defined in (2.6.1). Each orbit of Φ_λ^t intersects S exactly once since, for any $z \in W$, the function $h : \mathbb{R} \to \mathbb{R}$, $t \mapsto \varphi(\Phi_\lambda^t(z))$ verifies $\lim_{t\to-\infty} h(t) = -\infty$, $\lim_{t\to\infty} h(t) = \infty$ and h is strictly increasing. Indeed, we have:

$$h'(t) = d\varphi(C_\lambda)_{\Phi_\lambda^t(z)} = -\frac{2}{|\Phi_\lambda^t(z)|^2} \sum_{k=1}^{n} \ln a_k |a_k z_k|^2 > 0. \tag{2.6.3}$$

Hence, the map

$$\begin{aligned} \Phi_\lambda : S \times \mathbb{R} &\to W \\ (y, t) &\mapsto \Phi_\lambda^t(y) \end{aligned}$$

is a diffeomorphism. Let $\Psi_\lambda : W \to S \times \mathbb{R}$ be its inverse, and let $\varphi_\lambda := p_\mathbb{R} \circ \Psi_\lambda$, where $p_\mathbb{R} : S \times \mathbb{R} \to \mathbb{R}$ is the canonical projection, so that

$$\Psi_\lambda(z) = (\Phi_\lambda^{-\varphi_\lambda(z)}(z), \varphi_\lambda(z)).$$

Since Φ_λ is equivariant with respect to the $\mathbb{R}$-action, where $\mathbb{R}$ acts trivially on S and by translations on $\mathbb{R}$, it follows that $c^*\varphi_\lambda = \varphi_\lambda + 1$. In particular, $d\varphi_\lambda$ is $\mathbb{R}$-invariant, and as for any $y \in S$ and $t \in \mathbb{R}$ we have $\varphi_\lambda(\Phi_\lambda^t(y)) = t$, it follows that $d\varphi_\lambda(C_\lambda) = 1$. Moreover, if we let:

$$f_\lambda := \mathrm{e}^{-\varphi_\lambda}, \quad \Omega_\lambda = f_\lambda^{-1} dd^c f_\lambda, \quad \theta_\lambda = d\varphi_\lambda$$

we have $c^* f_\lambda = \mathrm{e}^{-1} f_\lambda$ and so the form Ω_λ descends to an $(1,1)$-form on H_λ verifying $d\Omega_\lambda = \theta_\lambda \wedge \Omega_\lambda$. In order to see that Ω_λ is an LCK form, we are left with showing that it is strictly positive. This was done in [GO98] by direct computations, but we will deduce it by more geometrical arguments.

Note that the tangent bundle of S, as a subbundle of TW, identifies with $\ker d\varphi|_S = \ker \theta|_S$. At the same time, we have $\varphi_\lambda(z) = 0 \Leftrightarrow z \in S$, hence S is also given by $\varphi_\lambda^{-1}(0)$ and $TS = \ker \theta_\lambda|_S$. Thus, there exists a function $l \in \mathcal{C}^\infty(S, \mathbb{R})$ so that $\theta_\lambda|_S = l\theta|_S$. Since:

$$1 = \theta_\lambda(C_\lambda) = l\theta(C_\lambda)$$

and $\theta(C_\lambda)$ is everywhere positive by (2.6.3), it follows that $l > 0$ on S.

Let us now consider the sub-bundle of TS:

$$H = \ker \theta|_S \cap \ker J\theta|_S = \ker \theta_\lambda|_S \cap \ker J\theta_\lambda|_S = TS \cap JTS$$

which is stable under J, the complex structure of W. On H we have:

$$\begin{aligned}\Omega_\lambda|_H &= (-dJ\theta_\lambda + \theta_\lambda \wedge J\theta_\lambda)|_H = J\theta_\lambda([\cdot,\cdot])|_H \\ &= lJ\theta([\cdot,\cdot])|_H = l(-dJ\theta + \theta \wedge J\theta)|_H = l\Omega|_H.\end{aligned}$$

In particular, as Ω is positive, it follows that Ω_λ is also positive when restricted to H.

Consider next the sub-bundle of TW given by $\mathcal{H} := \ker \theta_\lambda \cap \ker J\theta_\lambda$, which is again stable under J by definition. As both θ_λ and $J\theta_\lambda$ are Φ_λ^t-invariant, we have $\mathcal{H}_z = d_y \Phi_\lambda^t H_y$ for any $y \in S$, $t \in \mathbb{R}$ and $z = \Phi_\lambda^t(y)$. Therefore, it follows that Ω_λ is also positive when restricted to $\mathcal{H}_z$:

$$\Omega_\lambda|_{\mathcal{H}_z} = \Omega_\lambda|_{d_y\Phi_\lambda^t H_y} = (\Phi_\lambda^t)^* \Omega_\lambda|_{H_y} > 0.$$

Let us define the J-invariant sub-bundle of TW: $\mathcal{E} := \mathcal{H}^{\perp_{\Omega_\lambda}} = \mathcal{H}^{\perp_{-dJ\theta_\lambda}}$. Note that, as $\Omega_\lambda|_{\mathcal{H}}$ is non-degenerate, we have a $\mathcal{C}^\infty$ splitting $TW = \mathcal{H} \oplus \mathcal{E}$. Moreover, in the diagonal case, $\mathcal{E}$ is holomorphic, spanned by the Lee and Reeb vector fields, which we now exhibit.

To do this, let us note that Φ_λ^t given by (2.6.2) is the composite of a real dilatation D^t and a rotation R^t given by:

$$D^t(z) = (a_1^t z_1, \ldots, a_n^t z_n), \quad R^t(z) = (\mathrm{e}^{iu_1 t} z_1, \ldots, \mathrm{e}^{iu_n t} z_b), \quad z \in W,\ t \in \mathbb{R}.$$

For any $t \in \mathbb{R}$, S is invariant under R^t while $\Phi_\lambda^{\cdot}$ commutes with R^t, hence φ_λ and Ω_λ are also R^t-invariant. Thus, if we denote by B the real holomorphic vector field generating $\{D^t\}_{t\in\mathbb{R}}$, given by:

$$B = \sum_{k=1}^{n} 2\,\mathrm{Re}(\ln a_k z_k \frac{\partial}{\partial z_k})$$

it follows that $\theta_\lambda(B) = 1$. Also, we have:

$$\theta(JB) = \sum_{k=1}^{n} (z_k d\overline{z}_k + \overline{z}_k dz_k)(-2\,\mathrm{Im}(\ln a_k z_k \frac{\partial}{\partial z_k})) = 0.$$

Thus, as $\ker \theta|_S = \ker \theta_\lambda|_S$, $\theta_\lambda(JB) = 0$ on S, and so on the whole of W. Additionally, since θ_λ is D^t-invariant, we have:

$$\begin{aligned}\iota_B dJ\theta_\lambda &= \mathcal{L}_B J\theta_\lambda - d\iota_B J\theta_\lambda = 0 \\ \iota_{JB} dJ\theta_\lambda &= \mathcal{L}_{JB} J\theta_\lambda - d\iota_{JB} J\theta_\lambda \\ &= Jd\iota_{JB}\theta_\lambda = 0.\end{aligned}$$

In particular, $B, JB \in \mathcal{C}^\infty(\mathcal{E})$ and so $\mathcal{E} \subset \ker(-dJ\theta_\lambda)$. Thus, for any $X = X_\mathcal{H} + X_\mathcal{E} \in \mathcal{C}^\infty(\mathcal{H}) \oplus \mathcal{C}^\infty(\mathcal{E})$, we have:

$$\begin{aligned}\Omega_\lambda(X, JX) &= -dJ\theta_\lambda(X_\mathcal{H}, JX_\mathcal{H}) + \theta_\lambda \wedge J\theta_\lambda(X_\mathcal{E}, JX_\mathcal{E}) \\ &= -dJ\theta_\lambda(X_\mathcal{H}, JX_\mathcal{H}) + (\theta_\lambda(X_\mathcal{E}))^2 + (\theta_\lambda(JX_\mathcal{E}))^2 \geq 0.\end{aligned}$$

This quantity vanishes if and only if $X_\mathcal{H} = 0$ and $X_\mathcal{E} \in \ker J\theta_\lambda \cap \ker J\theta_\lambda = \mathcal{H}$, i.e. if $X = 0$, so Ω_λ is strictly positive.

Finally, we have $\iota_B \Omega_\lambda = J\theta_\lambda$, i.e. B is the Lee vector field of Ω_λ. As it is holomorphic and Ω_λ is an LCK metric with potential equal to 1, by Proposition 3.2.2, Ω_λ is Vaisman.

2.6.2 Non-diagonal Hopf surfaces

In general, two n-dimensional Hopf manifolds $H_c = W/_{<c>}$ and $H_{c'} = W/_{<c'>}$ are biholomorphic if and only if there exists an automorphism $A \in \mathrm{Aut}(W)$ with $AcA^{-1} = c'$. Unfortunately, there does not exist a classification of all the conjugacy classes of contractions fixing 0, and so a general classification of Hopf manifolds is lacking. However, Hopf surfaces were classified by K. Kodaira [Kod66] and Ma. Kato [Ka89], and their fundamental group is generated by:

$$c(z_1, z_2) := (\beta z_1, \alpha z_2 + \mu z_1^m)$$

with $\alpha, \beta, \mu \in \mathbb{C}, 0 < |\alpha|, |\beta| < 1, m \in \mathbb{N}^*$ and $\mu(\alpha - \beta^m) = 0$. In the realm of complex surfaces, these are usually called *primary* Hopf surfaces, while their smooth finite quotients are called *secondary* Hopf surfaces. In fact these are all the compact complex surfaces which are covered by $\mathbb{C}^2 - \{0\}$. All of them admit LCK metrics, and we only need to see this on the primary ones, as they will then descend to the secondary ones.

Note that if $\mu = 0$, then c is diagonal, a case we already presented. In the other case we have $\alpha = \beta^m$, and we will denote the corresponding non-diagonal Hopf surface by $H_{\beta,m,\mu}$. Moreover, for any $\mu_1, \mu_2 \in \mathbb{C}^*$, H_{β,m,μ_1} is biholomorphic to H_{β,m,μ_2} via the morphism induced by $A : W \to W$, $(z_1, z_2) \mapsto (z_1, \frac{\mu_2}{\mu_1} z_2)$.

LCK metrics on non-diagonal Hopf surfaces have been constructed less explicitely. This has been done by two different approaches in the literature, one by deformation, cf. [GO98], and the other one similar to the above construction, by F. Belgun in [Bel00]. We present the second one.

Let us fix in the sequel $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $m \in \mathbb{N}^*$. After writing β in polar coordinates $\beta = be^{iu}$, for any $\mu \in \mathbb{C}$, we can extend the action of $\Gamma :=< c >$ to a holomorphic action of $\mathbb{R}$ on W by:

$$\Phi_\mu^t(z) = (\beta^t z_1, \beta^{mt} z_2 + t\mu\beta^{m(t-1)} z_1^m).$$

Let C_μ denote the real vector field generating this action. As before, consider φ and θ as defined in (2.6.1), let $S = \mathbb{S}^3 = \varphi^{-1}(0) \subset W$ and consider the continuous function:

$$\begin{aligned} L : \mathbb{C} \times S &\to \mathbb{R} \\ (\mu, y) &\mapsto \frac{d}{dt}|_0(\varphi(\Phi_\mu^t)(y)) = \theta(C_\mu)_y. \end{aligned}$$

By (2.6.3), the function $L_0 = L(0, \cdot)$ is strictly positive, so by continuity and by the compactness of S, for $\mu \neq 0$ of module small enough, $L_\mu = \theta(C_\mu)$ also is positive. Let us fix a $\mu \in \mathbb{C}^*$

verifying this. Hence, the orbits of Φ_μ^t are transverse to S and intersect S exactly once, inducing thus an $\mathbb{R}$-equivariant diffeomorphism $S \times \mathbb{R} \to W$. Defining, as before, the corresponding function φ_μ and the form $\theta_\mu = d\varphi_\mu$, we obtain the Γ-invariant $(1,1)$ form on W:

$$\Omega_\mu = \mathrm{e}^{\varphi_\mu} dd^c \mathrm{e}^{-\varphi_\mu} = -dJ\theta_\mu + \theta_\mu \wedge J\theta_\mu.$$

We still have a splitting $TX = \mathcal{H} \oplus \mathcal{E}$, except that this time $\mathcal{E}$ is not contained in $\ker -dJ\theta_\mu$. As $\mathcal{H} \subset \ker J\theta_\mu$, $\mathcal{E} \cap \ker J\theta_\mu$ is a real one dimensional oriented vector bundle over W, so we can choose Z a positively oriented frame for it which is Φ_μ-invariant - for instance, the projection of C_μ.

Consider the Φ_μ-invariant function $\frac{dJ\theta_\mu(Z,JZ)}{\theta_\mu(Z)^2}$. It is bounded on S, and so on the whole of W, therefore there exists some positive constant $K > 0$ so that $K\theta_\mu(Z)^2 > dJ\theta_\mu(Z, JZ)$. Then the metric

$$\Omega_{\mu,K} := \mathrm{e}^{K\varphi_\mu} dd^c \mathrm{e}^{-K\varphi_\mu} = -KdJ\theta_\mu + K^2\theta_\mu \wedge J\theta_\mu$$

defines an LCK metric with potential on $H_{\beta,m,\mu}$ with Lee form $\theta_{\mu,K} = K\theta_\mu$. Indeed, for any $X = X_\mathcal{H} + X_\mathcal{E} \in \mathcal{C}^\infty(\mathcal{H}) \oplus \mathcal{C}^\infty(\mathcal{E})$, where $X_\mathcal{E} = fZ + gJZ$ with $f, g \in \mathcal{C}^\infty(W, \mathbb{R})$, we have:

$$\begin{aligned}\Omega_{\mu,K}(X, JX) &= -KdJ\theta_{\alpha,\beta}(X_\mathcal{H}, JX_\mathcal{H}) - KdJ\theta_\mu(X_\mathcal{E}, JX_\mathcal{E}) + K^2\theta_{\alpha,\beta} \wedge J\theta_{\alpha,\beta}(X_\mathcal{E}, JX_\mathcal{E}) \\ &= -KdJ\theta_{\alpha,\beta}(X_\mathcal{H}, JX_\mathcal{H}) + K(f^2 + g^2)(-dJ\theta_\mu(Z, JZ) + K\theta_\mu(Z)^2) \geq 0\end{aligned}$$

with equality if and only if $X_\mathcal{H} = 0$ and $f = g = 0$, i.e. if $X = 0$, so $\Omega_{\mu,K}$ is strictly positive. This time, the metric is not Vaisman, and in fact for $\mu \neq 0$, $H_{\beta,m,\mu}$ does not admit any Vaisman metric, cf. [Bel00]. We will show this later on (Example 3.4.6) as an application of a criterion for the existence of Vaisman metrics.

2.6.3 LCK manifolds obtained from ample vector bundles

The following construction is well-known, see for instance [Va76], [Va80] or [Ts99], and can be seen as a generalisation of the Hopf manifolds. Let N be a compact complex manifold admitting a negative holomorphic line bundle $L \to N$, in the sense that $c_1(L) < 0$. This means that there exists a Kähler metric ω_N on N so that $-\omega_N$ represents $c_1(L)$. At the same time, if we take a Hermitian structure h on L, consider the corresponding Chern connection D_h on L and denote by Θ_h its curvature, then $\frac{i}{2\pi}\Theta_h$ also represents $c_1(L)$. After an eventual conformal change of h, we can suppose that $\omega_N = -\frac{i}{2\pi}\Theta_h > 0$.

Consider $j : P = L - 0_N \to N$, which is a $\mathbb{C}^*$-principal bundle over N. The line bundle $j^*L \to P$ becomes holomorphically trivial over P, as it admits a global holomorphic frame $\sigma_P : P \to j^*L$ induced by the inclusion $i : P \to L$. Consequently, if we denote by $f : P \to \mathbb{R}$ the positive function defined by $f(s) = \|s\|^2_{h_x}$, $0 \neq s \in L_x$, then the corresponding connection form associated to the frame σ_P is defined globally on P by $\alpha = \partial \ln f$ and the curvature form of the induced Chern connection becomes exact on P: $j^*\Theta_h = \Theta_{j^*h} = \bar\partial\alpha = d\alpha$. If we let $\theta = -d\ln f$, which is a closed real one-form on P, we have: $\alpha = \frac{1}{2}(d\ln f + id^c \ln f) = -\frac{1}{2}(\theta + iJ\theta)$ and $4\pi j^*\omega_N = -2id\alpha = -dJ\theta$. Define the $(1,1)$-form on P:

$$\Omega = -dJ\theta + \theta \wedge J\theta = \mathrm{e}^\varphi dd^c \mathrm{e}^{-\varphi}, \quad \varphi(s) := -\ln \|s\|^2_h.$$

Letting $\mathcal{H} := \ker\alpha = \ker\theta \cap \ker J\theta \subset TP$ and $TF \subset TP$ be the tangent bundle of the fiber of $P \to N$ defined by $\ker j_*$, we have a $\mathcal{C}^\infty$ splitting $TP = \mathcal{H} \oplus TF$ which is Ω-orthogonal.

Moreover, we have $\Omega|_{\mathcal{H}} = 4\pi j^*\omega_N > 0$ and $\Omega|_{TF} = \theta \wedge J\theta > 0$. Thus Ω is a strictly positive $(1,1)$ form on P, and so an LCK metric with potential, with Lee form θ.

Now consider $\lambda \in \mathbb{C}^*$ with $|\lambda| < 1$ and let $\Gamma_\lambda \subset \mathbb{C}^*$ be the cyclic group generated by λ. The group Γ_λ acts on P fiber-wise, freely and properly discontinuously, so that the quotient $M := P/\Gamma_\lambda$ is a compact complex manifold and we have a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\Gamma_\lambda} & M \\ \downarrow{\scriptstyle \mathbb{C}^*} & \swarrow{\scriptstyle \mathbb{C}^*/\Gamma_\lambda \cong \mathbb{T}^2} & \\ N & & \end{array}$$

As Ω is $\mathbb{C}^*$-invariant, it is also Γ_λ-invariant, so descends to an LCK form with potential Ω on M. Also, note that if ξ denotes the holomorphic vector field on P induced by the action of $\mathbb{C}^*$, then as any $\mu \in \mathbb{C}^*$ acts on φ by $-2\ln|\mu| + \varphi$, we have $\theta(\xi) = -1$. Therefore it follows that, for $B = -\operatorname{Re}\xi$, we have $\theta(B) = 1$, $J\theta(B) = 0$ and $\iota_B j^*\omega_N = 0$, so B is the Lee vector field of Ω. Since it is real-holomorphic, Ω is then Vaisman by Proposition 3.2.2.

Let us note that P is the minimal Kähler cover of (X, Ω), with corresponding Kähler form $\Omega_K = dd^c f^{-\varphi}$, and $M \to N$ is a $\mathbb{T}^2 = \mathbb{C}^*/\Gamma_\lambda$-principal bundle. By a result of [Va80], all Vaisman manifolds whose canonical foliation $\mathcal{F}$ is strongly regular are obtained in this way. Finally, remark that if one takes $N = \mathbb{P}^n$ and $L = \mathcal{O}(-1)$, one obtains the standard Hopf manifold.

A generalisation of this construction starting from an anti-ample rank r holomorphic vector bundle $E \to N$ was given in [Ts97], as follows. Consider $p : \mathbb{P}E \to N$, where $\mathbb{P}E$ is the bundle of lines in E, i.e. for $x \in N$, $(\mathbb{P}E)_x = \mathbb{P}E_x = \{d \subset E_x \text{ complex line}\}$. Let $L_E \subset p^*E$ be the tautological sub-bundle of rank one of p^*E, so that for $d \in \mathbb{P}E$, $(L_E)_d = d$. Then E is called anti-ample, or equivalently E^* is called ample, if L_E^* is an ample line bundle over $\mathbb{P}E$. One can now repeat the above construction for $P = L_E - 0_{\mathbb{P}E}$ and $M = P/\Gamma_\lambda$, where $\lambda \in \mathbb{C}^*$ is of module smaller than 1. Let us note that we have a $\mathbb{C}^*$-equivariant biholomorphism $L_E^* - 0_{\mathbb{P}E} \cong L_E - 0_{\mathbb{P}E} \cong E - 0_N$, hence we can also view M as $M = E - 0_N/\Gamma_\lambda$. Thus, $M \to N$ is a fiber bundle over N with fiber the r-dimensional standard Hopf manifold $H_\lambda := H_{(\lambda,\dots,\lambda)}$, fitting in the diagram:

$$\begin{array}{ccc} M & & \\ \downarrow{\scriptstyle \mathbb{T}^2} & \searrow{\scriptstyle H_\lambda} & \\ \mathbb{P}E & \xrightarrow[\mathbb{P}^{r-1}]{} & N \end{array}$$

We can push this generalisation even further and construct fiber-bundles of fiber any diagonal Hopf manifold. As a general Hopf manifold $H_{(\lambda_1,\dots,\lambda_r)}$ does not fiber in tori, neither will our fiber bundle factor through $\mathbb{P}E$. So let N be a compact Kähler manifold admitting r (possibly isomorphic) holomorphic line bundles $L_1, \dots, L_r$, so that $E := L_1 \oplus \dots \oplus L_r$ is an anti-ample vector bundle of rank r over N. Let $\lambda_1, \dots \lambda_r \in \mathbb{C}^*$ be of module smaller than one, and consider the action of $\Gamma = \mathbb{Z}$ on $E - 0_N$ by: $1.(s_1, \dots, s_r) = (\lambda_1 s_1, \dots, \lambda_r s_r)$, where $x \in N$ and $s_i \in (L_i)_x$ for $i = \overline{1,r}$. Again, Γ acts freely and properly and the quotient $X = E - 0_N/\Gamma$ is a compact complex manifold. By the method used above, we can construct a Vaisman metric $\Omega = \mathrm{e}^{\varphi} dd^c \mathrm{e}^{-\varphi}$ on $E - 0_N$, but it will not be Γ-invariant. However, we can repeat the method of Subsection 2.6.1 in order to construct a new Γ-invariant Vaisman metric on $E - 0_N$.

Indeed, after choosing log-determinations of $\lambda_1, \dots, \lambda_r$, we can again extend the action of Γ to a holomorphic action of $\mathbb{R}$ on $E - 0_N$ by $\Phi^t(x, s_1, \dots, s_r) = (x, \lambda_1^t s_1, \dots, \lambda_r^t s_r)$. Then, defining $S_N := \varphi^{-1}(0) \subset E - 0_N$, which is compact as diffeomorphic to an $\mathbb{S}^{2r-1}$-bundle over N, we obtain a diffeomorphism $\Phi : S_N \times \mathbb{R} \to E - 0$, $(z, t) \mapsto \Phi^t(z)$. Thus we can define $\varphi_\lambda := p_\mathbb{R} \circ \Phi^{-1}$, where $p_\mathbb{R} : S_N \times \mathbb{R} \to \mathbb{R}$ is the natural projection. Then, in the same way as before, it can be shown that $\Omega_\lambda := \mathrm{e}^{\varphi_\lambda} dd^c \mathrm{e}^{-\varphi_\lambda}$ defines a Vaisman metric on X with Lee form $\theta = d\varphi_\lambda$. Note that the manifold X is a fiber bundle over N, with fiber $H_{\lambda_1,\dots,\lambda_r}$.

Finally, in the same way we can also construct LCK manifolds with positive potential which are not Vaisman. Let N be a compact Kähler manifold, let $L = L_1$ be a negative holomorphic line bundle over N, let $L_2 := L_1^m$ and let $E := L_1 \oplus L_2$, where $m \in \mathbb{N}^*$. Consider $P = E - 0_N$, let $\beta \in \mathbb{C}^*$ be of module smaller than 1, and $\mu \in \mathbb{C}^*$. Let $\mathbb{Z}$ act on P fiberwise by $1.(s_1, s_2) = (\beta s_1, \beta^m s_2 + \mu s_1^m)$, where $(s_1, s_2) \in E_x - 0_N$, $x \in N$. Again we can extend this action to a holomorphic action of $\mathbb{R}$ on P given by Φ_μ^t. If h denotes the Hermitian structure on L whose Chern curvature is negative, it induces naturally a Hermitian structure on E which we also denote by h, and we take $S := \{s \in E - 0_N | h(s, s) = 1\}$ and choose μ close enough to 0 so that the map $\Phi : S \times \mathbb{R} \to E - 0_N$, $(s, t) \mapsto \Phi_\mu^t(s)$ defines a diffeomorphism. Finally, we define $\varphi_\mu = p_\mathbb{R} \circ \Phi^{-1}$, and as in Subsection 2.6.2, there exists a positive constant $K > 0$ so that $\Omega_{\mu,K} = \mathrm{e}^{K\varphi_\mu} dd^c \mathrm{e}^{-K\varphi_\mu}$ is a positive $(1, 1)$-form. Consequently, it descends to an LCK metric with positive potential on the compact manifold $M := P/\mathbb{Z}$. Note that this time M fibers over N with fiber the non-diagonal Hopf manifold $H_{\beta,m,\mu}$. We will see in Example 3.4.7 that these manifolds cannot admit Vaisman metrics.

2.6.4 LCK metrics on blow-ups

The category of manifolds of LCK type is closed under certain blow-ups. Indeed, it was first shown by Tricerri [Tr82] and Vuletescu [Vu09] that a compact complex manifold admits an LCK metric if and only if its blow-up at a point admits one. Regarding blow-ups along manifolds of positive dimension, the following facts have been settled by Ornea, Verbitsky and Vuletescu. In what follows, (M, J) is a compact complex manifold, Z is a complex submanifold of M, $\mu : Bl_Z M \to M$ denotes the blow-up of M along Z and $E = \mu^{-1}(Z) \subset Bl_Z M$ denotes the exceptional divisor.

Theorem 2.6.1: *([OVV13]) If M admits some LCK metric and Z is of Kähler type, then $Bl_Z M$ admits an LCK metric as well.*

Theorem 2.6.2: *([OVV13]) Suppose $Bl_Z M$ admits some LCK metric. Then both E and Z are of Kähler type.*

However, in general it is unknown whether the existence of an LCK metric on $Bl_Z M$ implies the existence of an LCK metric on M. Note that this is not true for Kähler manifolds. Moreover, it was also shown in [OVV13] that if $Bl_Z M$ admits an LCK metric and $\dim_\mathbb{C} Z > 0$, then M cannot admit Vaisman metrics. Also we have the following simple remark, showing that in the end there are not so many cases left where one can perform blow-up on an LCK manifold:

Lemma 2.6.3: *Let (M, J) be a compact complex manifold and Z be a smooth compact complex submanifold of Kähler type. If $\dim_\mathbb{C} Z > 1$ or $Z \cong \mathbb{P}^1$, then M admits no exact LCK metrics.*

Proof. Suppose $\Omega = d_\theta \eta$ is LCK on M. Then it induces an LCK metric $j^*\Omega = d_{j^*\theta} j^*\eta$ on $j : Z \to M$. By hypotheses and by Theorem 2.2.8, we have $j^*\theta = df$, with $f \in \mathcal{C}^\infty(Z, \mathbb{R})$. The form $\Omega' := e^{-f} j^*\Omega$ is still LCK on Z, and at the same time is exact: $\Omega' = d(e^{-f} j^*\eta)$. As Z is compact, this is impossible. ∎

Note that this remark also shows that there do not exist exact LCK metrics (in particular Vaisman or with potential) on $Bl_Z M$ for any Z, as $Bl_Z M$ contains at least one smooth rational curve.

2.6.5 Complex surfaces

It is well known that a compact complex surface is not of Kähler type if and only if its first Betti number is odd. At the same time, most of these surfaces admit an LCK metric. As we have seen in 2.6.4, a surface admits an LCK metric if and only if some blow-up of it in a point admits one. Also, a blow-up never admits Vaisman, nor exact LCK metrics. Thus, we only need to look at minimal surfaces.

F. Belgun in [Bel00] classified all LCK surfaces of zero Euler characteristic. In particular, he obtained the list of all the surfaces of Vaisman type:

1. *Diagonal Hopf surfaces* and their finite quotients;
2. *Kodaira surfaces*, which are principal elliptic fiber bundles over an elliptic curve, and their finite quotients, the *secondary Kodaira surfaces*;
3. Minimal *properly elliptic surfaces*, which are surfaces M of Kodaira dimension $\mathrm{kod}(M) = 1$ admitting a proper holomorphic map to a smooth complex curve $p : M \to S$, such that a generic fiber of p is a (smooth) elliptic curve, and no fiber of p contains a smooth rational curve with auto-intersection number -1.

Note that the above list contains all the minimal surfaces of class VI, i.e. with odd b_1 and $p_g := \dim_{\mathbb{C}} H^0(M, K_M) > 0$. The remaining non-Kähler minimal surfaces form the class VII_0, defined by the conditions $b_1 = 1$ and $\mathrm{kod}(M) = -\infty$, and this class is much more complicated. We invite the reader to check the expository paper [Po14] and the references therein concerning the LCK metric structure of class VII_0-surfaces.

A class VII_0-surface with $b_2 = 0$ is either Hopf or Inoue-Bombieri. The latter were introduced in [In74] and [Bm73], and are obtained as quotients of $\mathbb{H} \times \mathbb{C}$ by discrete groups of affine biholomorphisms. They all have the strucure of a solvmanifold, and are separated in three classes: S^0, $S^+_t (t \in \mathbb{C})$ and S^-. LCK metrics were constructed on all of them by F. Tricerri [Tr82], except for the surfaces S^+_t with $t \in \mathbb{C} - \mathbb{R}$. Later, F. Belgun showed that in fact the ones that were left out do not admit LCK metrics. Moreover, he used this class of surfaces to show that the category of LCK manifolds is not closed under small deformations. The class S^0 has a higher-dimensional analogue given by the OT manifolds, which we present in a separated chapter. Concerning the special LCK metrics on these surfaces, as we have seen, Hopf manifolds admit metrics with positive potential, while Inoue-Bombieri surfaces do not admit exact LCK metrics, as shown in [O16].

As for the class VII_0 surfaces with $b_2 > 0$, all known examples are *Kato surfaces*, i.e. surfaces which admit a *global spherical shell* (GSS). A GSS in M is a neighbourhood of the sphere $\mathbb{S}^3$ in $\mathbb{C}^2$ which is embedded holomorphically in M in such a way that $M - V$ is connected. It is conjectured that all class VII_0 surfaces with $b_2 > 0$ are Kato surfaces. LCK metrics on certain classes of these were constructed by LeBrun [LeB91] - on certain *parabolic Inoue* surfaces, and later by Fujiki-Pontecorvo [FP10] - on *hyperbolic* and *half Inoue* surfaces - using a twistor construction. Shortly after, Brunella in [Bru10] and [Bru11] showed that all Kato surfaces admit LCK metrics. As was noted in [FP16, Remark 3.2], Kato surfaces cannot admit LCK metrics with potential, because their universal covers contain compact complex curves. In fact, by this argument it follows that they do not admit exact LCK metrics.

Chapter 3

Existence Criteria for LCK Metrics

3.1 Introduction

In the present chapter, we investigate the relation between the group of biholomorphisms of a compact complex manifold of LCK type and the existence of certain special LCK metrics. We find conditions on this group that imply or are equivalent to the existence of a particular type of metric. In particular, we also find obstructions to the existence of any kind of LCK metric.

A particular role in this discussion is played by the Lee vector field, which we study in Section 3.2. Recently, in [MMO17] there were given sufficient conditions for an LCK metric with holomorphic Lee vector field to be Vaisman. We add one more such condition in Proposition 3.2.2, which should be particularly useful when constructing examples, as it is easy to check. Moreover, in [MMO17] the authors constructed an example of a non-Vaisman LCK metric with holomorphic Lee vector field. We remark that in this construction, one can even find a positive potential for the metric, showing that the condition in Proposition 3.2.2 cannot be sharpened.

In Section 3.3, we review the proof of Ornea-Verbitsky [OV12] concerning the existence of an LCK metic with potential, given the presence of a holomorphic action of $\mathbb{S}^1$ which lifts to an effective $\mathbb{R}$-action on the minimal cover. We show that in their proof, one can find an explicit positive potential, by means of an ODE, without the need of invoking the more recent paper [OV17].

The next section 3.4 generalizes a criterion of Kamishima-Ornea [KO05] for the existence of a Vaisman metric. We show that a compact manifold of LCK type admits a Vaisman metric if and only if its group of biholomorphisms contains a torus which is not purely real. As a corollary, we obtain that a compact complex manifold whose group of biholomorphisms contains a compact torus whose Lie algebra $\mathfrak{t}$ verifies $\dim_{\mathbb{C}} \mathfrak{t} \cap i\mathfrak{t} > 1$ does not admit any LCK metric. An application of this is the characterisation of all manifolds of LCK type among the torus principal bundles, given in Section 3.5, analogous to a theorem of Blanchard [Bl54] in the Kähler context.

