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Sur le problème à deux corps et le rayonnement gravitationnel en théories scalaire-tenseur et Einstein-Maxwell-dilaton

THÈSE DE DOCTORAT - SPÉCIALITÉ : PHYSIQUE THÉORIQUE

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Résumé

Avec la naissance de l'"astronomie gravitationnelle", vient l'opportunité inédite de tester la relativité générale et ses alternatives dans un régime de champ fort jamais observé jusqu'alors : celui de la coalescence d'un système binaire d'objets compacts. Cette thèse propose d'étudier le problème du mouvement ainsi que du rayonnement gravitationnel d'un tel système en gravités modifiées, en y adaptant et en généralisant certains développements analytiques clés de la relativité générale.

On montre d'abord comment étendre le formalisme "effective-one-body" (EOB) à une large classe de gravités modifiées, parmi lesquelles les théories scalaire-tenseur. Dans ces dernières, l'interaction gravitationnelle est modifiée par l'ajout d'un degré de liberté scalaire (sans masse) à la relativité générale. Le lagrangien à deux corps correspondant étant connu à l'ordre post-post-keplerien, nous construisons un hamiltonien EOB associé, décrivant le mouvement d'une particule test dans des champs effectifs. Ceci permet de simplifier la dynamique à deux corps et d'en définir une resommation ; et ainsi, d'en explorer le régime de champ fort, près de la coalescence du système.

On "s'attaque" ensuite, et pour la première fois, à la description analytique d'un système binaire de trous noirs "chevelus", afin d'obtenir les formes d'ondes gravitationnelles (EOB) associées ; et ce, sur l'exemple simple des théories Einstein-Maxwell-dilaton, qui généralisent les théories scalaire-tenseur par l'ajout d'un champ vectoriel (sans masse). Pour ce faire, on calcule le lagrangien à deux corps à l'ordre post-keplerien ainsi que le flux d'énergie rayonnée à l'infini à l'ordre quadrupolaire. Tout comme en relativité générale, ces développements reposent sur la description de la trajectoire des trous noirs par les lignes d'univers de particules ponctuelles, décrites par une action "skeleton" généralisant celle, géodésique, de la relativité générale.

Enfin, à l'aide des "superpotentiels" de Katz, que l'on généralise pour définir la masse (nœtherienne) d'un trou noir à "cheveux" vectoriel et scalaire, on montre que la première loi de la thermodynamique qui en découle est particulièrement adaptée, lorsqu'un trou noir est membre d'un système binaire, pour en décrire les réajustements éventuels sous l'influence d'un compagnon lointain. La thermodynamique des trous noirs est alors utilisée pour interpréter et discuter du domaine de validité de leur "skeletonisation".

<u>Mots-clés</u> : ondes gravitationnelles ; relativité générale ; gravités modifiées ; développements post-newtoniens ; trous noirs "chevelus" ; thermodynamique des trous noirs ; charges globales ; superpotentiels de Katz.

Abstract

With the birth of "gravitational wave astronomy" comes the opportunity to test general relativity and its alternatives in a strong field regime that had never been observed so far : that of the coalescence of a compact binary sytem. This thesis studies the problem of motion and gravitational radiation from such systems in modified gravities, by adapting some of the key analytical tools that were first developed in the context of general relativity.

First, we show how to widen the "effective-one-body" (EOB) formalism to a large class of modified gravities, including, e.g., scalar-tensor theories. In the latter, the gravitational interaction is described by supplementing general relativity with a (massless) scalar degree of freedom. The corresponding two-body lagrangian being known at post-post-keplerian order, we build an associated EOB hamiltonian, which describes the motion of a test particle orbiting in effective external fields. This enables to simplify and resum the two-body dynamics ; and hence, to explore the strong-field regime near merger.

We then "tackle", for the first time, the analytical description of "hairy" binary black hole systems, and obtain their (EOB) gravitational waveform counterparts in Einstein-Maxwell-dilaton theories, which generalize scalar-tensor theories by means of a (massless) vector field. To that end, we derive the two-body lagrangian at post-keplerian order as well as the energy flux radiated at infinity at quadrupolar order. As in general relativity, our developments rely on the phenomenological description of the black hole's trajectories as worldlines of point particles that are, in turn, described by a "skeleton" action generalizing that of general relativity.

Finally, we develop a formalism based on Katz' "superpotentials" to define the mass (as a nœther charge) of a black hole that is endowed with vector and scalar "hair". We then deduce the first law of thermodynamics, which is particularly suitable to describe its readjustments when interacting with a faraway companion. Black hole thermodynamics is lastly shown to be a powerful tool to interpret and discuss the scope of their "skeletonization".

Keywords : gravitational waves ; general relativity ; modified gravities ; post-newtonian developments ; "hairy" black holes ; black hole thermodynamics ; global charges ; Katz superpotentials.

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Introduction

Le 12 février 2016, alors que débutait cette thèse, la collaboration LIGO-Virgo annonça la détection des ondes gravitationnelles émises par la coalescence de deux trous noirs, de 36 et 29 masses solaires respectivement, situés à un redshift $z \sim 0,09$ [1], près de cinquante ans après les tentatives infructueuses de J. Weber et ses barres résonnantes [2]. Il ne faudra pas plus de deux années pour que s'ensuivent les détections de quatre autres événements de même nature [3–6], ainsi que celle, "multi-messager", de la fusion d'un système binaire d'étoiles à neutrons [7, 8]. Autant dire que le sujet choisi pour cette thèse tombait à point...

On l'aura déjà beaucoup entendu, ou lu, mais l'événement "GW150914" ouvre bel et bien une nouvelle ère de l'astronomie. D'abord, parce qu'il représente la première confirmation observationnelle forte de l'existence des trous noirs (au delà de Cygnus X-1, Sagittarius A* et leurs homologues [9, 10]), un siècle après les travaux de K. Schwarzschild, qui présentait en 1916 sa solution exacte des équations d'Einstein [11]. Ensuite, parce que, si l'existence des ondes gravitationnelles n'était plus à prouver depuis la découverte des pulsars binaires par l'observation de leurs émissions d'impulsions radio [12–14], les interféromètres LIGO-Virgo permettent d'observer le passage sur Terre de l'onde gravitationnelle elle-même, qui contrairement à une onde radio, est un objet conceptuellement issu de la relativité générale. Enfin, parce que les ondes gravitationnelles ouvrent une nouvelle fenêtre sur l'univers, qui, à l'instar de celles ouvertes par les rayons X, gamma, UV, IR, radio, les neutrinos et autres rayons cosmiques, laisse derrière elle quatre siècles de découvertes astronomiques ouverts par Galilée [15], reposant sur l'observation seule de la lumière visible.

L'avenir de l'"astronomie gravitationnelle" ne peut être qu'enthousiasmant quand on apprend que, lorsque les détecteurs LIGO et Virgo atteindront leur sensibilité maximale (trois fois supérieure à la sensibilité actuelle), il faudra s'attendre à la détection d'une coalescence de système binaire par jour [16]. De plus, ces détecteurs seront bientôt rejoints par leurs homologues KAGRA au Japon [17], puis LIGO India [18], ainsi que, à plus long terme, par les interféromètres spatiaux DECIGO [19] et LISA [20]. Ce dernier permettra, par ses dimensions imposantes (de l'ordre de 10^6km, contre quelques kilomètres pour un interféromètre terrestre), d'observer des sources rayonnant dans une gamme de fréquences comprises entre $0,1\,\mathrm{mHz}$ et $1\,\mathrm{Hz}$ (contre $10\,\mathrm{Hz}$ à $1\,\mathrm{kHz}$ pour LIGO et Virgo), tels que les systèmes binaires impliquant des trous noirs supermassifs, et situées à des redshifts bien supérieurs, $z \sim 10$. Mentionnons enfin les "Pulsar Timing Arrays", complétant le spectre observable dans ses plus basses fréquences (jusqu'à 10^{-9}Hz) [21], ainsi que le projet d'interféromètre terrestre "Einstein Telescope", en phase d'étude, dont la sensibilité permettrait d'observer jusqu'aux détails du "ringdown", dernier soubresaut du trou noir final issu de la fusion d'un système binaire [22].

Gageons donc que le futur offrira de nombreuses opportunités nouvelles d'explorer la relativité générale et de la tester dans un régime de champ fort jamais observé jusqu'alors : celui de la coalescence des systèmes binaires compacts.

0.1 Une (très) brève histoire des ondes gravitationnelles

À la réflexion, il est impossible, en ces quelques pages, de présenter de manière exhaustive les cent ans d'histoire des ondes gravitationnelles, faisant déjà l'objet de nombreux ouvrages tels que [23, 24] ou, plus récemment, [25]. Nul besoin, non plus, d'introduire ici les détails techniques de leur description : la linéarisation des équations d'Einstein, le principe des développements post-newtoniens, le choix de jauge, l'obtention d'un Lagrangien à deux corps et d'un flux, etc... seront de toute façon développés dans la suite de ce manuscrit dans le cadre "élargi" des gravités modifiées.

Dans cette section, tentons plutôt de rappeler les étapes conceptuelles qui semblent être, dans l'optique de cette thèse, les plus marquantes dans l'histoire des ondes gravitationnelles.

0.1.1 1916-1974 : la genèse

La première étape remonte aux travaux fondateurs de A. Einstein de 1916 qui, dans les mois suivant l'introduction de sa théorie de la relativité générale [26–29], en linéarise les équations du champ et introduit le concept d'onde gravitationnelle en établissant les célèbres "formules du quadrupôle" [30, 31]. L'une relie l'onde gravitationnelle émise par une source matérielle à l'accélération de son quadrupôle $Q^{ij} = \int d^3x\, \rho (3x^i x^j - \delta^{ij} x^2)$, $\rho = T_{00}/c^2$ s'identifiant à cet ordre à sa densité de masse newtonienne ; l'autre montre que le flux d'énergie rayonnée par la source est quadratique en la sur-accélération de ce même quadrupôle.

La validité de ces formules sera cependant questionnée pendant six décennies. Dès 1922 en effet, A. Eddington montre que des changements de coordonnées suffisent à éliminer la plupart des composantes des ondes gravitationnelles d'Einstein, à l'exception de deux modes transverses. Il montre de plus que, les formules du quadrupôle faisant apparaitre l'accélération newtonienne des sources, leur justification pour des systèmes liés gravitationnellement nécessite en fait de résoudre les équations d'Einstein à l'ordre post-linéaire [32]. A. Einstein, de son côté, doute même de l'existence des ondes gravitationnelles, comme il l'écrit à M. Born [33], puisque, étant issues de la version linéarisée de la relativité générale, il n'est pas garanti qu'elles soient une approximation crédible de quelconques solutions de sa théorie. Avec N. Rosen, il ira jusqu'à conclure à leur inexistence en 1936, avant de se raviser en 1937, sous l'influence "discrète" de H. Robertson [34, 35].

Les années 1950 marquent un tournant dans la compréhension mathématique et physique des ondes gravitationnelles. D'abord, les travaux de mathématiciens portant sur le problème de Cauchy en relativité générale, tels que ceux de A. Lichnerowicz ou de Y. Choquet-Bruhat et son formalisme "$1+3$", montrent que les ondes de 1916 sont bien des solutions approchées des équations d'Einstein [36]. Ensuite, en 1957, H. Bondi, F. Pirani, I. Robinson et A. Trautman comprennent qu'une onde gravitationnelle propage une perturbation de la courbure de l'espace-temps, ne pouvant donc être mise en évidence localement -principe d'équivalence oblige-, mais que son passage se manifesterait, par exemple, par la fluctuation de la distance propre entre deux points éloignés [37–39]... Enfin, la conférence de Chapel Hill de 1957 [40] voit apparaitre l'argument des "perles visqueuses" (en anglais, "sticky bead"), proposé par R. Feynman et H. Bondi, montrant que si le passage d'une onde gravitationnelle peut fournir un travail à un système de deux perles coulissant sur une tige rigide, elle doit bel et bien emporter avec elle une énergie [41].

Encore faut-il relier cette onde à une source plausible. Celles que J. Weber, jusqu'au début des années 1970, disait observer ne l'étaient pas [2]...

0.1.2 Problème à deux corps et développements post-newtoniens

En 1974, R. Hulse et J. Taylor découvrent le premier pulsar binaire [12], dont la diminution de la période orbitale traduit une perte d'énergie mécanique en accord avec la seconde formule du quadrupôle d'Einstein [42]. À la même époque, R. Weiss, inspiré par V. Pustovoit et R. Forward, présente en 1972 les caractéristiques d'un interféromètre de Michelson à laser de sensibilité relative $\delta L/L = 10^{-21}$ pour des fréquences environnant les 100 Hz [43]. Lui comme d'autres, R. Drever et A. Brillet en particulier, ont en tête de détecter le "chirp" ("gazouillement", en français) gravitationnel émis par la coalescence des systèmes binaires d'objets compacts.

Le "problème à deux corps" de la relativité générale se retrouve ainsi, pour les théoriciens, sur le devant de la scène. Il avait en fait déjà été abordé, entre autres, par W. De Sitter, J. Droste et H. Lorentz dès 1916 [44, 45], puis par A. Einstein, L. Infeld et B. Hoffmann [46], qui en obtiennent en 1938 les premières corrections relativistes "post-newtoniennes" (1 PN), i.e. d'ordre $\mathcal{O}(G_* m/c^2 r) = \mathcal{O}(v/c)^2$ (G_* désignant la constante de Newton), valables dans l'approximation du champ faible et des petites vitesses relatives. Mais leurs calculs, basés sur des méthodes de raccordements asymptotiques, étaient d'une complexité trop importante pour pouvoir être poursuivis.

En 1954, L. Infeld ré-obtient le même Lagrangien en seulement quelques pages de calcul, en remplaçant les corps par des particules ponctuelles en mouvement géodésique dans la métrique qu'elles génèrent, et munies de "masses" constantes [47]. La grande simplification apportée par cette "skeletonisation", qui ignore la structure interne des corps en vertu du "principe d'effacement" [23], permet aux théoriciens d'aborder les ordres PN suivants [48–50]. En particulier, T. Damour et N. Deruelle en 1981, et T. Damour en 1982 obtiennent les équations du mouvement complètes à l'ordre 2.5 PN (n PN signifiant des corrections d'ordre $(v/c)^{2n}$ au-delà de Newton), ordre où apparait la force de réaction de rayonnement [51–53]. En établissant alors un bilan rigoureux entre l'énergie mécanique perdue par le système et le travail de cette force dissipative, ils démontrent la validité de la seconde formule du quadrupôle d'Einstein, mettant définitivement fin à sa controverse, et confirment que R. Hulse et J. Taylor ont bel et bien mis en évidence l'existence des ondes gravitationnelles.

Mais, ce qui suffisait pour décrire les pulsars binaires ne suffit plus aux interféromètres de R. Weiss. En effet, pour extraire un signal gravitationnel infime du bruit présent dans ces détecteurs, il est nécessaire d'en connaitre la forme détaillée (en particulier, la phase, à laquelle les méthodes de "filtrage adapté" sont les plus sensibles [54]).