On the other hand, we discuss the issue of irreducibility in LCK geometry. From early time [Va80], it was known that if one takes two compact LCK manifolds (M_i, Ω_i), $i = 1, 2$, the product metric is not LCK on $M_1 \times M_2$. However, whether there might exist some other LCK metric on $M_1 \times M_2$ has remained an open question, and in Section 3.6, we extend the known cases ([Ts99], [OPV14]) in which this fails.

A related question concerns the irreducibility of the natural connections determined by an LCK metric. It turns out that the Levi-Civita connection is as irreducible as it can be: it was shown in [MMP16] that any strict LCK metric on a compact manifold of complex dimension n has irreducible holonomy $\mathrm{SO}(n)$ of the Levi-Civita connection, unless the metric is Vaisman, in which case the holonomy is $\mathrm{SO}(2n-1)$. Thus we turn to the Weyl connection, which is also the Levi-Civita connection of the local Kähler metrics, and appears to have more interesting properties. We first adapt a result of Kourganoff [Kou15] to the LCK context regarding the structure of Weyl-reducible compact manifolds (Theorem 3.7.2), from which we then derive Theorem 3.7.7, implying in particular that an exact LCK metric is Weyl-irreducible. Note that Weyl-reducible LCK metrics do exist on the OT manifolds of type $(s, 1)$.

3.2 The Lee vector field

Recall that a Vaisman metric (Ω, θ) of volume 1 verifies the formula

$$\Omega = -dJ\theta + \theta \wedge J\theta \tag{3.2.1}$$

and its Lee vector field, defined by $\iota_B\Omega = J\theta$, is holomorphic, Killing, symplectic and of constant norm 1, cf. Proposition 2.4.5. Also, any metric which is conformal to one verifying (3.2.1) is called *LCK with positive potential*.

A natural question one could ask is what kind of conditions should one impose on an LCK metric with positive potential to ensure that it is a Vaisman one. Bellow we give a list of the equivalent conditions, many of them already classical. The less direct implications are based on a result of A. Moroianu and S. Moroianu:

Theorem 3.2.1: *([MM17]) Let (M, J, Ω) be a compact LCK manifold so that the symmetric endomorphism $\nabla B \in \mathrm{End}(TM)$ anticommutes with J. Then Ω is Vaisman.*

Proposition 3.2.2: *Let (Ω, θ) be an LCK structure on a compact complex manifold (M, J) with potential equal to 1, i.e. $\Omega = -dJ\theta + \theta \wedge J\theta$. Let B denote the corresponding Lee vector field. Then the following are equivalent:*

(i) Ω *is Vaisman;*

(ii) B *is real-holomorphic;*

(iii) B *is of constant norm* $a \in \mathbb{R}_+$*;*

(iv) B *is of constant norm* 1*;*

(v) B *is an infinitesimal symplectomorphism;*

(vi) B *is Killing*

(vii) Ω *is Gauduchon.*

Proof. Let us start by noting that (iv) is equivalent to (v) via formula (2.4.5). We have seen that if Ω is Vaisman, then the facts from (ii) to (vii) hold. Moreover, we have seen that (vi) always implies (i). It will be enough then to show that $(iii) \Rightarrow (iv)$, $(ii) \Rightarrow (i)$, $(vii) \Rightarrow (iv)$ and (v) implies that ∇B anticommutes with J, and so implies (i) by Theorem 3.2.1.

The implication $(iii) \Rightarrow (iv)$: note more generally that a metric of the form $\Omega = -dJ\theta + \theta \wedge J\theta$ verifies $\int_M \|B\|^2 \frac{\Omega^n}{n!} = \int_M \frac{\Omega^n}{n!}$, from which then (iv) follows. Indeed:

$$\int_M \|B\|^2 \frac{\Omega^n}{n!} = \int_M \theta \wedge J\theta \wedge \frac{\Omega^{n-1}}{(n-1)!} = \int_M \theta \wedge J\theta \wedge \frac{(-dJ\theta)^{n-1}}{(n-1)!} = \int_M \frac{\Omega^n}{n!}.$$

The implication $(ii) \Rightarrow (i)$: if B is real-holomorphic, then also $A = JB$ is. The Cartan formula and $\mathcal{L}_A\theta = 0$ imply:

$$\begin{aligned} 0 =& \mathcal{L}_A J\theta = d\iota_A J\theta + \iota_A dJ\theta = \\ =& -d(\theta(JA)) + \iota_A(\theta \wedge J\theta - \Omega) = \\ =& d(\|B\|^2) - \theta\|B\|^2 + \theta = \\ =& d_\theta(\|B\|^2 - 1). \end{aligned}$$

Now Lemma 2.2.9 implies that $\|B\|^2 = 1$, thus (2.4.5) gives $\mathcal{L}_B\Omega = 0$. Finally, since B is holomorphic and preserves the symplectic form, it is also Killing, hence Ω is Vaisman.

The implication $(vii) \Rightarrow (iv)$: the metric Ω is Gauduchon if and only if $dd^c\Omega^{n-1} = 0$. As $[J, d] = d^c$, we have $dd^c = dJd$, so:

$$\begin{aligned} dd^c\Omega^{n-1} &= dJ((n-1)\theta \wedge \Omega^{n-1}) \\ &= (n-1)(dJ\theta \wedge \Omega^{n-1} - J\theta \wedge \theta \wedge \Omega^{n-1}(n-1)). \end{aligned}$$

Hence $dd^c\Omega^{n-1} = 0$ is equivalent to $-dJ\theta \wedge \Omega^{n-1} = \theta \wedge J\theta \wedge \Omega^{n-1}$, or also, using the formula (3.2.1) of Ω, to $\Omega^n = n\theta \wedge J\theta \wedge \Omega^n = \|B\|^2\Omega^n$, so $dd^c\Omega^n = 0$ is equivalent to $\|B\|^2 = 1$.

The implication $(v) \Rightarrow (i)$: let us denote by $K := \mathcal{L}_B J$. We have $0 = \mathcal{L}_B(J^2) = JK + KJ$. The hypothesis $\mathcal{L}_B\Omega = 0$ writes:

$$\begin{aligned} 0 = \mathcal{L}_B g(J\cdot, \cdot) &= (\mathcal{L}_B g)(J\cdot, \cdot) + g(K\cdot, \cdot) \\ &\Leftrightarrow \mathcal{L}_B g = g(KJ\cdot, \cdot). \end{aligned}$$

At the same time we also have, for any $X, Y \in \Gamma(TM)$: $\mathcal{L}_B g(X, Y) = g(\nabla_X B, Y) + g(X, \nabla_Y B)$. But $d\theta = 0$ implies that $\nabla\theta$ is symmetric, hence also ∇B is, which thus gives $\mathcal{L}_B g = 2g(\nabla_. B, \cdot)$. With the above, we obtain:

$$2\nabla B = KJ.$$

In particular, $J\nabla B = \frac{1}{2}JKJ = \frac{1}{2}K = -\nabla_{J.}B$, and so we can apply Theorem 3.2.1. ■

Note that for the last implication we did not use the hypothesis that Ω is a metric with potential, so in fact we have, via Theorem 3.2.1:

Proposition 3.2.3: *Let (M, J, Ω) be a compact LCK manifold with Lee vector field B. Then Ω is Vaisman if and only if $\mathcal{L}_B\Omega = 0$.*

As will be seen in the last chapter, Lemma 5.2.1, OT manifolds provide examples of LCK metrics which are Gauduchon and whose Lee vector field is of constant norm, showing that in Proposition 3.2.2, condition (vi) together with (vii) alone do not imply that the metric is Vaisman. On the other hand, one could ask whether the holomorphicity of the Lee vector field implies that the metric is Vaisman. In the recent paper [MMO17], it is shown:

Theorem 3.2.4: *([MMO17]) Let (M, Ω, J) be a compact LCK manifold with holomorphic Lee vector field B. If B is of constant norm, or if Ω is Gauduchon, then Ω is Vaisman.*

In the same paper, the authors also construct an example of an LCK metric which is not Vaisman, but which has holomorphic Lee vector field, thus showing that also condition (*ii*) in Proposition 3.2.2 alone is not enough to imply the Vaisman condition. We now present this example, with the remark that in the construction of [MMO17], the metric can in fact be chosen with positive potential, i.e. conformal (but not homothetic) to a metric of the form 3.2.1. This shows that, regarding condition (*ii*), the hypotheses in Proposition 3.2.2 cannot be relaxed.

Example 3.2.5: ([MMO17]) Let (M, J, Ω, θ) be a compact Vaisman manifold with $\|\theta\|^2 = 1$, and let B be its Lee vector field. Suppose there exists a non-constant smooth function $f \in \mathcal{C}^\infty(M, \mathbb{R})$ verifying $f > -1$ everywhere on M and such that df is colinear with θ. After taking the interior product with B, this last condition is more precisely $df = B(f)\theta$. Such functions exist any time B generates an $\mathbb{S}^1$-action on M, for instance on the standard Hopf manifold.

Consider next the form:

$$\Omega' := \Omega + f\theta \wedge J\theta = d_{(1+f)\theta}(-dJ\theta).$$

As $f > -1$, Ω' is a strictly positive real $(1,1)$-form on M, and verifies $d\Omega' = (1+f)\theta \wedge \Omega'$. Thus Ω' is the fundamental form of an LCK metric with Lee form $\theta' = (1+f)\theta$.

Lemma 3.2.6: *The Lee vector field of Ω' is B, and so also holomorphic. The metric Ω' is not conformal to any Vaisman metric.*

Proof. As $\iota_B\Omega' = (1+f)J\theta = J\theta'$, B is also the Lee vector field of Ω'. Now suppose that there exists a Vaisman metric Ω'' on M so that $\Omega'' = \mathrm{e}^h\Omega'$. By a theorem of K. Tsukada [Ts97], the Lee vector field of a Vaisman metric is unique on the manifold M up to multiplication by a constant. Thus, we can suppose right from the beginning that the Lee vector field of Ω'' is also B. Now this reads:

$$\mathrm{e}^h J\theta' = \mathrm{e}^h \iota_B\Omega' = \iota_B\Omega'' = J\theta' + d^c h$$

that is: $dh + \theta'(1 - \mathrm{e}^h) = 0$, or also, after multiplying by $-\mathrm{e}^{-h}$: $d_{\theta'}(\mathrm{e}^{-h} - 1) = 0$. As θ' has no zero, it is non-exact, so Lemma 2.2.9 implies that $\mathrm{e}^{-h} = 1$, i.e. $h = 0$ and Ω' is Vaisman. But this last fact is impossible, as the norm of B is non-constant: $\Omega'(B, JB) = \theta'(B) = 1 + f$. ∎

Suppose now that the flow of B on M is periodic: $\Phi_B^{2\pi} = \mathsf{id}_M$, and that we have a diffeomorphism $M \cong N \times \mathbb{S}^1$, where N is a compact Sasaki manifold. Any non-constant function on $\mathbb{S}^1$, bounded bellow by -1, induces a function f on M verifying the desired properties.

Lemma 3.2.7: *Under the above hypothesis, the metric Ω' admits a positive potential.*

Proof. We think of f as a function on $\mathbb{R}$ which is 2π-periodic, and we are looking for another positive function $g : \mathbb{R} \to \mathbb{R}$, also 2π-periodic, verifying, when seen as a function on M:

$$\Omega' = d_{\theta'} d^c_{\theta'} g. \tag{3.2.2}$$

The function g we are looking for verifies that both dg and $d\mathcal{L}_B g$ are colinear with θ, which implies the following relations:

$$dg = \mathcal{L}_B g \cdot \theta, \quad d^c g = \mathcal{L}_B g \cdot J\theta, \quad dd^c g = \mathcal{L}_B^2 g \cdot \theta \wedge J\theta + \mathcal{L}_B g \cdot dJ\theta.$$

With this in mind, (3.2.2) writes:

$$-dJ\theta + (1+f)\cdot\theta\wedge J\theta = (\mathcal{L}_B g - g(1+f))dJ\theta + \\ +(\mathcal{L}_B^2 g - \mathcal{L}_B f\cdot g - 2(1+f)\mathcal{L}_B g + g(1+f)^2)\theta\wedge J\theta.$$

Now, the two forms $-dJ\theta$ and $\theta\wedge J\theta$ are linearly independent, which implies that in the above equation, the corresponding coefficients preceding them must be equal. We denote by t the variable on $\mathbb{R}$, and identify B with the vector field $\frac{d}{dt}$ on $\mathbb{R}$. Seeing f and g as functions on $\mathbb{R}$, (3.2.2) now becomes equivalent to:

$$\frac{d}{dt}g - g(1+f) + 1 = 0 \tag{3.2.3}$$

$$\frac{d^2}{dt^2}g - 2(1+f)\frac{d}{dt}g - g\frac{d}{dt}f + g(1+f)^2 - (1+f) = 0. \tag{3.2.4}$$

By differentiating the first equation, one obtains the second one, while the first ODE has a solution of the form:

$$g(t) = (c - \int_0^t \mathrm{e}^{-F(s)}ds)\mathrm{e}^{F(t)}, \quad \text{with } F(t) = a + \int_0^t (f(s)+1)ds, \quad a, c\in\mathbb{R}.$$

Thus a solution g of the above system exists, and now it is left for us to show that we can choose the constants a and c such that g is moreover strictly positive and 2π-periodic.

Let us note that, because f is 2π-periodic, we have, for any $t\in\mathbb{R}$:

$$F(t+2\pi) = F(t) + b, \quad \text{where } b = \int_0^{2\pi}(f(s)+1)ds > 0.$$

Thus we obtain:

$$\begin{aligned} g(t+2\pi) &= (c - \int_0^{2\pi}\mathrm{e}^{-F(s)}ds - \int_{2\pi}^{2\pi+t}\mathrm{e}^{-F(s)}ds)\mathrm{e}^{F(t)}\mathrm{e}^b \\ &= (c - K - \int_0^t \mathrm{e}^{-F(u)}\mathrm{e}^{-b}du)\mathrm{e}^{F(t)}\mathrm{e}^b \\ &= g(t) + \mathrm{e}^{F(t)}((c-K)\mathrm{e}^b - c) \end{aligned}$$

where $K = \int_0^{2\pi}\mathrm{e}^{-F(s)} > 0$ and, for the second equality, we made the change of variable $s = u + 2\pi$. Thus, in order for g to be 2π-periodic, we take $c := \frac{K\mathrm{e}^b}{\mathrm{e}^b - 1} > 0$. Finally, we need to see that g is in fact positive, which is also equivalent to saying that $v(t) := c - \int_0^t \mathrm{e}^{-F(s)}ds$ is positive. Note that $\frac{d}{dt}v(t) = -\mathrm{e}^{-F(t)} < 0$, so v can change sign at most once, and the same is then true for the function g. On the other hand, g is periodic and $g(0) = c\mathrm{e}^a > 0$, thus g is indeed everywhere positive. ■

Note that, although the above example shows that there can exist non-Vaisman metrics with holomorphic Lee vector field, it is however constructed out of a Vaisman metric. So a question remains open:

Question 3.2.8: Let (M, J, Ω) be a compact LCK manifold with holomorphic Lee vector field. Does there exist an LCK metric on M, not necessarily conformal to Ω, which is Vaisman?

This question is reminiscent to the criterion for the existence of LCK metrics with positive potential presented in the next section. However, this time we do not ask for B, the Lee vector field, to have closed orbits. Moreover, if the answer to the above question is yes, it would be interesting to know if one can recover the Vaisman metric (for example like in the proofs of Theorem 3.3.1 or of Theorem 4.4.1). Finally, recall that the Lee vector field of any Vaisman metric is uniquely determined up to multiplication by a positive constant, by [Ts97]. A related question is then:

Question 3.2.9: Suppose that the Lee vector field of an LCK metric on a manifold of Vaisman type is holomorphic. Is it then the Lee vector field of a Vaisman metric?

3.3 Existence of LCK metrics with potential

We present in this section a different proof of a result of L.Ornea and M.Verbitsky, [OV12]. It is a criterion for the existence of an LCK metric with positive potential in terms of the existence of a holomorphic vector field with a particular property. We note though that in the original proof, the positivity of the constructed potential is not clear, and only by invoking a recent, more difficult result from [OV17] does the proof become complete. However, the original proof can be made complete and self contained by a more careful analysis, which we will do next.

Theorem 3.3.1: *([OV12] and [OV17]) Let (M, J, Ω, θ) be a compact LCK manifold admitting a holomorhic action of $\mathbb{S}^1$ which, on the minimal cover $\hat{M}$, lifts to a faithful $\mathbb{R}$-action. Then there exists an LCK metric with positive potential whose Lee form is cohomologous to θ.*

Proof. Let us denote by D the real holomorphic vector field on M (and on $\hat{M}$) generating the $\mathbb{S}^1$ action. By a standard averaging argument which does not change the de Rham class of θ, we can suppose that both Ω and θ are preserved by D. In particular, $\mathcal{L}_D\theta = 0$ implies that $\theta(D)$ is constant, and as D generates an $\mathbb{R}$-action on $\hat{M}$, $\theta(D) = a \neq 0$ by Remark 2.5.5. Let C be the vector field on $\hat{M}$ defined by $C := \frac{1}{a}D$.

Let $\theta = d\varphi$ on $\hat{M}$, and let us denote by $\omega := \exp(-\varphi)\Omega$ the corresponding Kähler form. Then we have:

$$\mathcal{L}_C\omega = -\theta(C)\omega = -\omega. \tag{3.3.1}$$

Let us denote by η the real one-form on $\hat{M}$ defined by $\iota_C\omega = \eta$. Then (3.3.1) together with Cartan's formula imply:

$$\omega = -(d\iota_C + \iota_C d)\omega = -d\eta. \tag{3.3.2}$$

At the same time, using the fact that $\eta(JC) = \omega(C, JC) = \|C\|^2_\omega := f$, we have:

$$\mathcal{L}_{JC}\eta = d\iota_{JC}\eta + \iota_{JC}d\eta = df - J\eta,$$

from which it follows:

$$\mathcal{L}_{JC}\omega = -d(df - J\eta) = dJ\eta$$
$$\mathcal{L}^2_{JC}\omega = dJ\mathcal{L}_{JC}\eta = dd^cf + d\eta = dd^cf - \omega.$$

If we let Φ_t denote the one-parameter group generated by JC and denote by $\omega_t := \Phi_t^*\omega$ and by $f_t = \Phi_t^*f$, the last equation reads:

$$\frac{d^2}{dt^2}\omega_t = -\omega_t + dd^cf_t. \tag{3.3.3}$$

Let now g_t be the real-valued functions on $\hat{M}$ defined by the second order linear differential equation:

$$\frac{d^2}{dt^2}g_t + g_t = f_t, \quad g_0 = 0, \quad \frac{d}{dt}|_{t=0}g_t = 0. \tag{3.3.4}$$

We want to show that $\omega_t = \cos t\omega + \sin tdJ\eta + dd^c g_t$. For this, consider the forms $\beta_t := \omega_t - (\cos t\omega + \sin tdJ\eta + dd^c g_t)$, $t \in \mathbb{R}$. Using (3.3.3) and the definition (3.3.4) of the functions g_t, we have:

$$\begin{aligned}\frac{d^2}{dt^2}\beta_t &= \frac{d^2}{dt^2}\omega_t + \cos t\omega + \sin tdJ\eta - dd^c(\frac{d^2}{dt^2}g_t)\\ &= -\omega_t + dd^c f_t + \cos t\omega + \sin tdJ\eta - dd^c f_t + dd^c g_t\\ &= -\beta_t.\end{aligned}$$

Thus, the forms β_t verify the following homogeneous second order linear differential equation with initial conditions:

$$\frac{d^2}{dt^2}\beta_t + \beta_t = 0, \quad \beta_0 = 0, \quad \frac{d}{dt}|_{t=0}\beta_t = 0.$$

By the uniqueness of the solution, we have then that for all $t \in \mathbb{R}$, β_t vanishes identically, and so:

$$\omega_t = \cos t\omega + \sin tdJ\eta + dd^c g_t, \quad t \in \mathbb{R}. \tag{3.3.5}$$

Define now, using (3.3.5), a new form $\hat{\omega}$ by:

$$\hat{\omega} := \frac{1}{2\pi}\int_0^{2\pi} \Phi_t^*\omega dt = dd^c\frac{1}{2\pi}\int_0^{2\pi} g_t dt$$

and let us denote by g the function $1/2\pi \int_0^{2\pi} g_t dt$. As $\{\Phi_t\}_{t\in\mathbb{R}}$ is a subgroup of biholomorphism of $\hat{M}$, $\hat{\omega}$ is a Kähler form on $\hat{M}$. We wish to show that g is a strictly positive function on $\hat{M}$. Note first that, as $\theta(C) = 1$, C has no zeroes so the function f is everywhere positive. Moreover, as JC is real holomorphic, we have $[C, JC] = 0$, so Φ_t preserves both C and JC. This gives, for any $x \in \hat{M}$:

$$f_t(x) = \omega_{\Phi_t(x)}(C, JC) = \omega_{\Phi_t(x)}((d_x\Phi_t)C, (d_x\Phi_t)JC) = (\Phi_t^*\omega)_x(C, JC)$$

thus also the function f_t is strictly positive for any $t \in \mathbb{R}$.
Fix $x \in \hat{M}$ and define the functions $f_x, g_x : \mathbb{R} \to \mathbb{R}$ by $f_x(t) = f_t(x)$ and $g_x(t) = g_t(x)$. By (3.3.4), they satisfy:

$$g_x'' + g_x = f_x, \quad g_x(0) = 0, \quad g_x'(0) = 0. \tag{3.3.6}$$

Then we have:

$$\int_0^{2\pi} g_x(t)dt = \int_0^{2\pi} f_x(t)dt - \int_0^{2\pi} g_x''(t)dt = \int_0^{2\pi} f_x(t)dt - g_x'(2\pi). \tag{3.3.7}$$

On the other hand, integrating by parts and using (3.3.6) we compute:

$$\begin{aligned} g_x'(2\pi) &= g_x'(t)\cos t\Big|_0^{2\pi} \\ &= \int_0^{2\pi} g_x''(t)\cos t dt + \int_0^{2\pi} g_x'(t)(-\sin t)dt \\ &= \int_0^{2\pi} g_x''(t)\cos t dt - \left(g_x(t)\sin t\Big|_0^{2\pi} - \int_0^{2\pi} g_x(t)\cos t dt \right) \\ &= \int_0^{2\pi} f_x(t)\cos t dt. \end{aligned}$$

Thus, it follows from (3.3.7):

$$\int_0^{2\pi} g_x(t)dt = \int_0^{2\pi} f_x(t)(1-\cos t)dt > 0$$

implying that the function g is indeed everywhere positive.

Hence we can define $\hat{\theta} := d\ln g$. Note that by the uniqueness of the solution of (3.3.4), the functions g_t have the same Γ-equivariance as the functions f_t, or also as the function f. Here, Γ denotes the deck group of the cover $\hat{M} \to M$. Also we should note that, as C and JC are Γ-invariant, being lifts of vector fields from M, then the Γ-equivariance of $f = \omega(C, JC)$ is exactly the equivariance of ω. Thus it follows that $\hat{\theta}$ has the same Γ-equivariance as θ, and so the two one-forms are cohomologous. Hence the form

$$\hat{\Omega} := g^{-1}dd^c g \tag{3.3.8}$$

descends to M to an LCK metric with positive potential with Lee form $\hat{\theta}$, and the proof is finished. ■

Remark 3.3.2: Let us note that the above construction of an LCK metric with potential is natural and only depends on Ω and on C. In particular, if Ω is already JC-invariant, which will imply that the metric is Vaisman, then we have $f_t = f$ and the solution of (3.3.4) is then $g_t = (1-\cos t)f$, so in particular the potential $g = f$ remains unchanged.

Remark 3.3.3: On the other hand, for a metric Ω which is not JC-invariant, the above construction gives us a countable set of metrics with potential associated to the de Rham class of θ. Indeed, we considered the potential $g_{[1]} := g$, but for any $n \in \mathbb{N}^*$, the potential $g_{[n]} := 1/2n\pi \int_0^{2n\pi} g_t dt$ works as well.

3.4 Existence of Vaisman metrics

In this section we are interested in giving a criterion for the existence of Vaisman metrics on a complex manifold of LCK type only in terms of its group of holomorphic automorphisms. In particular, we will find that the existence of an LCK metric imposes restrictions on this group, and so our result can also be used as a criterion of non-existence of LCK metrics. This has interest in itself, as no obstructions coming from the existence of LCK metrics are known so far other than the inequality $b_1 < 2h^{0,1}$ (Theorem 2.2.8, Remark 2.3.2).

A criterion for deciding whether a given LCK conformal class is Vaisman or not was obtained by Y. Kamishima and L. Ornea in [KO05], but of course it involves already knowing the

Vaisman class. Its proof is quite involved, but it can also be obtained as a corollary of Theorem 3.4.3.

Theorem 3.4.1: *([KO05]) Let (M, J, g) be a compact connected strict LCK manifold. Then g is conformal to a Vaisman metric if and only if* $\mathrm{Aut}(M, J, [g])$ *contains a one-dimensional complex Lie group which does not act isometrically on the corresponding Kähler metric.*

We start by giving the main proposition, which will directly imply the general criterion. We will call a torus $\mathbb{T} \subset \mathrm{Aut}_0(M, J)$ with Lie algebra $\mathfrak{t}$ *purely real* if $\mathfrak{t} \cap i\mathfrak{t} = \mathfrak{t} \cap J\mathfrak{t} = \{0\}$.

Let (M, J, Ω, θ) be a Vaisman manifold with corresponding fundamental vector fields B and $A = JB$. Then $A, B \in \mathfrak{aut}(M, J, \Omega)$ generate a holomorphic $\mathbb{R} \times \mathbb{R}$ action on M, and we will denote by G the image of $\mathbb{R} \times \mathbb{R}$ in $\mathrm{Aut}_0(M, J, \Omega)$. Since the Lie group $\mathrm{Aut}_0(M, J, \Omega)$ is compact, we can take the closure of G in it, obtaining thus a compact torus $\mathbb{T} \subset \mathrm{Aut}_0(M, J, \Omega)$. The torus $\mathbb{T}$ is not purely real, since both A and B are in $i\mathfrak{t} \cap \mathfrak{t}$. In fact, we have:

Proposition 3.4.2: *Let $(M, J, [\Omega], [\theta]_{dR})$ be a strict LCK manifold and $\mathbb{T} \subset \mathrm{Aut}_0(M, J, [\Omega])$ be a compact torus. If $\mathbb{T}$ is not purely real, then $[\Omega]$ is Vaisman and $\mathfrak{t} \cap i\mathfrak{t} = \mathbb{R}\{A, B\}$, where $B = -JA$ is the Lee vector field of some Vaisman metric in $[\Omega]$.*

Proof. Choose a $\mathbb{T}$-invariant LCK structure (Ω, θ) in the conformal class $[\Omega]$, so that for any $X \in \mathfrak{t}$, $d(\theta(X)) = \mathcal{L}_X\theta = 0$. Let $0 \neq C \in \mathfrak{t}$ with $D := JC \in \mathfrak{t}$. Then both $\theta(C)$ and $\theta(D)$ are constant. However, we cannot have $\theta(C) = \theta(D) = 0$. Indeed, if it was the case, then:

$$\begin{aligned} 0 =& \iota_{[C,D]}\Omega = \mathcal{L}_C\iota_D\Omega - \iota_D\mathcal{L}_C\Omega = \\ =& d\iota_C\iota_D\Omega + \iota_C d\iota_D\Omega = \\ =& d(-\|C\|^2) + \theta\iota_C\iota_D\Omega = d_\theta(-\|C\|^2) \end{aligned}$$

implying, by Lemma 2.2.9, that $\|C\|^2 = 0$, contradiction. Hence, if $\theta(C) = a$ and $\theta(D) = b$, then $X := aD - bC \neq 0$ still verifies $X \in \mathfrak{t}$ and $JX \in \mathfrak{t}$ and, moreover, $\theta(X) = 0$, so $\theta(JX) \neq 0$. Therefore, we can suppose from the beginning that $\theta(C) = 1$ and $\theta(D) = 0$.

Let $f := \|C\|^2_\Omega$, which is an everywhere positive function since C cannot have any zeros. Take $\Omega' := \frac{1}{f}\Omega$, with corresponding Lee form $\theta' = \theta - d\ln f$. Then, since f is preserved by both C and D, we still have $\theta'(C) = 1$ and $\theta'(D) = 0$, and $C, D \in \mathfrak{aut}(M, J, \Omega')$.

Let $\eta := \iota_C\Omega'$. Then we have:

$$d\eta = \mathcal{L}_C\Omega' - \iota_C d\Omega' = -\theta'(C)\Omega' + \theta' \wedge \eta$$

or also $\Omega' = d_{\theta'}(-\eta)$. Since D preserves both C and Ω', it also preserves η. Moreover, we have $1 = \|C\|^2_{\Omega'} = \eta(D)$. Hence we get:

$$\begin{aligned} 0 =& \mathcal{L}_D\eta = d\iota_D\eta + \iota_D d\eta = \iota_D(-\Omega' + \theta' \wedge \eta) = \\ =& -J\eta + \theta'(D)\eta - \theta'\eta(D) = -J\eta - \theta'. \end{aligned}$$

Finally, this implies that $\eta = J\theta'$, so that C is actually the Lee vector field B of Ω'. Since C is holomorphic and preserves Ω', it is also Killing, so $\nabla\theta' = d\theta' = 0$, that is, Ω' is Vaisman.

Finally, since a Vaisman metric is unique in its conformal class up to multiplication by constants, it follows that $\mathfrak{t} \cap i\mathfrak{t} = \mathbb{R}\{C, D\} = \mathbb{R}\{A, B\}$. ■

Theorem 3.4.3: *A connected compact complex manifold (M, J) of LCK type admits a Vaisman metric if and only if* $\mathrm{Aut}_0(M, J)$ *contains a torus which is not purely real.*

Proof. As we already noted at the beginning of the section, if M admits a Vaisman metric then the corresponding holomorphic vector fields B and $A = JB$ sit in the Lie algebra of a torus in $\mathrm{Aut}_0(M, J)$.

Conversely, suppose $\mathbb{T} \subset \mathrm{Aut}_0(M, J)$ is not purely real. Take any LCK metric (Ω, θ) and average it over $\mathbb{T}$, in order to get a $\mathbb{T}$-invariant LCK metric. Hence we have $\mathbb{T} \subset \mathrm{Aut}_0(M, J, \Omega)$, and we can apply Proposition 3.4.2 in order to get the conclusion. ∎

A direct consequence of Proposition 3.4.2 is:

Corollary 3.4.4: *Let $(M, J, [\Omega])$ be an LCK manifold and $\mathbb{T} \subset Aut_0(M, J, [\Omega])$ be a maximal torus. Then $\dim_{\mathbb{C}} \mathfrak{t} \cap i\mathfrak{t} \leq 1$, with equality if and only if $[\Omega]$ is Vaisman.*

As a consequence of this, we also obtain:

Corollary 3.4.5: *Let (M, J) be a complex manifold so that $\mathrm{Aut}(M, J)$ contains a compact torus $\mathbb{T}$ whose Lie algebra $\mathfrak{t}$ verifies $\dim_{\mathbb{C}} \mathfrak{t} \cap i\mathfrak{t} > 1$. Then (M, J) admits no LCK metric.*

Let us illustrate this criterion in some examples. Of course, the examples that follow have already been settled ([Bel00]), but the arguments based on Theorem 3.4.3 are simpler.

Example 3.4.6: Consider the non-diagonal Hopf surface $H_{\beta,m,\lambda}$ defined in Section 2.6.1. We want to show that it admits no Vaisman metric, so we need to prove that there is no real-holomorphic vector field X so that both X and JX have closed orbits. Indeed, cf. [Bel00] and [MMP17], the complex Lie algebra of holomorphic vector fields on $H_{\beta,m,\lambda}$ identifies with the Lie algebra of Γ-invariant vector fields on W: $\mathfrak{g} = \mathbb{C}\{Z_1 = z_1\frac{\partial}{\partial z_1} + mz_2\frac{\partial}{\partial z_2}, Z_2 = z_1^m\frac{\partial}{\partial z_2}\}$, which is commutative. The complex flow of $W = aZ_1 + bZ_2$ is

$$\Phi_W^u(z_1, z_2) = (\mathrm{e}^{au}z_1, \mathrm{e}^{amu}(z_2 + buz_1^m)), u \in \mathbb{C}.$$

So in order for $\mathrm{Re}\, W$ to have closed orbits, we must have either $\Phi_W^1 = \gamma$ or $\Phi_W^1 = \mathsf{id}$. The first condition gives $a = \log\beta$ and $b = \frac{\lambda}{\beta^m}$ and the second one gives $a = 2\pi i$ and $b = 0$. We obtain thus that a maximal torus acting holomorphically on $H_{\beta,m,\lambda}$ is generated by $\mathfrak{t} = \mathbb{C}\{\xi_1 = \mathrm{Re}(\log\beta Z_1 + \frac{\lambda}{\beta^m}Z_2), \xi_2 = \mathrm{Re}(2\pi i Z_1)\}$. But clearly $\mathfrak{t} \cap J\mathfrak{t} = 0$, so applying Theorem 3.4.3 it follows that $H_{\beta,m,\lambda}$ admits no Vaisman metric.