Au prix d'efforts importants durant ces trente dernières années, les théoriciens poursuivirent donc leurs travaux sur le problème à deux corps aux ordres PN élevés. Citons simplement ici les deux approches les plus avancées qui, à l'aide de méthodes de régularisation dimensionnelle (liées au champ propre des particules ponctuelles) [55, 56], ont permis d'obtenir récemment les équations du mouvement complètes à l'ordre 4 PN : l'approche hamiltonienne "ADM", de T. Damour, P. Jaranowski et G. Schäfer [57], et celle du "Lagrangien de Fokker", développée par le groupe de L. Blanchet et G. Faye [58, 59]. Le secteur dissipatif de la dynamique, quant à lui, est aujourd'hui traité par l'obtention du flux d'énergie rayonnée par le système, connu actuellement à l'ordre 3.5 PN au-delà de la seconde formule du quadrupôle d'Einstein (alors dite "0 PN"), via l'approche "multipolaire post-minkowskienne" de L. Blanchet, T. Damour et B. Iyer [60, 61] et "DIRE" de C. Will, A. Wiseman et M. Pati [62]. On trouvera dans, e.g., [63] une présentation détaillée des autres approches au problème à deux corps (notamment, celle de la "force propre" dans le cas du rapport de masses petit [64]), ainsi que de l'inclusion des spins.

0.1.3 1998-... : l'ère de la relativité numérique et de la resommation

Il est d'usage de séparer la coalescence d'un système binaire compact en trois phases distinctes : la phase spiralante ("inspiral" en anglais), lorsque les corps orbitent à grande distance, $r \gg G_* m/c^2$, et se rapprochent lentement sous l'effet de la contre-réaction de rayonnement ; la phase de fusion ("merger" en anglais), lorsque $r \sim G_* m/c^2$ et $v \sim c$, correspondant aux dernières orbites stables puis au "plongeon" des deux corps l'un vers l'autre ; et enfin, la phase de relaxation ("ringdown" en anglais), pendant laquelle le trou final retourne vers une configuration stationnaire de Kerr.

La phase spiralante est bien décrite par le formalisme post-newtonien mentionné plus haut. La phase de relaxation, elle, est aussi connue dans le cadre de la théorie des perturbations des trous noirs, introduite en 1957 par T. Regge et J. Wheeler [65], et de leurs modes quasi-normaux, mis en évidence par C. Vishveshwara en 1970 [66], voir aussi [67] pour une revue. En revanche, la phase intermédiaire ne peut pas être décrite par l'approche post-newtonienne dont la convergence dans cette zone laisse à désirer [68], si bien que cette phase de champ fort reste inconnue jusque dans la fin des années 1990 [69], comme l'illustre non sans humour la figure ci-dessous, attribuée à K. Thorne.

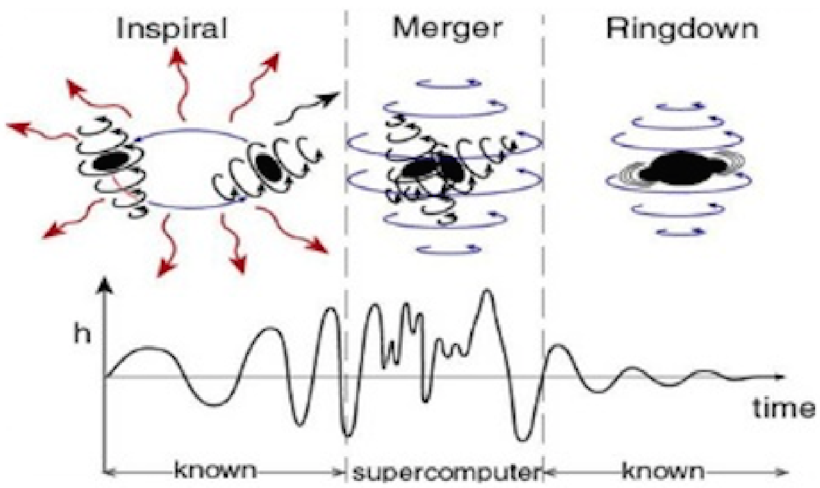

FIGURE 1 – L'état de l'art en 1998 : la forme d'onde gravitationnelle issue de la coalescence de deux trous noirs de masses comparables est inconnue dans le régime de champ fort, près de leur fusion ; figure extraite de [70].

Afin de décrire la dynamique d'un système à deux corps incluant sa fusion, deux stratégies se développent alors parallèlement. La première est celle de la relativité numérique, qui s'appuie sur la décomposition "3+1" [71] des équations d'Einstein (non sans rappeler le formalisme "1+3" de Y. Choquet-Bruhat mentionné plus haut), pour les intégrer numériquement. Une avancée importante dans ce domaine est réalisée par F. Pretorius en 2005 [72], qui stabilise l'évolution numérique du système et génère, pour la première fois, une forme d'onde gravitationnelle complète incluant la phase de fusion de deux trous noirs. On trouvera une revue des dernières avancées dans, e.g., [73].

La seconde est une approche analytique introduite par A. Buonanno et T. Damour en 1998 [68, 74], du nom de méthode "effective à un corps" (en anglais, "effective-one-body" ou EOB), à laquelle nous consacrerons le chapitre 1 de ce manuscrit. Elle consiste à définir, à l'aide de transformations canoniques, une resommation de l'information post-newtonienne sur la dynamique d'un système binaire, permettant d'en décrire l'évolution jusqu'aux dernières orbites avant sa coalescence, où la forme d'onde associée est raccordée à celle des modes quasi-normaux du trou noir final. Pour une revue de ses développements les plus récents, voir, e.g., [75]. Dans cette thèse, nous allons montrer comment étendre le formalisme EOB à certaines alternatives à la relativité générale, à commencer par l'exemple simple des théories scalaire-tenseur, afin d'en décrire le régime de champ fort, et obtenir les formes d'onde associées.

Les développements post-newtoniens, la théorie de perturbation des trous noirs, la relativité numérique et le formalisme "effective-one-body" sont autant de travaux cruciaux qui ont permis aux théoriciens d'être présents au rendez-vous donné par la collaboration LIGO-Virgo, pour qu'elle soit munie, au moment venu, de banques de quelques centaines de milliers de patrons d'ondes gravitationnelles, à la fois précis et complets, à comparer aux premières données du 14 septembre 2015.

0.2 Problème à deux corps et théories scalaire-tenseur

Qu'elles aient été introduites par P. Jordan en 1946, puis Y.-R. Thiry et M. Fierz comme conséquence de la théorie de Kaluza-Klein [76–78], ou redécouvertes par C. Brans et R. Dicke en 1961 pour des raisons phénoménologiques liées au principe de Mach [79], voir aussi [80], les théories scalaire-tenseur occupent une place importante sur la scène des gravités modifiées depuis la fin des années 1960.

En effet, elles suggèrent un ensemble de tests des propriétés de la relativité générale, dont deux particulièrement célèbres. Le premier, proposé par K. Nordtvedt dès 1968 [81, 82], est celui de l'universalité de la chute des corps auto-gravitants (c.-à-d., ayant un champ de gravitation propre), qui est brisée en théories scalaire-tenseur, comme nous le discutons ci-dessous. Le second porte sur la perte d'énergie par rayonnement gravitationnel d'un système binaire de corps auto-gravitants : comme le développa D. Eardley en 1974, en présence d'un champ scalaire, ce rayonnement peut avoir une composante de nature dipolaire, absente en relativité générale [83].

Dans cette thèse, nous étudions le problème du mouvement des systèmes binaires compacts dans la classe des théories scalaire-tenseur introduites par P. Bergman, K. Nordtvedt et R. Wagoner [84–86], et présentées sous forme moderne par C. Will et Zaglauer [87] et T. Damour et G. Esposito-Farèse [88], en nous limitant au cas simple d'un champ scalaire unique et sans masse.

0.2.1 "Jordan frame" vs "Einstein frame"

Il existe deux formulations usuelles et équivalentes des théories scalaire-tenseur [89, 90]. Dans la première, dite de "Jordan" ("Jordan frame") pour des raisons historiques, elles sont décrites par l'action (on pose $G_* = c = 1$) :

$$S_{\rm JF}[\tilde{g}_{\mu\nu}, \phi, \Psi] = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left(\phi \tilde{R} - \frac{\omega(\phi)}{\phi} \tilde{g}^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \right) + S_{\rm m} \left[\Psi, \tilde{g}_{\mu\nu} \right] \ , \tag{1}$$

où $\tilde{R}$ est le scalaire de Ricci associé à la métrique de Jordan $\tilde{g}_{\mu\nu}$, de métrique inverse $\tilde{g}^{\mu\nu}$, où $\tilde{g} = det \, \tilde{g}_{\mu\nu}$, et où les champs de matière Ψ sont minimalement couplés à la métrique de Jordan, garantissant la satisfaction du "principe d'équivalence faible" (qui a d'ailleurs été récemment contraint à être satisfait à une précision relative de 2.10^{-14} par la mission Microscope, cf. [91]).

La présence du champ scalaire sans masse ϕ modifie l'action de Einstein-Hilbert, et une théorie scalaire-tenseur est entièrement caractérisée par la donnée de la fonction $\omega(\phi)$. Les théories "Jordan-Fierz-Brans-Dicke", par exemple, correspondent à la sous-classe $\omega(\phi) = cste$, parmi laquelle se trouve la relativité générale, dans la limite $\omega \to \infty$.

Les équations du champ dérivées de la variation de l'action (1) s'écrivent :

$$\tilde{G}_{\mu\nu} = \frac{\omega(\phi)}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\tilde{g}_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi\right) + \frac{1}{\phi}\left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi - \tilde{g}_{\mu\nu}\Box_{\tilde{g}}\phi\right) + \frac{8\pi}{\phi}\tilde{T}_{\mu\nu}\,, \tag{2a}$$

$$\Box_{\tilde{g}}\phi = \frac{1}{3+2\omega(\phi)}\left(8\pi\tilde{T} - \frac{d\omega}{d\phi}\partial^\alpha\phi\partial_\alpha\phi\right), \tag{2b}$$

où $\tilde{G}_{\mu\nu}$ est le tenseur d'Einstein et $\tilde{\nabla}_\mu$ la dérivée covariante associés à $\tilde{g}_{\mu\nu}$, où $\Box_{\tilde{g}} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu$, et où $\tilde{R}$ a été éliminé du membre de droite de (2b) à l'aide de la trace de (2a). Enfin, $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_{\rm m}}{\delta\tilde{g}^{\mu\nu}}$ est le tenseur énergie-impulsion de la matière dans le "Jordan frame", avec $\tilde{T} = \tilde{T}^\mu{}_\mu$. Notons qu'il est conservé en conséquence du couplage minimal de la matière à la métrique de Jordan, $\tilde{\nabla}_\mu\tilde{T}^\mu{}_\nu = 0$, comme on peut aussi s'en assurer à l'aide de l'identité de Bianchi ($\tilde{\nabla}^\mu\tilde{G}_{\mu\nu} = 0$) et par manipulation des équations du champ (2) ; voir, e.g., [92].

Dans cette thèse, il sera cependant plus commode d'employer les conventions de T. Damour et G. Esposito-Farèse [88], et de reformuler les théories scalaire-tenseur à l'aide de la transformation conforme et des redéfinitions suivantes :

$$\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu} \quad , \quad 3+2\omega(\phi) = (d\ln\mathcal{A}/d\varphi)^{-2} \quad , \quad \mathcal{A}(\varphi) = 1/\sqrt{\phi} \quad , \tag{3}$$

telles que l'action (1) devient

$$S_{\rm EF}[g_{\mu\nu},\varphi,\Psi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right) + S_{\rm m}[\Psi,\mathcal{A}^2(\varphi)g_{\mu\nu}]\,, \tag{4}$$

où R est le scalaire de Ricci associé à la métrique d'Einstein $g_{\mu\nu}$, de métrique inverse $g^{\mu\nu}$, et où $g = det\, g_{\mu\nu}$.

Dans cette nouvelle représentation, dit d'"Einstein" ("Einstein frame"), la dynamique libre de la métrique $g_{\mu\nu}$ est décrite par l'action de Einstein-Hilbert, et celle du champ scalaire φ par un terme cinétique usuel, tandis que les champs de matière, Ψ, sont non-minimalement couplés à la métrique d'Einstein et au champ scalaire.

La théorie est à présent déterminée par la donnée de la fonction $\mathcal{A}(\varphi)$, et on retrouvera les théories "Jordan-Fierz-Brans-Dicke" pour $\mathcal{A}(\varphi) = e^{\alpha\varphi}$ où $\alpha = 1/\sqrt{3+2\omega}$ est une constante, ainsi que la relativité générale lorsque $\mathcal{A}(\varphi) = cste$. Notons aussi qu'en se restreignant à un espace-temps plat, (4) redonne la théorie de G. Nordström de la gravitation [93] et ses extensions non linéaires [94].

Un intérêt majeur de la formulation d'Einstein des théories scalaire-tenseur est que les équations du champ y prennent une forme familière :

$$G_{\mu\nu} = 2\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi\right) + 8\pi T_{\mu\nu}\,, \tag{5a}$$

$$\Box_g\varphi = -4\pi\alpha(\varphi)T\,, \tag{5b}$$

toutes les quantités étant cette fois calculées à l'aide de la métrique d'Einstein $g_{\mu\nu}$, et où

$$\alpha(\varphi) = \frac{d \ln \mathcal{A}(\varphi)}{d\varphi} \tag{6}$$

mesure le couplage de la matière au champ scalaire. En revanche, le tenseur énergie-impulsion de la matière dans le "Einstein frame", $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}$, n'est pas conservé : $\nabla_\mu T^\mu_\nu = \alpha(\varphi) T \partial_\nu \varphi$.

Que l'on utilise les formulations de Jordan ou d'Einstein, le champ scalaire ne doit pas être interprété comme un champ de matière ordinaire. En effet, il touche toujours à la géométrie : dans une représentation, sa présence modifie l'action de Einstein-Hilbert (1), tandis que dans l'autre, elle brise le mouvement géodésique (4). Le statut du champ scalaire est donc celui d'un degré de liberté supplémentaire de l'interaction gravitationnelle.

Cependant, contrairement à la métrique, cette nouvelle "patte" n'est pas un champ "effaçable" localement. Comme nous allons le voir, les conséquences sur le mouvement des corps auto-gravitants (i.e. ayant un champ de gravitation propre) en sont considérables.

0.2.2 Le principe d'équivalence fort : une expérience de pensée

Considérons, dans le cadre de la relativité générale, une cabine d'ascenseur en chute libre dans un champ gravitationnel externe, $g^{\mathrm{ext}}_{\mu\nu}$. Supposons ses dimensions négligeables, de sorte qu'elle s'identifie, dans ce champ, à un point. On peut alors construire un référentiel lié à la cabine dit "quasi-localement" inertiel et minkowskien, tel que, partout dans la cabine et sur ses parois, la métrique externe $g^{\mathrm{ext}}_{\mu\nu}$ s'identifie à celle de Minkowski, et ses dérivées premières $\partial_\alpha g^{\mathrm{ext}}_{\mu\nu}$ s'annulent.

Plaçons à présent à l'intérieur de la cabine un corps auto-gravitant, e.g., une étoile, dont les dimensions sont supposées négligeables devant celles de la cabine. Pour connaitre le champ de gravitation dans la cabine, il suffit d'y résoudre les équations d'Einstein, à condition que la solution (asymptotiquement plate) soit raccordée à l'infini, c.-à-d., sur ses parois, au champ externe. Mais, les équations d'Einstein étant du second ordre, elles n'y imposent que la continuité de la métrique et de ses dérivées premières... Dans notre référentiel "quasi-localement" inertiel, le champ externe n'a donc aucune influence sur l'étoile et son champ de gravitation ; et, tout comme une particule test, une étoile initialement immobile au centre de la cabine (e.g., une étoile statique et à symétrie sphérique) le restera.

Nous venons de mettre en évidence une des facettes du "principe d'équivalence fort" [95], stipulant que dans un champ de gravitation externe, le mouvement de tout corps de dimensions négligeables, même auto-gravitant, est universel : de fait notre étoile, immobile au centre de la cabine, suit, tout comme cette dernière, les géodésiques de $g^{\mathrm{ext}}_{\mu\nu}$ (rappelons que dans la limite où l'on néglige la contribution de l'étoile au champ externe, son rayonnement est, lui aussi, négligeable).

Reproduisons à présent notre expérience en théorie scalaire-tenseur. Dans le "Jordan frame", la cabine d'ascenseur est plongée dans une métrique $\tilde{g}^{\mathrm{ext}}_{\mu\nu}$ et un champ scalaire ϕ_{ext} externes. Supposons-la cette fois-ci en mouvement géodésique dans la métrique de Jordan externe, de façon à construire, comme ci-dessus, un référentiel "quasi-localement" inertiel, dans lequel $\tilde{g}^{\mathrm{ext}}_{\mu\nu}$ s'identifie à la métrique de Minkowski, et $\partial_\alpha \tilde{g}^{\mathrm{ext}}_{\mu\nu}$ s'annule partout dans la cabine et sur ses parois. La matière étant minimalement couplée à la métrique de Jordan, cf. (1), une particule dont on néglige le champ de gravitation qu'elle crée, autrement dit, d'auto-gravité négligeable, en suit les géodésiques ; donc, si elle est initialement immobile au centre de la cabine, elle le restera.

La situation est bien différente dans le cas d'un corps auto-gravitant : pour connaitre ses champs, il faut résoudre les équations d'Einstein et de Klein-Gordon dans la cabine, en en raccordant les solutions aux champs externes sur ses parois. L'équation de Klein-Gordon étant, tout comme celle d'Einstein, du second ordre, il faut là encore y assurer la continuité du champ scalaire et de ses dérivées premières. Or, ni le champ scalaire externe $\varphi_{\rm ext}$, ni ses dérivées $\partial_\alpha\varphi_{\rm ext}$ ne peuvent y être effacées.

Ainsi, dès lors que notre étoile contribue au champ de gravitation, elle devient sensible à son environnement scalaire : non seulement la valeur $\varphi_{\rm ext}$ a une influence directe sur les solutions des équations du champ (2), et donc, par exemple, sur la configuration à l'équilibre de l'étoile, mais les gradients $\partial_\alpha\varphi_{\rm ext}$ aussi, en définissant une direction privilégiée dans la cabine. L'étoile, initialement immobile au centre de la cabine, en "dérape" plus ou moins selon sa structure interne (e.g., son équation d'état), intervenant dans les équations du champ (2) et dans les équations du mouvement de la matière via $\tilde{T}_{\mu\nu}$.

Même en négligeant tout effet de taille finie, le mouvement des corps auto-gravitants n'est donc plus géodésique dans $\tilde{g}^{\rm ext}_{\mu\nu}$, et est encore moins universel : le "principe d'équivalence fort" est brisé en théories scalaire-tenseur.

0.2.3 La "skeletonisation" en théories scalaire-tenseur

Le raisonnement ci-dessus illustre comment, en théories scalaire-tenseur, l'environnement scalaire d'un corps grave A peut le dévier, selon sa structure interne, du mouvement géodésique de la métrique de Jordan.

Mais, plutôt que d'intégrer des equations du champ lourdes (2) pour remonter à sa trajectoire, D. Eardley propose en 1974 de généraliser la stratégie de la "skeletonisation" aux théories scalaire-tenseur, en remplaçant $S_{\rm m}$ dans (1) par l'action suivante, décrivant une particule ponctuelle [83] :

$$S^{\rm pp}_{\rm m,A} = -\int \tilde{m}_A(\varphi)\, d\tilde{s}_A\ , \tag{7}$$

avec $d\tilde{s}_A = \sqrt{-\tilde{g}_{\mu\nu}dx^\mu_A dx^\nu_A}$, où $x^\mu_A[\tilde{s}_A]$ est la ligne d'univers du corps A, et où la fonction $\tilde{m}_A(\varphi)$, appelée sensibilité du corps dans le "Jordan frame", dépend de la valeur du champ scalaire évaluée à sa position.

L'ansatz (7) mérite quelques remarques introductives : d'abord, on se convaincra aisément que, si l'on suppose pouvoir réduire un corps à une particule ponctuelle, (7) est bien l'action la plus générale pouvant le décrire, qui dépende à la fois de la métrique $\tilde{g}_{\mu\nu}$ et du champ scalaire φ, tout en préservant les symétries de l'action fondamentale (1). Ensuite, les équations du mouvement déduites de (7), pour une particule A plongée dans des champs externes $\tilde{g}^{\rm ext}_{\mu\nu}$ et $\varphi_{\rm ext}$, s'écrivent, avec $\tilde{m}'_A = d\tilde{m}_A/d\varphi$ et $\tilde{u}^\mu_A = dx^\mu_A/d\tilde{s}_A$:

$$\tilde{u}^\alpha_A\tilde{\nabla}_\alpha\left(\tilde{m}_A(\varphi_{\rm ext})\tilde{u}^\mu_A\right) = -\tilde{m}'_A(\varphi_{\rm ext})\,\partial^\mu\varphi_{\rm ext}\ , \tag{8}$$

soit encore, dans son référentiel localement inertiel (de Jordan) tangent, tel que $u^\mu_A = (1,0,0,0)$ initialement,

$$\tilde{m}_A(\varphi_{\rm ext})\frac{d^2x^i_A}{dt^2} = -\tilde{m}'_A(\varphi_{\rm ext})\,\partial_i\varphi_{\rm ext}\ .$$

On retrouve, conformément à notre expérience de pensée, que les gradients locaux du champ scalaire externe la dévient de son mouvement géodésique, selon sa structure interne encodée dans sa sensibilité $\tilde{m}_A(\phi_{\rm ext})$. Enfin, notons que (7) ne dépend pas des gradients $\partial_\mu = \{\partial_t, \partial_i\}$ des champs, qui auraient fait apparaître des effets de taille finie (e.g., de marée, ou hors équilibre) dans ses équations du mouvement (8), et que nous négligeons ici.

Dans cette thèse, nous adopterons cette approche "à la Eardley" pour décrire le mouvement des corps compacts. Une question laissée cependant en suspens, à ce stade, est comment calculer la fonction $\tilde{m}_A(\phi)$, pour un corps A donné : nous y reviendrons au chapitre 4. Notons simplement ici que $\tilde{m}_A(\phi)$ doit se réduire à une constante pour un corps d'autogravité négligeable, de sorte que son mouvement redevienne géodésique dans le "Jordan frame".

0.2.4 Le problème à deux corps

Nous avons à présent en mains tous les outils nécessaires au traitement du problème à deux corps en théories scalaire-tenseur. Dans le "Einstein frame", il est en effet décrit par l'action "skeleton" suivante :

$$S_{\rm EF}[g_{\mu\nu}, \varphi, \{x^\mu_A\}] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right) - \sum_A \int m_A(\varphi)\, ds_A\,, \tag{9}$$

avec $ds_A = \sqrt{-g_{\mu\nu}dx^\mu_A dx^\nu_A}$, et où nous avons utilisé la relation $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu}$ pour introduire les sensibilités $m_A(\varphi)$ des corps dans le "Einstein frame",

$$m_A(\varphi) = \mathcal{A}(\varphi)\tilde{m}_A(\varphi)\,. \tag{10}$$

Le problème à deux corps (sans spins) dépend donc de deux fonctions $m_A(\varphi)$ et $m_B(\varphi)$, prenant en compte à la fois la théorie et les effets d'auto-gravité des corps, au lieu de deux constantes seulement (m_A et m_B) en relativité générale.

Les théories scalaire-tenseur sont un exemple rare de théories pour lesquelles le problème à deux corps a été étudié à des ordres post-kepleriens (PK) élevés [1] : le lagrangien correspondant a été obtenu à l'ordre 1PK par T. Damour et G. Esposito-Farèse en 1992 [88], puis à l'ordre 2PK par S. Mirshekari et C. Will en 2013 [96]. Enfin, les équations du mouvement ont été obtenues récemment à l'ordre 3PK par L. Bernard dans [97].

Pour ce faire, il est nécessaire de résoudre par itération les équations du champ issues de la variation de l'action avec sources distributionnelles (9), dans l'approximation des petites vitesses relatives et du champ faible, autour de la métrique de Minkowski et de la valeur φ_0 du champ scalaire qui, on l'a vu, ne peut pas être choisie nulle, mais est imposée par l'environnement cosmologique du système binaire. Nous rencontrerons ainsi les quantités :

1. Nous adoptons ici la terminologie de T. Damour et G. Esposito-Farèse, et désignons par "nPK" des corrections relativistes à la dynamique à deux corps d'ordre $\mathcal{O}\left((v/c)^{2n}\right) \sim O\left((G_* m/r)^n\right)$ au-delà de l'approximation keplerienne. L'emploi du terme "keplerien", au lieu de "newtonien", permet quant à lui de rappeler que les effets d'auto-gravité, qui sont non-perturbatifs pour des objets compacts tels que les étoiles à neutrons et les trous noirs, sont encodés dans les fonctions $m_{A/B}(\varphi)$.

$$\alpha_A(\varphi) = \frac{d\ln m_A}{d\varphi} \quad \left(= \frac{d\ln\mathcal{A}}{d\varphi} + \frac{d\ln\tilde{m}_A}{d\varphi} \right), \tag{11a}$$

$$\beta_A(\varphi) = \frac{d\alpha_A}{d\varphi}, \tag{11b}$$

$$\beta'_A(\varphi) = \frac{d\beta_A}{d\varphi}, \tag{11c}$$

permettant de développer les sensibilités à l'ordre 2PK (auquel nous nous restreindrons) selon :

$$\ln m_A(\varphi) = \ln m_A^0 + \alpha_A^0(\varphi - \varphi_0) + \frac{1}{2}\beta_A^0(\varphi-\varphi_0)^2 + \frac{1}{6}\beta'^0_A(\varphi-\varphi_0)^3 + \cdots, \tag{12}$$

un indice 0 indiquant une quantité évaluée à l'infini, $\varphi = \varphi_0$. Notons que pour un corps d'auto-gravité négligeable, i.e. $\tilde{m}_A(\varphi) = cste$, les quantités (11) deviennent universelles,

$$\alpha(\varphi) = \frac{d\ln\mathcal{A}}{d\varphi} \quad , \quad \beta(\varphi) = \frac{d\alpha}{d\varphi} \quad , \quad \beta'(\varphi) = \frac{d\beta}{d\varphi} \quad , \tag{13}$$

tandis que la relativité générale est retrouvée lorsque les sensibilités du "Einstein frame" sont constantes : $m_{A/B}(\varphi) = cste$, soit $\alpha_A(\varphi) = \beta_A(\varphi) = \beta'_A(\varphi) = 0$.

L'objet de la première partie de cette thèse est d'étendre le formalisme EOB aux théories scalaire-tenseur. Pour cela, notre point de départ sera le Lagrangien 2PK de S. Mirshekari et C. Will (traduit dans le langage présenté ci-dessus), que le formalisme EOB nous permettra ensuite de resommer, comme en relativité générale, afin d'explorer le régime de champ fort de la dynamique à deux corps.

0.2.5 Ondes gravitationnelles : vers de nouvelles contraintes des théories scalaire-tenseur ?

Les théories scalaire-tenseur sont soumises à d'importantes contraintes expérimentales. Dans le système solaire par exemple, où l'auto-gravité des corps peut être ignorée, la fonction de couplage universelle $\mathcal{A}(\varphi)$ est contrainte par la mesure des paramètres $\beta_{\rm Edd}$ et $\gamma_{\rm Edd}$ d'Eddington, tous deux égaux à 1 en relativité générale, mais s'écrivant, en théorie scalaire-tenseur [88] :

$$\beta_{\rm Edd} = 1 + \frac{1}{2}\frac{\beta_\odot\alpha_\odot^2}{(1+\alpha_\odot^2)^2} \quad , \quad \gamma_{\rm Edd} = 1 - \frac{2\alpha_\odot^2}{1+\alpha_\odot^2} \quad , \tag{14}$$

où $\alpha_\odot = \alpha(\varphi_\odot)$ et $\beta_\odot = \beta(\varphi_\odot)$ sont les paramètres fondamentaux (13), évalués dans le système solaire. En particulier, la mesure de $\gamma_{\rm Edd}$ obtenue par la sonde Cassini donne la meilleure contrainte sur $\alpha_\odot$ à ce jour : $\alpha_\odot^2 \lesssim 2.10^{-5}$ [98], tandis que $\beta_\odot$ est plus difficilement contraint [99].

Cette contrainte forte dans le système solaire ne s'applique cependant pas au cas d'un système binaire compact. En effet, sa dynamique est alors déterminée par les paramètres "habillés" (11-12), dont les valeurs numériques peuvent a priori être largement augmentées par les effets d'auto-gravité des corps. On pensera, par exemple, au phénomène de "scalarisation spontanée" des étoiles à neutrons, découvert par T. Damour et G. Esposito-Farèse : pour une théorie telle que $\beta_0 = \beta(\varphi_0) \lesssim -4$ et $\alpha_0 = \alpha(\varphi_0)$ même nul (où φ_0 est l'environnement cosmologique du système), ils montrent dans [100] en intégrant les équations du champ à l'intérieur d'une étoile que le paramètre $\alpha_A^0 = \alpha_A(\varphi_0)$ la caractérisant peut être de l'ordre de l'unité.

Mais, là encore, l'observation de la diminution de la période orbitale des pulsars binaires, notamment celle de PSR J1738+0333, contraignant fortement la présence de rayonnement dipolaire, écarte tout effet de "scalarisation" importante : $|\alpha_A^0| \lesssim 2.10^{-3}$ [101]. Or, comme nous le verrons au chapitre 2, les corrections scalaire-tenseur au Lagrangien à deux corps sont toutes au moins quadratiques en leurs paramètres α_A^0, et ce, à tous les ordres post-kepleriens...

Si toutes les observations dans le système solaire et des pulsars binaires semblent rejeter les théories scalaire-tenseur, pourquoi donc continuer à les étudier aujourd'hui ? La réponse, que nous allons tenter d'illustrer dans cette thèse, est que les ondes gravitationnelles permettent d'observer des systèmes binaires compacts dans des régimes nouveaux, qui peuvent échapper aux contraintes existantes.