Example 3.4.7: Consider the example we constructed at the end of Subsection2.6.3: the holomorphic fiber bundle $p : M \to N$, where N is a compact Kähler manifold, M has an LCK metric with positive potential which restricts to the fibers of p, $F \cong H_{\beta,m,\nu}$, to the LCK metric constructed in 2.6.2. Then M cannot admit any Vaisman metric. Indeed, if it did, then this metric would induce, by Theorem 2.4.6, a Vaisman metric on the submanifold $F_x = p^{-1}(x)$, $x \in N$. But we saw just now that the non-diagonal Hopf surfaces do not admit Vaisman metrics.

Example 3.4.8: The Inoue surfaces S^0 and S^- have no holomorphic vector field, so clearly they cannot admit Vaisman metrics. As for S^+, by [In74] we have $\mathfrak{aut}(S^+, J) = \mathfrak{aut}(\mathbb{H}\times\mathbb{C})^{\Gamma^+} = \mathbb{C}Z$, where $Z = \frac{\partial}{\partial z}$ with corresponding complex flow $\Phi_Z^\lambda(w, z) = (w, z + \lambda)$. Hence the only solution $\lambda \in \mathbb{C}$ to $\Phi_{\lambda Z}^1 \in \Gamma^+$ is $\lambda_0 = \frac{c_3}{r}$, in which case $\Phi_{\lambda_0 Z}^1 = \sigma_3$. Hence there exists only one torus $\mathbb{S}^1 \subset \mathrm{Aut}^0(S^+, J)$ which is one-dimensional, generated by $\xi = \lambda_0 Z$. So, again, S^+ cannot admit any Vaisman metrics. As already stated, by [O16], Inoue surfaces admit no exact LCK metrics, but the proof uses the solvmanifold structure of the surfaces.

3.5 Torus principal bundles

Let $\mathbf{T} = \mathfrak{t}/\Lambda$ be a compact complex torus of dimension n, let N be a compact complex manifold and let $\pi : M \to N$ be a holomorphic $\mathbf{T}$-principal bundle over N. Its Chern class is an element:

$$c^{\mathbb{Z}}(\pi) \in H^2(N, \Lambda) \cong H^2(N, \mathbb{Z}) \otimes \Lambda.$$

The inclusion $\Lambda \subset \mathfrak{t}$ induces a natural map $H^2(N, \Lambda) \to H^2(N, \mathfrak{t}) \cong H^2(N, \mathbb{C}) \otimes \mathfrak{t}$, and we will denote by $c(\pi)$ the image of $c^{\mathbb{Z}}(\pi)$ under this map. The class $c(\pi)$ has a well defined rank. If we choose $\mathbb{C}$-bases for both $\mathfrak{t}$ and $H^2(N, \mathbb{C})$, then $c(\pi)$ can be represented by a $2n \times b_2(N)$ matrix over $\mathbb{C}$, and then the rank of $c(\pi)$ is the rank of this matrix.

Note that if the rank of $c(\pi)$ is 1, then there exists a minimal element $a \in \Lambda$, unique modulo sign, such that the non-torsion part of $c^{\mathbb{Z}}(\pi)$ writes $c^{\mathbb{Z}}(\pi)_0 = c_1^{\mathbb{Z}}(\pi) \otimes a$ with $c_1^{\mathbb{Z}}(\pi) \in H^2(N, \mathbb{Z})$. If $c_1(\pi)$ is the image of $c_1^{\mathbb{Z}}(\pi)$ under $H^2(N, \mathbb{Z}) \to H^2(N, \mathbb{C})$, then we will have $c(\pi) = c_1(\pi) \otimes a$, and again $c_1(\pi)$ is uniquely defined modulo sign. So it makes sense to ask weather $c_1(\pi)$ is a positive or negative class, i.e. weather $c_1(\pi)$ or $-c_1(\pi)$ can be represented by a Kähler form on N. In the affirmative case, we will call the class $c(\pi)$ *definite*.

By a theorem of Blanchard [Bl54], when N is of Kähler type, M carries a Kähler metric if and only if the rank of $c(\pi)$ is 0. On the other hand, a theorem of Vuletescu [Vu10] states that if $n = 1$ and the rank of $c(\pi)$ is 2, then M cannot admit LCK metrics.

As a direct application of our existence criterion for Vaisman metrics and of Corollary 3.4.5, we obtain a characterisation of manifolds of LCK-type among all the compact torus principal bundles over compact complex manifolds.

Proposition 3.5.1: *Let $\mathbf{T}$ be a complex compact n-dimensional torus and $\pi : M \to N$ be a $\mathbf{T}$-principal bundle over a compact complex manifold N. Then M admits a (strict) LCK metric if and only if $n = 1$ and the Chern class of π is of rank 1 and definite. In this case, M is of Vaisman type.*

Proof. Suppose that M admits a strict LCK metric. The complex torus $\mathbf{T}$ acts holomorphically and effectively on M, so, by Theorem 3.4.3, M admits a Vaisman metric (Ω, θ). Let B be the Lee vector field with $\theta(B) = 1$ and $A := JB$. By Proposition 3.4.2, $n = 1$ and $\mathfrak{t} = \mathfrak{Lie}(\mathbf{T})$ is spanned by A and B. Here, we identify $\mathfrak{t}$ with its isomorphic image as a subalgebra of $\Gamma(TM)$.

Since the $\mathbf{T}$-invariant 1-forms $\theta_1 = J\theta$ and $\theta_2 = \theta$ verify $\theta_i(X_j) = \delta_{ij}$, for $i, j = 1, 2$, where $X_1 = A$ and $X_2 = B$, there will exist some linear combination of them giving a connection form $\alpha \in \mathcal{C}^\infty(T^*M \otimes \mathfrak{t})$ in π. More precisely, if we denote by ξ_1, ξ_2 the fundamental vector fields of the action, and let $G = (g_{ij})$ be the matrix of $\{X_1, X_2\}$ in the basis $\{\xi_1, \xi_2\}$ of $\mathfrak{t}$, then the connection form will be given by:

$$\alpha := (g_{11}\theta_1 + g_{21}\theta_2) \otimes \xi_1 + (g_{12}\theta_1 + g_{22}\theta_2) \otimes \xi_2.$$

Indeed, it is $\mathbf{T}$-invariant and we have $\alpha(\xi_i) = \xi_i$ for $i = 1, 2$. Moreover, since $d\theta = 0$, its curvature is:

$$\Theta := d\alpha = d\theta_1 \otimes g_{11}\xi_1 + d\theta_1 \otimes g_{12}\xi_2 = dJ\theta \otimes A.$$

It is a basic form, so given by $\Theta = \pi^*\eta \otimes A$, with $\eta \in \Omega^2(N)$, and $\eta \otimes A$ represents the Chern class $c(\pi) \in H^2(N, \mathfrak{t})$. Then clearly $c(\pi)$ is of rank 1, and moreover, it is definite since the form $-\eta$ is a Kähler form on N. The last assertion comes from the fact that, as

Ω is Vaisman, we have $-dJ\theta = \Omega - \theta \wedge J\theta$, so the $(1,1)$-form $-dJ\theta$ is strictly positive on $Q := \ker\theta \cap \ker J\theta \subset TM$. But Q is exactly the horizontal distribution given by the connection α, and so identifies with TN via π_*.

The converse statement is a well known result, see 2.6.3. ■

3.6 Analytic irreducibility of complex manifolds of LCK type

It is not very difficult to see that a product metric cannot be LCK ([Va80]), but whether an LCK manifold must be analytically irreducible is still an open question. Under additional hypotheses, the answer is known to be positive ([Ts99], [OPV14]). In this section we wish to enlarge the list of hypotheses implying the analytic irreducibility of the manifold.

One of the results in this direction is due to Tsukada [Ts99], which we can also obtain as a direct consequence of Theorem 3.4.3:

Proposition 3.6.1: *([Ts99]) Let M_1 and M_2 be two compact complex manifolds of Vaisman type. Then $M := M_1 \times M_2$ admits no LCK metric.*

Proof. By Theorem 3.4.3, the groups of biholomorphisms $\mathrm{Aut}(M_i)$ contain tori $\mathbb{T}_i$ which are not purely real, for $i = 1, 2$. Then the Lie algebra $\mathfrak{t}$ of the torus $\mathbb{T} := \mathbb{T}_1 \times \mathbb{T}_2 \subset \mathrm{Aut}(M)$ verifies $\dim_{\mathbb{C}} \mathfrak{t} \cap i\mathfrak{t} = 2$. Hence, by Corollary 3.4.5, M cannot admit an LCK metric. ■

Tsukada obtained Proposition 3.6.1 as a corollary to the following result:

Theorem 3.6.2: *([Ts99]) Let (M, Ω) be a compact Vaisman manifold and let $\mathcal{F}$ be the canonical foliation on M generated by the Lee and the Reeb vector fields. Then $\mathcal{F}$ has a compact leaf.*

We can further exploit this and obtain the following, more general, result:

Theorem 3.6.3: *Let M_1, M_2 be two compact complex manifolds and suppose that M_1 is of Vaisman type. Then $M := M_1 \times M_2$ admits no LCK metric.*

Proof. Suppose M admits some LCK metric. Then, for any $x \in M_1$, this metric restricted to $\{x\} \times M_2 \cong M_2$ gives an LCK metric on M_2.

Since M_1 is of Vaisman type, there exists $\mathbb{T}_1 \subset \mathrm{Aut}(M_1)$ whose Lie algebra $\mathfrak{t}_1$ verifies $\dim_{\mathbb{C}} \mathfrak{t}_1 \cap i\mathfrak{t}_1 = 1$. The induced torus $\mathbb{T} = \mathbb{T}_1 \times \{\mathsf{id}_{M_2}\} \subset \mathrm{Aut}(M)$ is still not purely real, so by Theorem 3.4.3, M is of Vaisman type and $\mathfrak{t} := \mathfrak{Lie}(\mathbb{T})$ contains the corresponding Lee vector field B.

Let Ω be a Vaisman metric on M which, possibly after averaging, is $\mathbb{T}$-invariant. Then for any $y \in M_2$, Ω restricted to $M_1 \times \{y\} \cong M_1$ must be Vaisman. Indeed, by construction, the Lee vector field B is tangent to M_1, and [Va82, Theorem 5.1] states that any complex submanifold of a Vaisman manifold that is tangent to the Lee vector field is again Vaisman with the induced metric. Let now $E \subset M_1$ be a closed leaf of the canonical foliation on the Vaisman manifold M_1, as in the above theorem. Clearly, after choosing $O \in E$, E has the structure of an elliptic curve whose tangent bundle is generated by B and JB restricted to E. Hence, the submanifold $i : Y = E \times M_2 \to M$ together with $i^*\Omega$ is Vaisman. At the same time, $Y \to M_2$ is a trivial E-principal bundle, so we arrive at a contradiction via Proposition 3.5.1.

■

Also, using the result which states that a compact complex submanifold of a Vaisman manifold must contain the leaves of the canonical foliation, one has:

Proposition 3.6.4: *A compact complex manifold of Vaisman type is holomorphically irreducible.*

Proof. Let $M = M_1 \times M_2$ be the compact complex manifold with the product complex structure, and suppose it admits a Vaisman metric Ω with corresponding canonical foliation $\mathcal{F}$ generated by B, JB. Then, by Theorem 2.4.6, for any $(x_1, x_2) \in M$, both the submanifolds $M_1 \times \{x_2\}$ and $\{x_1\} \times M_2$ of M contain the leaves of $\mathcal{F}$, which is impossible.

■

On the other extreme, we have the following result, also obtained in [OPV14] in a different manner:

Theorem 3.6.5: *Let M_1, M_2 be two connected complex manifolds, and suppose that M_1 is compact and verifies the $\partial\bar{\partial}$-lemma. Moreover, if $\dim_{\mathbb{C}} M_1 = 1$, then suppose its genus g is 1 and M_2 is compact, or that $g = 0$. Then $M := M_1 \times M_2$ admits no (strict) LCK metric.*

Proof. Suppose M admits an LCK form Ω with corresponding Lee form θ. Denote by $p_i : M \to M_i$, $i = 1, 2$ the canonical projections. We have, by the Künneth formula, an isomorphism $p_1^* \oplus p_2^* : H^1(M_1, \mathbb{R}) \oplus H^1(M_2, \mathbb{R}) \to H^1(M, \mathbb{R})$, meaning that there exist two closed forms $\theta_i \in \mathcal{C}^\infty(T^*M_i)$, $i = 1, 2$, such that θ is cohomologous to $p_1^*\theta_1 + p_2^*\theta_2$. After a conformal change of Ω, we can suppose that $\theta = p_1^*\theta_1 + p_2^*\theta_2$.

Since an LCK metric on M induces one on M_1, by Theorem 2.2.8, the induced metric is globally conformal to a Kähler metric. Suppose moreover that $n := \dim_{\mathbb{C}} M_1 > 1$. Then this implies that θ_1 is exact on M_1. Again, by a global conformal change of Ω, we can suppose that $\theta_1 = 0$. Then the conclusion follows from Lemma 3.6.6 bellow.

If $\dim_{\mathbb{C}} M_1 = 1$, then the induced LCK form on M_1 is automatically Kähler, so we know nothing about θ_1. However, if $g = 1$, then $M \to M_2$ is a trivial principal elliptic bundle, so by Proposition 3.5.1, M cannot admit any strict LCK metric. If $g = 0$ then M_1 is simply connected, so we can again apply Lemma 3.6.6. ■

Lemma 3.6.6: *Let M_1 and M_2 be two connected complex manifolds, with M_1 compact. Suppose that (Ω, θ) is an LCK form on the manifold $M = M_1 \times M_2$ such that $i^*[\theta] = 0$ in $H^1(M_1, \mathbb{R})$, where $i : M_1 \to M$ is the inclusion $x \mapsto (x, y)$ for some $y \in M_2$. Then M (and thus also both M_1 and M_2) are of Kähler type.*

Proof. As before, after an eventual conformal change of Ω, we have $\theta = p_1^*\theta_1 + p_2^*\theta_2$ with $\theta_i \in \mathcal{C}^\infty(T^*M_i), i = 1, 2$. By hypotheses, $\theta_1 = df$, with $f \in \mathcal{C}^\infty(M_1)$, so again, by replacing Ω with $\mathrm{e}^{-p_1^*f}\Omega$, we can suppose that $\theta = p_2^*\theta_2$.

The algebra of differential forms on M, $\mathcal{C}^\infty(\bigwedge T^*M)$, has two compatible gradings: one given by the degree of the forms, and the second one induced by the splitting $T^*M = p_1^*T^*M_1 \oplus p_2^*T^*M_2$. With respect to this second splitting, write the differential $d = d_1 + d_2$, and write $\Omega = \Omega_1 + \Omega_{12} + \Omega_2 \in \mathcal{C}^\infty(\bigwedge^2 T^*M)$, where:

$$\bigwedge\nolimits^2 T^*M = \bigwedge\nolimits^2 p_1^*T^*M_1 \oplus p_1^*T^*M_1 \otimes p_2^*T^*M_2 \oplus \bigwedge\nolimits^2 p_2^*T^*M_2.$$

Then the equation $d\Omega = p_2^*\theta_2 \wedge \Omega$ gives, in the homogeneous parts $\bigwedge^2 p_1^*T^*M_1 \otimes \bigwedge^1 p_2^*T^*M_2$ and $\bigwedge^3 p_1^*T^*M_1$:

$$\begin{aligned} d_2\Omega_1 + d_1\Omega_{12} &= p_2^*\theta_2 \wedge \Omega_1 \\ d_1\Omega_1 &= 0. \end{aligned} \tag{3.6.1}$$

Let n be the complex dimension of M_1. If we take the wedge of the first equation in (3.6.1) with Ω_1^{n-1}, we obtain:

$$\frac{1}{n}d_2(\Omega_1^n) + d_1(\Omega_{12} \wedge \Omega_1^{n-1}) = p_2^*\theta_2 \wedge \Omega_1^n. \tag{3.6.2}$$

On the other hand, the compactness of M_1 implies that p_2 is a proper submersion, so it induces a push forward map on forms given by fiberwise integration:

$$\begin{aligned} (p_2)_* : \mathcal{C}^\infty(\textstyle\bigwedge^{2n} p_1^*T^*M_1 \otimes \bigwedge^k p_2^*T^*M_2) &\to \mathcal{C}^\infty(\textstyle\bigwedge^k T^*M_2) \\ ((p_2)_*\alpha)_y &:= \int_{M_1\times\{y\}} \alpha, \quad y \in M_2. \end{aligned}$$

Applying this map to (3.6.2) and using Stokes' theorem, we obtain the following relation on M_2:

$$dg = n\theta_2 g, \quad \text{where} \quad g := (p_2)_*\Omega_1^n \in \mathcal{C}^\infty(M_2).$$

This relation also reads $d_{n\theta_2}g = 0$, with $g \neq 0$ since it is in fact everywhere positive. By Lemma 2.2.9 we obtain that $n\theta_2$ is exact, so also θ is and Ω is globally conformal to a Kähler metric. ■

Remark 3.6.7: In [OPV14], the authors claim a proof of Theorem 3.6.5 also for the case when M_1 is a Riemann surface of genus ≥ 2, but we believe that their argument does not hold. However, we are only able to find restrictions on the manifold M_2 under the hypothesis that $M_1 \times M_2$ admits an LCK metric:

Proposition 3.6.8: *Let M_1 be a compact complex curve, let M_2 be a complex manifold and suppose that $M := M_1 \times M_2$ admits an LCK metric. Then M_2 admits an LCK metric with positive potential.*

Proof. We keep the same notations as before. If (Ω, θ) is the LCK form on M, we can suppose that $\theta = \theta_1 + \theta_2$ with each θ_i being the pullback of a closed one form from M_i, $i = 1, 2$. Moreover, up to a conformal change of Ω, as M_1 is Kählerian, we can choose θ_1 to be the real part of a holomorphic one form, so that $dJ\theta_1 = 0$, where J is the product complex structure on M.
As before, on the $\bigwedge^2 T^*M_1 \otimes \bigwedge^1 T^*M_2$-part, $d\Omega = \theta \wedge \Omega$ gives:

$$d_1\Omega_{12} + d_2\Omega_1 = \theta_2 \wedge \Omega_1 + \theta_1 \wedge \Omega_{12}. \tag{3.6.3}$$

Extend J as a derivation acting on forms, and let $d^c = i(\bar{\partial} - \partial)$. Then, on M we have the commutation relation:

$$[J, d] = d^c. \tag{3.6.4}$$

The formula $Jd\Omega = J(\theta \wedge \Omega)$, together with $J\Omega = 0$ and (3.6.4) gives, on the $\bigwedge^1 T^*M_1 \otimes \bigwedge^2 T^*M_2$-part:

$$d_2^c\Omega_{12} + d_1^c\Omega_2 = J\theta_1 \wedge \Omega_2 + J\theta_2 \wedge \Omega_{12}. \tag{3.6.5}$$

Now we apply the push forward map $(p_2)_*$ to equation (3.6.3) and Stoke's theorem in order to obtain:

$$(p_2)_*d_2\Omega_1 = (p_2)_*(\theta_2 \wedge \Omega_1 + \theta_1 \wedge \Omega_{12}).$$

If we denote by g the strictly positive function on M_2 given by $(p_2)_*\Omega_1$, this also reads:

$$d_{\theta_2}g = (p_2)_*(\theta_1 \wedge \Omega_{12}). \tag{3.6.6}$$

We apply d^c to this identity and use equation (3.6.5) together with (3.6.6) to get:

$$\begin{aligned} d^c d_{\theta_2}g &= -(p_2)_*(\theta_1 \wedge d_2^c\Omega_{12}) \\ &= -(p_2)_*(\theta_1 \wedge J\theta_1 \wedge \Omega_2) + J\theta_2 \wedge d_{\theta_2}g + (p_2)_*(\theta_1 \wedge d_1^c\Omega_2). \end{aligned}$$

Since we chose θ_1 so that $dJ\theta_1 = 0$, equation (3.6.4) implies that $d_1^c\theta_1 = 0$, hence the above simply gives:

$$d^c d_{\theta_2}g - J\theta_2 \wedge d_{\theta_2}g = -(p_2)_*(\Omega_2 \wedge \theta_1 \wedge J\theta_1).$$

Note that $\alpha := \Omega_2 \wedge \theta_1 \wedge J\theta_1$ is a semipositive $(2,2)$-form on M which is strictly positive on a non-empty open subset of M of the form $U \times M_2$, where $U \subset M_1$ is the open set where θ_1 does not vanish. Then $\eta := (p_2)_*\alpha$ is a strictly positive $(1,1)$-form on M_2 verifying:

$$\eta = d_{\theta_2}d_{\theta_2}^c g. \tag{3.6.7}$$

Finally, this implies that $\eta = -dJ\omega + \omega \wedge J\omega$, where $\omega := \theta_2 - d\ln g$, so (η, ω) is an LCK metric with positive potential on M_2. ■

3.7 Weyl reducible manifolds

In this section, we are interested in LCK metrics whose corresponding Weyl connection is reducible, or equivalently, the Levi-Civita connection of the Kähler metric on the minimal cover is reducible. The starting point of our discussion is a topological and metric description of the more general class of compact conformal manifolds with a reducible closed Weyl connection, given by M. Kourganoff in [Kou15]. From it, we will easily infer also the complex-analytic description of Weyl-reducible LCK manifolds.

Let us first recall that on a conformal manifold (M, c), a Weyl connection is a torsion free linear connection preserving the conformal structure. If the Weyl connection is locally the Levi-Civita connection of a local metric in the conformal class c, then the connection is called *closed*. This is equivalent to the existence of a global Riemannian metric $\tilde{g}$ on the universal cover $\tilde{M}$ in the conformal class given by the pull-back of c, on which $\pi_1(M)$ acts by homotheties. Although defined on $\tilde{M}$, $\tilde{g}$ is called a *similarity structure on M*.

Theorem 3.7.1: *(M. Kourganoff [Kou15]) Let (M, c, D) be a compact conformal manifold endowed with a closed Weyl connection, and let $\tilde{g}$ be the corresponding similarity structure. Then we are in one of the three cases:*

1. *$(\tilde{M}, \tilde{g})$ is flat;*

2. *D has irreducible holonomy and $\dim M > 1$;*

3. *$(\tilde{M}, \tilde{g})$ is isometric to $(\mathbb{R}^q, g_0) \times (N, g_N)$, where g_0 is the flat metric on $\mathbb{R}^q$, and the holonomy of the Levi-Civita connection corresponding to g_N is irreducible.*

Let us now suppose that (M, J, g) is a compact LCK manifold. There exists a unique Weyl connection preserving the conformal class of g and the complex structure J. Moreover, as g is LCK, the connection is closed. We are thus in the setting of the above theorem. If we are in the first case, a theorem of Vaisman [Va82] states that (M, J, g) is a Hopf manifold with the standard metric. Let us suppose that we are in the third case. We wish to show that the Riemannian manifolds appearing in the decomposition are in fact naturally Kähler, and the isometry is biholomorphic. The arguments for this are standard.

We will denote also by D the pull back connection to $\tilde{M}$, and note that D is the Levi-Civita connection of $\tilde{g}$. Let us denote by T^0 and T^1 respectively the pullback of $T\mathbb{R}^q$ and TN to $\tilde{M}$ by the natural projections. Fix a point $x \in \tilde{M}$, and recall that the holonomy group of D in x, $G := \mathrm{Hol}(x)$, is identified with a subgroup of $\mathrm{Aut}(T_x\tilde{M})$, given by the parallel transport of vectors in $T_x\tilde{M}$ along loops in x. We have the G-invariant decomposition $T_x\tilde{M} = T_x^0 \oplus T_x^1$, and G acts trivially on T_x^0. At the same time, as the pull-back complex structure J on $\tilde{M}$ is D-parallel, the elements of G commute with $J_x \in \mathrm{Aut}(T_x\tilde{M})$. Thus, for any $v \in T_x^0$ and any $g \in G$ we have: $gJ_xv = J_xgv = J_xv$. But the elements of T_x^0 are characterized by the fact that G acts trivially on them, so $J_xv \in T_x^0$, implying thus that $JT_x^0 = T_x^0$. At the same time, T_x^1 is the orthogonal of T_x^0 in $T_x\tilde{M}$ with respect to $\tilde{g}_x$, which is J_x-invariant, so also $JT_x^1 = T_x^1$. Finally, as T^0 and T^1 are obtained by the parallel transport of T_x^0 and T_x^1, and as J is D-parallel, it follows that $JT^0 = T^0$ and $JT^1 = T^1$. In particular, J splits as $J = J_0 + J_1$, where for $i = 0, 1$, $J_i \in \mathrm{Aut}(T^i)$ is defined by $J_i := J|_{T^i}$ and is an almost complex structure. Moreover, since J is integrable, its Nijenhuis tensor $N^J : \bigwedge^2 T\tilde{M} \to T\tilde{M}$ vanishes. But for $i = 0, 1$, the Nijenhuis tensor corresponding to J_i is just the restriction of N^J to $\bigwedge^2 T^i$, and so also vanishes, thus J_0 and J_1 are integrable and the isometry in (3) is holomorphic. Now clearly $\omega_0 := g_0(J_0\cdot, \cdot)$ and $\omega_1 := g_N(J_1\cdot, \cdot)$ are Kähler forms on the corresponding factors.

Theorem 3.7.2: *(LCK version) Let (M, J, g) be a compact LCK manifold, and let $(\tilde{M}, \tilde{J}, \tilde{g})$ be its universal cover with the corresponding Kähler metric. Then we are in one of the three cases:*

1. *$(\tilde{M}, \tilde{g})$ is biholomorphic and isometric to a flat Hopf manifold with the standard metric;*

2. *The Levi-Civita connection of $\tilde{g}$ has irreducible holonomy;*

3. *$(\tilde{M}, \tilde{J}, \tilde{g})$ is biholomorphic and isometric to $(\mathbb{C}^t, g_0) \times (N, J_N, g_N)$, where g_0 is the flat metric on $\mathbb{C}^t$, and (N, J_N, g_N) is a Kähler manifold with irreducible holonomy.*

Let us note that the OT manifolds of type $(s, 1)$ with the LCK metric defined in (5.2.5) verify the third case of the above theorem, but we do not know if these are all the Weyl-reducible LCK manifolds. However, we can generalize a metric property showed by A. Otiman in [O16] for OT manifolds to the (supposedly larger) class of Weyl-reducible LCK manifolds. It concerns the non-existence of exact LCK metrics on such manifolds. For this, in addition to the above theorem, we will also need a few preliminary results from [Kou15] concerning the structure of some groups of automorphisms appearing in the description of the manifold.

We introduce first the notations. From now on, we fix a Weyl-reducible compact LCK manifold (M, J, g) so that its universal cover with the Kähler metric $(\tilde{M}, \tilde{J}, \tilde{g})$ is $(\mathbb{R}^{2t}, J_0, g_0) \times (N, J_N, g_N)$. For a Hermitian structure (g, J), denote by

$$\mathrm{Isom}(g, J) \subset \mathrm{Sim}(g, J) \subset \mathrm{Conf}(g, J)$$

respectively the groups of biholomorphic isometries, homotheties, and conformal automorphisms of the given structure. They are all closed subgroups of the respective groups

$$\mathrm{Isom}(g) \subset \mathrm{Sim}(g) \subset \mathrm{Conf}(g).$$

The group $\pi_1(M)$ acts diagonally by holomorphic homotheties on $\tilde{M}$, so it is a subgroup of $\mathrm{Sim}(g_0, J_0) \times \mathrm{Sim}(g_N, J_N)$. Denote by Γ_0 the image of $\pi_1(M)$ in $\mathrm{Sim}(g_0, J_0)$, and by Γ_N the image of the same group in $\mathrm{Sim}(g_N, J_N)$. Let G be the closure of Γ_N in $\mathrm{Sim}(g_N, J_N)$, and G^0 its connected component. Finally, let $\Gamma^0 := \pi_1(M) \cap (\mathrm{Sim}(g_0) \times G^0)$. Recall that $\mathrm{Sim}(g_0, J_0) = \mathbb{R}_{>0} \cdot U(t) \ltimes \mathbb{C}^t$. Note that the definition of the groups Γ_0, Γ_N, G, G^0 and Γ^0 is in fact independent of the complex structures. We have the following facts concerning them, proven in [Kou15]:

Fact 3.7.3: *[Kou15, Lemma 4.8] The group G is a Lie group acting properly on N.*

Fact 3.7.4: *[Kou15, Lemma 4.1] The group G^0 is abelian.*

Fact 3.7.5: *[Kou15, Lemma 4.12] The group G^0 sits in* $\mathrm{Isom}(g_N)$.

Fact 3.7.6: *[Kou15, Lemma 4.16] The group* Γ^0 *is a lattice in* $\mathbb{C}^t \times G^0$.

These facts give a somewhat more particular description of the structure of a Weyl-reducible compact LCK manifold. Note first that the group $L := \mathbb{C}^t \times G^0$ is normal in $\mathrm{Sim}(g_0) \times G$, since $\mathbb{C}^t$ is normal in $\mathrm{Sim}(g_0)$ and G^0 is normal in G, being its connected component. This implies that Γ^0 is a normal subgroup of $\pi_1(M)$. Indeed, let $\gamma \in \pi_1(M) \subset \mathrm{Sim}(g_0) \times G$ and $\tau \in \Gamma^0 \subset L$. Then $u := \gamma\tau\gamma^{-1} \in L$. But Γ^0 is a lattice in L, by Fact 3.7.6, so if $u \notin \Gamma^0$ then $\mathrm{span}{<} \Gamma^0, u {>} \subset \pi_1(M)$ is not discrete in L. In particular, $\pi_1(M)$ cannot act properly discontinuously on $\tilde{M}$, which is a contradiction. So $u \in \Gamma^0$, meaning that Γ^0 is a normal subgroup of $\pi_1(M)$. Denote by $\hat{M} := \tilde{M}/\Gamma^0$. It is a Galois cover of M, of deck group $U := \pi_1(M)/\Gamma^0$. By Fact 3.7.5, $\Gamma^0 \subset \mathrm{Isom}(\tilde{g})$, so $\tilde{g}$ descends to $\hat{M}$ to a Kähler metric. Note also that, by Fact 3.7.4 and Fact 3.7.6, $T := L/\Gamma^0$ is a compact abelian Lie group, acting by biholomorphisms on $\hat{M}$.

Theorem 3.7.7: *A non-flat Weyl-reducible compact LCK manifold M does not admit any exact LCK metric.*

Proof. As we are in the third case of Theorem 3.7.2, we will keep the same notation as before. So we can identify $\tilde{M}$ with $\mathbb{C}^t \times N$, where $t > 1$ and N is some Kähler manifold.

Suppose we have an exact LCK metric $\Omega = d\eta - \theta \wedge \eta$ on M, and let $\varphi : \mathbb{C}^t \times N \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function so that $\theta = d\varphi$ on $\tilde{M}$. Denote by $\rho : \pi_1(M) \to \mathbb{R}$ the morphism $\gamma \mapsto \gamma^*\varphi - \varphi$. By Fact 3.7.5, $\Gamma^0 \subset \mathrm{Isom}(\tilde{g})$, but at the same time $\Gamma^0 \subset \mathrm{Isom}(g)$, as Γ^0 is a subgroup of $\pi_1(M)$. Hence the function φ is Γ^0-invariant and descends to $\hat{M}$ to a function denoted also by φ.

Now let μ be a constant volume form on T with $\int_T \mu = 1$, and define the T-invariant function on $\hat{M}$:

$$\varphi' := \int_T h^*\varphi\mu(h).$$

The function φ' has the same U-equivariance as φ, which can be seen as follows. As L was normal in $\mathrm{Sim}(g_0)\times G$, U acts on T by conjugation. For $\gamma \in U$ and $h \in T$, let $h_\gamma := \gamma^{-1}h\gamma \in T$. Then, for any $\gamma \in U$, we have:

$$\gamma^*\varphi' = \int_T (h\gamma)^*\varphi\mu(h) = \int_T (\gamma h_\gamma)^*\varphi\mu(h) = \int_T h_\gamma^*(\varphi + \rho(\gamma))\mu(h_\gamma) = \varphi' + \rho(\gamma).$$

In particular, $\theta' := d\varphi'$ has the same de Rham cohomology class as θ, and is also T-invariant. Let f be a $\mathcal{C}^\infty$-function on M verifying $\theta' = \theta + df$, and consider the LCK form $\Omega_f := \mathrm{e}^f\Omega$. Note that, if we let $\eta_f := \mathrm{e}^f\eta$, then $\Omega_f = d\eta_f - \theta' \wedge \eta_f$, so Ω_f is also an exact LCK metric.

Define now the T-invariant form on $\hat{M}$:

$$\eta' := \int_T h^*\eta_f\mu(h).$$

Just as before, as η_f was U-invariant, also η' is, and so descends to a well defined one-form on M. Moreover, using the T-invariance of θ', we have on $\hat{M}$:

$$\begin{aligned} d_{\theta'}\eta' &= d\int_T h^*\eta_f\mu(h) - \theta' \wedge \int_T h^*\eta'\mu(h) \\ &= \int_T h^*(d\eta_f - \theta' \wedge \eta_f)\mu(h) \\ &= \int_T h^*\Omega_f\mu(h) := \Omega'. \end{aligned}$$

Thus, $\Omega' = d_{\theta'}\eta'$ is an exact LCK metric on M, which is additionally T-invariant.

Now let us go back to $\tilde{M}$. If we denote, as before, by T^0 and T^1 the pull-back of $T\mathbb{C}^t$ and TN to $\tilde{M}$ by the natural projections, we have:

$$T^*\tilde{M} = (T^0)^* \oplus (T^1)^* \qquad \textstyle\bigwedge^2 T^*\tilde{M} = \bigwedge^2(T^0)^* \oplus (T^0)^* \otimes (T^1)^* \oplus \bigwedge^2(T^1)^*.$$

With respect to these splittings, let us write $d = d_0 + d_1$ and $\eta' = \eta_0 + \eta_1$. Note that, as φ' is T-invariant, it is in particular constant in the $\mathbb{C}^t$-variables. This implies that $\theta' = d\varphi' \in \mathcal{C}^\infty(\tilde{M}, (T^1)^*)$. Moreover, as η' is T-invariant, we also have $d_0\eta_0 = 0 = d_0\eta_1$. Therefore, we obtain that

$$\Omega' = d_1\eta_0 + d_1\eta_1 - \theta' \wedge \eta_0 - \theta' \wedge \eta_1$$

has no $\bigwedge^2(T^0)^*$-component, so it cannot be non-degenerate. This contradicts the fact that Ω' was an LCK form and concludes the proof. ■

As a corollary, we obtain a claim made in [MO09]:

Corollary 3.7.8: *Let (M, J, Ω) be a compact Vaisman manifold, or more generally, an LCK manifold with potential, which is not a flat Hopf manifold. Then its Weyl connection is irreducible.*

Chapter 4

Toric LCK Manifolds

4.1 Introduction

In the present chapter, which is part of [Is17], we are interested in the incarnation of toric geometry for locally conformally Kähler (LCK), or more generally, for locally conformally symplectic (LCS) manifolds. The beginnings of this study can be traced down to the article of I. Vaisman [Va85], where he argues that LCS manifolds are the natural phase spaces for Hamiltonian mechanics and is the first to give a good notion of Hamiltonians in this context.

General Hamiltonian group actions and the corresponding reduction procedure in the LCS and LCK context have been considered by S. Haller and T. Rybicki in [HR01], or by R. Gini, L. Ornea and M. Parton in [GOP05]. But only recently were Hamiltonian actions of maximal tori on LCK manifolds studied towards a classification, by M. Pilca in [Pi16] and by F. Madani, A. Moroianu and M. Pilca in [MMP17]. The program is as follows: there exists a class of LCK manifolds, called Vaisman manifolds, which is better understood via its many geometric properties. In particular, the universal cover of a Vaisman manifold is a Kähler cone over a Sasaki manifold. In [Pi16], toric Vaisman manifolds are studied and it is shown that for every known existing equivalence of categories between them and some other class of manifolds, the Hamiltonian toric action also is equivalent to a natural Hamiltonian toric action in the given category. Then, in [MMP17], it is shown that the toric Sasaki manifold corresponding to a toric Vaisman manifold is actually compact. But Sasaki manifolds are in particular contact, and compact toric contact manifolds have been classified by E. Lerman in [Ler03].

On the other hand, in [MMP17] toric LCK manifolds of complex dimension 2 have been given a classification, and it turns out that they all admit toric Vaisman metrics. Hence the question was raised of whether this is always the case, regardless of dimension. The main result of the chapter, Theorem 4.4.1, is an affirmative answer to it, and so, together with the above cited papers, amounts to a classification of toric LCK manifolds as complex manifolds with a torus action.

Remark that the universal cover of a LCK manifold is a non-compact Kähler manifold, so one might want to use the theory of toric symplectic manifolds in order to prove the result. However, in the non-compact world the theorems of convexity and connectedness for moment maps of Atiyah and Guillemin-Sternberg fail, and one no longer has a characterisation of the symplectic manifold in terms of the image of the moment map. As proven by E. Lerman and S. Tolman in [KL15], classification results still are possible, but in terms of more complicated objects. Hence we chose to give a direct proof, not relying on the known facts from toric symplectic geometry.

The proof occupies Section 4.4 and roughly goes as follows. First we remark that the holomorphic action of the compact torus $\mathbb{T}$ on the manifold M naturally extends to a holomorphic action of the complexified torus $\mathbb{T}^c$. In particular, on the minimal Kähler cover $\hat{M}$ of M, $\mathbb{T}^c$ has a dense connected open orbit, since the $\mathbb{T}$-action is Hamiltonian. This allows us to view the deck group Γ of $\hat{M}$ as a subgroup of $\mathbb{T}^c$, and to extend it to a one-parameter subgroup of $\mathbb{T}^c$. However, there is no reason for this group to act conformally on the LCK form, so at this point we have to construct, by averaging, a new LCK form, still compatible with the $\mathbb{T}$-action. Finally, we are able to explicitly write down a toric Vaisman metric in the conformal class of the averaged metric.

The rest of the chapter is organised as follows: in Section 4.2 we introduce Hamiltonian group actions in the LCS context, and we base our discussion on Section 2.5. Section 4.3 puts together the results we use for our proof. In particular, we show that for a compact toric LCS manifold, the action of the torus lifts to a Hamiltonian action on the minimal symplectic cover. In Section 4.5 we discuss a few examples of toric LCK manifolds. In particular, we exhibit an example showing that the class of compact toric LCS manifolds strictly contains the compact toric LCK manifolds, unlike in the symplectic context.

4.2 Twisted Hamiltonian Vector Fields

In this section we study the corresponding notions of Hamiltonian vector field and Hamiltonian group action to the LCS context. The definitions, as presented, were introduced by I.Vaisman in [Va85], where one can also see a number of reasons for why these are the natural analogues to the ones from the symplectic world. Let us fix in this section an LCS manifold (M, Ω, θ).

Definition 4.2.1: A vector field $X \in \Gamma(TM)$ is called *twisted Hamiltonian* if there exits a function $f \in \mathcal{C}^\infty(M)$ such that $\iota_X \Omega = d_\theta f$.

Remark 4.2.2: Although it is not apparent from the definition, the above notion is actually conformally invariant. Indeed, if $X = X_f$ is a twisted Hamiltonian vector field for Ω with corresponding function $f \in \mathcal{C}^\infty(M)$ and $\Omega' := \mathrm{e}^u \Omega$ is another conformal form with corresponding Lee form $\theta' = \theta + du$, then we have:

$$\iota_X \Omega' = \mathrm{e}^u(df - \theta f) = d^{\theta'}(f\mathrm{e}^u). \tag{4.2.1}$$

As in the symplectic setting, an LCS form Ω defines on $\mathcal{C}^\infty(M)$ a Poisson bracket:

$$\{f, g\} := \Omega(X_g, X_f) \quad \forall f, g \in \mathcal{C}^\infty(M)$$

and by straightforward calculations it can be seen that $X_{\{f,g\}} = [X_f, X_g]$. Hence the set of twisted Hamiltonian vector fields

$$\mathfrak{ham}(M, [\Omega]) := \{X \in \Gamma(TM) | \exists f \in \mathcal{C}^\infty(M) \ \iota_X \Omega = d_\theta f\}$$

forms a Lie subalgebra of $\Gamma(TM)$. Actually, $\mathfrak{ham}(M, [\Omega]) \subset \mathfrak{aut}'(M, [\Omega])$, where we recall that the latter algebra, defined in (2.5.2), is formed by the vector fields X verifying $\mathcal{L}_X \Omega = \theta(X)\Omega$.

Indeed, for $X = X_f \in \mathfrak{ham}(M, [\Omega])$ we have:

$$\mathcal{L}_{X_f}\Omega = \iota_{X_f} d\Omega + d\iota_{X_f}\Omega = \iota_{X_f}(\theta \wedge \Omega) + d(df - \theta f) = \theta(X_f)\Omega. \tag{4.2.2}$$

Remark 4.2.3: If $\pi : \hat{M} \to M$ is the minimal cover of deck group Γ and symplectic form Ω_K, the pull-back morphism π^* establishes an isomorphism between $\mathfrak{ham}(M, [\Omega])$ and the Lie algebra of Γ-invariant Hamiltonian vector fields of the symplectic form on the minimal cover $\mathfrak{ham}(\hat{M}, \Omega_K)^\Gamma$. Indeed, if $X \in \mathfrak{aut}'(M, [\Omega])$ and $\hat{X} = \pi^* X$ is the pull-back vector field to $\hat{M}$, by writing $\Omega_K = \mathrm{e}^{-\varphi}\Omega$ we have, on $\hat{M}$:

$$\iota_{\hat{X}}\Omega = d_\theta f \Leftrightarrow \iota_{\hat{X}}\Omega_K = \mathrm{e}^{-\varphi}(df - f d\varphi) = d(\mathrm{e}^{-\varphi} f).$$

Definition 4.2.4: Let $(M, [\Omega])$ be an LCS manifold. We say that an action of a Lie group G on M is *twisted Hamiltonian* if $\mathfrak{g} := \mathfrak{Lie}(G) \subset \mathfrak{ham}(M, [\Omega])$.

Remark 4.2.5: If the Lie group G is compact and acts conformally on $[\Omega]$, then we can find an LCS form in the given conformal class that is G-invariant. Indeed, take any LCS form $\Omega \in [\Omega]$. Then, for any g in G, we have $g^*\Omega = \mathrm{e}^{f_g}\Omega$, with $f_g \in \mathcal{C}^\infty(M)$. Let dv be a normalised Haar measure on G, and take $h := \int_G f_g dv(g)$, so that $\Omega^G := \int_G g^*\Omega dv(g) = \mathrm{e}^h\Omega$. Then $\Omega^G \in [\Omega]$ is, by definition, a G-invariant LCS form with corresponding Lee form $\theta^G = dh + \theta$.

For $\Omega \in [\Omega]$, define the map $A^\Omega : \mathcal{C}^\infty(M) \to \mathfrak{ham}(M, [\Omega])$ by sending a function f to its corresponding Hamiltonian vector field X_f with respect to Ω. If we consider on $\mathcal{C}^\infty(M)$ the Lie algebra structure given by Ω, A^Ω is a Lie algebra morphism. On the other hand, if Ω is strict LCS, then $d_\theta : \mathcal{C}^\infty(M) \to \Omega^1_M$ is injective by Lemma 2.2.9. Thus also A^Ω is injective, hence A^Ω is actually an isomorphism of Lie algebras. Note that under a conformal change of Ω, by (4.2.1) this map changes by the rule $A^{\mathrm{e}^u\Omega}(f) = A^\Omega(\mathrm{e}^u f)$.

Suppose that a compact Lie group G has a twisted Hamiltonian action on the LCS manifold $(M, [\Omega])$. As soon as we choose an LCS form $\Omega \in [\Omega]$, there automatically exists a Lie algebra morphism $\rho^\Omega : \mathfrak{g} \to \mathcal{C}^\infty(M)$ which is a section of A^Ω. Indeed, as A^Ω is an isomorphism of Lie algebras, we simply have $\rho^\Omega = (A^\Omega)^{-1}|_{\mathfrak{g}}$. In particular, we have a moment map $\mu^\Omega : M \to \mathfrak{g}^*$ given by

$$< \mu^\Omega(x), X >= \rho^\Omega(X)(x), \quad x \in M, X \in \mathfrak{g}.$$

Remark 4.2.6: If $(M, [\Omega])$ is an exact LCS manifold, and G is a compact Lie group that acts conformally on it such that $\mathfrak{g} = \mathfrak{Lie}(G) \subset \mathfrak{aut}'(M, [\Omega])$, then this action is automatically twisted Hamiltonian. Indeed, as before, we can choose from the beginning, in the given conformal class, $\Omega = d_\theta\eta$ and θ G-invariant. Now define $\eta^G := \int_G g^*\eta dv(g)$. Then, since θ is G-invariant, we have:

$$d_\theta\eta^G := \int_G g^*(d\eta)dv(g) - \theta \wedge \int_G g^*\eta dv(g) = \int_G g^*(d\eta - \theta \wedge \eta)dv(g) = \int_G \Omega dv(g) = \Omega.$$

Hence, there exists a momentum map given by the G-invariant form $-\eta^G$. More precisely, we have, for any $X \in \mathfrak{g}$:

$$\iota_X\Omega = \iota_X d\eta^G + \theta\eta^G(X) = \mathcal{L}_X\eta^G - d(\eta^G(X)) + \theta\eta^G(X) = d_\theta(-\eta^G(X)).$$

Remark 4.2.7: Let us note that, like in the symplectic case, the moment map μ^Ω is G-equivariant if and only if Ω is G-invariant, which can be seen directly by computations, following the same proof as in the symplectic case.

4.3 Torus actions on LCS manifolds

In this section we assemble mostly already known results concerning tori actions that we will need in the sequel. In particular we will make use of the following well-known general result about the orbits of smooth actions of compact Lie groups, which is a consequence of the slice theorem. We refer the reader to [Bre] or to [DK] for a proof of the result and for a detailed presentation of the subject.

Theorem 4.3.1: *Let N be a connected smooth manifold and G be a compact Lie group which acts effectively by diffeomorphisms on N. For any $x \in N$, denote by $G_x := \{g \in G | g.x = x\}$ the stabiliser of x in G, and let $r = \inf_{x\in N} \dim G_x$. Then $N_r := \{x \in N | \dim G_x = r\}$, called the set of principal G-orbits, is a dense connected open submanifold of N, and $N - N_r$ is a union of submanifolds of codimension ≥ 2. Moreover, if G is abelian and acts effectively on N, then $r = 0$.*

Remark 4.3.2: In general, if $G = \mathbb{T}$ is the compact torus and acts effectively on N as in the above theorem, the stabilisers G_x need not be connected. However, if in addition we have a symplectic form on N which is preserved by G and such that the orbits of the action are isotropic, then indeed all the stabilisers are connected tori. As we will see soon, cf. Proposition 4.3.4, this hypothesis will be verified in our context. For a proof of the connectedness of the stabilizers, see for instance [Ben02, Lemma 6.7], but we will also give a self-contained argument of this in our context when the time comes. In particular, in this case, the set N_0 is acted upon freely.

Recall that, by Proposition 2.5.4, if $(M, [\Omega])$ is an LCS manifold and a compact torus $\mathbb{T}$ acts conformally it so that $\mathfrak{Lie}(\mathbb{T}) = \mathfrak{t} \subset \mathfrak{aut}'(M, [\Omega])$, then the action of $\mathbb{T}$ lifts to the minimal cover $\hat{M}$. Thus we have:

Corollary 4.3.3: *Any twisted Hamiltonian action of a compact torus $\mathbb{T}$ on an LCS manifold $(M, [\Omega])$ lifts to a Hamiltonian action of $\mathbb{T}$ to the minimal symplectic cover $(\hat{M}, \Omega_K)$.*

Proof. Indeed, by (4.2.2), $\mathfrak{t}$ sits in $\mathfrak{aut}'(M, [\Omega])$, so the $\mathbb{T}$-action lifts to $\hat{M}$. Moreover, the lifted action is still Hamiltonian, since it admits the moment map $\hat{\mu} : \hat{M} \to \mathfrak{t}^*$, $\hat{\mu}(\hat{x}) = \mathrm{e}^{-\varphi(\hat{x})}\mu^\Omega(\pi(\hat{x}))$. Remark that we chose a form $\Omega \in [\Omega]$ in order to define $\hat{\mu}$, but actually $\hat{\mu}$ is conformally invariant. ∎

For a symplectic manifold, the maximal dimension of a torus acting symplectically and effectively on it is bounded from above only by the dimension of the manifold and, moreover, in many cases the orbits are not isotropic. The next proposition shows that things are different in the LCS setting. A variant of this result can again be found as Proposition 3.9 in [MMP17].

Proposition 4.3.4: *Suppose that a real torus $\mathbb{T}^m$ acts conformally end effectively on an LCS manifold $(M^{2n}, [\Omega])$. Then $m \leq n + 1$ and, moreover, if $\mathfrak{t} = \mathfrak{Lie}(\mathbb{T}^m) \subset \mathfrak{aut}'(M, [\Omega])$, then $m \leq n$ and the orbits are isotropic with respect to any representative in $[\Omega]$.*

Proof. Denote by $\mathcal{T} \subset TM$ the distribution generated by $\mathbb{T}^m$ on M, and by $\hat{\mathcal{T}}$ the one on $\hat{M}$. Suppose first that $\mathfrak{t} \subset \mathfrak{aut}'(M, [\Omega])$. By (2.5.3) it follows that $\Gamma(\hat{\mathcal{T}}) \subset \mathfrak{aut}(\hat{M}, \Omega_K)$. Hence, using the formula:

$$\iota_{[X,Y]} = \mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X \tag{4.3.1}$$

we have, for any $\hat{X}$ and $\hat{Y}$ in $\Gamma(\hat{\mathcal{T}})$:

$$0 = \iota_{[\hat{X},\hat{Y}]}\Omega_K = \mathcal{L}_{\hat{X}}\iota_{\hat{Y}}\Omega_K = d\iota_{\hat{X}}\iota_{\hat{Y}}\Omega_K + \iota_{\hat{X}}d\iota_{\hat{Y}}\Omega_K.$$

But we also have:

$$d\iota_{\hat{Y}}\Omega_K = \mathcal{L}_{\hat{Y}}\Omega_K = 0$$

implying that $d(\Omega_K(\hat{X}, \hat{Y})) = 0$, or also that $\Omega_K(\hat{X}, \hat{Y}) = c \in \mathbb{R}$. It follows that $e^{\varphi}c = \pi^*(\Omega(X, Y))$, and since e^{φ} is not Γ-invariant, $c = 0$. Therefore, for any $\hat{x} \in \hat{M}$, $\hat{\mathcal{T}}_{\hat{x}}$ is isotropic with respect to $(\Omega_K)_{\hat{x}}$, so $\hat{\mathcal{T}}$ and also $\mathcal{T}$ have maximal rank at most n.

On the other hand, let $M_0 \subset M$ be the dense open set composed by all the m-dimensional orbits, as in Theorem 4.3.1. Then $M_0 \times \mathfrak{t}$ injects into $\mathcal{T}|_{M_0}$ as a vector subbundle in a natural way, hence $m \leq n$.

In the general case, if $\mathfrak{t} \not\subset \mathfrak{aut}'(M, [\Omega])$, then by (2.5.1) there exists $C \in \mathfrak{t} - \mathfrak{aut}'(M, [\Omega])$ such that $l(C) = 1$. Then we have a splitting $\mathfrak{t} = \mathbb{R}C \oplus \mathfrak{t}'$ with $\mathfrak{t}' \subset \mathfrak{aut}'(M, [\Omega])$ and by the above, $\mathfrak{t}'$ has dimension at most n, hence the conclusion follows.■

Definition 4.3.5: An LCS manifold $(M^{2n}, [\Omega])$ is called *toric LCS* if the maximal compact torus $\mathbb{T}^n$ acts effectively in a twisted Hamiltonian way on it. An LCK manifold $(M^{2n}, J, [\Omega])$ is called *toric LCK* if $(M^{2n}, [\Omega])$ is toric LCS with respect to an action of $\mathbb{T}^n$ which is moreover holomorphic.

4.4 Proof of the Main Theorem

We are now ready to give the proof of the main result of the chapter:

Theorem 4.4.1: *Let $(M, J, [\Omega])$ be a compact toric LCK manifold. Then there exists an LCK form Ω' (possibly nonconformal to Ω) with respect to which the same action is still twisted Hamiltonian, and such that the corresponding metric g' is Vaisman.*

Proof. Denote by $\mathbb{T}$ the n-dimensional compact torus that acts on the LCK manifold as in the hypotheses of the theorem. Then the holomorphic action of $\mathbb{T}$ naturally extends to a holomorphic action of the complexified torus $\mathbb{T}^c = (\mathbb{C}^*)^n$ on M. Indeed, on one hand the induced inclusion homomorphism $\tau : \mathfrak{t} \to \mathfrak{aut}(M, J)$ extends to a Lie algebra morphism $\tau^c : \mathfrak{Lie}(\mathbb{T}^c) = \mathfrak{t} \otimes \mathbb{C} \to \mathfrak{aut}(M, J)$ by:

$$\tau^c(\xi_1 + i\xi_2) = \tau\xi_1 + J\tau\xi_2 = X_{\xi_1} + JX_{\xi_2}.$$

On the other hand, we have the Cartan decomposition $\mathbb{T}^c = \mathbb{T} \times (\mathbb{R}_{>0})^n$ and $i\mathfrak{t} \subset \mathfrak{t}^c$ is isomorphic to $(\mathbb{R}_{>0})^n$ under the exponential map. Hence, if $\xi_1, \dots, \xi_n$ form a basis of the Lie algebra $\mathfrak{t}$, then $JX_{\xi_1}, \dots, JX_{\xi_n} \in \mathfrak{aut}(M, J)$, being complete vector fields, generate the holomorphic action of $(\mathbb{R}_{>0})^n$ on M.

Let us note that although the action of $\mathbb{T}^c$ might not be effective on M, the morphism τ^c is injective. Indeed, this is equivalent to saying that $\tau(\mathfrak{t}) \cap J\tau(\mathfrak{t}) = \{0\} \subset \mathfrak{aut}(M, J)$. But this last assertion follows from the fact that the action of $\mathbb{T}$ is Hamiltonian with respect to $[\Omega]$, and by choosing $\Omega \in [\Omega]$ $\mathbb{T}$-invariant, also the corresponding moment map $\mu : M \to \mathfrak{t}^*$ is $\mathbb{T}$-invariant. Therefore, for $X \in \tau(\mathfrak{t})$ so that $JX \in \tau(\mathfrak{t})$, we have:

$$0 \leq \Omega(X, JX) = \iota_{JX}(d\mu_X - \theta\mu_X) = \mathcal{L}_{JX}\mu_X = 0$$

which implies that $X = 0$.

Let $(\hat{M}, \hat{J}, \Omega_K)$ be the minimal Kähler cover of $(M, J, [\Omega])$ of deck group Γ. The action of $\mathbb{T}^c$ evidently lifts to $\hat{M}$, and $\mathbb{T}$ also acts in a Hamiltonian way with respect to Ω_K. Denote by $\hat{\mu} : \hat{M} \to \mathbb{R}^n$ the moment map of this action, and let $\hat{M}_0 \subset \hat{M}$ be the corresponding connected dense open set of principal $\mathbb{T}$-orbits, as in Theorem 4.3.1. The group $\mathbb{T}$ acts freely on $\hat{M}_0$. Indeed, let $g \in \mathbb{T}$ and $x \in \hat{M}_0$ with $g.x = x$. Then $d_x g$ is a $\mathbb{C}$-linear automorphism of $T_x\hat{M}$. On the other hand, if we denote by $\mathfrak{t}_x \subset T_x\hat{M}$ the image of the evaluation map $\mathrm{ev}_x : \mathfrak{t} \to T_x\hat{M}$, $\xi \mapsto (\tau\xi)_x$, then, because ev_x is injective, we have the decomposition $T_x\hat{M} \cong \mathfrak{t}_x \oplus J\mathfrak{t}_x$. As g is a biholomorphism of $\hat{M}$, $d_x g$ preserves this decomposition. Moreover, since $g \in \mathbb{T}$, it follows that $d_x g|_{\mathfrak{t}_x} = \mathsf{id}$, and so $d_x g = \mathsf{id}$ on the whole of $T_x\hat{M}$. Finally, as g is an isometry of the Kähler metric, it follows that g is the trivial element in $\mathbb{T}$.

Fact 1: $\mathbb{T}^c$ preserves $\hat{M}_0$ and acts freely on it.

By the above, $\mathbb{T}^c = \mathbb{T} \times (\mathbb{R}_{>0})^n$ preserves $\hat{M}_0$ iff $\forall u \in (\mathbb{R}_{>0})^n$, $\forall \hat{x} \in \hat{M}_0$, $\forall t \in \mathbb{T} - \{1\}$, $tu.\hat{x} \neq u.\hat{x}$. But this is obvious since t and u commute.

To show that the action of $\mathbb{T}^c$ is free on $\hat{M}_0$, let $g \in \mathbb{T}^c$ and $\hat{x} \in \hat{M}_0$ with $g.\hat{x} = \hat{x}$. With the above remarks on $\mathbb{T}^c$, we have $g = tu$ with $t \in \mathbb{T}$ and $u = \exp(i\xi)$, $\xi \in \mathfrak{t}$. By letting $\hat{y} := t.\hat{x}$, it follows that $u.\hat{y} = \hat{x} \in \mathbb{T}\hat{x} = \mathbb{T}\hat{y}$. Let $c : \mathbb{R} \to \hat{M}$ be the curve $c(s) = \exp(is\xi).\hat{y}$. Since $\hat{\mu}^\xi$ is constant on the orbits of $\mathbb{T}$, it follows that:

$$\hat{\mu}^\xi(c(0)) = \hat{\mu}^\xi(\hat{y}) = \hat{\mu}^\xi(\hat{x}) = \hat{\mu}^\xi(c(1)). \tag{4.4.1}$$

On the other hand, the vector field $\tau(i\xi) = JX_\xi$ is, by definition, the gradient of the Hamiltonian $\hat{\mu}^\xi$. So, if $\xi \neq 0$, then $\hat{\mu}^\xi$ would be strictly increasing along c, but this contradicts (4.4.1). Thus $\xi = 0$ and we have $t.\hat{x} = \hat{x}$, implying again that t is the trivial element in $\mathbb{T}$, hence g is the trivial element in $\mathbb{T}^c$.

Fact 2: $\mathbb{T}^c$ acts transitively on $\hat{M}_0$.

For any $\hat{x} \in \hat{M}_0$, the map $\mathbb{T}^c \to \hat{M}_0$, $g \mapsto g.\hat{x}$ is a holomorphic open embedding. Therefore, the connected open set $\hat{M}_0$ is a union of disjoint open orbits of $\mathbb{T}^c$, hence it must contain (and be equal to) a sole orbit.

In conclusion, for any choice of a point $\hat{x}_0 \in \hat{M}_0$ we have a $\mathbb{T}^c$-equivariant biholomorphism $F_{\hat{x}_0} : (\mathbb{C}^*)^n \to \hat{M}_0$, $g \mapsto g.\hat{x}_0$, where $(\mathbb{C}^*)^n$ acts on itself by (left) multiplication. On the other hand, Γ preserves $\hat{M}_0$, hence we can view Γ as a subgroup of biholomorphisms of $(\mathbb{C}^*)^n$ acting freely.

Fact 3: $\Gamma \subset \mathbb{T}^c$.

Let $\hat{y} = F_{\hat{x}_0}(g)$, with $g \in \mathbb{T}^c$, be any element of $\hat{M}_0$ and let $\gamma \in \Gamma$. Denote by $g_\gamma \in \mathbb{T}^c$ the element verifying $\gamma(\hat{x}_0) = F_{\hat{x}_0}(g_\gamma)$. Since the action of $\mathbb{T}^c$ on $\hat{M}$ is the lift of the action of $\mathbb{T}^c$ on M, Γ commutes with $\mathbb{T}^c$. We thus have:

$$\gamma(\hat{y}) = \gamma(g.\hat{x}_0) = g.\gamma(\hat{x}_0) = g.g_\gamma.\hat{x}_0 = g_\gamma.\hat{y}$$

implying that $\gamma = g_\gamma \in \mathbb{T}^c$.

Remark that, if $g_\gamma^j \in \mathbb{C}^*$ are the components of g_γ, then for at least one $1 \leq j \leq n$, $|g_\gamma^j| \neq 1$. Otherwise we would have $\Gamma \subset \mathbb{T}$ and so $\mathbb{T}$ would not act effectively on M.

Let now $\gamma \in \Gamma \cong \mathbb{Z}^k$ be a nontrivial primitive element and denote by Γ' the subgroup generated by γ. With the same notations as before, we can extend the action of Γ' on $\hat{M}_0 \cong (\mathbb{C}^*)^n$ to a holomorphic action of $\mathbb{R}$ on $\hat{M}_0$. Indeed, if γ expresses, as an automorphism of $(\mathbb{C}^*)^n$, as:

$$\gamma(z_1, \ldots, z_n) = (\alpha_1 z_1, \ldots, \alpha_n z_n),$$

with $\alpha_j = \rho_j e^{i\theta_j}$ in polar coordinates, then define the one-parameter group:

$$\begin{gathered}\mathbb{R} \ni t \mapsto \Phi_t \in \mathrm{Aut}((\mathbb{C}^*)^n) \\ \Phi_t(z_1, \ldots, z_n) = (\rho_1^t e^{it\theta_1} z_1, \ldots, \rho_n^t e^{it\theta_n} z_n).\end{gathered}$$

Remark that $\mathbb{R} \cong \{\Phi_t\}_{t\in\mathbb{R}}$ is a subgroup of $\mathbb{T}^c \subset \mathrm{Aut}((\mathbb{C}^*)^n)$, hence its action on $\hat{M}_0$ actually extends to the whole of $\hat{M}$. Moreover, this also implies that Γ commutes with $\mathbb{R}$, so the action of $\mathbb{R}$ descends on M to an effective action of $\mathbb{R}/\Gamma' \cong S^1$. Let $C \in \Gamma(TM)$ be the real holomorphic vector field generating this action.

Lemma 4.4.2: *There exists on M an LCK form Ω^C compatible with the complex structure J, with corresponding Lee form θ^C, so that C preserves both Ω^C and θ^C. Moreover, the given action of $\mathbb{T}$ is still Hamiltonian with respect to this new form.*

Proof. For any $t \in \mathbb{R}$ let $f_t := \Phi_t^*\varphi - \varphi \in \mathcal{C}^\infty(\hat{M})$ and define $h := \int_0^1 f_t dt \in \mathcal{C}^\infty(\hat{M})$. Note that the functions $\{f_t\}_{t\in\mathbb{R}}$ are Γ-invariant:

$$\delta^* f_t = \Phi_t^*\delta^*\varphi - \delta^*\varphi = \Phi_t^*(\varphi + \rho(\delta)) - (\varphi + \rho(\delta)) = f_t, \qquad \forall \delta \in \Gamma \tag{4.4.2}$$

hence so is h and they all descend to M. Moreover, since $\mathfrak{t} \subset \ker\theta$, φ is $\mathbb{T}$-invariant. As $\mathbb{T}$ commutes with $\{\Phi_t\}_{t\in\mathbb{R}}$, it follows that also the function h is $\mathbb{T}$-invariant.

Let the new Lee form be:

$$\theta^C := \int_{\mathbb{R}/\Gamma'} \Phi_t^*\theta dt = d\int_0^1 \Phi_t^*\varphi dt = d(\varphi + h) = \theta + dh. \tag{4.4.3}$$

By definition, it is C-invariant, but also $\mathbb{T}$-invariant since $\mathfrak{t}$ commutes with C.

Let now $\Omega_h := e^h\Omega \in \Omega^2(M)$ and define the new LCK form as:

$$\Omega^C := \int_{\mathbb{R}/\Gamma'} \Phi_t^*\Omega_h dt.$$

Since $d\Omega_h = \theta^C \wedge \Omega_h$ by (4.4.3), we see that the Lee form of Ω^C is indeed θ^C:

$$\begin{aligned} d\Omega^C &= \int_{\mathbb{R}/\Gamma'} \Phi_t^*(d\Omega_h)dt = \int_{\mathbb{R}/\Gamma'} \Phi_t^*\theta^C \wedge \Phi_t^*\Omega_h dt = \\ &= \theta^C \wedge \int_{\mathbb{R}/\Gamma'} \Phi_t^*\Omega_h dt = \theta^C \wedge \Omega^C. \end{aligned}$$

Again, the C-invariance of Ω^C follows from its definition. Moreover, since h is $\mathbb{T}$-invariant and $\mathbb{T}$ commutes with $\mathbb{R}/\Gamma'$, also Ω^C is $\mathbb{T}$-invariant.

Finally, $\mathcal{L}_C\theta^C = 0$ implies that $\theta^C(C)$ is constant. On the other hand $\theta^C(C) = \mathcal{L}_C\varphi$, and since φ is not even Γ-invariant, it follows that $\theta^C(C) = \lambda \neq 0$. Hence, by Lemma 2.5.2, the form $\eta = -\frac{1}{\lambda}\iota_C\Omega^C \in \Omega^1_M$ verifies $\Omega^C = d^{\theta^C}\eta$. Moreover, η is automatically $\mathbb{T}$-invariant, since both C and Ω^C are. Therefore, cf. Remark 4.2.6, we have a moment map for the action of $\mathbb{T}$ on (M, J, Ω^C) given by $\mu^C(X) = -\eta(X)$, implying that the action is still Hamiltonian. ∎

Lemma 4.4.3: *The minimal cover corresponding to the form Ω^C is $\hat{M}$.*

Proof. Let $p_C : \hat{M}_C \to M$ be the minimal Kähler cover corresponding to Ω^C with deck group Γ_C, and denote by $p : \hat{M} \to M$ the projection corresponding to Ω. We have $p\Phi_t = \Phi_t p$ for any $t \in \mathbb{R}$, by making no distinction of notation between objects on M and on $\hat{M}$. We see, by (4.4.3) in Lemma 4.4.2, that $p^*\theta^C = d\varphi^C$ is exact, where $\varphi^C = \varphi + p^*h$. So $\hat{M}$ is a covering of $\hat{M}_C$ and Γ_C is a subgroup of Γ'. On the other hand, by the same lemma, h is Γ-invariant, so for any $\delta \in \Gamma'$ we have $\delta^*\varphi^C = \rho(\delta) + \varphi^C$. Thus no element of Γ' preserves φ^C, therefore $\hat{M}_C = \hat{M}$. ∎

We also give here a lemma since it follows directly from the above considerations, but we will not make use of it in the sequel.