En effet, les interféromètres LIGO-Virgo (et dans le futur, LISA) sont conçus pour observer des sources situées à des redshifts importants : jusqu'à $z \sim 2$ pour LIGO-Virgo, et même $z \sim 10$ pour LISA [20]. Or, à ces redshifts, la valeur φ_0 de l'environnement scalaire d'un système binaire peut être très différente de celle qu'elle a dans le système solaire, $\varphi_\odot$, ou celle de notre environnement galactique ; voir, par exemple, [102] pour une étude de son évolution cosmologique. En conséquence, les contraintes mentionnées ci-dessus ne s'appliquent plus à ces redshifts, et même les phénomènes de "scalarisation spontanée" y deviennent envisageables.

De plus, les ondes gravitationnelles permettent de tester les théories scalaire-tenseur dans le régime de champ fort de la dynamique de deux corps compacts, près de leur coalescence, où contribuent les ordres post-kepleriens élevés. Mentionnons par ailleurs le phénomène de "scalarisation dynamique" mis en évidence numériquement par E. Barausse, C. Palenzuela, M. Ponce et L. Lehner [103, 104], qui, lorsqu'elle se produit, implique l'augmentation soudaine du paramètre α_A d'une étoile sous l'influence de son compagnon, lors des dernières orbites précédant leur coalescence. Pour des approches analytiques sur la "scalarisation dynamique", voir, e.g., [105, 106].

La fenêtre observationnelle ouverte par l'astronomie gravitationnelle justifie donc une étude approfondie du rayonnement émis par deux corps orbitant l'un autour de l'autre en théories scalaire-tenseur.

La partie dissipative de la dynamique à deux corps a été étudiée par T. Damour et G. Esposito-Farèse qui ont obtenu dans [88] le flux d'énergie rayonnée à l'ordre 0PK (le flux dipolaire étant considéré d'ordre "-1PK" relativement à celui de la seconde formule du quadrupole d'Einstein), tandis que R. Lang a obtenu son expression à l'ordre suivant, i.e. 1PK, dans [107, 108]. Les premières formes d'ondes, elles, ont été obtenues par N. Sennett, S. Marsat et A. Buonanno [109] à l'ordre 1PK (relativement aux formules du quadrupole d'Einstein), en s'appuyant sur le Lagrangien de S. Mirshekari et C. Will (mentionné plus haut) ainsi que le flux de R. Lang.

Dans cette thèse, notre objectif sera de construire des formes d'ondes scalaire-tenseur dans le cadre "resommé" du formalisme EOB.

0.3 Trous noirs "chevelus" en théories Einstein-Maxwell-dilaton

Comme nous l'avons mentionné en section 0.1, le formalisme EOB permet de décrire les systèmes binaires compacts jusqu'aux dernières orbites précédant leur coalescence. Ceci le rend particulièrement adapté à la description des trous noirs binaires, qui sont d'ailleurs les principales sources d'ondes gravitationnelles observées jusqu'à présent par les interféromètres LIGO et Virgo.

Or, si notre objectif est de généraliser le formalisme EOB aux gravités modifiées, il faut garder à l'esprit que dans les théories scalaire-tenseur présentées ci-dessus, les trous noirs (statiques et à symétrie sphérique tout du moins) s'identifient à ceux de la relativité générale. En effet, même en munissant le champ scalaire d'un potentiel $V(\varphi)$ (convexe), la seule solution statique et à symétrie sphérique, régulière sur l'horizon, et asymptotiquement plate des équations du champ (5) dans le vide, est celle de Schwarzschild, ainsi que le montre, e.g., J. Bekenstein dans [110], voir aussi [111] pour une démonstration simple.

La sensibilité $m_A(\varphi)$ d'un trou noir se réduisant alors à une constante (comme nous le montrerons aux chapitres 3 et 4), l'action (9) se réduit, pour un système de deux trous noirs, à celle de la relativité générale ; et, ni la dynamique, ni le rayonnement associé n'en sont alors distinguables.

Si l'on souhaite étudier les signatures d'éventuels "cheveux" dans la dynamique d'un système binaire formé de deux trous noirs, et non d'étoiles à neutrons, il faut donc étendre nos considérations par-delà les théories scalaire-tenseur.

0.3.1 Systèmes binaires de trous noirs en gravités modifiées : un domaine de recherche naissant

Il existe une vaste littérature consacrée à la recherche de solutions de type trou noir en gravités modifiées : trous noirs avec "cheveux" scalaires, vectoriels, trous noirs en théories de Lovelock, à D dimensions, etc. [112, 113]. Mais, si les exemples de solutions ne manquent pas, celles qui parmi elles sont asymptotiquement plates et à quatre dimensions se font en revanche plus rares.

Par ailleurs, même ayant en mains quelques exemples de trous noirs "chevelus" et isolés, leur mouvement et leur rayonnement au sein d'un système binaire reste à ce jour une question largement inexplorée [114, 115].

Pour ne citer que les développements les plus récents, on trouvera des travaux sur les trous noirs en théories scalaire-tenseur généralisées, telle que Horndeski, par E. Babichev, C. Charmousis et A. Lehébel [116], voir aussi [117], ou encore sur les trous noirs des théories scalaire-tenseur couplées à l'invariant topologique de Gauss-Bonnet, qui exhibent des propriétés de "scalarisation spontanée" mise en évidence par E. Berti *et al.* [118] et dont les modes quasi-normaux sont étudiés par J. Blázquez-Salcedo *et al.* [119] ; mais, dans les deux cas, ils s'agit d'étudier les propriétés de trous noirs isolés, et non leur mouvement autour d'un compagnon. Ajoutons à cela l'étude, en relativité numérique, des trous noirs binaires en théorie Chern-Simons par M. Okounkova *et al.* [120], ainsi que l'approche analytique à la gravité massive dans la limite du rapport de masses extrême développée par Y. Decanini, A. Folacci et M. Ould El Hadj [121, 122], et nous avons probablement "fait le tour" des travaux récents d'un domaine de recherche encore naissant... à l'exception de développements portant sur les théories Einstein-Maxwell-dilaton, dont ceux présentés dans cette thèse.

0.3.2 Les théories Einstein-Maxwell-dilaton

Les théories Einstein-Maxwell-dilaton (EMD), introduites par G. Gibbons et K.I. Maeda [123], consistent à généraliser les théories scalaire-tenseur par l'ajout d'un champ de jauge vectoriel sans masse. Dans la représentation d'Einstein ("Einstein frame"), elles sont décrites par l'action (avec $G_* = c = 1$) :

$$S[g_{\mu\nu}, A_\mu, \varphi] = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi} F^{\mu\nu}F_{\mu\nu} \right) + S_{\rm m}[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}, A_\mu] \, , \tag{15}$$

où R est le scalaire de Ricci associé à $g_{\mu\nu}$, où $g = det\, g_{\mu\nu}$, et où $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Les champs scalaire et vectoriel sont couplés non-minimalement via le paramètre fondamental a, de sorte que A_μ ne sera pas considéré comme un champ de matière ordinaire (e.g., le champ de Maxwell de l'électrodynamique), mais, au même titre que φ, comme un degré de liberté supplémentaire de la gravitation [2].

Les champs de matière Ψ, quant à eux, sont supposés minimalement couplés à ce "graviphoton" A_μ ainsi qu'à la métrique de Jordan $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu}$, à l'instar des théories scalaire-tenseur, de sorte que la fonction $\mathcal{A}(\varphi)$ et le paramètre a définissent la théorie considérée.

Enfin, notons que les théories EMD préservent la symétrie de jauge $U(1)$, $A_\mu \to A_\mu + \partial_\mu\chi$ (χ étant un champ scalaire arbitraire), tandis que dans le vide, i.e. lorsque $S_{\rm m} = 0$, la "shift symmetry" des théories scalaire-tenseur (4), $\varphi \to \varphi + \Delta\varphi$, où $\Delta\varphi$ est une constante, devient

$$\varphi \to \varphi + \Delta\varphi \quad , \quad A^\mu \to e^{a\Delta\varphi} A^\mu \, , \tag{16}$$

ce qui sera crucial, comme on le verra, pour munir les trous noirs, c.-à-d., les solutions des équations du vide de la théorie, d'une sensibilité $m_A(\varphi)$.

Les théories EMD retiendront notre attention puisque, contrairement aux théories scalaire-tenseur, elles prédisent l'existence de trous noirs qui diffèrent de ceux de la relativité générale : les trous noirs statiques, à symétrie sphérique, et chargés électriquement (ou magnétiquement) ont été introduits par G. Gibbons et K.I. Maeda [127], puis redécouverts par D. Garfinkle, G. Horowitz, et A. Strominger [124] ; leurs homologues en rotation ont été étudiés par V. Frolov *et al.* en théorie $a = \sqrt{3}$ [125], et dans l'approximation de la rotation lente par J. Horne et G. Horowitz [126]. Dans cette thèse, nous considérerons l'exemple simple des trous noirs statiques et à symétrie sphérique, et munis de "cheveux" scalaire et "graviphotonique", dont les champs sont donnés par :

$$\begin{aligned}
&ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2 + r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} d\Omega^2 \, , \\
&A_t = -\frac{Q\, e^{2a\varphi_\infty}}{r} \, , \quad A_i = 0 \, , \qquad \text{où} \qquad Q^2 = \frac{r_+ r_-}{1+a^2}\, e^{-2a\varphi_\infty} \, , \\
&\varphi = \varphi_\infty + \frac{a}{1+a^2} \ln\left(1 - \frac{r_-}{r}\right) \, ,
\end{aligned} \tag{17}$$

avec $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. La solution (17) sera discutée en détail au chapitre 4 et en servira de point de départ. Elle dépend de trois paramètres : le rayon r_+ de son horizon, la position r_- de la singularité de courbure (avec $r_- < r_+$), et enfin, la valeur φ_∞ du champ scalaire à l'infini.

L'objectif de la seconde partie de cette thèse est de montrer, sur cet exemple simple, comment traiter analytiquement le problème du mouvement et du rayonnement d'un trou noir "chevelu" lorsqu'il est placé au sein d'un système binaire. Pour cela, nous procéderons en plusieurs étapes : nous montrerons d'abord comment "skeletoniser" le trou noir (17) et en calculer la sensibilité $m_A(\varphi)$. Nous en déduirons ensuite la dynamique de deux tels trous noirs par le calcul du lagrangien post-keplerien correspondant (chapitre 4), avant de les

2. Notons que l'action (15) est motivée par la limite de basse énergie de théories des cordes [124], tandis que $a = \sqrt{3}$ correspond à réduction dimensionnelle de la théorie de Kaluza-Klein [125, 126], et $a = 0$, $\mathcal{A}(\varphi) = cste$ redonne la théorie de Einstein-Maxwell minimalement couplée à un champ scalaire.

inclure au sein du formalisme EOB ; enfin, nous en étudierons le rayonnement gravitationnel et obtiendrons les formes d'ondes EOB associées (chapitre 5).

Là encore, nous mettrons en évidence le rôle crucial joué par la valeur φ_0 de leur environnement cosmologique, pouvant notamment amplifier les effets de leur charge "gravipho-tonique" Q sur leur dynamique, même lorsqu'elle est arbitrairement petite (mais toutefois non nulle).

La dynamique à deux corps des trous noirs (17) a été recemment étudiée par E. Hirschmann *et al.* [128] dans le cadre de la relativité numérique, en se limitant toutefois à une valeur unique pour φ_0 ($|\varphi_0| = 10^{-10}$). On trouvera aussi des approches analytiques restreintes au cas de deux trous noirs extrémaux, i.e. (17) avec $r_+ = r_-$, par N. Kan et K. Shiraishi [129–131], dont les résultats sont à prendre avec précaution puisque décrivant le mouvement de deux singularités nues. Nos résultats, valables pour tout φ_0 et toutes configurations (17) y seront confrontés et comparés.

0.4 Charges nœtheriennes et thermodynamique des trous noirs

Considérons un trou noir de Reissner-Nordström [93, 132], solution bien connue des équations du champ de la théorie Einstein-Maxwell dans le vide (i.e. (15) avec $a = 0$ et $S_\mathrm{m} = 0$) , dont la métrique et le champ vectoriel peuvent s'écrire :

$$ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{18a}$$

$$\text{et} \qquad A_t = \pm\frac{\sqrt{r_+ r_-}}{r} \quad , \quad A_i = 0 \quad , \tag{18b}$$

avec $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, et où A_t est défini à une constante près choisie de sorte qu'il s'annule à l'infini. La solution (18) décrit un trou noir chargé, statique et à symétrie sphérique, et dépend de deux paramètres indépendants : le rayon r_+ de l'horizon des événements, et la position r_- (avec $r_- < r_+$) d'un second horizon.

Introduisons quelques quantités définies sur son horizon externe r_+. Notons par exemple Φ la valeur qu'y prend le potentiel électrique,

$$\Phi = A_t(r_+) - A_t(r \to \infty) = \pm\sqrt{r_-/r_+}\,, \tag{19}$$

ainsi que κ la gravité de surface du trou noir, définie par $\kappa^2 = -\frac{1}{2}(\nabla_\mu\xi_\nu\nabla^\mu\xi^\nu)_{r_+}$, voir, e.g., [133], avec $\xi^\mu = (1,0,0,0)$ le vecteur de Killing associé à la stationnarité de la solution (18) :

$$\kappa = \frac{1}{2r_+}\left(1-\frac{r_-}{r_+}\right) . \tag{20}$$

Enfin, notons A_H l'aire du trou noir, i.e. de la 2-sphère de rayon $r = r_+$ (cf. (18a) avec $dt = 0$ et $dr = 0$), donnée par

$$A_H = 4\pi {r_+}^2 \,. \tag{21}$$

Sans plus de cérémonie, caractérisons maintenant le trou noir (18) par le comportement asymptotique de ses champs, soit $g_{\mu\nu} = \eta_{\mu\nu} + \left(\frac{2M}{r}\right)\delta_{\mu\nu} + \mathcal{O}(1/r^2)$ en coordonnées cartésiennes, et $A_t = \frac{Q}{r} + \mathcal{O}(1/r^2)$, avec :

$$M = \frac{r_+ + r_-}{2}\,, \tag{22a}$$

$$Q = \pm\sqrt{r_+ r_-}\,. \tag{22b}$$

C'est alors un exercice simple de vérifier que les variations de A_H, M, et Q par rapport aux deux constantes d'intégration r_+ et r_- sont reliées par l'identité suivante :

$$\frac{\kappa}{8\pi}\,\delta A_H = \delta M - \Phi\delta Q\,. \tag{23}$$

A priori, nous n'avons rien fait de plus ici que de donner une identité reliant cinq combinaisons distinctes de r_+ et r_-. En réalité, (23) témoigne de l'analogie profonde entre les propriétés des trous noirs et les lois de la thermodynamique, (23) en étant un exemple de "première loi".

Pour s'en convaincre, il faut d'abord montrer que M et Q sont les charges nœtheriennes (conservées) du trou noir ; nous y reviendrons plus bas. Mais, même en l'admettant pour le moment, il reste à justifier que le membre de gauche de (23) peut s'identifier au "$T\delta S$" de la première loi de la thermodynamique... [3]

0.4.1 La thermodynamique des trous noirs

Au début des années 1970, D. Christodoulou et R. Ruffini montrent que lorsqu'un trou noir chargé et en rotation (i.e. de Kerr-Newman) interagit avec un environnement de particules, son aire A_H (ou plutôt, dans leur langage, sa "masse irreductible" $M_{\rm irr} = \sqrt{A_H/16\pi}$) ne peut qu'augmenter, $\delta A_H \geq 0$, même lorsqu'on en extrait de l'énergie par "processus de Penrose" [134–136]. Ce résultat est ensuite généralisé par S. Hawking qui montre, à l'aide des équations d'Einstein, que pour tout processus impliquant l'interaction de trous noirs entre eux ou avec de la matière (satisfaisant la condition dite faible sur l'énergie, i.e. $T_{\mu\nu}u^\mu u^\nu > 0$ pour tout vecteur u^μ de genre temps), la somme de leurs aires doit toujours augmenter, $\sum \delta A_H \geq 0$ [137].