Lemma 4.4.4: *The rank of* Γ *is 1.*

Proof. With the same notations as before, suppose there existed some $\gamma' \in \Gamma$ independent (over $\mathbb{Z}$) of γ. Then, in the same way, γ' would generate another real holomorphic vector field $C' \in \Gamma(TM)$, independent of C. Indeed, if this was not the case, then suppose we have $C = aC'$ with $a \in \mathbb{R}$. Then the corresponding flows would verify $\Phi^t_C = \Phi^t_{aC'} = \Phi^{at}_{C'}$. In particular, for any $m \in \mathbb{Z}$ we would have that $\Phi^m_C = \Phi^{am}_{C'} \in \Gamma$. From the independence of γ and γ' it follows that $a \notin \mathbb{Q}$, so the additive subgroup Λ generated by 1 and a in $(\mathbb{R}, +)$ is not discrete. Now, if we fix some $\hat{x} \in \hat{M}$, the map $F : \mathbb{R} \to \hat{M}$, $t \mapsto \Phi^t_{C'}(\hat{x})$ is continuous, so also $F(\Lambda) \subset \hat{M}$ is not discrete. But $F(\Lambda)$ is contained in the fiber of the covering map through $\hat{x}$ which must be discrete, hence we have a contradiction.

Now let Ω' be the LCK form obtained by averaging Ω^C with respect to C', as in Lemma 4.4.2. We would thus have an effective holomorphic action of $\mathbb{T}^{n+2}$ on M generated by $\mathfrak{t} \oplus \mathbb{R}C \oplus \mathbb{R}C'$, which is moreover conformal with respect to Ω'. But by Proposition 4.3.4 this is impossible. ∎

From now on, to simplify notation, denote by Ω and by θ the forms Ω^C and θ^C obtained in Lemma 4.4.2. Let $\pi_{i\mathfrak{t}} : \mathfrak{t}^c \to i\mathfrak{t}$ and $\pi_{\mathfrak{t}} : \mathfrak{t}^c \to \mathfrak{t}$ be the natural projections, and consider the vector field $B' := \pi_{i\mathfrak{t}}(C) \in i\mathfrak{t}$. Since $\mathfrak{t} \subset \ker\theta$, we have $\theta(B') = \theta(C) = \lambda$, so let $B = -\frac{1}{\lambda}B' \in i\mathfrak{t}$. Moreover, since $J\pi_{i\mathfrak{t}} = \pi_{\mathfrak{t}}J$, we also have $JB = -\frac{1}{\lambda}\pi_{\mathfrak{t}}(JC) \in \mathfrak{t}$. Since B is a difference of vector fields preserving Ω, it also preserves Ω and so does JB, being in $\mathfrak{t}$.

Consider, on the minimal cover $\hat{M}$, the Kähler form $\Omega_K = \mathrm{e}^{-\varphi}\Omega$ with corresponding metric g_0, where $d\varphi = \theta$. We have:

$$\mathcal{L}_B\Omega_K = -\theta(B)\Omega_K = \Omega_K \text{ and } \mathcal{L}_{JB}\Omega_K = 0.$$

Let $\eta_0 := \iota_B\Omega_K$ and $f_0 := \|B\|^2_{g_0} = \eta_0(JB)$. It follows, by the above:

$$d\eta_0 = \mathcal{L}_B\Omega_K = \Omega_K.$$

Since JB commutes with B and preserves Ω_K, it also preserves η_0. Hence we have:

$$0 = \mathcal{L}_{JB}\eta_0 = d\iota_{JB}\eta_0 + \iota_{JB}d\eta_0 = df_0 + J\eta_0$$

implying:

$$\eta_0 = Jdf_0 \text{ and } \Omega_K = d\eta_0 = dd^c f_0. \tag{4.4.4}$$

Now, since both B and JB are Γ-invariant, we have, for any $\gamma \in \Gamma$:

$$\gamma^* f_0 = (\gamma^* \Omega_K)(B, JB) = \mathrm{e}^{-\rho(\gamma)} f_0$$

hence f_0 is a Γ-equivariant function, which is moreover strictly positive. Therefore,

$$\Omega' := \frac{1}{f_0} dd^c f_0 \tag{4.4.5}$$

is an LCK form on M, in the same conformal class as Ω, with corresponding Lee form $\theta' = -d \ln f_0$.

Finally, we have, by (4.4.4):

$$\iota_{JB}\Omega' = \frac{1}{f_0} J\eta_0 = -\frac{1}{f_0} df_0 = \theta'$$

which implies that $-B$ is the fundamental vector field $(\theta')^{\#}$. In particular, since B is both holomorphic and an infinitesimal automorphism of Ω', it is also Killing, so:

$$\nabla \theta' = d\theta' = 0 \tag{4.4.6}$$

implying that (Ω', θ') is the Vaisman structure that we have been looking for. This ends the proof of the theorem. ■

Now, as a consequence of the main result, of [MMP17, Proposition 5.4] and of [Da78, Theorem 9.1] we have:

Corollary 4.4.5: *Let $(M, J, [\Omega])$ be a toric LCK manifold, strict or not. If (M, J) is Kählerian, then M is simply connected, and in particular $b_1(M)$, the first Betti number of M, is 0. If it is not strict, then $b_1(M) = 1$.*

Remark 4.4.6: Lemma 4.4.4 can also be seen, a posteriori, as a consequence of a result of [MMP17, Proposition 5.4], where it is shown that a compact toric Vaisman manifold has first Betti number 1.

4.5 Examples

As we have seen, all examples of toric LCK manifolds are of Vaisman type. In complex dimension 2, by [MMP17, Theorem 7.2], any toric LCK manifold is a diagonal Hopf surface. More generally, any diagonal Hopf manifold is a toric manifold:

Example 4.5.1: Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be of module smaller than 1. Recall from Section 2.6.1 that a diagonal Hopf manifold is $H_\lambda := \mathbb{C}^n - \{0\}/(z_1, \ldots, z_n) \sim (\lambda_1 z_1, \ldots, \lambda_n z_n)$. We have an $\mathbb{R}$-action on $\mathbb{C}^n - \{0\}$ given by $\Phi^t(z) = (\lambda_1^t z_1, \ldots, \lambda_n^t z_n)$, well-defined after writing each λ_i in polar coordinates, which allows us to define a real-valued function φ on $\mathbb{C}^n - \{0\}$. For each $z \in \mathbb{C}^n - \{0\}$, $\varphi(z)$ is the unique solution $s \in \mathbb{R}$ to the equation $\Phi^{-s}(z) \in S := \{z \in \mathbb{C}^n - \{0\} | |z|^2 = 1\}$. With this, one constructs a Vaisman metric on the

Hopf manifold given by $\Omega = \mathrm{e}^{\varphi} dd^c \mathrm{e}^{-\varphi}$ of Lee form $\theta = d\varphi$. On the other hand, $\mathbb{T}^n$ acts in a Hamiltonian way on the Kähler manifold $(\mathbb{C}^n - \{0\}, \Omega_K = dd^c \mathrm{e}^{-\varphi})$ by:

$$t.z = (\mathrm{e}^{it_1} z_1, \ldots, \mathrm{e}^{it_n} z_n), \quad t = (t_1, \ldots, t_n) \in \mathbb{T}^n.$$

The fundamental vector fields of this action are given by $X_j = iz_j \frac{\partial}{\partial z_j} - i\bar{z}_j \frac{\partial}{\partial \bar{z}_j}$, and the moment map is $\hat{\mu}_{X_j} := \mathrm{e}^{-\varphi} d^c \varphi(X_j)$, $1 \leq j \leq n$. Note that, as Φ^t commutes with the $\mathbb{T}^n$-action, the function φ is constant on the orbits of the torus, and the torus action descends to an effective action of $\mathbb{T}^n$ on H_λ. Moreover, the action is Hamiltonian, with moment map $\mu_{X_j} := J\theta(X_j)$.

Example 4.5.2: The following is an example from [MMO17], see also Example 3.2.5, and gives a toric LCK manifold $(M, \Omega, \mathbb{T})$ which is not conformal to a Vaisman metric, showing that the formulation of Theorem 4.4.1 is sharp. Let $\lambda \in \mathbb{C}$ with $|\lambda| \neq 0, 1$ and let $(H_\lambda, \Omega, \mathbb{T})$ be the toric diagonal Hopf manifold with the Vaisman metric $\Omega = -dJ\theta + \theta \wedge J\theta$ constructed in the above example. Consider the Γ_λ-invariant function on $\mathbb{C}^n - \{0\}$ given by $f(z) = \frac{1}{2}\sin(2\pi \frac{\ln|z|}{\ln|\lambda|})$. As $\theta = -2d\ln|z|$, f verifies $df \wedge \theta = 0$. Moreover, as $f > -1$ everywhere, one can easily check that $\Omega' = \Omega + f\theta \wedge J\theta$ is a strictly positive form, and so defines an LCK metric on H_λ with Lee form $\theta' = (1+f)\theta$. Moreover, as f is $\mathbb{T}$-invariant, the new LCK form Ω' is also $\mathbb{T}$-invariant and we have $\mathfrak{t} \subset \ker\theta'$. Thus, by Remark 4.2.6, $(H_\lambda, \Omega', \mathbb{T})$ is a toric LCK manifold. On the other hand, by Lemma 3.2.6, Ω' is not conformal to any Vaisman metric.

Let us note that in this example, the two LCK metrics have the same moment map: $\mu = \mu' = J\theta|_{\mathfrak{t}}$, because we have $\Omega = d_\theta(-J\theta)$ and $\Omega' = d_{\theta'}(-J\theta)$. On the minimal cover, with respect to the basis $X_1 \ldots X_n$ of $\mathfrak{t}$ defined in the first example, we will then have that the moment map of the Kähler metric corresponding to the Vaisman case is given by:

$$\hat{\mu} : \mathbb{C}^n - \{0\} \to \mathbb{R}^n, \quad \hat{\mu}(z) = -2(|z_1|^2, \ldots, |z_n|^2).$$

On the other hand, as $\theta' = d\varphi'$, with

$$\varphi'(z) = -\ln|z|^2 + \frac{1}{c}\cos(c\ln|z|) \in \mathcal{C}^\infty(\mathbb{C}^n - \{0\}, \mathbb{R}), \quad c = \frac{2\pi}{\ln|\lambda|}$$

we have the second Kähler moment map corresponding to Ω'_K:

$$\hat{\mu}' : \mathbb{C}^n - \{0\} \to \mathbb{R}^n, \quad \hat{\mu}'(z) = -2\exp\left(\frac{\cos(c\ln|z|)}{c}\right)(|z_1|^2, \ldots, |z_n|^2).$$

In particular, although $\hat{\mu} \neq \hat{\mu}'$, we have

$$\operatorname{Im}\hat{\mu} = \operatorname{Im}\hat{\mu}' = \{(x_1, \ldots, x_n) \in \mathbb{R}^n - \{(0, \ldots, 0)\} | x_1 \leq 0, \ldots, x_n \leq 0\}.$$

This shows two things: the moment image (and even the moment map itself) of the LCK metric does not determine the LCS form. Moreover, neither does the moment image of the corresponding Kähler metric determine the LCS form.

Example 4.5.3: Following [Pi16, Theorem 5.1], given a compact toric Hodge manifold with moment map $(N, \omega_N, \mathbb{T}^{n-1}, \mu_N)$ of complex dimension $n-1$ and a complex number λ with $|\lambda| \neq 0, 1$, one can uniquely associate to it an n-dimensional toric Vaisman manifold $(M, \Omega, \mathbb{T}^n, \mu)$. The manifold M is given as in Example 2.6.3, namely if (L, h) is the positive Hermitian line bundle over N associated with ω_N, then $M = L^* - 0_N/\Gamma_\lambda$ where $\Gamma_\lambda \cong \mathbb{Z}$ is the group generated by the dilatation by λ along the fiber of $L^* - 0_N \to N$. The Reeb vector field A of this metric is a twisted Hamiltonian vector field with closed orbits, so generates an $\mathbb{S}^1$-action on M. Consider the Chern connection on (L^*, h_{L^*}), and endow the $\mathbb{C}^*$-principal bundle $\pi : P := L^* - 0_N \to N$ with the induced connection. For any infinitesimal generator X of the $\mathbb{T}^{n-1}$-action on N, we can consider its horizontal lift $\hat{X}$ to P. The vector field $X' := \hat{X} - \pi^*\mu_N(X)A$ is Γ_λ-invariant, twisted Hamiltonian and has closed orbits. Moreover, A commutes with all vector fields X' constructed in this way. Hence, all these vector fields together with A generate a Hamiltonian action of $\mathbb{T}^n$ on M. The corresponding moment map $\mu : M \to \mathfrak{Lie}(\mathbb{T}^n)$ is given as follows: $\mu(X') = p^*\mu_N(X)$ for $X \in \mathfrak{Lie}(\mathbb{T}^n)$ and $\mu(A) = 1$, where $p : M \to N$ is the natural map. In particular, the image of the moment map $\mu(\mathfrak{Lie}(\mathbb{T}^n))$ sits in the affine hyperplane $H := \{\alpha \in \mathfrak{Lie}(\mathbb{T}^n)^* | \alpha(A) = 1\}$ and is a Delzant polytope in $H \cong \mathfrak{Lie}(\mathbb{T}^{n-1})^*$.

Remark 4.5.4: In the symplectic setting it follows, by using the Delzant classification, that every compact toric symplectic manifold is in fact a toric Kähler manifold, i.e. there exists an integrable complex structure, preserved by the torus action, compatible with the given symplectic form. It turns out that this does not happen in the LCS setting, i.e. a toric LCS manifold is not necessarily toric LCK. We illustrate this by the following simple example.

Example 4.5.5: Consider on the compact manifold $M = (S^1)^4$ the action of $T = \mathbb{T}^2$ given by:

$$(e^{it_1}, e^{it_2}).(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) = (e^{i(t_1+\theta_1)}, e^{i(t_2+\theta_2)}, e^{i\theta_3}, e^{i\theta_4})$$

where $\theta_1, \ldots, \theta_4$ and t_1, t_2 are the polar coordinates on M, respectively T, given by the exponential map $\exp : \mathbb{R} \to S^1, \theta \mapsto e^{i\theta}$. Let ν be the volume form on S^1 such that $\exp^* \nu = d\theta$, let $p_j : M \to S^1$ be the canonical projection on the j-th factor of M and let $\nu_j := p_j^*\nu$, where $j \in \{1, 2, 3, 4\}$. Define the T-invariant 1-forms on M:

$$\theta := \nu_4 \text{ and } \eta := \sin\theta_3\nu_1 + \cos\theta_3\nu_2.$$

Then we have:

$$d\eta = \cos\theta_3\nu_3 \wedge \nu_1 - \sin\theta_3\nu_3 \wedge \nu_2 \text{ and } d\eta \wedge \eta = \nu_3 \wedge \nu_1 \wedge \nu_2$$

implying that η induces a contact form on $p_1^*S^1 \times p_2^*S^1 \times p_3^*S^1$, hence $\Omega = d^\theta\eta \in \Omega^2(M)$ is a T-invariant LCS form on M. Moreover, clearly $\mathfrak{Lie}(T) \subset \ker\theta$, so $\mathfrak{Lie}(T) \subset \mathfrak{aut}'(M, [\Omega])$. Thus, by Remark 4.2.6, the T-invariant form $-\eta$ gives a moment map for the T-action, hence $(M, [\Omega])$ is a toric LCS manifold. On the other hand, $b_1(M) = 4 \notin \{0, 1\}$, so by Corollary 4.4.5 M cannot admit a toric LCK structure (strict or not).

4.6 Final remarks and questions

During the proof of Theorem 4.4.1, we have constructed explicitly a projection corresponding to a complex manifold $(M^n, J, \mathbb{T}^n)$ which is toric LCK for some LCK structure:

$$P : \{\text{toric LCK structure } ([\Omega], [\theta]_{dR})\} \to \{\mathbb{T}^n - \text{invariant toric Vaisman structure } (\Omega', \theta')\}.$$

Recall that the extension of $H_1(M, \mathbb{Z}) \subset \mathbb{R} \subset \mathrm{Aut}(\hat{M}, J)$ on $\hat{M}$ induces an effective holomorphic action of $\mathbb{S}^1$ on M which commutes with $\mathbb{T}^n$ but is not part of the torus. As the Lee vector field of any Vaisman metric belongs to $\mathfrak{Lie}(\mathbb{T}^n \times \mathbb{S}^1)$, but not to $\mathfrak{Lie}(\mathbb{T}^n)$, any $\mathbb{T}^n$-invariant Vaisman metric is also $\mathbb{S}^1$-invariant, so P is indeed a projection.

Question 4.6.1: How big is the pre-image of a point of P? Can one express all toric LCK metrics in terms of toric Vaisman metrics?

Concerning the image of the moment map of a toric LCK metric, we can say the following. Fix a toric Vaisman metric $(M, J, \Omega, \theta, \mathbb{T}^n)$ which is $\mathbb{T}^n$-invariant, with moment map μ. Let $\hat{\mu} = \mathrm{e}^{-\varphi}\mu$ be the moment map of the Kähler metric on $\hat{M}$, where $\theta = d\varphi$. Then $S := \varphi^{-1}(0)$ is a compact toric Sasaki manifold, and letting Φ_t denote the $\mathbb{R}$-action on $\hat{M}$ which extends the deck group action, we have an $\mathbb{R}$-equivariant diffeomorphism $S \times \mathbb{R} \to \hat{M}$, $(y, t) \to \Phi_t(y)$, descending to a diffeomorphism $S \times \mathbb{S}^1 \cong M$.

Now μ is Φ_t-invariant by the above discussion, while $\rho(t) := \Phi_t^*\varphi - \varphi \in \mathbb{R}$ gives a diffeomorphism of $\mathbb{R}$. Thus we have, for any $(y, t) \in S \times \mathbb{R}$:

$$\hat{\mu}(y, t) = \mathrm{e}^{-\varphi(y) - \rho(t)}\mu(\Phi_t(y)) = \mathrm{e}^{-\rho(t)}\mu(y)$$

so $P := \hat{\mu}(S) = \mu(S) = \mu(M)$. But, by [Ler03], P is a convex polytope and $C := \hat{\mu}(\hat{M}) \cup \{0\}$ is a *good cone* over P. Thus, if A denotes the Reeb vector field of Ω, $\mu(P)$ is a convex polytope with non-empty interior in the hyperplane $H_A := \{l \in \mathfrak{t}^* | l(A) = 1\} \subset \mathfrak{t}^*$. It is a Delzant polytope in H_A if the foliation generated by A and B is strongly regular, i.e. if M fibers holomorphically over a smooth Kähler manifold in elliptic curves.

As explained in the introduction, by the results of [Ler03] and [MSY06], any compact toric Vaisman manifold can be recovered from the image of the moment map of the corresponding Kähler metric. It follows then that it can also be recovered from the image of the LCK moment map. However, as we have seen in Example 4.5.2, the same cannot be said about an arbitrary LCK metric.

Question 4.6.2: Given a toric LCK metric on M, what happens to the image of the moment map of the LCK/ Kähler metric when we apply P? Are these necessarily moment images corresponding to Vaisman metrics?

Finally, let us note that Example 4.5.5 of a toric LCS manifold admitting no compatible toric LCK structure is pathological: the torus action we consider is free, which cannot be the case for a toric LCK manifold, and the manifold we consider admits symplectic and even Kähler structures, although not toric. We could then ask:

Question 4.6.3: Is there a simple characterisation of toric LCS manifolds which admit compatible toric LCK structures?

At least for the LCS manifolds *of the first kind* (cf. [Va85]), this question should not be too difficult, via Lerman's classification of toric contact manifolds.

Chapter 5

Cohomological properties of OT manifolds

5.1 Introduction

This chapter presents the results of the paper [IO17], which is in collaboration with Alexandra Otiman. It is concerned with the cohomological properties of Oeljeklaus-Toma manifolds. These manifolds were introduced by K. Oeljeklaus and M. Toma in [OT05], and are quotients of $\mathbb{H}^s \times \mathbb{C}^t$ by discrete groups of affine transformations arising from a number field K and a particular choice of a subgroup of units U of K. They are commonly referred to as OT manifolds of type (s,t), and denoted by $M(K,U)$. They have been of particular interest for LCK geometry. When they were introduced, OT manifolds of type $(s,1)$ were shown to carry LCK metrics and they constituted the first examples of manifolds to disprove a conjecture of Vaisman, according to which the odd index Betti numbers of an LCK manifold should be odd.

We start by presenting the construction of OT manifolds and some of their metric properties, in Section 5.2. Moreover, in this section we determine the set of all the possible Lee classes of LCK metrics on OT manifolds (Proposition 5.2.2).

Next, after some technical preliminaries, we turn to the computation of the de Rham cohomology algebra (Theorem 5.4.1) and of the twisted cohomology (Theorem 5.6.1) of any OT manifold. This is done by two different approaches, one by reducing to the invariant cohomology with respect to a certain compact Lie group, in Section 5.4, and the other one using the Leray-Serre spectral sequence, in Section 5.5. This last approach is also used to prove Theorem 5.6.1 in Section 5.6.

The result we obtain is given in terms of numerical invariants coming from $U \subset K$. We specialise it, in the last section, to OT manifolds of LCK type (Proposition 5.7.4, Proposition 5.7.5), and for some OT manifolds associated to a certain family of polynomials (Example 5.7.3). Additionally, we determine all the possible twisted classes of LCK forms on OT manifolds (Corollary 5.7.8), generalizing a result of [O16] showing that this class cannot vanish. They all turn out to induce a non-degenerate Lefschetz map in cohomology. A final application (Proposition 5.7.10) concerns the vanishing of certain real Chern classes of vector bundles on OT manifolds.

5.2 Oeljeklaus-Toma manifolds

5.2.1 The construction

We start by recalling the construction of Oeljeklaus-Toma manifolds, following [OT05], and some of their properties that we will need.

Given two positive numbers $s, t > 0$, an OT manifold X of type (s,t) is a compact quotient of $\tilde{X} := \mathbb{H}^s \times \mathbb{C}^t$ by a discrete group Γ of rank $2(s+t)$ arising from a number field. More specifically, let $m = s+t$ and $n = 2t+s$ and let K be a number field with n embeddings in $\mathbb{C}$, s of them real and $2t$ complex conjugate. We shall denote these embeddings by $\sigma_1, \dots, \sigma_n$, with the convention that the first s are real and $\sigma_{s+t+i} = \overline{\sigma_{s+i}}$, for any $1 \leq i \leq t$. The ring of integers of K, O_K, which as a $\mathbb{Z}$-module is free of rank n, acts on $\mathbb{H}^s \times \mathbb{C}^t$ via the first m embeddings. If $(w,z) = (w_1, \dots, w_s, z_1, \dots, z_t)$ denote the holomorphic coordinates on $\mathbb{H}^s \times \mathbb{C}^t$, the action is given by translations:

$$T_a(w,z) = (w_1 + \sigma_1(a), \dots, w_s + \sigma_s(a), z_1 + \sigma_{s+1}(a), \dots z_t + \sigma_{s+t}(a)), \quad a \in O_K.$$

It is a free and proper action, and as a smooth manifold, the quotient is given by:

$$\hat{X} := \mathbb{H}^s \times \mathbb{C}^t / O_K \cong (\mathbb{R}_{>0})^s \times \mathbb{T}^n.$$

Next, one defines inside the group of units O_K^* the subgroup of positive units $O_K^{*,+}$ as:

$$O_K^{*,+} = \{u \in O_K^* \mid \sigma_i(u) > 0, 1 \leq i \leq s\}.$$

This group acts on $\mathbb{H}^s \times \mathbb{C}^t$ by dilatations as:

$$R_u(w,z) = (\sigma_1(u)w_1, \dots, \sigma_s(u)w_s, \sigma_{s+1}(u)z_1, \dots, \sigma_{s+t}(u)z_t), \quad u \in O_K^{*,+}.$$

This action is free, but not properly discontinuous. However, as shown in [OT05], one can choose a rank s subgroup U in $O_K^{*,+}$ which embeds as a lattice in $(\mathbb{R}_{>0})^s$ via:

$$\begin{aligned} j : U &\to (\mathbb{R}_{>0})^s \\ u &\mapsto (\sigma_1(u), \dots, \sigma_s(u)). \end{aligned}$$

We will denote by $U_{\mathbb{H}} \cong U$ the lattice $j(U)$. In particular, U acts properly discontinuously on $\mathbb{H}^s \times \mathbb{C}^t$. Clearly, U also acts on O_K, so that one gets a free, properly discontinuous action of the semi-direct product $\Gamma := U \ltimes O_K$ on $\mathbb{H}^s \times \mathbb{C}^t$. The quotient of this action $X := \mathbb{H}^s \times \mathbb{C}^t / \Gamma$, denoted by $X(K,U)$, is the Oeljeklaus-Toma manifold of type (s,t) associated to K and U.

Since the action of Γ on $\tilde{X}$ is holomorphic, X is a complex manifold. Moreover, it is compact, because it has in fact the structure of a torus fiber bundle over another torus:

$$\mathbb{T}^n \to X(K,U) \xrightarrow{\pi} \mathbb{T}^s. \tag{5.2.1}$$

Indeed, this last assertion can be seen as follows: the natural projection

$$\hat{\pi} : \hat{X} = (\mathbb{R}_{>0})^s \times \mathbb{T}^n \to (\mathbb{R}_{>0})^s$$

is a trivial $\mathbb{T}^n$-fiber bundle over $(\mathbb{R}_{>0})^s$. The group U acts on $\hat{X}$, but also on $(\mathbb{R}_{>0})^s$ by translations via j, and $\hat{\pi}$ is equivariant for this action. As $X = \hat{X}/U$ and $\mathbb{T}^s = (\mathbb{R}_{>0})^s / U$, $\hat{\pi}$ descends to the $\mathbb{T}^n$-fiber bundle (5.2.1).

For later use, it is important to note that π is a flat fiber bundle, meaning that it has locally constant transition functions. This is equivalent to saying that it is given by a representation $R : \pi_1(\mathbb{T}^s) = U_{\mathbb{H}} \to \mathrm{Diff}(\mathbb{T}^n)$, $u \mapsto \Phi_u$. We can make R explicit, after identifying $U_{\mathbb{H}}$ with U. Recalling that $\mathbb{T}^n = \mathbb{R}^s \times \mathbb{C}^t/O_K$ and denoting by $r = (r_1, \dots, r_s)$ the real coordinates on $\mathbb{R}^s$, for any $u \in U$, Φ_u is given by:

$$\Phi_u((r,z) \mod O_K) = (\sigma_1(u)r_1, \dots, \sigma_s(u)r_s, \sigma_{s+1}(u)z_1, \dots, \sigma_m(u)z_t)) \mod O_K \tag{5.2.2}$$

which is clearly well defined, since $UO_K \subset O_K$.
The tangent bundle of $\hat{X}$ splits smoothly as $T\hat{X} = E \oplus V$, where E is the pullback of $T(\mathbb{R}_{>0})^s$ and V is the pullback of $T\mathbb{T}^n$ on $\hat{X}$ by the natural projections. These bundles are trivial, and for later use we will need to fix a global frame of $V^* \otimes \mathbb{C}$ over $\hat{X}$. If $z_1, \dots, z_t$ denote the holomorphic coordinates on $\mathbb{C}^t$ and $w_1 = r_1 + iv_1, \dots, w_s = r_s + iv_s$ are holomorphic coordinates on $\mathbb{H}^s$, also viewed as local coordinates on $\hat{X}$, we choose as a basis of $\mathcal{C}^\infty(\hat{X}, V^* \otimes \mathbb{C})$ over $\mathcal{C}^\infty(\hat{X}, \mathbb{C})$:

$$e_j = dr_j \text{ for } 1 \leq j \leq s, e_{s+j} = dz_j \text{ and } e_{s+t+j} = d\bar{z}_j \text{ for } 1 \leq j \leq t.$$

In particular, for any $0 \leq l \leq n$, a frame for $\bigwedge^l V^* \otimes \mathbb{C}$ is given by:

$$\{e_I = e_{i_1} \wedge \ldots \wedge e_{i_l} | I = (0 < i_1 < \ldots < i_l \leq n)\}. \tag{5.2.3}$$

For any multi-index $I = (0 < i_1 < \ldots < i_l \leq n)$, let us denote by $\sigma_I : U \to \mathbb{C}^*$ the representation:

$$\sigma_I(u) = \sigma_{i_1}(u) \cdots \sigma_{i_l}(u). \tag{5.2.4}$$

Then an element $u \in U$ acts on e_I by $u^* e_I = \sigma_I(u) e_I$.

5.2.2 Metric properties

It was shown in [OT05] that OT manifolds of type $(s, 1)$ admit an LCK metric:

$$\Omega = v_1 \cdots v_s (\frac{i}{2} dz \wedge d\bar{z} + dd^c(v_1 \cdots v_s)^{-1}) \tag{5.2.5}$$

with Lee form $\theta = \sum_{k=1}^s d \ln v_k$. Note that this metric can also be written as

$$\Omega = \omega_0 + d_\theta(-J\theta)$$

with $\omega_0 = 2v_1 \cdots v_s i dz \wedge d\bar{z}$. In this way, it can be easily seen that the corresponding Lee vector field, defined by $\iota_B \Omega = J\theta$, is given by $B = \frac{1}{s+1} \sum_{k=1}^s v_k \frac{\partial}{\partial v_k}$. In particular, it is of constant norm $\frac{s}{s+1}$. At the same time, Ω is Gauduchon, which is equivalent to $dd^c(\Omega^{m-1}) = 0$. Indeed, on the one hand we always have

$$dd^c(\Omega^{m-1}) = (m-1)(dJ\theta + (m-1)\theta \wedge J\theta) \wedge \Omega^{m-1}.$$

Let us denote by $\omega_{\mathbb{H}} := d_\theta(-J\theta)$. Then we have

$$\Omega^{m-1} = \Omega^s = \omega_{\mathbb{H}}^s + s\omega_{\mathbb{H}}^{s-1} \wedge \omega_0 \qquad \omega_{\mathbb{H}}^{s-1} = (-dJ\theta)^{s-1} + (s-1)(-dJ\theta)^{s-2} \wedge \theta \wedge J\theta.$$

Moreover, we compute:

$$(-dJ\theta)^s = s!(v_1\cdots v_s)^{-2}dr_1\wedge dv_1\wedge\ldots\wedge dr_s\wedge dv_s = (-dJ\theta)^{s-1}\theta\wedge J\theta.$$

Thus we find:

$$\begin{aligned}\frac{1}{s^2}dd^c(\Omega^{m-1}) &= (dJ\theta\wedge\omega_{\mathbb{H}}^{s-1} + s\theta\wedge J\theta\wedge\omega_{\mathbb{H}}^{s-1})\wedge\omega_0\\ &= (-(-dJ\theta)^s - (s-1)(-dJ\theta)^{s-1}\wedge\theta\wedge J\theta + s(-dJ\theta)^{s-1}\wedge\theta\wedge J\theta)\wedge\omega^0\\ &= (-(-dJ\theta)^s + (-dJ\theta)^{s-1}\wedge\theta\wedge J\theta)\wedge\omega_0 = 0.\end{aligned}$$

Thus we have found:

Lemma 5.2.1: *The LCK metric given in (5.2.5) on an OT manifold of type $(s,1)$ is Gauduchon, and its Lee vector field has constant norm.*

On the other hand, by [OT05, Proposition 2.5], OT manifolds admit no holomorphic vector fields. In particular, they cannnot admit Vaisman metrics. In fact, in [O16] it was shown that they do not even admit exact LCK metrics.

In general, the existence of an LCK metric on an OT manifold $X(K,U)$ of type (s,t) is equivalent to a condition on (K,U), namely:

$$r(u)^2 := |\sigma_{s+1}(u)|^2 = \ldots = |\sigma_{s+t}(u)|^2 = (\sigma_1(u)\cdots\sigma_s(u))^{-1/t},\qquad u\in U \tag{5.2.6}$$

as shown in [OT05] and in [Du14]. It is still unknown whether examples exist with $t>1$, but many of the pairs (s,t) have been eliminated from the discussion in [Vu14] and [Du14].

Next we address the problem of determining the set of Lee classes on any OT manifold. This generalizes the result in [O16], where it is proven that the set of possible Lee classes of LCK metrics on Inoue surfaces of type $\mathcal{S}^0$, i.e. OT manifolds of type $(1,1)$, has only one element.

Proposition 5.2.2: *Let $X = X(K,U)$ be an OT manifold of type (s,t). There exists at most one Lee class of an LCK metric on X, namely the one represented by the $U\ltimes O_K$-invariant form on $\mathbb{H}^s\times\mathbb{C}^t$, $\theta = \frac{1}{t}d\ln(\prod_{k=1}^s v_k)$.*

Proof. First note that, as $H^1(X,\mathbb{R})\cong \mathrm{Hom}(\pi_1(X),\mathbb{R})$, we can identify a de Rham class $[\eta]_{dR}$ with a group morphism $\tau:\pi_1(X)\to\mathbb{R}$. The corresponding morphism τ is precisely the automorphy representation: if $\eta = d\varphi$ on the universal cover $\tilde{X}$, then τ is given by $\tau(\gamma) = \gamma^*\varphi - \varphi$, for any $\gamma\in\pi_1(X)$.

Moreover, if X admits some LCK metric (Ω,η) with $\eta = d\varphi$ on $\tilde{X}$, and if $\Omega_K := \mathrm{e}^{-\varphi}\Omega$ is the corresponding Kähler form on $\tilde{X}$, then $\tau = \tau_{[\eta]}$ is also determined by: $\gamma^*\Omega_K = \mathrm{e}^{-\tau(\gamma)}\Omega_K$ for any $\gamma\in\pi_1(X)$. Hence, it suffices to show that for any Kähler metric Ω_K on $\tilde{X}$ inducing an LCK metric on X, the automorphy representation determined by Ω_K is precisely the one corresponding to θ, namely:

$$\begin{aligned}\tau_\theta(a) &= 0 \text{ for } a\in O_K\\ \tau_\theta(u) &= \frac{1}{t}\sum_{k=1}^s \ln\sigma_k(u) \text{ for } u\in U.\end{aligned}$$

Let now Ω_K be a Kähler metric on $\tilde{X}$ on which $\pi_1(X)$ acts by homotheties, and denote by τ the corresponding representation described before. We recall that under the abelianisation morphism $U \ltimes O_K \to H_1(X, \mathbb{Z})$, O_K maps to a finite group. This implies that O_K will act by isometries on Ω_K. Hence, the form Ω_K descends to the manifold $\hat{X} := \mathbb{H}^s \times \mathbb{C}^t / O_K$. But the torus $\mathbb{T} := \mathbb{R}^{2t+s}/O_K$ acts holomorphically by translations on $\hat{X}$, so we can average Ω_K over $\mathbb{T}$ to get a new $\mathbb{T}$-invariant Kähler form on $\hat{X}$:

$$\Omega'_K := \int_{\mathbb{T}} a^* \Omega_K \mu(a)$$

where μ is the constant volume form on $\mathbb{T}$ with $\int_{\mathbb{T}} \mu = 1$. The automorphy of Ω'_K is also τ, as for any $u \in U$ we have:

$$u^*\Omega'_K = \int_{\mathbb{T}} (au)^* \Omega_K \mu(a) = \int_{\mathbb{T}} (uc_u(a))^* \Omega_K \mu(a) = \int_{\mathbb{T}} c_u^*(a)(\mathrm{e}^{-\tau(u)}\Omega_K)\mu(c_u(a)) = \mathrm{e}^{-\tau(u)}\Omega'_K.$$

Now write $\Omega'_K = \Omega_0 + \Omega_{01} + \Omega_1$ with respect to the splitting

$$\textstyle\bigwedge^2_{\hat{X}} = \bigwedge^2_{\mathbb{C}^t} \oplus (\bigwedge^1_{\mathbb{C}^t} \otimes \bigwedge^1_{\mathbb{H}^s}) \oplus \bigwedge^2_{\mathbb{H}^s} \tag{5.2.7}$$

and also split $d = d_0 + d_1$, with d_0 being the differentiation with respect to the $\mathbb{C}^t$-variables and d_1, the $\mathbb{H}^s$-variables. The $\mathbb{C}^t$-invariance of Ω'_K implies that $d_0\Omega'_K = 0$. The condition $d\Omega'_K = 0$ then gives, on the $\bigwedge^2_{\mathbb{C}^t} \otimes \bigwedge^1_{\mathbb{H}^s}$-component, $d_1\Omega_0 = 0$. So $\Omega_0 = \sum_{ij} f_{ij} dz_i \wedge d\bar{z}_j$, with $f_{ij} \in \mathbb{C}$ for any $1 \le i, j \le t$.

Now, if $u \in U$, $u^*\Omega'_K = \mathrm{e}^{-\tau(u)}\Omega'_K$ implies that:

$$f_{ij}\sigma_{s+i}(u)\overline{\sigma_{s+j}(u)} = f_{ij}\mathrm{e}^{-\tau(u)} \text{ for any } 1 \le i, j \le t.$$

In particular, since $f_{ii} \neq 0$ for any $1 \le i \le t$, we have: $\tau(u) = -\ln|\sigma_{s+1}(u)|^2 = \ldots = -\ln|\sigma_{s+t}(u)|^2$. But we also have $\prod_{k=1}^s \sigma_k \prod_{j=s+1}^{s+t} |\sigma_j|^2 = 1$. This implies that $\tau = \tau_\theta$, and the conclusion follows. ∎

This result has an immediate corollary concerning the stability of LCK metrics on OT manifolds, as studied by R. Goto in [Go14].

Corollary 5.2.3: *On an OT manifold of LCK type, the LCK structure is not stable under small deformations of flat line bundles. More specifically, if (Ω, L, ∇) is an LCK structure on an OT manifold X, $\epsilon > 0$ and $\{L_v\}$ is a non-trivial analytic deformation of flat line bundles for $|v| < |\epsilon|$ with $L_0 = L$, then for any $0 < |v| < \epsilon$, there are no L_v-valued LCK structures.*

5.3 Technical Preliminaries

5.3.1 Leray-Serre spectral sequence of a locally trivial fibration

In this section, we review the general properties of the Leray-Serre spectral sequence associated to a fiber bundle. For a thorough presentation of spectral sequences and the Leray-Serre sequence we refer to [GH] and [BT]. Let $F \to X \xrightarrow{\pi} B$ be a locally trivial fibration. For a trivializing open set $U \subset B$ for π, we denote by φ_U the isomorphism $\varphi_U : \pi^{-1}(U) \to U \times F$, and for two trivialising open sets U, V, we denote by $g_{UV} = \varphi_U \circ \varphi_V^{-1}$ the corresponding transition function. Let us also denote by $\mathcal{X}^v$ the sheaf of vertical vector fields on X, i.e. the vector fields tangent to the fiber F.

If $\mathcal{E}^k$ is the sheaf of $\mathbb{C}$-valued smooth k-forms on X, we have the de Rham complex of X:

$$K^\bullet: \qquad \dots \xrightarrow{d} \mathcal{E}^k(X) \xrightarrow{d} \mathcal{E}^{k+1}(X) \xrightarrow{d} \dots$$

which is endowed with the following descending filtration:

$$F^pK^{p+q} := \{\omega \in \mathcal{E}^{p+q}(X) \mid \iota_{X_{q+1}} \dots \iota_{X_1}\omega = 0, \forall X_1, \dots, X_{q+1} \in \mathcal{X}^v(X)\}. \tag{5.3.1}$$

By the theory of spectral sequences, this filtration determines a sequence of double complexes $(E_r^{\bullet,\bullet}, d_r)_{r\geq 0}$ with $d_r : E_r^{p,q} \to E_r^{p+1,q-r+1}$ of bidegree $(r, -r+1)$, which computes the cohomology of the complex $(K^\bullet, d)$. More precisely, if we denote by $E_\infty^{p,q} := \lim_{r\to\infty} E_r^{p,q}$, we have:

$$H^k(X, \mathbb{C}) := H^k(K^\bullet) = \oplus_{p+q=k} E_\infty^{p,q} \quad 0 \leq k \leq \dim_{\mathbb{R}} X.$$

The complex E_{r+1}, called the $(r+1)$-th page of E, is defined recurrently as the cohomology of (E_r, d_r). We now make explicit the definition of each page of the spectral sequence. The pages E_0 and E_1 are simply given by:

$$E_0^{p,q} = \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}}, \quad d_0 : E_0^{p,q} \to E_0^{p,q+1}$$

$$d_0(\hat{\eta}_{p,q}) = \widehat{d\eta}_{p,q+1}$$

$$E_1^{p,q} = \frac{\text{Ker } d_0^{p,q}}{\text{Im } d_0^{p,q-1}}, \quad d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

$$d_1([\hat{\eta}]_{d_0}^{p,q}) = [\widehat{d\eta}]_{d_0}^{p+1,q}.$$

The second page is again:

$$E_2^{p,q} = \frac{\text{Ker } d_1^{p,q}}{\text{Im } d_1^{p-1,q}}, \quad d_2 : E_2^{p,q} \to E_2^{p+2,q-1}.$$

In order to write down d_2, one needs now to make sense of the objects of E_2. If $[\hat{\eta}]_{d_0}^{p,q} \in \text{Ker } d_1^{p,q}$, then there exists $\hat{\xi} \in E_0^{p+1,q-1}$ such that $\widehat{d\eta} = d_0\hat{\xi} = \widehat{d\xi}$, hence $\widehat{d\eta - d\xi} = 0$, meaning that $d\eta - d\xi \in F^{p+2}K^{p+q+1}$. Then:

$$d_2([[\hat{\eta}]_{d_0}]_{d_1}) = [[\widehat{d\eta - d\xi}]_{d_0}]_{d_1}. \tag{5.3.2}$$

In general, by induction, one can show that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is given by:

$$d_r([\dots[[\hat{\eta}]_{d_0}]_{d_1}\dots]_{d_{r-1}}) = [\dots[[\widehat{d\eta - d\delta}]_{d_0}]_{d_1}\dots]_{d_{r-1}}, \tag{5.3.3}$$

where $\delta = \xi_1 + \dots + \xi_{r-1}$, and the elements $\xi_1 \in F^{p+1}K^{p+q}, \dots, \xi_{r-1} \in F^{p+r-1}K^{p+q}$ are chosen via diagram chasing such that $d\eta - d\delta \in F^{p+r}K^{p+q+1}$.

When the fiber bundle π is flat, one has a $\mathcal{C}^\infty$ splitting of $TX = T_B \oplus T_F$, where, locally, T_B is the tangent space of B and T_F, of the fiber. The differential d also splits as $d_B + d_F$ with $d_B^2 = 0$ and $d_F^2 = 0$, where d_B is the derivation in the direction of the basis and d_F is the derivation along the fiber. In this case, the spectral sequence becomes more explicit. We have an induced splitting of the vector bundle of k-forms:

$$\bigwedge^k T^*X = \oplus_{p+q=k} \bigwedge^p T_B^* \otimes \bigwedge^q T_F^*$$

and then $E_0^{p,q} = \mathcal{C}^\infty(X, \bigwedge^p T_B^* \otimes \bigwedge^q T_F^*)$ for any $0 \leq p, q \leq k$. Moreover, one has:

$$d_0 = d_F, \quad d_1([\alpha]_{d_F}) = [d_B \alpha]_{d_F}.$$

For $a \in \text{Ker } d_1^{p,q}$ represented by $\eta \in \text{Ker } d_F \subset E_0^{p,q}$, there exists $\xi \in E_0^{p+1,q-1}$ so that $d_B \eta = d_F \xi$. One then has:

$$d\eta - d\xi = d_F \eta + d_B \eta - d_F \xi - d_B \xi = -d_B \xi \in \ker d_F \subset E_0^{p+2,q-1}$$

so that, by (5.3.2), d_2 is given by:

$$d_2([[\eta]_{d_F}]_{d_B}) = -[[d_B \xi]_{d_F}]_{d_B}.$$

In general, for a given element $[\ldots[[\eta]_{d_F}]_{d_B} \ldots]_{d_{r-1}} \in E_r^{p,q}$, (5.3.3) tells us that:

$$d_r([\ldots[[\eta]_{d_F}]_{d_B} \ldots]_{d_{r-1}}) = -[\ldots[[d_B \xi_{r-1}]_{d_F}] \ldots]_{d_{r-1}}$$

for some $\xi_{r-1} \in E_0^{p+r-1,q-r+1}$ such that there exist $\xi_1 \in E_0^{p+1,q-1}, \ldots, \xi_{r-2} \in E_0^{p+r-2,q-r+2}$ that satisfy $d\eta - d\xi_1 - \ldots d\xi_{r-1} \in E_0^{p+r,q-r+1}$, obtained by chasing diagrams.

5.3.2 Twisted cohomology

Let M be a compact differentiable manifold, let θ be a complex valued closed one-form on M and let d_θ be the differential operator $d_\theta = d - \theta \wedge \cdot$. Since $d_\theta^2 = 0$, we have a complex:

$$0 \longrightarrow \mathcal{E}_M^0(M) \xrightarrow{d_\theta} \mathcal{E}_M^1(M) \xrightarrow{d_\theta} \ldots$$

whose cohomology $H_\theta^\bullet(M) := \frac{\text{Ker } d_\theta}{\text{Im } d_\theta}$ is called the *twisted* cohomology associated to $[\theta]_{dR}$. Indeed, it depends only on the de Rham cohomology class of θ. If M is orientable, there is a version of Poincaré duality that holds for $H_\theta^\bullet(M)$, see for instance [Di, Corollary 3.3.12], and that is: $H_\theta^\bullet(M)^* \cong H_{-\theta}^{N-\bullet}(M)$, where $N = \dim_{\mathbb{R}} M$. Moreover, we have the following result:

Lemma 5.3.1: *Let $\tau \in H^1(M, \mathbb{C})$ be a de Rham class. Then the following are equivalent:*

1. $H_\tau^0(M) \neq 0$;
2. $\tau \in H^1(M, 2\pi i\mathbb{Z}) \subset H^1(M, \mathbb{C})$;
3. *For any* $k \in \mathbb{Z}$, $H_\tau^k(M, \mathbb{C}) \cong H_{dR}^k(M, \mathbb{C})$.

Proof. Clearly, if (3) holds then $H_\tau^0(M, \mathbb{C}) = H^0(M, \mathbb{C}) \neq 0$, so we have (1).
Now suppose (1) holds, meaning that if we choose a representative $\theta \in \tau$, there exists a smooth function $h = M \to \mathbb{C}$ so that $h\theta = dh$, with h not identically zero. Then we also have $\overline{h\theta} = d\overline{h}$, which implies $d|h|^2 = |h|^2 2\,\text{Re}\,\theta$. This is a linear first order differential system, so if $|h|^2$ has some zero, then h would vanish everywhere on M. Thus, we have $2\,\text{Re}\,\theta = d \ln |h|^2$, and without any loss of generality, we can now suppose that $\text{Re}\,\theta = 0$.
On the universal cover $\tilde{M}$, there exists $f \in \mathcal{C}^\infty(\tilde{M}, \mathbb{C})$ so that $\theta = df$. Then we find:

$$d(\mathrm{e}^{-f} h) = \mathrm{e}^{-f}(-df h + dh) = 0$$

thus $h = c\mathrm{e}^f$, with $c \in \mathbb{C}$ a constant. On the other hand, by the universal coefficient theorem and Hurewicz theorem we have $H^1(M,\mathbb{C}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{C})$, and the homomorphism $\tau : \pi_1(M) \to \mathbb{C}$ corresponding to θ is precisely given by $\tau(\gamma) = \gamma^* f - f$, $\gamma \in \pi_1(M)$. Thus, as $\mathrm{e}^f = c^{-1}h$ is defined on M, it is $\pi_1(M)$-invariant as a function on $\tilde{M}$, so that we have

$$\gamma^* \mathrm{e}^f = \mathrm{e}^f \mathrm{e}^{\tau(\gamma)} = \mathrm{e}^f, \quad \forall \gamma \in \pi_1(M).$$

Therefore τ takes values in $2\pi i\mathbb{Z}$, from which assertion (2) follows.

Similarly, if (2) holds and we choose θ a representative of τ and write $\theta = df$ on $\tilde{M}$, then as $\tau(\gamma) = \gamma^* f - f \in 2\pi i \mathbb{Z}$ for any $\gamma \in \pi_1(M)$, the function $h = \mathrm{e}^f$ is $\pi_1(M)$-invariant and descends to a well-defined function $h : M \to \mathbb{C}^*$ satisfying $dh = h\theta$. Finally, let us note that in this case $d_\theta(\cdot) = hd(h^{-1}\cdot)$, which establishes an isomorphism between the twisted cohomology $H^\bullet_\tau(M)$ and $H^\bullet(M,\mathbb{C})$. ■

Remark 5.3.2: A result of [LLMP03] states that if $\theta \in \mathcal{E}^1_M(M,\mathbb{R})$ is a non-zero closed form, and there exists a Riemannian metric on M so that θ is parallel for the corresponding Levi-Civita connection, then we have $H^\bullet_\theta(M) = 0$. Note that this is not true if θ is complex valued.

The twisted cohomology can also be seen as the cohomology of certain flat line bundles. In general, these are parametrised by elements $\rho \in \mathrm{Hom}(\pi_1(M),\mathbb{C}^*)$ as follows: we let L_ρ be the induced complex line bundle over M, that is the quotient of $\tilde{M} \times \mathbb{C}$ by the action of $\pi_1(M)$ given by:

$$\gamma(x,\lambda) = (\gamma(x), \rho(\gamma)\lambda), \quad \gamma \in \pi_1(M), \quad (x,\lambda) \in \tilde{M} \times \mathbb{C}.$$

Moreover, we endow L_ρ with the unique flat connection ∇ whose corresponding parallel sections are exactly the locally constant sections of L_ρ. Denote by d^∇ the differential operator acting on $\mathcal{E}^\bullet_M \otimes L_\tau$ which is induced by ∇ by the Leibniz rule. Then the cohomology of (L_ρ,∇) is the cohomology denoted by $H^\bullet(M,L_\rho)$ of the complex:

$$0 \longrightarrow \mathcal{E}^0_M(M,L_\rho) \xrightarrow{d^\nabla} \mathcal{E}^1_M(M,L_\rho) \xrightarrow{d^\nabla} \dots.$$

Equivalently, if we let $\mathcal{L}_\rho$ be the sheaf of parallel sections of (L_ρ,∇), then we also have a natural isomorphism $H^\bullet(M,L_\rho) \cong H^\bullet(M,\mathcal{L}_\rho)$, where the latter is the sheaf cohomology. $\mathcal{L}_\rho$ is called a local system, and determines and is completely determined by (L_ρ,∇).

On the other hand, the exponential induces an exact sequence:

$$0 \longrightarrow H^1(M,2\pi i\mathbb{Z}) \longrightarrow H^1(M,\mathbb{C}) \xrightarrow{\exp} H^1(M,\mathbb{C}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \tag{5.3.4}$$

and all elements $\rho \in \ker c_1 \subset H^1(M,\mathbb{C}^*) \cong \mathrm{Hom}(\pi_1(M),\mathbb{C}^*)$ are of the form $\exp\tau$, with $\tau \in \mathrm{Hom}(\pi_1(M),\mathbb{C})$. For the corresponding flat line bundle (L_ρ,∇), the connection has an explicit form. We choose $\theta \in \tau$ a representative, write $\theta = d\varphi$ on $\tilde{M}$, so that $s = \mathrm{e}^\varphi$ determines a global trivialising section of L_ρ. Then ∇ is given by $\nabla s = \theta \otimes s$, and it can be easily seen that this construction does not depend on the chosen $\tau \in \exp^{-1}(\rho)$, nor on $\theta \in \tau$. Moreover, we have a natural isomorphism:

$$H^\bullet_\theta(M) \cong H^\bullet(M,L^*_\rho).$$

Also note that if $H^2(M,\mathbb{Z})$, or also $H_1(M,\mathbb{Z})$, has no torsion, then the map c_1 in (5.3.4) is zero, and then all flat line bundles on M are of this form.

In particular, by Lemma 5.3.1 we have the following result:

Lemma 5.3.3: $H^\bullet_\tau(\mathbb{S}^1, \mathbb{C}) = 0$ *if and only if* $\tau \notin H^1(\mathbb{S}^1, 2\pi i\mathbb{Z})$.

Proof. The only if part is assured by Lemma 5.3.1. On the other hand, if $\tau \notin H^1(\mathbb{S}^1, 2\pi i\mathbb{Z})$, then also $-\tau \notin H^1(\mathbb{S}^1, 2\pi i\mathbb{Z})$, thus Lemma 5.3.1 implies $H^0_\tau(\mathbb{S}^1) = H^0_{-\tau}(\mathbb{S}^1) = 0$. Finally, by Poincaré duality we find $H^1_\tau(\mathbb{S}^1) \cong H^0_{-\tau}(\mathbb{S}^1)^* = 0$, which concludes the proof. ∎

This allows us to prove the following, which we will use a number of times in the sequel:

Lemma 5.3.4: *Let $\mathbb{T}^s$ be the compact s-dimensional torus, let $\rho : \pi_1(\mathbb{T}^s) \to \mathbb{C}^*$ be any representation of $\pi_1(\mathbb{T}^s)$ on $\mathbb{C}$ and let $(L_\rho, \nabla) \to \mathbb{T}^s$ be the associated flat complex line bundle. Then $H^\bullet(\mathbb{T}^s, L_\rho) = 0$ if and only if ρ is not trivial.*

Proof. Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ be a non-trivial element. As $H^2(\mathbb{T}^s, \mathbb{Z})$ is a free abelian group, there exists $0 \neq \tau \in H^1(\mathbb{T}^s, \mathbb{C})$ so that $\rho = \exp \tau$. We write then $L_\rho = L_\tau$.

Let us identify $\mathbb{T}^s$ with $(\mathbb{S}^1)^s$, and let $p_k : \mathbb{T}^s \to \mathbb{S}^1$ the the projection on the k-th component, for $k \in \{1, \dots, s\}$. If we denote by ν a generator of $H^1(\mathbb{S}^1, \mathbb{Z})$, then τ writes $\tau = \sum_{k=1}^s a_k p_k^* \nu$, with $a_1, \dots, a_s \in \mathbb{C}$, not all $2\pi i\mathbb{Z}$-valued. In particular, it follows that

$$L_\tau \cong p_1^* L_{a_1\nu} \otimes \ldots \otimes p_s^* L_{a_s\nu}.$$

Now, by the Künneth formula for local systems (see [Di, Corollary 2.3.31]) it follows that:

$$H^\bullet(\mathbb{T}^s, L_\tau) \cong H^\bullet(\mathbb{S}^1, L_{a_1\nu}) \otimes \cdots \otimes H^\bullet(\mathbb{S}^1, L_{a_s\nu}).$$

Since there exists at least one $k \in \{1, \dots, s\}$ with $a_k \notin 2\pi i\mathbb{Z}$, Lemma 5.3.3 implies that $H^\bullet(\mathbb{S}^1, L_{a_k\nu})$ vanishes, and the conclusion follows. ∎

Notation

In all that follows, we will denote by X a complex manifold (which will usually be an OT manifold). Only in this chapter, the sheaf of complex valued smooth l-forms on X will be denoted by $\mathcal{E}^l_X$ or simply by $\mathcal{E}^l$, if there is no ambiguity about the manifold X, and its global sections will be denoted by $\mathcal{E}^l_X(X)$ or by $\mathcal{E}^l(X)$. Also, for a given OT manifold X of type (s,t) corresponding to (K, U), we will sometimes denote by $\Gamma := U \ltimes O_K$ its fundamental group and by $\hat{X} := \mathbb{H}^s \times \mathbb{C}^t / O_K$. Concerning the compact tori that will appear in our discussion, we will use the notation $\mathbb{T}^k$ for the k-dimensional torus viewed as a smooth manifold (without any additional structure), and $\mathbb{T}$ for the n-dimensional abelian compact Lie group which, in our case, acts on $\hat{X}$, where $n = 2t + s$. As already mentioned, U acts on $\mathbb{T}$ by conjugation, and for any $u \in U$, we will denote by $c_u \in \mathrm{Aut}(\mathbb{T})$ the automorphism $c_u(a) = u^{-1}au$. For any $q \in \mathbb{N}^*$ we will denote by $\mathcal{I}_q$ the set of multi-indexes $I = (0 < i_1 < \ldots < i_q \leq n)$ and for $I \in \mathcal{I}_q$ we will denote by $|I|$ the length of I which is q. Finally, for a given representation $\rho : \pi_1(\mathbb{T}^s) = U_\mathbb{H} \to \mathbb{C}^*$, we denote by L_ρ the induced flat complex line bundle over $\mathbb{T}^s$, and for a closed one-form θ on X, we denote by $\rho^\theta \in \mathrm{Hom}(U, \mathbb{C}^*) = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ the representation it induces.

5.4 The de Rham cohomology

In the next two sections, we will prove in two different ways the following result:

Theorem 5.4.1: *Let $X = X(K, U)$ be an OT manifold of type (s, t) of complex dimension m. For any $l \in \{0, \dots, 2m\}$ we have:*

$$H^l(X, \mathbb{C}) \cong \bigoplus_{\substack{p+q=l \\ |I|=q \\ \sigma_I = 1}} \bigwedge\nolimits^p \mathbb{C}\{d \ln \operatorname{Im} w_1, \dots, d \ln \operatorname{Im} w_s\} e_I.$$

In particular, the Betti numbers of X are given by:

$$b_l = \sum_{p+q=l} \binom{s}{p} \cdot \rho_q,$$

where ρ_q is the cardinal of the set $\{I \mid |I| = q, \sigma_I = 1\}$.

In this section, we compute the de Rham cohomology of an OT manifold X by identifying it with the cohomology of invariant forms on X with respect to a certain compact torus action. In order to be precise, let us fix an OT manifold $X = X(K, U)$ of type (s, t) and of complex dimension $m = s + t$. Recall that $\mathbb{T} = \mathbb{T}^n$ acts holomorphically by translations on $\hat{X} = \mathbb{H}^s \times \mathbb{C}^t / O_K$, but not on X. However, by identifying smooth forms on X with smooth U-invariant forms on $\hat{X}$, it makes then sense to speak of $\mathbb{T}$-invariant forms on X: these will be exactly the $U \ltimes \mathbb{T}$-invariant forms on $\hat{X}$. Let us denote by $A^\bullet$ the graded sheaf of such invariant forms, which is a subsheaf of $\mathcal{E}_X^\bullet$. The differential d acting on $\mathcal{E}_X^\bullet$ fixes $A^\bullet$, so $(A^\bullet, d)$ is a subcomplex of the de Rham complex of X. As in the usual setting of a manifold endowed with a compact group action, we have the following:

Lemma 5.4.2: *There exists a projection graded morphism $\pi : \mathcal{E}_X^\bullet \to A^\bullet$ commuting with the differential d.*

Proof. The projection morphism will be given by averaging over the torus action. Let us fix $0 \leq l \leq 2m$, and consider a (local) smooth l-form η on X, identified with a U-invariant form on $\hat{X}$. Let μ be the $\mathbb{T}$-invariant n-form on $\mathbb{T}$ with $\int_{\mathbb{T}} \mu = 1$, and let:

$$\pi\eta := \int_{\mathbb{T}} a^* \eta \mu(a).$$

Clearly, $\pi\eta$ is a $\mathbb{T}$-invariant form on $\hat{X}$. In order to see that it descends to a form on X, we have to show that $\pi\eta$ is U-invariant. Indeed, for any $u \in U$, if $c_u \in \operatorname{Aut}(\mathbb{T})$ is the conjugation $a \mapsto u^{-1} a u$ as before, then we have:

$$\begin{aligned} u^*(\pi\eta) &= \int_{\mathbb{T}} (au)^* \eta \mu(a) = \int_{\mathbb{T}} (u c_u(a))^* \eta \mu(a) = \\ &= \int_{\mathbb{T}} c_u(a)^* \eta \mu(a) = \int_{\mathbb{T}} a^* \eta \mu(c_u^{-1}(a)) = \\ &= \int_{\mathbb{T}} a^* \eta \mu(a) = \pi\eta. \end{aligned}$$

Above, we made the change of variable $a \mapsto c_u^{-1}(a)$, and then used the fact that μ is a constant c_u-invariant form on $\mathbb{T}$, so that $\mu(c_u^{-1}(a)) = ((c_u^{-1})^*\mu)(a) = \mu(a)$.

Finally, it is clear from the definition of π that it commutes with d, and that π restricted to $A^\bullet$ is the identity. ■

In the context of a manifold endowed with a compact group action, a standard result states that the de Rham cohomology of the manifold is the cohomology of invariant forms. This is still true in our context, and the proof follows the same lines:

Lemma 5.4.3: *For any $0 \leq l \leq 2m$, any open set $O \subset X$ and any d-closed form $\eta \in \mathcal{E}_X^l(O)$ there exists some $\beta \in \mathcal{E}_X^{l-1}(O)$ so that $\pi\eta - \eta = d\beta$. In particular, we have an isomorphism $H^l[\pi] : H^l(X, \mathbb{C}) \to H^l(X, A^\bullet(X))$.*

Proof. Let $\eta \in \mathcal{E}_X^l(O)$ be a closed form, let $\hat{O}$ be the preimage of O in $\hat{X}$ and, as before, identify η with a form on $\hat{O}$. Let $a \in \mathbb{T}$ and let $\{\Phi_a^v\}_{v\in\mathbb{R}}$ be a one-parameter subgroup of $\mathbb{T}$ with $\Phi_a^1 = a$. Let ξ_a be the vector field on $\hat{X}$ generated by Φ_a, and consider the map $F_a : \mathbb{R} \times \hat{O} \to \hat{O}$, $(v, x) \mapsto \Phi_a^v(x) = x + va$. We then have $F_a^*\eta = \eta_1 + dv \wedge \eta_2$, with $\eta_1(v, \cdot) = (\Phi_a^v)^*\eta$ and $\eta_2(v, \cdot) = \iota_{\xi_a}(\Phi_a^v)^*\eta$. If we denote by d_X the differential with respect to the X-variables on $\mathbb{R} \times \hat{O}$, then $d\eta = 0$ implies:

$$0 = dF_a^*\eta = d_X\eta_1 + dv \wedge \frac{\partial v}{\partial \eta_1} - dv \wedge d_X\eta_2.$$

In particular, we have $\frac{\partial v}{\partial \eta_1} = d_X\eta_2$, or also:

$$a^*\eta - \eta = \int_0^1 \frac{\partial v}{\partial \eta_1} dv = \int_0^1 d_X\eta_2 dv = d\int_0^1 \eta_2 dv.$$

Denoting by β_a the form $\int_0^1 \eta_2 dv$ and by $\beta := \int_T \beta_a\mu(a)$, we have $\pi\eta - \eta = d\beta$, and we are then left with showing that the form β is U-invariant. Let $u \in U$ and $a \in \mathbb{T}$. Upon noting that

$$u_*^{-1}\xi_a = \frac{d}{dv}|_{v=0}(u^{-1}\Phi_a^v) = \frac{d}{dv}|_{v=0}(\Phi_{c_u(a)}^v u^{-1}) = \xi_{c_u(a)}$$

we have:

$$\begin{aligned} u^*\beta_a &= \int_0^1 u^*\iota_{\xi_a}(\Phi_a^v)^*\eta dv = \int_0^1 \iota_{u_*^{-1}\xi_a} u^*(\Phi_a^v)^*\eta dv \\ &= \int_0^1 \iota_{\xi_{c_u(a)}}(u\Phi_{c_u(a)}^v)^*\eta dv = \int_0^1 \iota_{\xi_{c_u(a)}}(\Phi_{c_u(a)}^v)^*\eta dv = \beta_{c_u(a)}. \end{aligned}$$

So, as in the previous lemma, the $\mathbb{T}$-invariance of μ implies then that $\int_{\mathbb{T}} \beta_{c_u(a)}\mu(a) = \beta$.

For the last assertion, it is enough to see that the inclusion $\iota : A^\bullet \to \mathcal{E}^\bullet$ induces an isomorphism $H(\iota)$ in cohomology. If $\eta \in A^l(X)$ verifies $\eta = d\alpha$, $\alpha \in \mathcal{E}^{l-1}(X)$, then $\eta = \pi\eta = d\pi\alpha$, so $H(\iota)$ is injective. If $\eta \in \mathcal{E}^l$ is a closed form, then by the above we have $H(\iota)[\pi\eta] = [\pi\eta]_{dR} = [\eta]_{dR}$, so $H(\iota)$ is surjective. ■

For the sequel, we will fix some $l \in \{0, \ldots, 2m\}$. Recalling that the tangent bundle of $\hat{X} \cong (\mathbb{R}_{>0})^s \times \mathbb{T}^n$ splits smoothly as $T\hat{X} = E \oplus V$, where E is the pullback of $T(\mathbb{R}_{>0})^s$ and V is the pullback of $T\mathbb{T}^n$ on $\hat{X}$, we have:

$$\bigwedge^l T^*\hat{X} = \oplus_{p=0}^l \bigwedge^p E^* \otimes \bigwedge^{l-p} V^*.$$

If we denote by A^l_p the sheaf which associates to any open set $O \subset X$

$$A^l_p(O) := A^l(O) \cap \mathcal{C}^\infty(\hat{O}, \textstyle\bigwedge^p E^* \otimes \bigwedge^{l-p} V^* \otimes \mathbb{C}),$$

where $\hat{O}$ is the pre-image of O in $\hat{X}$, then we also have:

$$A^l = \oplus_{p=0}^l A^l_p. \tag{5.4.1}$$

At the same time, A^l can be seen as:

$$A^l(O) = \{\eta \in \mathcal{E}^l_{\hat{X}}(\hat{O}) | \eta \text{ is } U \text{ invariant and } \mathcal{L}_Z \eta = 0 \;\; \forall Z \in \mathcal{C}^\infty(\hat{O}, V)\}$$

which implies that the differential d is compatible with the grading of $A^\bullet$ given by (5.4.1), in the sense that $d(A^l_p) \subset A^{l+1}_{p+1}$ for any $0 \leq p \leq l$.