En 1972, J. Bekenstein voit là une similitude avec la "seconde loi" de la thermodynamique des systèmes fermés, $\delta S \geq 0$, et propose d'attribuer une entropie S aux trous noirs, proportionnelle à leur aire [138] :

$$S = \alpha\frac{A_H}{\hbar}\,, \tag{24}$$

où α est une constante sans dimension. Mais, ce qui convaincra de l'analogie entre trous noirs et thermodynamique est la mise en évidence, par S. Hawking en 1974, et par un raisonnement semi-classique, de leur rayonnement de "corps gris" de température T donnée par [139, 140] :

$$T = \frac{\kappa\hbar}{2\pi}\,. \tag{25}$$

On en déduit que $\alpha = 1/4$ dans (24), de sorte que notre relation (23), par exemple, se réécrit

$$T\delta S = \delta M - \Phi\delta Q\,, \tag{26}$$

et s'interprète aujourd'hui comme la première loi de la thermodynamique des trous noirs de Reissner-Nordström. Enfin, (25) montre que la température d'un trou noir est proportionnelle à sa gravité de surface qui, comme le remarque B. Carter, est homogène sur son horizon (même lorsqu'il est en rotation), conformément à la "loi zéro" [141]. Pour une présentation historique détaillée et étendue aux trous noirs en rotation, voir, e.g., [142, 143].

La thermodynamique des trous noirs est aujourd'hui un aspect important de l'étude des trous noirs "chevelus" en gravités modifiées : trous noirs à D dimensions, à "cheveux"

3. Dans la suite, nous posons $G_* = c = 1$ ainsi que $k_B = 1$, où k_B est la constante de Boltzmann, de sorte qu'une entropie S est sans dimension, et qu'une température T a la dimension d'une masse.

de jauge (autres que celui de Maxwell), mais aussi asymptotiquement anti-de Sitter (AdS), d'intérêt premier dans le contexte de la correspondance AdS-CFT [144, 145]... la question "redondante" étant de vérifier s'ils satisfont, eux aussi, une première loi de la thermodynamique généralisant (26).

Pour ce faire, il suffit d'en calculer la température, $T = \kappa\hbar/2\pi$, ainsi que l'entropie, $S = A_H/4\hbar$ (ou plus généralement, celle de R. Wald lorsque l'action décrivant la métrique n'est plus celle de Einstein-Hilbert [146]). En revanche, l'évaluation du membre de droite de (26) est plus délicate : en effet, elle soulève le problème de la définition des charges globales associées aux symétries (parfois uniquement asymptotiques) d'un espace-temps. On trouvera d'ailleurs un nombre conséquent de définitions de la masse M (et du moment cinétique J pour un trou noir en rotation) d'un espace-temps, la plus courante aujourd'hui étant probablement celle, hamiltonienne, de R. Arnowitt, S. Deser et C. Misner [71] ; voir, e.g. [147] pour une revue des différentes approches. Mais ces dernières ne sont pas toujours équivalentes, surtout dans le cas de trous noirs asymptotiquement (A)dS, qui nécessitent des procédés supplémentaires de régularisation, voir, e.g., [148].

Que viennent faire de telles considérations thermodynamiques au sein d'une thèse consacrée au problème à deux corps et aux ondes gravitationnelles ?

Nous allons voir que, si la thermodynamique permet de décrire la façon dont un trou noir se réajuste lorsqu'il interagit avec son environnement, elle en est encore capable lorsque ledit environnement est un compagnon en orbite autour de lui.

Dans la troisième partie de cette thèse, nous généraliserons donc la première loi de la thermodynamique (26) au trou noir "chevelu" de G. Gibbons et K.I. Maeda, introduit ci-dessus (cf. (17)). Mais, si la prise en compte d'un "graviphoton" est un exercice aisé, elle est plus subtile pour un champ scalaire qui, n'étant pas un champ de jauge, ne permet pas de définir de "charge scalaire" associée. Dans cette thèse, nous montrerons comment obtenir la première loi en définissant les charges globales du trou noir via l'approche dite des "superpotentiels de Katz", que nous présentons ci-dessous.

0.4.2 Les superpotentiels de Katz

Dans cette section, nous introduisons brièvement le formalisme dit des "superpotentiels de Katz", qui permet de définir les charges nœtheriennes d'un espace-temps dans un cadre lagrangien. Pour ce faire, nous nous basons sur les approches developpées dans [133, 149, 150].

Considérons l'action de Einstein-Hilbert de la gravitation :

$$I_{\rm EH} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\, \hat{R}\,, \tag{27}$$

où R est le scalaire de Ricci associé à $g_{\mu\nu}$, de métrique inverse $g^{\mu\nu}$, et où dorénavant, un chapeau dénote la multiplication par $\sqrt{-g}$, avec $g = \det g_{\mu\nu}$. Il est facile de montrer que sa variation s'écrit [4] :

$$16\pi\, \delta I_{\rm EH} = \int_{\mathcal{M}} d^4x\, \big(\hat{G}_{\mu\nu}\, \delta g^{\mu\nu} + \partial_\mu \hat{V}^\mu_{\rm EH}\big) \quad \text{avec} \quad \hat{V}^\mu_{\rm EH} = \hat{g}^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - \hat{g}^{\mu\alpha}\delta\Gamma^\beta_{\alpha\beta}\,, \tag{28}$$

où $G_{\mu\nu}$ est le tenseur d'Einstein, et où $\Gamma^\mu_{\nu\rho}$ désigne les symboles de Christoffel.

4. Pour cela on utilise $\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$, $\delta R_{\mu\nu} = \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha}$, et $\nabla_\mu \hat{V}^\mu = \partial_\mu \hat{V}^\mu$ si V^μ est un vecteur.

Ainsi, bien que I_{EH} contienne des dérivées secondes de la métrique, son extrémalisation, $\delta I_{\text{EH}} = 0$ $(\forall \delta g^{\mu\nu})$, mène à des équations du champ du second ordre uniquement, celles d'Einstein, $G_{\mu\nu} = 0$; à condition cependant d'ignorer la divergence $\partial_\mu \hat{V}^\mu_{\text{EH}}$ dans (28). En effet, le théorème de Gauss ne permet de l'annuler que si l'on fixe non seulement la métrique sur $\Sigma = \partial\mathcal{M}$, i.e. $\delta g^{\mu\nu}|_\Sigma = 0$ "à la Dirichlet", mais aussi ses dérivées, contenues dans les symboles de Christoffel, $\delta(\partial_\alpha g^{\mu\nu})|_\Sigma = 0$: le principe variationnel issu de l'action de Einstein-Hilbert est donc mal posé.

Afin d'y remédier, on peut remarquer que la modification de l'action de Einstein-Hilbert selon

$$I_{\text{E}} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\left(\hat{R} + \partial_\mu \hat{v}^\mu\right) \quad \text{avec} \quad \hat{v}^\mu = -\left(\hat{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \hat{g}^{\mu\alpha}\Gamma^\beta_{\alpha\beta}\right) , \tag{29}$$

permet d'éliminer toutes les dérivées secondes de la métrique présentes dans $\hat{R}$, pour obtenir (en utilisant $\partial_\alpha g^{\mu\nu} = -\Gamma^\mu_{\alpha\beta}g^{\beta\nu} - \Gamma^\nu_{\alpha\beta}g^{\beta\mu}$) :

$$I_{\text{E}} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\hat{g}^{\mu\nu}\left(\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}\right) . \tag{30}$$

Cette seconde action, dite d'Einstein [151], conduit aux mêmes équations du champ que I_{EH} puisque leurs lagrangiens ne diffèrent que par la dérivée totale $\partial_\mu \hat{v}^\mu$, voir (29). De plus, elle est maintenant compatible avec des conditions au bord de Dirichlet, la variation $\delta\hat{v}^\mu$ éliminant les variations des symboles de Christoffel présents dans $\hat{V}^\mu_{\text{EH}}$, cf. (28) :

$$16\pi\,\delta I_{\text{E}} = \int_{\mathcal{M}} d^4x\,\left(\hat{G}_{\mu\nu}\,\delta g^{\mu\nu} + \partial_\mu \hat{V}^\mu_{\text{E}}\right) \quad \text{avec} \quad \hat{V}^\mu_{\text{E}} = -\left(\delta\hat{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \delta\hat{g}^{\mu\alpha}\Gamma^\beta_{\alpha\beta}\right) , \tag{31}$$

de sorte que seule la métrique doit être fixée sur le bord, $\delta g^{\mu\nu}|_\Sigma = 0$.

Cependant, l'action d'Einstein (30) n'est pas covariante. La raison en est que pour l'obtenir, nous avons ajouté à $\hat{R}$ la divergence de la quantité v^μ qui, puisqu'elle dépend des symboles de Christoffel, cf. (29), n'est pas un vecteur.

Afin de bâtir une action menant aux équations d'Einstein, compatible avec les conditions au bord de Dirichlet, tout en préservant la covariance de l'action de Einstein-Hilbert, une stratégie possible est de "covariantiser" v^μ dans (29), en utilisant le fait que des différences de symboles de Christoffel sont, elles, des tenseurs. Plus précisément, introduisons une métrique $\bar{g}_{\mu\nu}$ non dynamique, dite "de référence", pour construire l'action de Einstein-Katz :

$$\begin{gathered} I_{\text{EK}} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\left(\hat{R} - \overline{\hat{R}} + \partial_\mu \hat{k}^\mu\right) \\ \text{où} \quad k^\mu = -\left(g^{\alpha\beta}\Delta^\mu_{\alpha\beta} - g^{\mu\alpha}\Delta^\beta_{\alpha\beta}\right) \quad \text{avec} \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \overline{\Gamma}^\mu_{\nu\rho} , \end{gathered} \tag{32}$$

une barre dénotant une quantité évaluée avec $\bar{g}_{\mu\nu}$, comme dans $\overline{\hat{R}} = \sqrt{-\bar{g}}\overline{R}$, et où k^μ est le "vecteur de Katz", introduit par J. Katz [149], puis J. Katz, J. Bičak, et D. Lynden-Bell [150].

À présent, la variation de (32) par rapport à $g_{\mu\nu}$ s'écrit :

$$16\pi\,\delta_{(g)} I_{\text{EK}} = \int_{\mathcal{M}} d^4x\,\left(\hat{G}_{\mu\nu}\,\delta g^{\mu\nu} + \partial_\mu \hat{V}^\mu_{\text{EK}}\right) \quad \text{avec} \quad \hat{V}^\mu_{\text{EK}} = -\left(\delta\hat{g}^{\alpha\beta}\Delta^\mu_{\alpha\beta} - \delta\hat{g}^{\mu\alpha}\Delta^\beta_{\alpha\beta}\right) , \tag{33}$$

si bien qu'il suffit de fixer la métrique au bord, $\delta g^{\mu\nu}|_\Sigma = 0$, pour que l'extrémalisation de

(33) conduise aux équations d'Einstein. De plus, k^μ étant un vecteur (contrairement à v^μ), l'action $I_{\rm EK}$ construite à partir de sa divergence est covariante. Il est d'ailleurs facile de la réécrire selon [5] :

$$I_{\rm EK} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x \left[\hat{g}^{\mu\nu}\left(\Delta^\alpha_{\mu\beta}\Delta^\beta_{\nu\alpha} - \Delta^\alpha_{\mu\nu}\Delta^\beta_{\alpha\beta}\right) + (\hat{g}^{\mu\nu} - \bar{\hat{g}}^{\mu\nu})\bar{R}_{\mu\nu}\right] , \tag{34}$$

de sorte qu'elle est une généralisation covariante de l'action d'Einstein, cf. (30). [6]

L'invariance sous les difféomorphismes, i.e. $x^\mu \to x^\mu + \xi^\mu$, de l'action de Katz-Bičak-Lynden-Bell est cruciale, car elle permet de définir, dans le cadre lagrangien, des charges nœtheriennes associées. En effet, un calcul techniquement simple mais quelque peu fastidieux de la dérivée de Lie de (32) selon ξ^μ, détaillé, e.g., dans [133], permet de déduire de sa covariance l'existence d'un courant conservé $\hat{j}^\mu$, dérivant lui-même du "superpotentiel de Katz" $\hat{J}^{[\mu\nu]}$:

$$\partial_\mu \hat{j}^\mu = 0 \quad \text{avec} \quad \hat{j}^\mu = \partial_\nu \hat{J}^{[\mu\nu]} \quad \text{et} \quad 8\pi\, J^{[\mu\nu]} = \nabla^{[\mu}\hat{\xi}^{\nu]} - \overline{\nabla^{[\mu}\hat{\xi}^{\nu]}} + \xi^{[\mu}\hat{k}^{\nu]} , \tag{35}$$

où les crochets désignent l'antisymétrisation, $J^{[\mu\nu]} = (J^{\mu\nu} - J^{\nu\mu})/2$.

Le premier terme de $\hat{J}^{[\mu\nu]}$ s'identifie au superpotentiel de Komar [154], tandis que le second, qui dépend de la métrique de référence, joue un rôle régularisateur, comme nous le verrons dans cette thèse, notamment au chapitre 6. Enfin, le troisième terme dépend du vecteur de Katz k^μ, et sa contribution corrige notamment un facteur 2 indésirable apparaissant dans la définition à la Komar du moment angulaire des trous noirs de Kerr [149].