Hence, a form $\eta = \sum_{p=0}^l \eta_p \in A^l(X)$ decomposed with respect to the grading (5.4.1) is closed if and only if each η_p is closed. As a consequence, the complex $0 \to A^\bullet(X)$ splits in the subcomplexes:

$$C^\bullet_p : 0 \xrightarrow{d} A^p_0(X) \xrightarrow{d} A^{p+1}_1(X) \xrightarrow{d} A^{p+2}_2(X) \xrightarrow{d} \ldots \tag{5.4.2}$$

for $0 \leq p \leq s$. Moreover, if a form $\eta = \sum_{p=0}^l \eta_p \in A^l(X)$ is exact: $\eta = d\beta$, then writing again $\beta = \sum_{q=0}^{l-1} \beta_q$, we must have $\eta_{p+1} = d\beta_p$ for any $0 \leq p \leq l-1$ and $\eta_0 = 0$. So we see that η is exact if and only if each η_p is. Hence, if we let:

$$H^l_p(X, A) := \frac{\ker d : A^l_p(X) \to A^{l+1}_{p+1}(X)}{\operatorname{Im} d : A^{l-1}_{p-1}(X) \to A^l_p(X)} = H^p(C^\bullet_{l-p})$$

then we have:

$$H^l(X, A^\bullet(X)) = \oplus_{p=0}^l H^l_p(X, A). \tag{5.4.3}$$

Now let us take a closer look at the complex $C^\bullet_l$. Denoting by d_E the differentiation in the E direction, $(C^\bullet_l, d)$ is a subcomplex of:

$$(\mathcal{C}^\infty(\hat{X}, \textstyle\bigwedge^l V^* \otimes \bigwedge^\bullet E^* \otimes \mathbb{C}), d_E)$$

which, in turn, is just $\mathcal{C}^\infty(\hat{X}, \bigwedge^l V^* \otimes \mathbb{C})$ tensorized by:

$$0 \longrightarrow \mathcal{C}^\infty(\hat{X}, \mathbb{C}) \xrightarrow{d_E} \mathcal{C}^\infty(\hat{X}, E^* \otimes \mathbb{C}) \xrightarrow{d_E} \mathcal{C}^\infty(\hat{X}, \textstyle\bigwedge^2 E^* \otimes \mathbb{C}) \longrightarrow \ldots$$

But recall that, for any $0 \leq q \leq l$, $\bigwedge^q V^* \otimes \mathbb{C}$ is globally trivialised over $\hat{X}$ by $\{e_I\}_{I \in \mathcal{I}_q}$, where $\mathcal{I}_q$ denotes the set of all multi-indexes $I = (0 < i_1 < \ldots < i_q \leq n)$ and the forms e_I were defined in (5.2.3). Thus, for any $0 \leq p \leq l$, we have:

$$\mathcal{C}^\infty(\hat{X}, \textstyle\bigwedge^q V^* \otimes \bigwedge^p E^* \otimes \mathbb{C}) = \oplus_{I \in \mathcal{I}_q} \mathcal{C}^\infty(\hat{X}, \bigwedge^p E^*) \otimes \mathbb{C} e_I.$$

Moreover, a section $\eta = f \otimes e_I$ of $\bigwedge^p E^* \otimes \mathbb{C} e_I$ belongs to $A^{p+|I|}_p$ if and only if it is $\mathbb{T}$-invariant and

$$u^* f = \sigma_I(u)^{-1} f \text{ for any } u \in U. \tag{5.4.4}$$

If we denote by $E^p_{\sigma_I}$ the sheaf of $\mathbb{T}$-invariant sections f of $\bigwedge^p E^* \otimes \mathbb{C}$ which are σ_I^{-1} equivariant, i.e. verify (5.4.4), it follows that we have:

$$A_p^{q+p} = \oplus_{I\in\mathcal{I}_q} E^p_{\sigma_I} e_I.$$

Moreover, as the e_I's are closed forms, we have:

$$d : E^p_{\sigma_I} e_I \to E^{p+1}_{\sigma_I} e_I \quad \forall I \in \mathcal{I}_q.$$

So finally we get that the complex $C^\bullet_{l-p}$ splits into the complexes on X:

$$C^\bullet_{l-p}(I) : 0 \longrightarrow E^0_{\sigma_I}(\hat{X})e_I \xrightarrow{d} E^1_{\sigma_I}(\hat{X})e_I \xrightarrow{d} E^2_{\sigma_I}(\hat{X})e_I \xrightarrow{d} \dots \tag{5.4.5}$$

indexed after all $I \in \mathcal{I}_{l-p}$. So also the cohomology splits:

$$H^l_p(X, A) = \oplus_{I\in\mathcal{I}_{l-p}} H^p_{\sigma_I}(X, A) \tag{5.4.6}$$

where $H^p_{\sigma_I}(X, A) := H^p(C^\bullet_{l-p}(I))$.

At the same time, the $\mathbb{T}$-invariant sections of $\bigwedge^p E^* \otimes \mathbb{C}$ over $\hat{X}$ naturally identify with the sections of $\mathcal{E}^p_{(\mathbb{R}_{>0})^s}$ over $(\mathbb{R}_{>0})^s$. Hence, the sections of $E^p_{\sigma_I}$ coincide then with the sections of $\mathcal{E}^p_{\mathbb{T}^s} \otimes L^*_{\sigma_I}$, and we have:

$$H^p_{\sigma_I}(X, A) \cong H^p(\mathbb{T}^s, L^*_{\sigma_I}) \otimes e_I. \tag{5.4.7}$$

So, putting together (5.4.7), (5.4.6), (5.4.3), Lemma 5.4.3 and Lemma 5.3.4, we get:

$$H^l(X, \mathbb{C}) \cong \oplus_{p+q=l} \oplus_{I\in\mathcal{I}_q} H^p(\mathbb{T}^s, L^*_{\sigma_I}) \otimes e_I$$

leading, together with Lemma 5.3.4, to Theorem 5.4.1.

5.5 The Leray-Serre spectral sequence of OT manifolds

Let $X = X(K, U)$ be an OT manifold of type (s, t). In this section, we are interested in computing its de Rham cohomology using the Leray-Serre spectral sequence associated to the fibration depicted in (5.2.1):

$$\mathbb{T}^n \to X \xrightarrow{\pi} \mathbb{T}^s.$$

We endow the de Rham complex of X with the filtration described in (5.3.1). It turns out that the Leray-Serre sequence associated to this filtration degenerates at the page E_2 and we prove this by outlining the special properties of the OT fiber bundle.

Let us start by noting that we have two fiber bundles over $\mathbb{T}^s$ associated to this fibration:

$$\mathcal{E}^\bullet(\mathbb{T}^n) \to \mathcal{E}^\bullet(\mathbb{T}^{\mathbf{n}}) \to \mathbb{T}^s \tag{5.5.1}$$

$$H^\bullet(\mathbb{T}^n, \mathbb{C}) \to \mathbf{H}^\bullet(\mathbb{T}^{\mathbf{n}}) \to \mathbb{T}^s. \tag{5.5.2}$$

Indeed, recall that we have an action of $U_{\mathbb{H}}$ on $\mathbb{T}^n$ defined in (5.2.2), with respect to which π is defined as $(\mathbb{R}_{>0})^s \times \mathbb{T}^n / U_{\mathbb{H}} \to \mathbb{T}^s$. But then we also have an induced action of $U_{\mathbb{H}}$ on $\mathcal{E}^\bullet(\mathbb{T}^n)$ by push-forward, which defines $\mathcal{E}^\bullet(\mathbb{T}^{\mathbf{n}}) := (\mathbb{R}_{>0})^s \times \mathcal{E}^\bullet(\mathbb{T}^n)/U_{\mathbb{H}} \to \mathbb{T}^s$ as an infinite-dimensional vector bundle over $\mathbb{T}^s$. Also we have an induced action of $U_{\mathbb{H}}$ on $H^\bullet(\mathbb{T}^n, \mathbb{C})$, which then defines the vector bundle $\mathbf{H}^\bullet(\mathbb{T}^{\mathbf{n}}) := (\mathbb{R}_{>0})^s \times H^\bullet(\mathbb{T}^n, \mathbb{C})/U_{\mathbb{H}} \to \mathbb{T}^s$.

Fact 1: The fibration is locally constant, meaning that if $U_\alpha \cap U_\beta$ is a connected open subset of $\mathbb{T}^s$, then $g_{\alpha\beta} : U_\alpha \cap U_\beta \times \mathbb{T}^n \to U_\alpha \cap U_\beta \times \mathbb{T}^n$ only depends on the $\mathbb{T}^n$-variables. This allows us to make the following identification:

$$E_0^{p,q} \simeq \mathcal{E}^p(\mathbb{T}^s, \mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})). \tag{5.5.3}$$

Indeed, recall that we have $TX = E \oplus V$, where, locally, E is the tangent bundle of the base $\mathbb{T}^s$ and V is the tangent bundle of the fiber $\mathbb{T}^n$, and we have identified $E_0^{p,q}$ with $\mathcal{C}^\infty(X, \bigwedge^p E^* \otimes \bigwedge^q V^* \otimes \mathbb{C})$. Consider $\eta \in E_0^{p,q}$ and suppose that U_α is an open set of $\mathbb{T}^s$ trivializing π via $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{T}^n$. Write $(\varphi_\alpha)_*\eta = \sum_i a_i^\alpha \wedge b_i^\alpha$, where, for each i, a_i^α is a p-form on U_α and b_i^α is an element of $\mathcal{C}^\infty(U_\alpha \times \mathbb{T}^n, \bigwedge^q T^*\mathbb{T}^n \otimes \mathbb{C})$ which may depend on both the coordinates of U_α and of $\mathbb{T}^n$. Of course, the forms a_i^α and b_i^α are not unique. If (U_β, φ_β) is another trivializing open set for π intersecting U_α, then we have:

$$(\varphi_\beta)_*\eta = (\varphi_\beta \circ \varphi_\alpha^{-1})_* \circ (\varphi_\alpha)_*\eta = (g_{\beta\alpha})_* \sum_i a_i^\alpha \wedge b_i^\alpha.$$

As $g_{\beta\alpha}$ is locally constant on $U_\alpha \cap U_\beta$, $(g_{\beta\alpha})_* a_i^\alpha = a_i^\alpha$, therefore $(\varphi_\beta)_*\eta = \sum_i a_i^\alpha \wedge (g_{\beta\alpha})_* b_i^\alpha$. In particular, for each i, the forms $\{a_i^\alpha\}_\alpha$ glue up to a well-defined global p-form a_i on $\mathbb{T}^s$ and η is then an element of $\mathcal{E}^p(\mathbb{T}^s, \mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$.

Fact 2: $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$ *is a completely reducible local system.* Indeed, as already mentioned, $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$ is a flat vector bundle defined by the induced representation $[R] : U_{\mathbb{H}} \to \mathrm{Aut}(H^q(\mathbb{T}^n, \mathbb{C}))$. In order to determine $[R]$, recall that we have fixed a frame for V^* over $(\mathbb{R}_{>0})^s \times \mathbb{T}^n$ given by $\{e_1, \dots, e_n\}$ in (5.2.3). As this frame does not depend on $(\mathbb{R}_{>0})^s$, it induces a frame for $T^*\mathbb{T}^n$ over $\mathbb{T}^n$ which we will denote the same, and we have $H^q(\mathbb{T}^n, \mathbb{C}) = \bigwedge^q \mathbb{C}\{e_1, \dots, e_n\} = \oplus_{I \in \mathcal{I}_q} \mathbb{C} e_I$. Then, for any I and any $u \in U_{\mathbb{H}}$, we have $[R](u)e_I = [R(u)_* e_I] = \sigma_I^{-1}(u) e_I$, or also $[R] = \sum_{I \in \mathcal{I}_q} \sigma_I^{-1}$ under the above direct sum decomposition.

For any multi-index I, let us denote, as before, by $L_{\sigma_I} \to \mathbb{T}^s$ the flat line bundle defined by the representation σ_I, so that $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}) = \oplus_{I \in \mathcal{I}_q} L_{\sigma_I}^*$. If ∇^I denotes the induced connection on $L_{\sigma_I}^*$ and ∇_q denotes the induced flat connection on $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$, then also ∇_q splits with respect to the direct sum decomposition as $\nabla_q = \sum_{I \in \mathcal{I}_q} \nabla^I$. In particular, we also obtain:

$$H^p(\mathbb{T}^s, \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})) \cong \oplus_{I \in \mathcal{I}_q} H^p(\mathbb{T}^s, L_{\sigma_I}^*). \tag{5.5.4}$$

Fact 3: The base is a torus. This allows us to compute, via Lemma 5.3.4:

$$H^p(\mathbb{T}^s, \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})) \cong \oplus_{\substack{I \in \mathcal{I}_q \\ \sigma_I \equiv 1}} H^p(\mathbb{T}^s, \mathbb{C}) e_I. \tag{5.5.5}$$

Let us now describe the pages of the Leray-Serre sequence of the OT fibration.

Page 0: By Fact 1, we have $E_0^{p,q} \simeq \mathcal{E}^p(\mathbb{T}^s, \mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$. In order to determine $d_0 : E_0^{p,q} \to E_0^{p,q+1}$, which on X corresponds to differentiation in the V-direction, let us first identify the corresponding operator on $\mathcal{E}^\bullet(\mathbb{T}^{\mathbf{n}})$. Consider the differential of $\mathbb{T}^n$ which acts on $\mathcal{E}^\bullet(\mathbb{T}^n)$, and then define by d^v the operator acting on $(\mathbb{R}_{>0})^s \times \mathcal{E}^\bullet(\mathbb{T}^n)$ trivially on the first factor, and as the differential of $\mathbb{T}^n$ on the second one. Clearly, this operator commutes with the action of $U_{\mathbb{H}}$, and so descends to an operator d^v on $\mathcal{E}^\bullet(\mathbb{T}^{\mathbf{n}})$. Under the isomorphism (5.5.3), we have then $d_0 = d^v$, i.e. for $\hat{\eta} = \sum a_i \otimes b_i^\alpha \in \mathcal{E}^p(U_\alpha, \mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$ we have:

$$d_0\left(\sum a_i \otimes b_i^\alpha\right) = \sum (-1)^p a_i \otimes d^v b_i^\alpha. \tag{5.5.6}$$

Page 1: By (5.5.6) we have $E_1^{p,q} \simeq \mathcal{E}^p(\mathbb{T}^s, \mathbf{H^q}(\mathbb{T}^\mathbf{n}))$. The differential $d_1 : \mathcal{E}^p(\mathbb{T}^s, \mathbf{H^q}(\mathbb{T}^\mathbf{n})) \to \mathcal{E}^{p+1}(\mathbb{T}^s, \mathbf{H^q}(\mathbb{T}^\mathbf{n}))$ is identified then with $d_1 = d^\nabla$, where d^∇ is the differential operator on $\mathbb{T}^s$ induced by the flat connection ∇ on $\mathbf{H^q}(\mathbb{T}^\mathbf{n})$.

Page 2: From above, we deduce that $E_2^{p,q} \simeq H^p(\mathbb{T}^s, \mathbf{H^q}(\mathbb{T}^\mathbf{n}))$. Let $[[\eta]_{d_0}]_{d_1} \in E_2^{p,q}$ be a non-zero element and let $\eta = \sum a_i \otimes b_i$ locally. Then $[\eta]_{d_0} = \sum a_i \otimes [b_i]_{d^v}$. The fact that $[\eta]_{d_0} \in \ker d_1$ implies that there exists $\gamma \in E_0^{p+1,q-1}$ so that $d^\nabla \sum (a_i \otimes b_i) = d^v \gamma$. As in Section 2.2, we have $(d^\nabla + d^v)(\sum a_i \otimes b_i - \gamma) = -d^\nabla \gamma \in \ker d^v \subset E_0^{p+2,q-1}$, hence, according to (5.3.2), d_2 is given by:

$$d_2([\sum a_i \otimes [b_i]_{d^v}]_{d_1}) = [[-d^\nabla \gamma]_{d^v}]_{d_1}. \tag{5.5.7}$$

At the same time, by (5.5.5) in Fact 3, we have that any element $[[\eta]_{d^v}]_{d_1}$ of $E_2^{p,q}$ can be represented by a sum:

$$\eta = \sum_{\substack{I \in \mathcal{I}_q \\ \sigma_I \equiv 1}} \alpha_I \otimes e_I \in E_0^{p,q} \tag{5.5.8}$$

where for each I appearing in the sum, $\alpha_I \in \mathcal{E}^p(\mathbb{T}^s)$ is a closed form on $\mathbb{T}^s$, and e_I, given in (5.2.3), is U invariant on $\hat{X}$, and so descends to a global element of $\mathcal{E}^\mathbf{q}(\mathbb{T}^\mathbf{n})$ on $\mathbb{T}^s$, verifying $d^\nabla e_I = 0$. In particular, we have $d^\nabla \eta = 0 = d^v(0)$, so, by (5.5.7) it follows that $d_2[[\eta]_{d^v}]_{d_1} = [[-d^\nabla 0]_{d^v}]_{d_1} = 0$, so $d_2 \equiv 0$.

Finally, for any $r \geq 2$, any class in $E_r^{p,q}$ can be represented by $[\ldots[[\eta]_{d_0}]_{d_1} \ldots]_{d_{r-1}}$, where η is of the form (5.5.8). Since $d\eta = (d^\nabla + d^v)\eta = 0$, by (5.3.3) all $\xi_1, \ldots, \xi_{r-1}$ can be chosen to be zero, so $d_r \equiv 0$. Thus we have shown:

Theorem 5.5.1: *The Leray-Serre spectral sequence of OT manifolds degenerates at E_2.*

As a corollary of this, one immediately obtains Theorem 5.4.1.

5.6 Twisted cohomology of OT manifolds

Now we want to compute the twisted cohomology groups of OT manifolds with respect to any closed one-form. The exact statement that we will obtain is the following:

Theorem 5.6.1: *Let $X = X(K, U)$ be an OT manifold of type (s, t) and of complex dimension m, and let $\theta = \sum_{k=1}^s a_k d \ln v_k$ be a closed one-form on $X(K, U)$, where $a_1, \ldots, a_s \in \mathbb{C}$. Then for any $l \in \{0, \ldots, 2m\}$ we have:*

$$H_\theta^l(X, \mathbb{C}) \cong \bigoplus_{\substack{p+q=l \\ |I|=q \\ \rho^\theta \otimes \sigma_I = 1}} \bigwedge^p \mathbb{C}\{d \ln v_1, \ldots, d \ln v_s\}(v_1^{a_1} \cdot \ldots \cdot v_s^{a_s}) e_I.$$

In particular, the corresponding twisted Betti numbers are given by:

$$b_l^\theta = \sum_{p+q=l} \binom{s}{p} \cdot \rho_q^\theta,$$

where ρ_q^θ is the cardinal of the set $\{I \mid |I| = q, \rho^\theta \otimes \sigma_I = 1\}$.

It is already known from [OT05] that $b_1(X) = s$, hence any closed one form is cohomologous to one of the form $\pi^*\eta$, where η is closed one-form on $\mathbb{T}^s$. As the twisted cohomology $H_\theta^\bullet$

depends only on the de Rham cohomology class of θ, and not on θ itself, we can assume that θ is the pullback of a form from $\mathbb{T}^s$.

We are going to use the same approach as in the previous section. Consider the complex:

$$K_\theta^\bullet : \qquad \dots \xrightarrow{d_\theta} \mathcal{E}^p(X) \xrightarrow{d_\theta} \mathcal{E}^{p+1}(X) \xrightarrow{d_\theta} \dots$$

which we endow with the same descending filtration as before:

$$F^p K_\theta^{p+q} := \{\omega \in \mathcal{E}^{p+q}(X) \mid \iota_{X_{q+1}} \dots \iota_{X_1}\omega = 0, \forall X_1, X_2, \dots X_{q+1} \in \mathcal{X}^v(X)\}.$$

It is easy to see that it is indeed a filtration, i.e. $d_\theta F^p K_\theta^{p+q} \subset F^p K_\theta^{p+q+1}$, as a consequence of θ being the pullback of a form from $\mathbb{T}^s$. We study the spectral sequence associated to K_θ with this filtration, which we denote also by $E_\bullet$.

Again, we denote by $\mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$ and by $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$ the vector bundles described in (5.5.1) and (5.5.2), and as before we have the 0-th page:

$$E_0^{p,q} = \frac{F^p K_\theta^{p+q}}{F^{p+1} K_\theta^{p+q}} \simeq \mathcal{E}^p(\mathbb{T}^s, \mathcal{E}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$$

and via this isomorphism, $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is given over a trivializing open set U_α by:

$$d_0(\sum_i a_i \otimes b_i^\alpha) = \sum (-1)^p a_i \otimes d^v b_i^\alpha.$$

Thus, we again have:

$$E_1^{p,q} \simeq \mathcal{E}^p(\mathbb{T}^s, \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$$

but this time, $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is given over a trivializing open set U_α by:

$$d_1(\sum a_i \otimes [b_i^\alpha]_{d^v}) = \sum da_i \otimes [b_i^\alpha]_{d^v} + (-1)^p \sum a_i \wedge ([\nabla' b_i^\alpha]_{d^v} - \theta \otimes [b_i^\alpha]_{d^v})$$

where ∇' is the flat connection on $\mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$. Equivalently, if we see θ as a form on $\mathbb{T}^s$ and define L_θ to be the complex flat line bundle over $\mathbb{T}^s$ corresponding to $\exp[\theta]_{dR} \in H^1(\mathbb{T}^s, \mathbb{C}^*) \simeq \mathrm{Hom}(\pi_1(\mathbb{T}^s), \mathbb{C}^*)$, we have the following identification:

$$E_1^{p,q} \simeq \mathcal{E}^p(\mathbb{T}^s, L_\theta^* \otimes \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}}))$$
$$d_1 = d^\nabla$$

where d^∇ the differential operator induced by the corresponding flat connection on $L_\theta^* \otimes \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$. Thus we obtain the second page:

$$E_2^{p,q} \simeq \frac{\mathrm{Ker}(d^\nabla)^{p,q}}{\mathrm{Im}(d^\nabla)^{p-1,q}} = H^p(\mathbb{T}^s, L_\theta^* \otimes \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})).$$

Let $\theta = \sum_{k=1}^s a_k d \ln v_k$ with $a_k \in \mathbb{C}$, which induces the representation $\rho^\theta = \sigma_1^{a_1} \otimes \dots \otimes \sigma_s^{a_s} \in \mathrm{Hom}(\pi_1(\mathbb{T}^s), \mathbb{C}^*)$. The flat vector bundle $L_\theta^* \otimes \mathbf{H}^{\mathbf{q}}(\mathbb{T}^{\mathbf{n}})$ over $\mathbb{T}^s$ is then given by the representation $[R]_\theta : U_{\mathbb{H}} \to \mathrm{Aut}(H^q(\mathbb{T}^n))$, $[R]_\theta := (\rho^\theta)^{-1} \otimes [R]$.

We again have:

Theorem 5.6.2: *The spectral sequence associated to $K_\theta^\bullet$ degenerates at the second page.*

Proof. As before, we want to show that $d_r \equiv 0$ for $r \geq 2$. We notice that, as $\mathbf{H^q}(\mathbb{T}^\mathbf{n})$ is a completely reducible local system, then so is $L_\theta^* \otimes \mathbf{H^q}(\mathbb{T}^\mathbf{n})$. The same arguments as in Fact 2 and Fact 3 in Section 3 show that we have an isomorphism:

$$H^p(\mathbb{T}^s, L_\theta^* \otimes \mathbf{H^q}(\mathbb{T}^\mathbf{n})) \cong \oplus_{I \in \mathcal{I}_q} H^p(\mathbb{T}^s, L_\theta^* \otimes L_{\sigma_I}^*) \cong \oplus_{\substack{I \in \mathcal{I}_q \\ \rho^\theta \otimes \sigma_I \equiv 1}} H^p(\mathbb{T}^s, \mathbb{C}) e_I$$

where e_I is now identified with a global parallel frame of $L_\theta^* \otimes L_{\sigma_I}^*$. This means that any element $[[\eta]_{d^v}]_{d^\nabla} \in E_2^{p,q}$ can be represented, globally on $\mathbb{T}^s$, by:

$$\eta = \sum_{\substack{I \in \mathcal{I}_q \\ \rho^\theta \otimes \sigma_I \equiv 1}} \alpha_I \otimes e_I \in \mathcal{E}^p(\mathbb{T}^s, L_\theta^* \otimes \mathcal{E}^\mathbf{q}(\mathbb{T}^\mathbf{s})),$$

with α_I closed one forms on $\mathbb{T}^s$. Since we have $d^\nabla e_I = 0$ for any I, we obtain $d^\nabla \eta = 0 = d^v 0$, so $d_2[[\eta]_{d^v}]_{d^\nabla} = [[-d^\nabla 0]_{d^v}]_{d^\nabla} = 0$. Moreover, by (5.3.3) and by the same arguments used to prove Theorem 5.5.1, each $\xi_1, \ldots, \xi_{r-1}$ can step by step be chosen to be 0 and thus $d_r \equiv 0$, for $r \geq 2$. We proved thus that $E_2 = E_\infty$. ∎

We proceed now with the proof of Theorem 5.6.1:

Proof. Since $E_r^{\bullet,\bullet}$ converges to $H_\theta^\bullet(X, \mathbb{C})$ and $E_2 = E_\infty$, then

$$H_\theta^l(X, \mathbb{C}) \cong \oplus_{p+q=l} E_2^{p,q} \cong \oplus_{\substack{p+q=l \\ I \in \mathcal{I}_q \\ \rho^\theta \otimes \sigma_I \equiv 1}} H^p(\mathbb{T}^s, \mathbb{C}) \otimes e_I.$$

Finally, in order to represent $H_\theta^l(X, \mathbb{C})$ by U invariant forms on $\hat{X}$, we need to tensorize with a global frame s of L_θ. If $\theta = \sum_{k=1}^s a_k d \ln v_k$, then s is given by $s = \prod_{k=1}^s v_k^{a_k}$ on $\hat{X}$, and so the conclusion follows.

∎

Remark 5.6.3: We want to draw attention to the fact that for both spectral sequences involved in our proofs, the isomorphism $E_2^{\bullet,\bullet} \cong H^\bullet(B) \otimes H^\bullet(F)$ alone was not enough to imply the degeneracy of $E_r^{\bullet,\bullet}$ at page E_2. An example of fiber bundle $F \to X \to B$ for which this isomorphism at the second page holds, but whose corresponding Leray-Serre spectral sequence does not degenerate at E_2 is given by the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$.

5.7 Applications and Examples

Let us start this section by giving the immediate consequence of Theorem 5.4.1, which is the explicit cohomology of OT manifolds when there are no trivial representations other than the obvious ones:

Corollary 5.7.1: *Let (K, U) be a number field together with an admissible group of units $U \subset K$ so that U admits no trivial representations σ_I other than the ones corresponding to $I = \emptyset$ and $I = (1, 2, \ldots, n)$, and let X be the OT manifold associated to (K, U). The Betti numbers of X are:*

$$b_l = b_{2m-l} = \binom{s}{l} \text{ for } 0 \leq l \leq s$$

$$b_l = 0 \text{ for } s < l < n.$$

Corollary 5.7.2: *For an OT manifold of type (s,t), all Betti numbers b_l for $0 \leq l \leq s$ and for $2m - s \leq l \leq 2m$ are positive.*

Proof. For $0 \leq l \leq s$, $H^l(\mathbb{T}^s, \mathbb{C})$ is a summand of $H^l(X, \mathbb{C})$, corresponding to $p = l$ and $I = \emptyset$, so $\sigma_I \equiv 1$. Hence:

$$b_l(X) \geq b_l(\mathbb{T}^s) = \binom{s}{l} > 0.$$

The assertion follows for $2m - s \leq l \leq 2m$ by Poincaré duality. ■

We have computed the cohomology algebras of an OT manifold $X(K,U)$ in terms of numerical invariants associated to (K,U), namely in terms of the trivial representations σ_I of U. Clearly, if (K,U) is not simple, in the sense that there exists an intermediate field extension $\mathbb{Q} \subset K' \subset K$ so that $U \subset K'$, then there exist trivial representations $\sigma_I : U \to \mathbb{C}^*$ with $0 < |I| = [K' : \mathbb{Q}] < [K : \mathbb{Q}]$. It would be interesting to know whether the converse is true, i.e. if (K,U) is of simple type, is the set $\{I | \sigma_I : U \to \mathbb{C}^*, \sigma_I \equiv 1\}$ only formed by $\emptyset$ and $I = (1, \ldots, n)$? Let us note that in [OT05, Proposition 2.3], the second Betti number of an OT manifold of simple type was computed, and coincides with ours when there are no other trivial representations, implying an affirmative answer for the above question when $|I| = 2$. We do not address this problem in the present chapter, but we give an example where the answer is affirmative, allowing us to give the explicit Betti numbers of the corresponding manifold:

Example 5.7.3: Let p be any odd prime number and take the polynomial $f = X^p - 2 \in \mathbb{Q}[X]$. This polynomial has one real root $\sqrt[p]{2}$ and the complex roots $\sqrt[p]{2}\epsilon, \ldots, \sqrt[p]{2}\epsilon^{p-1}$, where ϵ is a p-th root of unity. Let $K = \mathbb{Q}(\sqrt[p]{2})$, which is of type $(1, \frac{p-1}{2})$. We notice first that $u = \sqrt[p]{2} - 1$ is a unit of O_K since its norm, which is the product of all the embeddings of u in $\mathbb{C}$, is equal to 1:

$$(\sqrt[p]{2} - 1) \ldots (\sqrt[p]{2}\epsilon^{p-1} - 1) = (-1)^p f(1) = 1.$$

Since u is also clearly positive, we can then take U to be generated by u. Let then $X = X(K,U)$ be the corresponding OT manifold. We claim that there is no index I with $p > |I| \geq 2$ and $\sigma_I \equiv 1$. By Corollary 5.7.1, this will imply that the Betti numbers of X will verify $b_0 = b_{p+1} = b_1 = b_p = 1$ and $b_i = 0$ for any $i \neq 0, 1, p, p+1$.

Let us assume by contradiction the existence of such $I = (1 \leq i_1 < \ldots < i_k \leq p)$. For any $1 \leq j \leq p$, we denote by σ_j the embedding of K into $\mathbb{C}$ mapping $\sqrt[p]{2}$ to $\sqrt[p]{2}\epsilon^{j-1}$. Then $\sigma_I \equiv 1$ rewrites as:

$$(\sqrt[p]{2}\epsilon^{i_1 - 1} - 1)(\sqrt[p]{2}\epsilon^{i_2 - 1} - 1) \ldots (\sqrt[p]{2}\epsilon^{i_k - 1} - 1) = 1,$$

equivalent to:

$$a_0 \sqrt[p]{2^k} - a_1 \sqrt[p]{2^{k-1}} + \ldots + (-1)^{k-1} a_{k-1} \sqrt[p]{2} + (-1)^k - 1 = 0 \tag{5.7.1}$$

where $a_l = \sum_{j_1 < \ldots < j_l} \epsilon^{i_1 + \ldots + \widehat{i_{j_1}} + \ldots + \widehat{i_{j_l}} + \ldots i_k - k + l}$, and the symbol $\hat{\cdot}$ over an element marks the fact that the element is missing. Let g be the polynomial:

$$g = a_0 X^k - a_1 X^{k-1} + \ldots + (-1)^{k-1} a_{k-1} X + (-1)^k - 1 \in \mathbb{Q}(\epsilon)[X].$$

Then (5.7.1) implies $g(\sqrt[p]{2}) = 0$, hence g is a multiple of the minimal polynomial of $\sqrt[p]{2}$ over the field $\mathbb{Q}(\epsilon)$. We prove next that this polynomial is actually $X^p - 2$. Indeed, we have the following two intermediate extensions:

$$\mathbb{Q} \subset \mathbb{Q}(\epsilon) \subset \mathbb{Q}(\epsilon, \sqrt[p]{2})$$

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt[p]{2}) \subset \mathbb{Q}(\epsilon, \sqrt[p]{2}).$$

We thus get

$$[\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}] = [\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\epsilon)] \cdot [\mathbb{Q}(\epsilon) : \mathbb{Q}] = [\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\sqrt[p]{2})] \cdot [\mathbb{Q}(\sqrt[p]{2}) : \mathbb{Q}]. \quad (5.7.2)$$

Since $X^p - 2 \in \mathbb{Q}(\epsilon)[X]$, we have $[\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\epsilon)] \leq p$. In general, if ϵ is an n-th root of unity, $[\mathbb{Q}(\epsilon) : \mathbb{Q}] = \varphi(n)$, where $\varphi(n)$ is Euler's function. In our case $\varphi(p) = p - 1$, whence (5.7.2) implies:

$$[\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}] = (p-1)[\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\epsilon)] = p \cdot [\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\sqrt[p]{2})].$$

As p and $p-1$ are relatively prime, we get moreover that p divides $[\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\epsilon)]$, therefore $p = [\mathbb{Q}(\epsilon, \sqrt[p]{2}) : \mathbb{Q}(\epsilon)]$. Thus the minimal polynomial of $\sqrt[p]{2}$ over the field $\mathbb{Q}(\epsilon)$ is $X^p - 2$, contradicting the fact that $k = \deg g < p$.