Ainsi, par l'intégration de $\partial_\mu \hat{j}^\mu = 0$ sur le 4-volume $\mathcal{M}$ et l'emploi du théorème de Gauss-Stokes (voir, e.g., [155]), la charge nœtherienne conservée, définie "à la Katz", est une intégrale sur la 2-sphère à l'infini (spatial) $\partial\Sigma$, soit en coordonnées adaptées :

$$Q = -\lim_{r\to\infty}\int d\theta d\phi \hat{J}^{[0r]} . \tag{36}$$

5. Pour ce faire, considérons l'identité $R^\alpha{}_{\mu\beta\nu} - \bar{R}^\alpha{}_{\mu\beta\nu} = \bar{\nabla}_\beta\Delta^\alpha_{\mu\nu} - \bar{\nabla}_\nu\Delta^\alpha_{\mu\beta} + \Delta^\alpha_{\beta\lambda}\Delta^\lambda_{\mu\nu} - \Delta^\alpha_{\nu\lambda}\Delta^\lambda_{\mu\beta}$, qui se prouve aisément puisqu'elle est de nature tensorielle et qu'elle est vraie dans le référentiel localement inertiel associé à $\bar{g}_{\mu\nu}$, i.e. tel que $\bar{\Gamma}^\mu_{\nu\rho} = 0$ (mais $\partial_\alpha\bar{\Gamma}^\mu_{\nu\rho} \neq 0$). Prenons-en la trace :

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\alpha\Delta^\alpha_{\mu\nu} - \bar{\nabla}_\nu\Delta^\alpha_{\mu\alpha} + \Delta^\alpha_{\alpha\lambda}\Delta^\lambda_{\mu\nu} - \Delta^\alpha_{\nu\lambda}\Delta^\lambda_{\mu\alpha} .$$

Il suffit alors de réécrire les deuxième et troisième termes du membre de droite de l'égalité ci-dessus pour éliminer les dérivées covariantes $\bar{\nabla}_\mu$ au profit de ∇_μ selon $\bar{\nabla}_\alpha\Delta^\mu_{\nu\rho} = \nabla_\alpha\Delta^\mu_{\nu\rho} - \Delta^\mu_{\alpha\beta}\Delta^\beta_{\nu\rho} + \Delta^\beta_{\alpha\nu}\Delta^\mu_{\beta\rho} + \Delta^\beta_{\alpha\rho}\Delta^\mu_{\nu\beta}$, puis de contracter l'égalité obtenue avec $g^{\mu\nu}$ pour obtenir

$$R = g^{\mu\nu}\overline{R}_{\mu\nu} + \nabla_\mu\left(g^{\alpha\beta}\Delta^\mu_{\alpha\beta} - g^{\mu\alpha}\Delta^\beta_{\alpha\beta}\right) + g^{\mu\nu}\left(\Delta^\alpha_{\mu\beta}\Delta^\beta_{\nu\alpha} - \Delta^\alpha_{\mu\nu}\Delta^\beta_{\alpha\beta}\right) .$$

On obtient finalement (34) en remplaçant R dans (32) par son expression ci-dessus.

6. Notons aussi qu'en coordonnées adaptées au bord Σ, telles que $ds^2 = \epsilon d\omega^2 + h_{ij}(x,\omega)dx^i dx^j$ et $d\bar{s}^2 = \epsilon d\omega^2 + \bar{h}_{ij}(x,\omega)dx^i dx^j$, avec $\epsilon = \pm 1$ selon le genre de la 3-surface Σ d'équation $\omega = cste$, on a $\int d^4x\, \partial_\mu\hat{k}^\mu = \int d^3x\sqrt{|h|}k^\omega$ avec $h = \det h_{ij}$ et

$$k^\omega = 2\epsilon[K - \frac{1}{2}\bar{K}_{ij}(\bar{h}^{ij} + h^{ij})] ,$$

où $K_{ij} = \frac{1}{2}\partial_\omega h_{ij}$ et $\bar{K}_{ij} = \frac{1}{2}\partial_\omega\bar{h}_{ij}$ sont les courbures extrinsèques, avec $K = h^{ij}K_{ij}$. Ainsi, à des termes près qui joueront un rôle régularisateur important dans cette thèse, le terme de bord de Katz est identique à celui de Gibbons-Hawking-York [152, 153], $2\epsilon\int d^3x\sqrt{|h|}K$, qui permet lui aussi de garantir qu'il suffise de fixer la métrique induite sur Σ à la Dirichlet, $\delta h_{ij}\,|_\Sigma = 0$, et non ses dérivées normales, $\partial_\omega h_{ij}$.

Enfin, si $\xi^\mu = (1,0,0,0)$ est un vecteur de Killing (même uniquement asymptotique) de l'espace-temps considéré, (36) s'interprète comme la charge associée à sa stationnarité, c.-à-d., sa masse M.

L'approche des superpotentiels de Katz a été appliquée à de nombreux exemples de trous noirs, notamment asymptotiquement AdS, à D dimensions, ou bien en rotation par N. Deruelle et J. Katz [156], mais aussi en théorie Gauss-Bonnet par N. Deruelle, Y. Morisawa et S. Ogushi [157, 158], et a plus récemment été généralisée aux théories de Lovelock par N. Deruelle, N. Merino et R. Olea [159, 160].

Dans la troisième partie de cette thèse, nous montrerons comment la généraliser pour inclure la contribution d'un champ scalaire dans la définition de la masse d'un espace-temps. Pour cela, nous considérerons dans un premier temps un exemple de trou noir asymptotiquement AdS avec "cheveux" scalaires (chapitre 6), avant de calculer la masse des trous noirs de G. Gibbons et K.I. Maeda (17). Ceci permettra d'en écrire la première loi de la thermodynamique (chapitre 7), et ainsi, lorsqu'il est membre d'un système binaire, d'interpréter ses réajustements éventuels sous l'influence de son compagnon.

0.5 Plan de la thèse

La présente thèse a mené à la publication de cinq articles et d'un chapitre de livre, ainsi qu'à un article récent ; soit, par ordre chronologique :

[1] Andrés Anabalón, Nathalie Deruelle, Félix-Louis Julié, "Einstein-Katz action, variational principle, Noether charges and the thermodynamics of AdS-black holes," *JHEP, 049, 08, 2016* [arXiv:1606.05849].

[2] Félix-Louis Julié, Nathalie Deruelle, "Two-body problem in scalar-tensor theories as a deformation of general relativity : an effective-one-body approach," *Phys.Rev. D95, 12, 124054, 2017* [arXiv:1703.05360].

[3] Félix-Louis Julié, "Reducing the two-body problem in scalar-tensor theories to the motion of a test particle : a scalar-tensor effective-one-body approach," *Phys.Rev. D97, 2, 024047, 2018* [arXiv:1709.09742].

[4] Félix-Louis Julié, "On the motion of hairy black holes in Einstein-Maxwell-dilaton theories," *JCAP 1801, 01, 026, 2018* [arXiv:1711.10769].

[5] Marcela Cárdenas, Félix-Louis Julié, Nathalie Deruelle, "Thermodynamics sheds light on black hole dynamics," *Phys.Rev. D97, 12, 124021, 2018* [arXiv:1712.02672].

[6] Nathalie Deruelle, Félix-Louis Julié, "The two-body problem : an effective-one-body approach," in "Relativity in Modern Physics," de Nathalie Deruelle et Jean-Philippe Uzan, *Oxford University Press.*

[7] Félix-Louis Julié, "Gravitational radiation from compact binary systems in Einstein-Maxwell-dilaton theories" [arXiv:1809.05041].

Le manuscrit est divisé en trois parties distinctes. La partie I porte sur le formalisme EOB et son extension aux gravités modifiées ; la partie II, sur le problème à deux corps et le rayonnement gravitationnel en théories Einstein-Maxwell-dilaton ; enfin, la partie III est consacrée à la thermodynamique des trous noirs "chevelus".

Dans le chapitre 1, nous présentons le formalisme EOB avec, en guise de "galop préliminaire", l'exemple de la relativité générale à l'ordre 2.5PN, maintenant suffisamment standard pour être inclus dans des livres de cours [6]. On y montre comment réduire la dynamique d'un système binaire au mouvement d'une particule test dans une métrique effective, pour en déduire la forme d'onde gravitationnelle émise jusqu'à la coalescence du système. On propose ensuite d'étendre le formalisme EOB aux théories scalaire-tenseur à l'ordre 2PK, de deux façons distinctes. La première consiste à réduire la dynamique à deux corps au mouvement géodésique dans une métrique qui est une simple déformation de celle de la relativité générale (chapitre 2 et [2]) ; la seconde, davantage "centrée" sur les spécificités des théories scalaire-tenseur, consiste à réduire la dynamique à deux corps au mouvement d'une particule dans une métrique et un champ scalaire effectifs (chapitre 3 et [3]).

Le chapitre 4 (et [4]) est consacré à la "skeletonisation" des corps compacts en théories Einstein-Maxwell-dilaton, et plus particulièrement, à celle des trous noirs "chevelus" de G. Gibbons et K.I. Maeda, introduits ci-dessus (cf. (17)). La "skeletonisation" des corps compacts est une étape essentielle, puisqu'elle permet ensuite de calculer le lagrangien à deux corps correspondant (à l'ordre post-keplerien), puis d'en inclure la dynamique au sein du formalisme EOB. L'incorporation de la force de réaction au rayonnement ainsi que l'obtention les formes d'ondes gravitationnelles émises par de tels systèmes binaires "chevelus" font, elles, l'objet du chapitre 5 (et [7]).

Le chapitre 6 (et [1]) porte sur le formalisme de Katz, et sa généralisation pour inclure la contribution d'un champ scalaire dans la définition des charges globales d'un trou noir. Pour cela, nous considérons un exemple de trou noir "chevelu" et asymptotiquement AdS, et montrons que sa masse, alors définie "à la Katz", est celle qui intervient dans la première loi de la thermodynamique d'un tel trou noir. La "technologie" ainsi obtenue est ensuite aisément appliquée à nos trous noirs EMD au chapitre 7 (et dans [5]), pour en obtenir les charges globales ainsi que la première loi de la thermodynamique. Nous montrons que cette dernière permet non seulement d'interpréter les réajustements d'un trou noir sous l'influence d'un compagnon lointain, mais aussi, de discuter du domaine de validité de sa "skeletonisation".

Première partie

L'approche effective à un corps et sa généralisation aux gravités modifiées

Chapitre 1

L'approche effective à un corps : l'exemple de la relativité générale à l'ordre 2.5PN

Comme nous l'avons mentionné dans l'introduction, le formalisme post-newtonien n'est pas conçu pour décrire la dynamique d'un système binaire (ainsi que son rayonnement gravitationnel) dans le régime de champ fort, où il devient délicat d'en extraire des prédictions. Pour décrire un système binaire proche de sa coalescence, il faut recourir, soit à la relativité numérique, soit à des méthodes analytiques telles que l'approche "Effective-One-Body" (EOB), que nous décrivons dans ce chapitre.

En théorie de la gravitation de Newton, il est bien connu que la dynamique de deux corps en interaction, de masses m_A et m_B, s'identifie à celle d'une particule fictive, de masse réduite $\mu = m_A m_B/(m_A + m_B)$, orbitant autour de la masse totale du système, $M = m_A + m_B$. L'approche EOB, proposée par A. Buonanno et T. Damour [68, 74], en est une généralisation relativiste. Elle consiste à réduire le problème à deux corps de la relativité générale à celui du mouvement d'une particule test, de masse μ, plongée dans une métrique effective.

Pour ce faire, le formalisme EOB s'appuie sur la puissance des transformations canoniques pour définir un Hamiltonien H_{EOB}, fonctionnelle d'un Hamiltonien H_e décrivant une particule test plongée dans une métrique effective simple, qui est une resommation du Hamiltonien à deux corps à un ordre PN donné. Les équations du mouvement découlant de H_{EOB} définissent alors une resommation de la dynamique conservative à deux corps, et permettent d'explorer le régime de champ fort. L'adjonction de la force de réaction de rayonnement (dont l'expression est éventuellement elle-même resommée à l'aide d'approximants de Padé) aux équations du mouvement permet de décrire l'évolution complète du système, jusqu'à sa coalescence. Dans le cas d'un système binaire de trous noirs, on peut alors prédire la masse ainsi que le moment cinétique du trou noir final, et ainsi raccorder la forme d'onde gravitationnelle à celle de ses modes quasi-normaux, connus pas ailleurs en théorie de perturbation des trous noirs, voir, e.g., [161] et [67].

La figure (1.1) montre la première forme d'onde gravitationnelle complète, obtenue en 2OOO dans [74], et ce, de façon purement analytique. La forme d'onde inclut la transition de la phase "spiralante" vers le "ring-down", inconnue jusqu'alors (voir figure 1), un régime qui devint accessible à la relativité numérique en 2005 [162].

Nous reproduisons ci-dessous le chapitre intitulé "The two-body problem : an effective-one-body approach", rédigé en collaboration avec Nathalie Deruelle, à paraître dans le livre "Relativity in Modern Physics" de Nathalie Deruelle et Jean-Philippe Uzan [163], et publié par *Oxford University Press*. Ce chapitre, basé sur les travaux originaux de A. Buonanno et T. Damour [68, 74], introduit de façon concise le formalisme EOB, en traitant en détail l'exemple du problème à deux corps de la relativité générale à l'ordre 2.5PN.

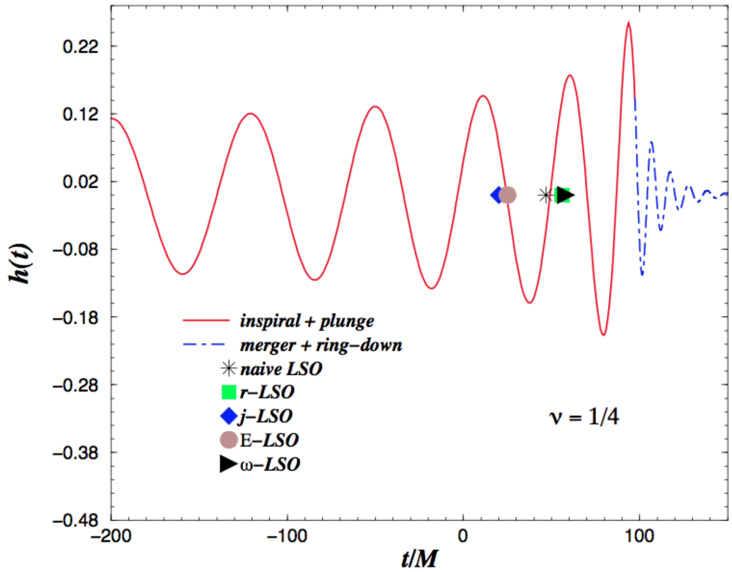

FIGURE 1.1 – Forme d'onde gravitationnelle prédite pour un système binaire de trous noirs de masse égale ($\nu = \mu/M = 1/4$) extraite de [74]. L'approche EOB permet de prolonger l'évolution du système jusqu'à sa coalescence, où la forme d'onde est raccordée aux modes quasi-normaux du trou noir final.

On y montre d'abord comment l'information contenue dans le Lagrangien à deux corps 2PN [164],

$$\begin{aligned}
L &= -(m_A + m_B) + \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2 + \frac{m_A m_B}{R} + L_{1PN} + L_{2PN} + \mathcal{O}(V^8) \\
L_{1PN} &= \frac{1}{8} m_A V_A^4 + \frac{1}{8} m_B V_B^4 + \frac{m_A m_B}{2R}\left[3(V_A^2 + V_B^2) - 7(V_A.V_B) - (N.V_A)(N.V_B)\right] - \frac{m_A m_B (m_A + m_B)}{2R^2} \\
L_{2PN} &= \frac{1}{16} m_A V_A^6 + \frac{m_A m_B}{R}\left[\frac{7}{8} V_A^4 + \frac{15}{16} V_A^2 V_B^2 - 2V_A^2 (V_A.V_B) + \frac{1}{8}(V_A.V_B)^2 - \frac{7}{8}(N.V_A)^2 V_B^2 \right. \\
&\qquad \left. + \frac{3}{4}(N.V_A)(N.V_B)(V_A.V_B) + \frac{3}{16}(N.V_A)^2 (N.V_B)^2\right] \\
&\quad + \frac{m_A^2 m_B}{R^2}\left[\frac{1}{4} V_A^2 + \frac{7}{4} V_B^2 - \frac{7}{4}(V_A.V_B) + \frac{7}{2}(N.V_A)^2 + \frac{1}{2}(N.V_B)^2 - \frac{7}{2}(N.V_A)(N.V_B)\right] \\
&\quad + m_A m_B \left[(N.A_A)\left(\frac{7}{8} V_A^2 - \frac{1}{8}(N.V_B)^2\right) - \frac{7}{4}(V_B.A_A)(N.V_B)\right] \\
&\quad + \frac{m_A^2 m_B}{R^3}\left[\frac{1}{2} m_A + \frac{19}{8} m_B\right] + (A \leftrightarrow B)
\end{aligned} \tag{1.1}$$

(où $R = |\vec{Z}_A - \vec{Z}_B|$, $\vec{N} = (\vec{Z}_A - \vec{Z}_B)/R$, et $\vec{V}_A = d\vec{Z}_A/dt$, $\vec{A}_A = d\vec{V}_A/dt$, $\vec{Z}_A$ étant la position du corps A dans un système de coordonnées harmoniques) peut être condensée, après transformations canoniques et dans le référentiel du centre de masse, sous la forme d'un Hamiltonien EOB,

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}, \quad \text{où} \quad H_e = \mu\sqrt{A\left(1 + \frac{p_r^2}{\mu^2 B} + \frac{p_\phi^2}{\mu^2 r^2}\right)}, \tag{1.2}$$

H_e étant le Hamiltonien décrivant le mouvement géodésique d'une particule test de masse μ dans la métrique effective, statique et à symétrie sphérique suivante :

$$ds_e^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\phi^2 \,, \tag{1.3a}$$

$$\text{avec} \quad A(r) = 1 - 2\left(\frac{M}{r}\right) + 2\nu\left(\frac{M}{r}\right)^3 \,, \quad B(r) = \frac{1}{A}\left[1 - 6\nu\left(\frac{M}{r}\right)^2\right] \,, \tag{1.3b}$$

$\nu = m_A m_B/(m_A + m_B)^2$ étant compris entre 0 (dans la limite $m_A \ll m_B$ ou $m_A \gg m_B$) et 1/4 (lorsque $m_A = m_B$). $H_{\rm EOB}$ définit ainsi une resommation du mouvement à deux corps se réduisant aux géodésiques de la métrique de Schwarzschild exacte dans la limite $\nu = 0$.

L'évolution du système est alors donnée par les équations de Hamilton, supplémentées de la force de freinage "quadrupolaire" $\mathcal{F}_\phi$, dont nous reparlerons au chapitre 5. Ainsi, à l'approximation des orbites quasi-circulaires (on définit $\dot{a} = da/dt$) :

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r} \quad , \quad \dot{\phi} = \frac{\partial H_{\rm EOB}}{\partial p_\phi} \quad , \quad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r} \quad , \quad \dot{p}_\phi = \mathcal{F}_\phi \tag{1.4a}$$

$$\text{avec} \quad \mathcal{F}_\phi = -\frac{32}{5} M \nu^2 (M\dot{\phi})^{7/3} \,. \tag{1.4b}$$

La forme d'onde gravitationnelle associée est donnée, en première approximation, par la première formule du quadrupôle d'Einstein (en jauge transverse et sans trace), pour des orbites quasi-circulaires :

$$h = \mathcal{C}(M\dot{\phi})^{2/3} \cos(2\phi) \,, \tag{1.5}$$

où $\mathcal{C}$ est un facteur sans dimension dépendant de la position du point d'observation par rapport au plan de l'orbite.

Enfin, lorsque la particule effective atteint "l'anneau de lumière", i.e. le rayon de l'orbite circulaire de genre lumière la plus externe de la métrique (1.3), l'onde (1.5) est raccordée à la forme d'onde associée aux modes quasi-normaux du trou noir final, dont la masse et le spin sont identifiés respectivement aux valeurs du Hamiltonien $H_{\rm EOB}$ et du moment cinétique total p_ϕ à cet instant.

Le formalisme EOB joue un rôle essentiel dans l'analyse des données des détecteurs LIGO et Virgo en permettant de générer rapidement plusieurs centaines de milliers de "patrons" d'ondes gravitationnelles à confronter aux observations et a, à cette fin, atteint un niveau d'élaboration bien plus élevé que celui présenté dans ce chapitre.

Dès l'ordre 3PN en effet, la métrique effective est resommée à l'aide d'approximants de Padé qui garantissent l'existence d'un horizon effectif, par continuité avec la limite de Schwarzschild (retrouvée lorsque $\nu = 0$) ; le Hamiltonien H_e doit par ailleurs être supplémenté de nouveaux termes non géodésiques [165]. L'ordre 4PN, connu analytiquement, est intégré dans [166], tandis que l'ordre 5PN peut être incorporé par calibration de nouveaux paramètres effectifs obtenus par comparaison avec les résultats de la relativité numérique [167]. La force de freinage $\mathcal{F}_\phi$ est obtenue par resommation des flux connus à l'ordre 3.5PN (au-delà de la formule du quadrupole), et même à des ordres plus élevés dans la limite $\nu = 0$ [168].

L'inclusion des trous noirs en rotation est abordée dans [169, 170] et s'inspire du mouvement géodésique dans une métrique de Kerr ; les étoiles à neutrons peuvent être décrites en incluant les effects de taille finie (e.g., de marée) dans la métrique effective, à l'ordre 5PN [171]. Enfin, l'approche EOB est actuellement étendue aux systèmes binaires non liés, à l'aide du formalisme post-Minkowskien (valide en champ faible, mais pour toutes vitesses) [172]. On trouvera une revue récente de l'état de l'art dans, e.g., [173] et [75].

Book III - Chapter 16
The Two-Body problem : an effective-one-body approach

(This chapter was written in collaboration with Félix-Louis Julié.)

In this chapter we present the basics of the "effective-one-body" approach to the two-body problem in general relativity and show that the 2PN equations of motion can be mapped, by means of an appropriate canonical transformation, to a geodesic motion in a static, spherically symmetric, spacetime, thus considerably simplifying the dynamics. Including then the 2.5PN radiation reaction force in the (resummed) equations of motion we give the waveform (that is, the time dependence or the radiative field) during the inspiral, merger and "ring-down" phases of the coalescence of two non-spinning black holes into a final, Kerr, one. We will also comment on the present developments of this approach, which is instrumental to build the libraries of waveform templates that are needed to analyze the data collected by the present gravitational wave detectors.

SECTION 69. The 2PN Hamiltonian

The two-body Lagrangian which describes the conservative part of the dynamics at 2PN order has been derived in the previous chapter in harmonic coordinates, see eq. (68.8). We shall denote here the positions, velocities and accelerations by capital letters, $Z^i(t) - Z'^i(t) = RN^i$ with $(N.N) = 1$, $V^i = \frac{dZ^i}{dt}$, $A^i = \frac{dV^i}{dt}$, and we recall its expression :

$$
\begin{aligned}
L &= -(m+m') + \frac{1}{2}mV^2 + \frac{1}{2}m'V'^2 + \frac{mm'}{R} + L_2 + L_4 + \mathcal{O}(V^8)\\
L_2 &= \frac{1}{8}mV^4 + \frac{1}{8}m'V'^4 + \frac{mm'}{2R}\Big[3(V^2+V'^2) - 7(V.V') - (N.V)(N.V')\Big] - \frac{mm'(m+m')}{2R^2}\\
L_4 &= \frac{1}{16}mV^6\\
&\quad + \frac{mm'}{R}\Big[\frac{7}{8}V^4 + \frac{15}{16}V^2V'^2 - 2V^2(V.V') + \frac{1}{8}(V.V')^2 - \frac{7}{8}(N.V)^2V'^2\\
&\qquad\qquad + \frac{3}{4}(N.V)(N.V')(V.V') + \frac{3}{16}(N.V)^2(N.V')^2\Big]\\
&\quad + \frac{m^2m'}{R^2}\left[\frac{1}{4}V^2 + \frac{7}{4}V'^2 - \frac{7}{4}(V.V') + \frac{7}{2}(N.V)^2 + \frac{1}{2}(N.V')^2 - \frac{7}{2}(N.V)(N.V')\right]\\
&\quad + mm'\Big[(N.A)\Big(\frac{7}{8}V^2 - \frac{1}{8}(N.V')^2\Big) - \frac{7}{4}(V'.A)(N.V')\Big]\\
&\quad + \frac{m^2m'}{R^3}\Big[\frac{1}{2}m + \frac{19}{8}m'\Big] + (m \leftrightarrow m')\,.
\end{aligned}
\tag{0.1}
$$

The Fokker Lagrangian

The two-body Lagrangian (0.1) can be derived in various ways. In section 55 we built it from a symmetrized free particle action (a la Fichtenholz) and in section 68 we inferred it from the equations of motion obtained beforehand. A third option, initiated by A. Fokker in 1929 and applied to general relativity by L. Infeld and J. Plebanski in 1960 (that we used in Book 2, section 81 to obtain the Darwin Lagrangian for electromagnetism) consists in starting from the total action describing gravitational and matter fields :

$$S[g_{\mu\nu}, \Psi] = S_g[g_{\mu\nu}] + S_m[g_{\mu\nu}, \Psi]\,.$$

Here the Einstein-Hilbert action for gravity $S_g[g_{\mu\nu}]$ is written "a la Einstein", that is up to a boundary term, see eq. (25.2), and includes a "gauge fixing term" when working in harmonic coordinates :

$$S_g[g_{\mu\nu}] = \frac{1}{16\pi}\int\sqrt{-g}\left[g^{\mu\nu}\left(\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}\right) - \frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\right] \quad \text{where} \quad \Gamma^\mu \equiv g^{\nu\rho}\Gamma^\mu_{\nu\rho}\,.$$

As for the matter action it reads, for point particles :

$$S_m[g_{\mu\nu}, \Psi] = -\sum m\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu}\,.$$

The "Fokker" action is

$$S_F[\Psi] \equiv S_g[\bar{g}_{\mu\nu}[\Psi]] + S_m[\bar{g}_{\mu\nu}[\Psi], \Psi]$$

where $\bar{g}_{\mu\nu}[\Psi]$ is a solution of Einstein's equations $\delta S/\delta g^{\mu\nu} = 0$ to the desired post-Newtonian order. In the case of point particles, $S_F = \int L_F[Z(t), Z'(t)]\, dt$ –where L_F is the Fokker Lagrangian– depends on the trajectories of the bodies only. The extremization of $S_F[\Psi]$ with respect to the matter fields Ψ yields, at least formally, the matter equations of motion $\delta S/\delta\Psi = 0$ since

$$\frac{\delta S_F[\Psi]}{\delta\Psi} = \frac{\delta\bar{g}[\Psi]}{\delta\Psi}\left(\frac{\delta S}{\delta g}\right)_{g=\bar{g}} + \left(\frac{\delta S}{\delta\Psi}\right)_{g=\bar{g}} ,$$

where the first term vanishes when $\bar{g}_{\mu\nu}[\Psi]$ is a solution of the field equations $\delta S/\delta g = 0$. After proper regularization, this method returns (0.1).[1]

The Lagrangian (0.1) depends on positions, velocities and (linearly) on accelerations. Let us add to it a 2PN total time derivative,

$$L \to L_{\rm f} = L + \frac{df}{dt} , \tag{0.2}$$

where f is the generic function,

$$\begin{aligned} \frac{f}{mm'} &= (f_1V^2 + f_2V.V' + f_3V'^2)(N.V) - (f_4V^2 + f_5V.V' + f_6V'^2)(N.V') \\ &+ f_7(N.V)^3 + f_8(N.V)^2(N.V') - f_9(N.V')^2(N.V) - f_{10}(N.V')^3 \\ &+ f_{11}\left(\frac{m}{R}\right)(N.V) + f_{12}\left(\frac{m'}{R}\right)(N.V) - f_{13}\left(\frac{m}{R}\right)(N.V') - f_{14}\left(\frac{m'}{R}\right)(N.V') , \end{aligned} \tag{0.3}$$

that depends on fourteen parameters f_i. This total derivative generates a 2PN boundary term in the action that does not affect the equations of motion which remain unchanged.

In order to reduce $L_{\rm f}$ to an "ordinary" Lagrangian depending only on positions and velocities, one may "naively" replace the accelerations by their leading order, that is Newtonian, on-shell expressions :[2]

$$L_{\rm f} \to L_{\rm f}^{\rm red} = L_{\rm f}\left(A^i \to -\frac{m'}{R^2}N^i\ ,\ A'^i \to \frac{m}{R^2}N^i\right) . \tag{0.4}$$

The equations of motion derived from $L_f^{\rm red}$ do differ from the ones derived from L_f (i.e. from L) at 2PN level. However the *dynamics* is unchanged. The reason is that the reduction (0.4) amounts in fact to implicitely switching from the harmonic coordinates to new ones, defined, as can be checked, by the 2PN-level, f-dependent, contact transformation $Y^i = Z^i + \delta Z^i$, $Y'^i = Z'^i + \delta Z'^i$:

$$\begin{aligned} \delta Z^i &= \frac{m'}{8}\left[14V'^i(N.V') - N^i\left(7V'^2 - (N.V')^2\right)\right] \\ &- m'\left[2V^i\big(f_1(N.V) - f_4(N.V')\big) + V'^i\big(f_2(N.V) - f_5(N.V')\big)\right] - \frac{m'N^i}{R}(mf_{11} + m'f_{12}) \\ &- m'N^i\left[f_1V^2 + f_2(V.V') + f_3V'^2 + 3f_7(N.V)^2 + 2f_8(N.V)(N.V') - f_9(N.V')^2\right] , \\ \delta Z'^i &= \frac{m}{8}\left[-14V^i(N.V) + N^i\left(7V^2 - (N.V)^2\right)\right] \\ &- m\left[V^i\big(f_2(N.V) - f_5(N.V')\big) + 2V'^i\big(f_3(N.V) - f_6(N.V')\big)\right] + \frac{mN^i}{R}(mf_{13} + m'f_{14}) \\ &+ mN^i\left[f_4V^2 + f_5(V.V') + f_6V'^2 - f_8(N.V)^2 + 2f_9(N.V)(N.V') + 3f_{10}(N.V')^2\right] . \end{aligned}$$

We thus now have on hand a whole class of ordinary Lagrangians $L_{\rm f}^{\rm red}$ (depending on the 14 parameters f_i), each one corresponding to a specific choice of a coordinate system ; the harmonic-coordinate, acceleration-dependent

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[1] For a detailed presentation of the Fokker action, up to 4PN, see L. Bernard, L. Blanchet et *al.*, *Fokker action of non-spinning compact binaries at the fourth post-Newtonian approximation,* Phys. Rev. D 93, 084037 (2016) or arXiv: 1512. 02876, together with T. Damour, P. Jaranowski and G. Schäfer, *Conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity*, Phys. Rev. D 93, 084014 (2016) or arXiv: 1601.01283.

[2] This reduction was done by T. Ohta *et al.* in 1973-1974 and shown to be correct –when realized that it implies a coordinate change– by G. Schäfer (1984) and T. Damour, G. Schäfer (1985). For details see T. Damour and G. Schäfer, *"Redefinition of position variables and the reduction of higher-order Lagrangians"*, J. of Math. Phys. 32, 127 (1991).