We can also obtain, via Corollary 5.7.1, the explicit de Rham cohomology algebra of OT manifolds of LCK type. Recall that an OT manifold $X(K, U)$ admits an LCK metric if and only if, for any $u \in U$, we have:

$$r(u)^2 := |\sigma_{s+1}(u)|^2 = \ldots = |\sigma_{s+t}(u)|^2 = (\sigma_1(u) \cdots \sigma_s(u))^{-1/t}. \quad (5.7.3)$$

In particular, all OT manifolds of type $(s, 1)$ admit such metrics.

Proposition 5.7.4: *Let X be an OT manifold of type (s, t) admitting some LCK metric. Its de Rham cohomology algebra $H^\bullet(X, \mathbb{C})$ is isomorphic to the graded algebra over $\mathbb{C}$ generated by:*

$$d \ln v_1, \ldots, d \ln v_s, dz_1 \wedge d\bar{z}_1 \wedge \ldots dz_t \wedge d\bar{z}_t \wedge dr_1 \wedge \ldots \wedge dr_s.$$

In particular, its Betti numbers are:

$$b_l = b_{2m-l} = \binom{s}{l} \text{ for } 0 \leq l \leq s$$

$$b_l = 0 \text{ for } s < l < n.$$

Proof. By Corollary 5.7.1, it suffices to show that U admits no trivial representations σ_I other than the two obvious ones. So let $I = (0 < i_1 < \ldots < i_k \leq n)$ with $k > 0$ and $\sigma_I \equiv 1$. After eventually renumbering the coordinates, we can suppose without loss of generality that I is of the form

$$I = (1, \ldots q, j_1, \ldots j_p, s+t+1, \ldots s+t+l),$$

with $0 \leq q \leq s < j_1 < \ldots < j_p \leq s+t$ and $0 \leq p, l \leq t$.
Since $\sigma_I = 1$ we have $|\sigma_I| = 1$ which, together with (5.7.3), gives the relation:

$$(\sigma_1 \cdots \sigma_q)^{-1} = r^{l+p} = (\sigma_1 \cdots \sigma_s)^{-\frac{l+p}{2t}}.$$

As $\sigma_1, \ldots, \sigma_s$ are $\mathbb{R}$-linearly independent, this relation must be the trivial one, implying that $l+p=2t$ and $q=s$, so $I=(1,\ldots,n)$, which finishes the proof. ■

In LCK geometry, it is interesting to know also the twisted cohomology with respect to the Lee form of the LCK metric. Recall that in Proposition 5.2.2, we have determined the uniqueness of the de Rham class of a Lee form on OT manifolds of LCK type.

Proposition 5.7.5: *Let X be an OT manifold of type (s,t) admitting an LCK metric and let $\theta = \frac{1}{t}\sum_{k=1}^{s} d\ln v_k$. Then for any $0 \leq l \leq 2m$, we have:*

$$H^l_\theta(X) \cong (v_1 \cdots v_s)^{\frac{1}{t}} \oplus_{j=1}^{t} \mathbb{C} dz_j \wedge d\bar{z}_j \otimes \bigwedge^{l-2} \mathbb{C}\{d\ln v_1, \ldots, d\ln v_s\}.$$

In particular, the corresponding twisted Betti numbers are given by $\dim_{\mathbb{C}} H^l_\theta(X) = t\binom{s}{l-2}$ for any $0 \leq l \leq 2m$.

Proof. In order to apply Theorem 5.6.1, we need to identify, for any $0 \leq k \leq n$, the set corresponding to $[\theta]_{dR}$:

$$\mathcal{J}_k = \{I = (1 \leq i_1 < i_2 \ldots < i_k \leq n) \mid \sigma_1^{\frac{1}{t}} \ldots \sigma_s^{\frac{1}{t}} \sigma_{i_1} \ldots \sigma_{i_k} = 1\}.$$

We shall prove that $\mathcal{J}_k = \emptyset$ for $k \neq 2$ and $\mathcal{J}_2 = \{(s+j, s+t+j) | 1 \leq j \leq t\}$.

Let us fix k and $I \in \mathcal{J}_k$. As before, we can assume that I is of the form:

$$I = (1, \ldots, q, j_1, \ldots, j_p, s+t+1, \ldots, s+t+l)$$

such that $0 \leq q \leq s < j_1 < \ldots < j_p \leq s+t$, $0 \leq p, l \leq t$ and $q+p+l=k$. Then $|(\sigma_1 \cdots \sigma_s)^{1/t} \sigma_I| = 1$ together with (5.7.3) implies:

$$\sigma_1^{\frac{1}{t}+1} \ldots \sigma_q^{\frac{1}{t}+1} \sigma_{q+1}^{\frac{1}{t}} \ldots \sigma_s^{\frac{1}{t}} = r^{-(p+l)} = (\sigma_1 \cdots \sigma_s)^{\frac{p+l}{2t}}.$$

By the $\mathbb{R}$ linear independence of $\sigma_1, \ldots, \sigma_s$, this must be the trivial relation. If $0 < q < s$, we would get that $\frac{1}{t} + 1 = \frac{p+l}{2t} = \frac{1}{t}$, which is a contradiction. If $q = s$, then we would get that $\frac{1}{t} + 1 = \frac{p+l}{2t}$, or also $p + l = 2t + 2$, contradicting the fact that $p + l \leq 2t$. Hence $q = 0$ and we get the relation $\frac{1}{t} = \frac{p+l}{2t}$, or also $p + l = 2 = k$. In particular, $\mathcal{J}_k = \emptyset$ for $k \neq 2$.

Let us note now that the set $\{(s+j, s+t+j) | j = 1, t\}$ is included in $\mathcal{J}_2$. In order to show that these are all the possible multi-indexes, let $I = (i_1 < i_2) \in \mathcal{J}_2$. We already showed that $i_1 > s$. Since $\sigma_{i_1}\sigma_{i_2} = (\sigma_1 \cdots \sigma_s)^{-1/t}$ is real, we get that $\sigma_{i_1}\sigma_{i_2} = \overline{\sigma_{i_1}\sigma_{i_2}}$. Combining with $|\sigma_{i_1}| = |\sigma_{i_2}|$, we obtain $\sigma_{i_1}^2 = \overline{\sigma}_{i_2}^2$, therefore $\sigma_{i_1} = \pm\overline{\sigma}_{i_2}$. The case $\sigma_{i_1} = -\overline{\sigma}_{i_2}$ is excluded, because this would give the following contradiction:

$$0 > -\overline{\sigma}_{i_2}\sigma_{i_2} = (\sigma_1 \ldots \sigma_s)^{-\frac{1}{t}} > 0.$$

So $\sigma_{i_1} = \overline{\sigma}_{i_2}$. But there exists $s+t \leq j \leq s+2t$ with $|i_1 - j| = t$ and $\sigma_{i_1} = \overline{\sigma}_j$, so $\sigma_{i_2} = \sigma_j$. We want to show that $i_2 = j$, i.e. $I = (j-t, j)$.

Consider M the $\mathbb{Z}$-submodule of O_K generated by U, which is a subring of O_K, and let K' be its fraction field. We have $U \subset M \subset K' \subset K$, and we showed in the above proposition that U

has no trivial representations, so in particular (K, U) is simple, thus $K' = K$. But the relation $\sigma_{i_2} = \sigma_j$ extends to M, and so also to $K' = K$. This last fact is possible only if $i_2 = j$. ∎

Remark 5.7.6: Notice that since $H^l_\theta(X)$ does not vanish and θ is real-valued, by the result of [LLMP03] θ is not parallel with respect to any metric g on X.

Remark 5.7.7: In [Kas13a], OT manifolds are given a solvmanifold structure, namely they are shown to be of the form $\Gamma \setminus G$, where G is a solvable Lie group and Γ is a co-compact lattice in G. Consequently, one can consider the cohomologies $H^\bullet(\mathfrak{g})$ and $H^\bullet_\theta(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G and θ is a closed G-invariant form. A natural question is then: does one have isomorphisms $H^\bullet_{dR}(X(K, U)) \cong H^\bullet(\mathfrak{g})$ and $H^\bullet_\theta(X(K, U)) \cong H^\bullet_\theta(\mathfrak{g})$? For a general solvmanifold, this does not always hold. However, H. Kasuya proved in [Kas13b, Example 4] that on OT manifolds of type $(s, 1)$, this isomorphism is valid for the de Rham cohomology. In [AOT17, Theorem 4.3], it is proved that in the twisted cohomology, the isomorphism holds for a subclass of $X(K, U)$ of type $(s, 1)$, satisfying the so-called *Mostow condition*. Finally, since in Theorem 5.6.1 we represented the corresponding cohomologies by invariant forms with respect to the action of G described in [Kas13a], we obtain as a consequence that for all OT manifolds X of type (s, t), we have the isomorphism $H^k_\theta(X) \cong H^k_\theta(\mathfrak{g})$, although they might not all satisfy the Mostow condition.

In [O16] it was proven that there are no d_θ-exact metrics on OT manifolds of type $(s, 1)$. We give next a generalization of this result, in which we determine all the possible LCK classes in H^2_θ. As a consequence of this, we also obtain a hard Lefschetz-type theorem associated to an LCK metric on an OT manifold.

Corollary 5.7.8: *Let X be an OT manifold of type (s, t) with an LCK structure (Ω, θ), where $\theta = \frac{1}{t}\sum_{k=1}^{s} d\ln v_k$. Then the twisted class of Ω in $H^2_\theta(X)$ is necessarily of the form:*

$$(v_1 \cdots v_s)^{\frac{1}{t}} \sum_{j=1}^{t} a_j i dz_j \wedge d\bar{z}_j, \quad a_j \in \mathbb{R}_{>0}\ \forall j \in \{1, \dots, t\}.$$

In particular, if we let Lef_Ω *denote the Lefschetz operator* $\mathrm{Lef}_\Omega = \Omega \wedge \cdot$, *then for any* $0 \le l \le 2m - 2$, Lef_Ω *induces a morphism in cohomology:*

$$[\mathrm{Lef}_\Omega] : H^l(X, \mathbb{C}) \to H^{l+2}_\theta(X)$$

which is injective for $0 \le l \le m$ *and surjective for* $m \le l \le 2m - 2$.

Proof. Let us start by noting that, as in the case of the de Rham cohomology, the twisted cohomology with respect to θ is the twisted cohomology of $\mathbb{T}$-invariant forms. This is a direct consequence of Theorem 5.6.1, but can also be seen by an argument completely analogous to Lemma 5.4.3 and using the fact that θ vanishes on vector fields tangent to $\mathbb{T}^n$. Hence, by averaging the form Ω to a $\mathbb{T}$-invariant LCK form Ω' as in Proposition 5.2.2, the twisted class does not change: $[\Omega]_\theta = [\Omega']_\theta \in H^2_\theta(X)$.

At the same time, we saw that the corresponding Kähler form Ω'_K writes with respect to the splitting (5.2.7) as $\Omega'_K = \Omega_0 + \Omega_{01} + \Omega_1$, with Ω_0 a constant positive form on $\mathbb{C}^t$. Also, given the expression of θ, we have $\Omega' = (v_1 \cdots v_s)^{1/t}\Omega'_K := \omega_0 + \omega_{01} + \omega_1$, where again Ω' was decomposed with respect to the splitting (5.2.7). Clearly, $d_\theta\omega_0 = 0$, so also $d_\theta(\omega_{01} + \omega_1) = 0$, thus we can write $[\Omega']_\theta = [\omega_0]_\theta + [\omega_{01} + \omega_1]_\theta \in H^2_\theta(X)$. Now, since by Proposition 5.7.5, we have:

$$H^2_\theta(X) \cong (v_1 \cdots v_s)^{\frac{1}{t}} \oplus_{j=1}^t \mathbb{C} dz_j \wedge \ d\bar{z}_j, \tag{5.7.4}$$

it follows that $[\omega_{01} + \omega_1] = 0 \in H^2_\theta(X)$. Indeed, otherwise we would have that on $\tilde{X}$, $\omega_{01} + \omega_1 + d_\theta \eta$ is valued in $\bigwedge^2_{\mathbb{C}^t}$ for some one-form $\eta \in \Omega^1_X(X)$, which is impossible. Hence $[\Omega]_\theta = [\omega_0]_\theta = \omega_0$ under the isomorphism (5.7.4), so the first assertion follows. The second assertion follows from the description of the cohomology groups given in Proposition 5.7.4 and Proposition 5.7.5 and from the non-degeneracy of $[\Omega]$. ∎

Remark 5.7.9: The fact that for any LCK form Ω on X, the operator $\mathrm{Lef}_\Omega : H^1(X, \mathbb{C}) \to H^3_\theta(X)$ is injective also implies Corollary 5.2.3 via [Go14, Theorem 2.4].

We end this section with one more application concerning the possible real Chern classes of vector bundles on OT manifolds:

Proposition 5.7.10: *Let $X(K, U)$ be an OT manifold of type (s, t) verifying that U admits no trivial representations σ_I unless $|I| \in \{0, n\}$. Then, for any $1 \leq k < n/2$, every d-closed real (k, k) form on X is exact. In particular, if E is some complex vector bundle on X, its first $[(n-1)/2]$ real Chern classes $c_k(E)^{\mathbb{R}} \in H^{2k}(X, \mathbb{R})$ vanish.*

Proof. By Corollary 5.7.1, we deduce that:

$$H^{2k}(X, \mathbb{R}) \cong \bigwedge\nolimits^{2k} \mathbb{R}\{f_1, \ldots, f_s\} \qquad \text{for } 2k < n$$

where $f_l := v_l^{-1} dv_l$ for $1 \leq l \leq s$. Let us also denote by $\varphi_l = -\frac{i}{2} v_l^{-1} dw_l = f_l^{1,0}$ for $1 \leq l \leq s$, so that $f_l = \varphi_l + \overline{\varphi}_l$.

Let α be a real closed (k, k) form on X. By the above, we can write: $\alpha = \sum_{I \in \mathcal{I}_{2k}} a_I f_I + d\beta$, where for every multi-index $I = (i_1 < \ldots < i_{2k})$, $f_I = f_{i_1} \wedge \ldots \wedge f_{i_{2k}}$, $a_I \in \mathbb{R}$ and $\beta \in \mathcal{E}^{2k-1}_X(X)$ is a real form. In particular, in bidegree $(2k, 0)$, this reads:

$$\alpha^{2k,0} = 0 = \sum_{I \in \mathcal{I}_{2k}} a_I \varphi_I + \partial \beta^{2k-1,0}.$$

But, for any I, φ_I is not ∂-exact, and neither is the sum $\sum_I a_I \varphi_I$, unless it is zero. In order to see this, one could for instance choose a hermitian metric on X defining an L^2 adjoint operator ∂^* with respect to which one would have $\partial^* \varphi_I = 0$ for any I. It would follow then that each α_I is L^2-orthogonal to $\operatorname{Im} \partial$, and so $\sum_I a_I \varphi_I = \partial \beta^{2k-1,0} = 0$. In particular, this implies that $a_I = 0$ for each $I \in \mathcal{I}_{2k}$, and so $\alpha = d\beta$. ∎

Remark 5.7.11: In the literature specialized on topology, there is a complex called Morse-Novikov, associated to a closed one-form θ of Morse type, i.e. locally given by the differential of a Morse function. It was first considered by Novikov in [N81] and [N82], and for a thorough description we refer to [F]. The construction of this complex is based on the number of zeros of θ, just as the Morse-Smale complex of a Morse function f is based on the number of zeros of f and actually these two complexes coincide when $\theta = df$. If θ is a nowhere vanishing one-form, as the Lee form in Proposition 5.7.5 is, the Morse-Novikov complex is trivial, therefore its cohomology vanishes. However, the twisted cohomology does not vanish, as our computation indicates; consequently, OT manifolds provide examples in all dimensions of spaces for which these two cohomologies differ.

Bibliography

[Ab10] M. Abreu, *Kähler-Sasaki geometry of toric symplectic cones in action-angle coordinates*, Port. Math. **67** (2010), no. 2, 121–153. (Cited on page xiv.)

[AD16a] V. Apostolov, G. Dloussky, *Locally Conformally Symplectic Structures on Compact Non-Kähler Complex Surfaces*, Int. Math. Res. Not. **9** (2016), 2717–2747. (Cited on page x.)

[AD16b] V. Apostolov, G. Dloussky, *On the Lee classes of locally conformally symplectic complex surfaces*, to appear in J. Symplectic Geom., arXiv:1611.00074. (Cited on page x.)

[AOT17] D. Angella, A. Otiman, N. Tardini, *Cohomologies of locally conformally symplectic manifolds and solvmanifolds*, Ann. Global Anal. Geom **53** (2018), 67–96. (Cited on page 87.)

[Bea83a] A. Beauville, *Some remarks on Kähler manifolds with* $c_1 = 0$. Prog. Math. **39** (1983) (Cited on page 12.)

[Bea83b] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle.* J. Diff. Geom. **18** (1983), 755–782. (Cited on pages vii, 1, 3, and 13.)

[Bel00] F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40. (Cited on pages x, 15, 29, 31, 32, 35, and 46.)

[BM16] F. Belgun, A. Moroianu, *On the irreducibility of locally metric connections*, J. Reine Angew. Math. **714** (2016), 123–150. (Cited on page 19.)

[Ben02] Y. Benoist, *Actions Symplectiques de Groupes Compacts*, Geometriae Dedicata **89** (2002), 181–245. (Cited on page 58.)

[Bl54] A. Blanchard, *Espaces fibrés kähleriens compacts*, Comptes Rendus Acad. Sci, Paris **238** (1954), 2281–2283. (Cited on pages xii, 37, and 47.)

[Bo96] F. A. Bogomolov, *On Guan's examples of simply connected non-Kähler compact complex manifolds*, Amer. J. Math. **118** (1996), 1037–1046. (Cited on page 7.)

[BNS11] S. Boissiére, M. Nieper-Wißkirchen, A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties.* J. Math. Pures Appl. **95** (2011), 553–563. (Cited on page 13.)

[Bm73] E. Bombieri, Letter to Kodaira, 1973. (Cited on page 35.)

[Bo54] W. M. Boothby, *Some fundamental formulas for Hermitian manifolds with vanishing torsion*, American J. Math. **76**(1954), 509–534. (Cited on page ix.)

[Bra17] O. Braunling, *Oeljeklaus-Toma manifolds and arithmetic invariants*, Math. Z. **286** (2017), 291–323. (Cited on page xv.)

[Bre] G. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics **46**, Academic Press, New York-London, 1972. (Cited on page 58.)

[Bru10] M. Brunella, *Locally conformally Kähler metrics on certain non-Kählerian surfaces*, Math. Ann. **346** (2010), 629–639. (Cited on page 36.)

[Bru11] M. Brunella, *Locally conformally Kähler metrics on Kato surfaces*, Nagoya Math. J **202** (2011), 77–81. (Cited on pages xi and 36.)

[BT] R. Bott, L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, Springer Verlag, 1982. (Cited on page 71.)

[Da78] V.I. Danilov, *The Geometry of Toric Varieties*, Russ. Math. Surveys **33**, 2 (1978) 97–154 = Uspekhi Mat. Nauk SSSR **33**, 2 (1988), 85–134. (Cited on page 63.)

[Di] A. Dimca, *Sheaves in topology*, Universitext, Springer Verlag, 2003. (Cited on pages 73 and 75.)

[Du14] A. Dubickas, *Nonreciprocal units in a number field with an application to OT manifolds*, New York J. Math. **20** (2014), 257–274. (Cited on pages xv and 70.)

[DK] J.J. Duistermaat, J.A.C. Kolk, *Lie Groups*, Universitext, Springer Verlag, Berlin, 2000. (Cited on page 58.)

[F] M. Farber, *Topology of closed one-form*, Mathematical surveys and monographs, Amer. Math. Soc, vol 108, 2004. (Cited on page 88.)

[FP10] A. Fujiki, M. Pontecorvo, *Anti-self-dual bihermitian structures on Inoue surfaces*, J. Diff. Geom. **85** (2010), 15–71. (Cited on pages x and 36.)

[FP16] A. Fujiki, M. Pontecorvo, *Bi-Hermitian metrics on Kato surfaces*, preprint (2016), `arXiv:1607.00192`. (Cited on page 36.)

[Ga76] P. Gauduchon, *La classe de Chern pluriharmonique d'un fibré en droites*, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 9, Aii, 479–482. (Cited on pages 21 and 22.)

[Ga77] P. Gauduchon, *Le théorème de l'excentricité nulle*, C.R. Acad. Sci. Paris **285** (1977), 387–390. (Cited on page 23.)

[Ga84] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518. (Cited on page 22.)

[GO98] P. Gauduchon, L. Ornea, *Locally Conformally Kähler Metrics on Hopf Surfaces*, Ann. Inst. Fourier **48** (1998), 1107–1127. (Cited on pages x, 15, 29, 30, and 31.)

[GOP05] R. Gini, L. Ornea, M. Parton, *Locally Conformal Kähler Reduction*, J. Reine Angew. Math. **581** (2005), 1–21. (Cited on pages xiii and 55.)

[GOP06] R. Gini, L. Ornea, M. Parton, P. Piccinni, *Reduction of Vaisman structures in complex and quaternionic geometry*, J. Geom. Phys. **56** (2006), no. 12, 2501–2522. (Cited on page xiii.)

[Go14] R. Goto, *On the stability of locally conformal Kähler structures*, J. Math. Soc. Japan **66** (2014), no. 4, 1375–1401. (Cited on pages x, 25, 71, and 88.)

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York, 1978. (Cited on page 71.)

[Gu94] D. Guan, *Examples of compact holomorphic symplectic manifolds which admit no Kähler structure*, Geometry and Analisys on Complex Manifolds – Festschrift for Professor Kobayashi S. 60th Birthday, World Scientific, Teaneck, NJ, 1994, 63–74. (Cited on page 7.)

[HR01] S. Haller, T. Rybicki, *Reduction for Locally Conformal Symplectic Manifolds*, J. Geom. Phys. **37** (2001), no. 3, 262–271. (Cited on pages xiii and 55.)

[In74] M. Inoue, *On surfaces of Class* VII_0, Invent. Math. **24** (1974), 269–310. (Cited on pages 35 and 46.)

[IKO80] M. Inoue, S. Kobayashi, T. Ochiai, *Holomorphic affine connections on compact complex surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no 2, 247–264. (Cited on pages viii and 1.)

[Is16] N. Istrati, *Twisted holomorphic symplectic forms*, Bull. London Math. Soc. **48** (2016), no. 5, 745–756. (Cited on page 1.)

[Is17] N. Istrati, *A characterisation of toric LCK manifolds*, preprint (2017), `arXiv:1612.03832` (to appear in J. Symplectic Geom.) (Cited on page 55.)

[IO17] N. Istrati, A. Otiman, *De Rham and twisted cohomology of Oeljeklaus-Toma manifolds*, preprint (2017), `arXiv:1711.07847`. (Cited on page 67.)

[JR05] P. Jahnke, I. Radloff, *Projective Threefolds with Holomorphic Conformal Structure*, Int. J. Math. **16** (2005), 595–607. (Cited on pages viii and 1.)

[KO05] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kähler manifolds*, Tôhoku Math. J. **(2)** (2005), 201–221. (Cited on pages xii, 37, 44, and 45.)

[KL15] Y. Karshon, E. Lerman, *Non-Compact Symplectic Toric Manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), Paper 055, 37. (Cited on page 55.)

[Ka89] Ma. Kato, *Topology of Hopf surfaces*, J. Math. Soc. Japan **27** (1975), 222–238; *Erratum to "Topology of Hopf surfaces"*, J. Math. Soc. Japan **41** (1989), 173–174. (Cited on page 31.)

[Kas13a] H. Kasuya, *Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds*, Bull. London Math. Soc. **45** (2013), no. 1, 15–26. (Cited on pages xv and 87.)

[Kas13b] H. Kasuya, *Minimal models, formality, and hard Lefschetz properties of solvmanifolds with local systems*, J. Differential Geom. **93** (2013), no. 2, 269–297. (Cited on pages xv and 87.)

[KO82] S. Kobayashi, T. Ochiai, *Holomorphic structures modeled after hyperquadrics*, Tôhoku Math. J. (2) **34** (1982), no. 4, 587–629. (Cited on pages viii and 1.)

[Kod66] K. Kodaira, *On the Structure of Compact Complex Analytic Surfaces, I,II*, American J. Math. **88** (1966), 682–721. (Cited on page 31.)

[Kou15] M. Kourganoff, *Transverse similarity structures on foliations, and De Rham decomposition*, preprint (2015), `arXiv:1612.05430`. (Cited on pages xi, 38, 51, and 53.)

[LeB91] C. R. LeBrun, *Anti-self-dual Hermitian metrics on blown-up Hopf surfaces*, Math. Ann **289** (1991), 383–392. (Cited on pages x and 36.)

[LLMP03] M. de León, B. López, J. C. Marrero, E. Padrón, *On the computation of the Lichnerowicz-Jacobi cohomology*, J. Geom. Phys. **44** (2003), no. 4, 507–522. (Cited on pages x, 74, and 87.)

[Ler03] E. Lerman, *Contact Toric Manifolds*, J. Symplectic Geom. **1** (2003), 785–828. (Cited on pages xiv, 55, and 66.)

[MMP16] F. Madani, A. Moroianu, M. Pilca, *Conformally related Kähler metrics and the holonomy of LCK manifolds*, to appear in J. Eur. Math. Soc., `arXiv:1511.09212`. (Cited on pages xi and 38.)

[MMP17] F. Madani, A. Moroianu, M. Pilca, *On Toric Locally Conformally Kähler Manifolds*, Ann. Glob. Anal. Geom. **51** (2017), 401–417. (Cited on pages xiii, 19, 27, 46, 55, 58, and 63.)

[MSY06] D. Martelli, J. Sparks, S.-T. Yau, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, Comm. Math. Phys. **268** (2006), 39–65. (Cited on pages xiv and 66.)

[M] A. Moroianu, *Lectures on Kähler Geometry*, London Mathematical Society Student Texts **69**, Cambridge University Press, Cambridge, 2007. (Cited on page 5.)

[MM17] A. Moroianu, S. Moroianu, *On Pluricanonical Locally Conformally Kähler manifolds*, IMRN **14** (2017), 4398–4405. (Cited on page 38.)

[MMO17] A. Moroianu, S. Moroianu, L. Ornea, *Locally conformally Kähler manifolds with holomorphic Lee field*, preprint (2017), `arXiv:1712.05821`. (Cited on pages xii, 37, 39, 40, and 64.)

[MO09] A. Moroianu, L. Ornea, *Transformations of locally conformally Kähler manifolds*, Manuscripta Math. **130** (2009), 93–100. (Cited on page 54.)

[MPS13] A. Moroianu, M. Pilca, U. Semmelmann, *Homogeneous almost quaternion-Hermitian manifolds*, Math. Ann. **357** (2013), 1205–1216. (Cited on page 6.)

[N81] S. P. Novikov, *Multi-valued functions and functionals. An analogue of Morse theory*, Soviet Math. Doklady, **24** (1981), 222–226. (Cited on page 88.)

[N82] S. P. Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys, **37** (1982), 1–56. (Cited on page 88.)

[OT05] K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier **55** (2005), 1291–1300. (Cited on pages ix, xiv, xv, 67, 68, 69, 70, 81, and 84.)

[OS11] K. Oguiso, S. Schröer, *Enriques manifolds*, J. Reine Angew. Math. **661** (2011), 215–235. (Cited on page 13.)

[OPV14] L. Ornea, M. Parton, V. Vuletescu, *Holomorphic submersions of locally conformally Kähler manifolds*, Annali di Matematica **193** (2014), 1345–1351. (Cited on pages xi, 37, 48, 49, and 50.)

[OV10] L. Ornea, M. Verbitsky, *Locally conformal Kähler manifolds with potential*, Math. Ann. **248** (2010), 25–33. (Cited on pages x and 25.)

[OV11] L. Ornea, M. Verbitsky, *Oeljeklaus-Toma manifolds admitting no complex subvarieties*, Math. Res. Lett. **18** (2011), no. 4, 747–754. (Cited on page xv.)

[OV12] L. Ornea, M. Verbitsky, *Automorphisms of locally conformally Kähler manifolds*, Int. Math. Res. Not. **2012** (2012), no.4, 894–903. (Cited on pages xii, 37, and 42.)

[OV17] L. Ornea, M. Verbitsky, *Positivity of LCK potential*, preprint (2017), `arXiv:1705.08477`. (Cited on pages xii, 25, 37, and 42.)

[OVV13] L. Ornea, M. Verbitsky, V. Vuletescu, *Blow-ups of locally conformally Kähler manifolds*, Int. Math. Res. Not. **12** (2013), 2809–2821. (Cited on pages xi and 34.)

[O15] A. Otiman, *Locally Conformally Symplectic Bundles*, preprint (2015), `arXiv:1510.02770`. (Cited on page xiii.)

[O16] A. Otiman, *Morse-Novikov cohomology of locally conformally Kähler surfaces*, to appear in Math. Z., DOI: 10.1007/s00209-017-1968-y. (Cited on pages x, xiv, 35, 46, 52, 67, 70, and 87.)

[PV12] M. Parton, V. Vuletescu, *Examples of non-trivial rank in locally conformal Kähler geometry*, Math. Z. **270** (2012), no. 1, 179–187. (Cited on page xv.)

[Pi16] M. Pilca, *Toric Vaisman Manifolds*, J. Geom. Phys. **107** (2016), 149–161. (Cited on pages xiii, 55, and 64.)

[Po14] M. Pontecorvo, *On a question of Vaisman concerning complex surfaces*, M. Annali di Matematica **193** (2014), 1283–1293. (Cited on page 35.)

[To15] V. Tosatti, *Non-Kähler Calabi-Yau manifolds*, Contemp. Math. **644** (2015), 261–277. (Cited on page 7.)

[Tr82] F. Tricerri, *Some examples of locally conformal Kähler manifolds*, Rend. Sem. Mat. Univers. Politecn. Torino **40** (1982), 81–92. (Cited on pages x, xi, 34, and 35.)

[Ts94] K. Tsukada, *Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds*, Compositio Math. **93** (1994), 1–22. (Cited on pages ix and x.)

[Ts97] K. Tsukada, *Holomorphic Maps of Compact Generalized Hopf Manifolds*, Geometriae Dedicata **68** (1997), 61–71. (Cited on pages xi, 15, 24, 33, 40, and 42.)

[Ts99] K. Tsukada, *The canonical foliation of a compact generalized Hopf manifold*, Differential Geom. Appl. **11** (1999), 13–28. (Cited on pages xi, 32, 37, and 48.)

[Va76] I. Vaisman, *On locally conformal almost Kahler manifolds*, Israel J. Math. **24** (1976), 338–351. (Cited on pages xi, 28, and 32.)

[Va80] I. Vaisman, *On Locally and Globally Conformal Kähler Manifolds*, Trans. Amer. Math. Soc. **262** (1980), no. 2, 553–542. (Cited on pages ix, xi, 18, 32, 33, 37, and 48.)

[Va82] I. Vaisman, *Generalized Hopf Manifolds*, Geometriae Dedicata **13** (1982), 231–255. (Cited on pages ix, xi, 23, 24, 48, and 52.)

[Va85] I. Vaisman, *Locally Conformal Symplectic Manifolds*, Internat. J. Math. & Math. Sci. **3** (1985), 521–536. (Cited on pages xiii, 19, 26, 55, 56, and 66.)

[Ve04] M. S. Verbitsky, *Theorems on the vanishing of cohomology for locally conformally hyperKähler manifolds*, (Russian) Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 64–91; translation in Proc. Steklov Inst. Math. **246** (2004), 54–78. (Cited on pages x and 24.)

[Vu09] V. Vuletescu, *Blowing-up points on l.c.K. manifolds*, Bull. Math. Soc. Sci. Math. Roumanie (N. S.) **52(100)** (2009), 387–390. (Cited on pages xi and 34.)

[Vu10] V. Vuletescu, *LCK metrics on elliptic principal bundles*, preprint (2010), `arXiv:1001.0936`. (Cited on page 47.)

[Vu14] V. Vuletescu, *LCK metrics on Oeljeklaus-Toma manifolds vs Kronecker's theorem*, Bull. Math. Soc. Sci. Math. Roumanie (N. S.) **57(105)** (2014), 225–231. (Cited on pages xv and 70.)