Lagrangians (0.2) do not belong to that class. The Arnowitt-Deser-Misner (ADM) coordinates, see section 21, turn out to correspond to the choice (the other f_i being zero) :

$$f_3 = f_4 = -\frac{1}{4}\ , \quad f_{12} = f_{13} = \frac{1}{4}\ , \quad f_{11} = f_{14} = \frac{7}{4}\ . \tag{0.5}$$

Now that the accelerations (A^i, A'^i) have been eliminated, it is a straightforward exercise to derive the associated class of Hamiltonians (see Book 1, section 39) :

$$P^i = \frac{\partial L_{\rm f}^{\rm red}}{\partial V^i}\ , \quad P'^i = \frac{\partial L_{\rm f}^{\rm red}}{\partial V'^i}\quad , \quad H = P.V + P'.V' - L_{\rm f}^{\rm red}\ .$$

In the centre of mass frame where $P^i + P'^i = 0$ (see section 56), the conjugate variables are $R^i = Z^i - Z'^i$ and P^i. The relative motion lies in the equatorial section and is described using polar coordinates (R, Φ), with conjugate momenta $P_R = (N.P)$, $P_\Phi = R(N \wedge P)_z$ and from now on we denote $(Q, P) = (R, \Phi; P_R, P_\Phi)$. Introducing the dimensionless quantities :

$$\hat{P}^2 = \hat{P}_R^2 + \frac{\hat{P}_\Phi^2}{\hat{R}^2}\ , \qquad \hat{P}_R = \frac{P_R}{\mu}\ ,\ \hat{P}_\Phi = \frac{P_\Phi}{\mu M}\ ,\ \hat{R} = \frac{R}{M}\ , \qquad M = m + m'\ ,\ \mu = \frac{mm'}{M}\ ,\ \nu = \frac{\mu}{M}\ ,$$

the two-body Hamiltonians H :

$$\frac{H - M}{\mu} = \left(\frac{\hat{P}^2}{2} - \frac{h^{\rm N}}{\hat{R}}\right) + \hat{H}^{\rm 1PN} + \hat{H}^{\rm 2PN} + \cdots \tag{0.6}$$

depend, generically, on 17 coefficients :

$$\begin{aligned}
\hat{H}^{\rm 1PN} &= \left(h_1^{\rm 1PN}\hat{P}^4 + h_2^{\rm 1PN}\hat{P}^2\hat{P}_R^2 + h_3^{\rm 1PN}\hat{P}_R^4\right) + \frac{1}{\hat{R}}\left(h_4^{\rm 1PN}\hat{P}^2 + h_5^{\rm 1PN}\hat{P}_R^2\right) + \frac{h_6^{\rm 1PN}}{\hat{R}^2}\ ,\\
\hat{H}^{\rm 2PN} &= \left(h_1^{\rm 2PN}\hat{P}^6 + h_2^{\rm 2PN}\hat{P}^4\hat{P}_R^2 + h_3^{\rm 2PN}\hat{P}^2\hat{P}_R^4 + h_4^{\rm 2PN}\hat{P}_R^6\right)\\
&\quad + \frac{1}{\hat{R}}\left(h_5^{\rm 2PN}\hat{P}^4 + h_6^{\rm 2PN}\hat{P}_R^2\hat{P}^2 + h_7^{\rm 2PN}\hat{P}_R^4\right) + \frac{1}{\hat{R}^2}\left(h_8^{\rm 2PN}\hat{P}^2 + h_9^{\rm 2PN}\hat{P}_R^2\right) + \frac{h_{10}^{\rm 2PN}}{\hat{R}^3}\ .
\end{aligned} \tag{0.7}$$

In general relativity, the seven coefficients $h^{\rm N}$ and $h_i^{\rm 1PN}$ are found to be

$$\begin{aligned}
&h^{\rm N} = 1\\
&h_1^{\rm 1PN} = -\frac{1}{8}(1 - 3\nu)\ , \quad h_2^{\rm 1PN} = h_3^{\rm 1PN} = 0\\
&h_4^{\rm 1PN} = -\frac{1}{2}(3 + \nu)\ , \quad h_5^{\rm 1PN} = -\frac{1}{2}\nu\ , \quad h_6^{\rm 1PN} = \frac{1}{2}\ .
\end{aligned} \tag{0.8}$$

At 2PN level, the coefficients $h_i^{\rm 2PN}$ depend on the 14 f_i parameters, i.e. on the coordinate systems introduced in (0.2) and (0.3). In general relativity, and in the ADM coordinates defined in (0.5), for example, they read :

$$\begin{aligned}
&h_1^{\rm 2PN} = \frac{1}{16}\left(1 - 5\nu + 5\nu^2\right)\ , \quad h_2^{\rm 2PN} = h_3^{\rm 2PN} = h_4^{\rm 2PN} = 0\ ,\\
&h_5^{\rm 2PN} = \frac{1}{8}(5 - 20\nu - 3\nu^2)\ , \quad h_6^{\rm 2PN} = -\frac{\nu^2}{4}\ , \quad h_7^{\rm 2PN} = -\frac{3}{8}\nu^2\ ,\\
&h_8^{\rm 2PN} = \frac{5}{2} + 4\nu\ , \quad h_9^{\rm 2PN} = \frac{3}{2}\nu\ , \quad h_{10}^{\rm 2PN} = -\frac{1}{4}(1 + 3\nu)\ .
\end{aligned} \tag{0.9}$$

The equations of motion derived from the Hamiltonians (0.6) *et seq.* through Hamilton's equations ($dQ/dt = \partial H/\partial P$, $dP/dt = -\partial H/\partial Q$) are strictly equivalent to those which can be obtained from the 2PN Lagrangian (0.1) after proper change of the coordinates given in (0.5).[3]

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[3] This 2PN Hamiltonian was first obtained by G. Schäfer in 1985 and is currently known to 4PN order, see Damour, Jaranowski and Schäfer, *Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems*, Phys. Rev. D89 (2014), 064058 or arXiv:1401.4548 .

SECTION 70. The equations of motion of a test particle in a SSS metric

In view of the mapping to come of the two-body dynamics to that of a test particle in the field of an effective single body, let us here consider a static, spherically symmetric (SSS) metric in the equatorial section, written in Schwarzschild-Droste coordinates (t_e, r, ϕ) :

$$ds_\mathrm{e}^2 = -A(r)\,dt_\mathrm{e}^2 + B(r)\,dr^2 + r^2 d\phi^2 \tag{0.10}$$

where the subscript "e" stands for "effective".

The action of a test particle in the metric (0.10), of mass μ which we will identify to the two-body reduced mass, that is $\mu = mm'/(m+m')$, is, see section 13 :

$$S_\mathrm{e} = \int L_\mathrm{e}\,dt_\mathrm{e} = -\mu \int \sqrt{-\frac{ds_\mathrm{e}^2}{dt_\mathrm{e}^2}}\,dt_\mathrm{e}$$

$$\text{so that}\quad L_\mathrm{e} = -\mu\sqrt{A - B\,\dot r_\mathrm{e}^2 - r^2\dot\phi_\mathrm{e}^2}\ ,\quad \text{with}\quad \dot r_\mathrm{e} = \frac{dr}{dt_\mathrm{e}}\ ,\ \dot\phi_\mathrm{e} = \frac{d\phi}{dt_\mathrm{e}}\ .$$

The conjugate momenta (p_r, p_ϕ) of (r, ϕ) and the Hamiltonian H_e are defined as, see Book 1, chapter 9 :

$$p_r = \frac{\partial L_\mathrm{e}}{\partial \dot r_\mathrm{e}}\quad ,\quad p_\phi = \frac{\partial L_\mathrm{e}}{\partial \dot\phi_\mathrm{e}}\quad ,\quad H_\mathrm{e} = p_r \dot r_\mathrm{e} + p_\phi \dot\phi_\mathrm{e} - L_\mathrm{e}$$

$$\text{so that}\quad \frac{H_\mathrm{e}}{\mu} = \sqrt{A(r)\left(1 + \frac{\hat p_r^2}{B(r)} + \frac{\hat p_\phi^2}{\hat r^2}\right)} \tag{0.11}$$

$$\text{where}\quad \hat r = \frac{r}{M}\ ,\quad \hat p_r = \frac{p_r}{\mu}\ ,\quad \hat p_\phi = \frac{p_\phi}{\mu M}\ ,\quad \hat p^2 = \hat p_r^2 + \frac{\hat p_\phi^2}{\hat r^2}\ ,$$

M being a mass which we will identify to the two-body total mass, that is $M = m + m'$.

If we restrict our attention to the dynamics of the particle at 2PN order only, the metric potentials $A(r)$ and $B(r)$ can be expanded as :

$$A(r) = 1 + \frac{a_1}{\hat r} + \frac{a_2}{\hat r^2} + \frac{a_3}{\hat r^3} + \cdots\ ,\quad B(r) = 1 + \frac{b_1}{\hat r} + \frac{b_2}{\hat r^2} + \cdots\ , \tag{0.12}$$

where a_1, a_2, a_3, b_1 and b_2 are five dimensionless coefficients characterizing the effective SSS spacetime at that order. The 2PN effective Hamiltonian then becomes

$$\frac{H_\mathrm{e}}{\mu} - 1 = \hat H_\mathrm{e}^\mathrm{N} + \hat H_\mathrm{e}^\mathrm{1PN} + \hat H_\mathrm{e}^\mathrm{2PN} + \cdots$$

$$\text{where}\quad \hat H_\mathrm{e}^\mathrm{N} = \frac{\hat p^2}{2} + \frac{a_1}{2\hat r}\quad ,\quad \hat H_\mathrm{e}^\mathrm{1PN} = -\frac{\hat p^4}{8} - b_1\frac{\hat p_r^2}{2\hat r} + a_1\frac{\hat p^2}{4\hat r} + \frac{a_2 - a_1^2/4}{2\hat r^2} \tag{0.13}$$

$$\hat H_\mathrm{e}^\mathrm{2PN} = \frac{\hat p^6}{16} - \frac{\hat p^2(a_1\hat p^2 - 4b_1\hat p_r^2)}{16\hat r} + \frac{(4a_2 - a_1^2)\hat p^2 + 4(2b_1^2 - 2b_2 - a_1 b_1)\hat p_r^2}{16\hat r^2} + \frac{a_1^3 - 4a_1a_2 + 8a_3}{16\hat r^3}\ .$$

The Hamilton equations of motion derived from (0.11), $dq/dt_\mathrm{e} = \partial H_\mathrm{e}/\partial p$, $dp/dt_\mathrm{e} = -\partial H_\mathrm{e}/\partial q$, with $q = (r, \phi)$ and $p = (p_r, p_\phi)$, yield back the geodesic equation and its first integrals, as obtained in section 47 :

$$H_\mathrm{e} = \mu E\quad ,\quad \hat p_\phi = j\quad ,\quad \frac{d\phi}{dt_\mathrm{e}} = \frac{j}{ME}Au^2\quad ,\quad \left(\frac{dr}{dt_\mathrm{e}}\right)^2 = \frac{A}{BE^2}F(u)$$

$$\text{where}\quad F(u) = E^2 - A(u)\left(\epsilon + j^2u^2\right)\quad \text{with}\quad u = \frac{M}{r}\ ,$$

and where E and j are the (dimensionless) energy and angular momentum of the particle (and where $\epsilon = 1$ or $\epsilon = 0$ for timelike or null geodesics).

Circular timelike orbits are such that the radial velocity vanishes, $F = 0$, while circularity also requires $F' = 0$, where $F' = dF/du$. Hence j^2 and E are related to u by :

$$j^2(u) = -\frac{A'}{(Au^2)'}\ ,\quad E(u) = A\sqrt{\frac{2u}{(Au^2)'}}\ . \tag{0.14}$$

The innermost stable circular orbit (ISCO) requires the third (inflection point) condition $F''=0$, and its position, $u_{\rm ISCO}$, is the root of the equation :

$$\frac{A''}{A'}=\frac{(Au^2)''}{(Au^2)'}\ . \tag{0.15}$$

As for the position $u_{\rm LR}$ of the null circular orbit, or "light-ring", it is determined by the conditions $F_{\epsilon=0}=E^2-j^2Au^2=0$ and $dF/du|_{\epsilon=0}=0$, and hence is given by the root of the equation

$$(Au^2)'=0 \tag{0.16}$$

which gives also the position of a circular timelike geodesic whose angular momentum formally goes to infinity. Note that in the Schwarzschild-Droste coordinates used here the circular orbits are described by the function $A(u)$ only, given in terms of three coefficients a_1,a_2,a_3 at 2PN order, see (0.12).

SECTION 71. The EOB mapping

In section 69, we have obtained a class of two-body (centre of mass) Hamiltonians, $H(Q,P)$, depending on 17 coefficients $h_i^{\rm NPN}$ at 2PN order, see (0.6) *et seq.* ; we contrasted them with a much simpler effective Hamiltonian, $H_{\rm e}(q,p)$, depending on 5 coefficients a_i, b_i at 2PN order, that describes the geodesic motion of a test particle in an effective static and spherically symmetric metric, see (0.13). The "effective-one-body" (EOB) mapping, proposed by A. Buonanno and T. Damour in 1998, consists in relating both problems by means of a canonical transformation together with a functional relation $H_{\rm e}=f_{\rm EOB}(H)$ between their Hamiltonians.[4]

• *The canonical transformation*

The first step of the mapping procedure is to relate the phase space coordinates (Q,P) of the two-body Hamiltonian $H(Q,P)$ to those, (q,p), of the effective one $H_{\rm e}(q,p)$ by means of a canonical transformation.

Recalling that $(Q,P)=(R,\Phi;P_R,P_\Phi)$ and $(q,p)=(r,\phi;p_r,p_\phi)$, the canonical transformation is defined by a time-independent function (since the Hamiltonians are conservative) $F(q,Q)$ which shifts the two-body Lagrangian by the total derivative $L_{\rm f}^{\rm red}(Q,\dot Q)=L'(q,\dot q)+dF/dt$ and $H'(q,p)=H(Q,P)$, see Book 1 section 40 :

$$L'dt+dF=(p_r\,dr{+}p_\phi\,d\phi-H dt)+dF=P_R\,dR+P_\Phi\,d\Phi-H dt\ ,$$
$$\text{and thus}\ :\quad dF=P_R\,dR+P_\Phi\,d\Phi-(p_r\,dr+p_\phi\,d\phi)\ .$$

To avoid cumbersome algebra we shall rather consider the generating function $G(Q,p)$ such that

$$G=F+(p_r\,r+p_\phi\,\phi)-(p_r\,R+p_\phi\,\Phi)\ ,$$
$$\text{so that}\quad dG=dR\,(P_R-p_r)+d\Phi\,(P_\Phi-p_\phi)+dp_r\,(r-R)+dp_\phi\,(\phi-\Phi)\ ,$$

which yields the canonical transformation relating (q,P) to (Q,p) :

$$r(Q,p)=R+\frac{\partial G}{\partial p_r}\ ,\quad \phi(Q,p)=\Phi+\frac{\partial G}{\partial p_\phi}\ ,\quad P_R(Q,p)=p_r+\frac{\partial G}{\partial R}\ ,\quad P_\Phi(Q,p)=p_\phi+\frac{\partial G}{\partial \Phi}\ . \tag{0.17}$$

The generic ansatz for G, which generates 1PN and 2PN phase coordinate changes depends on nine parameters :[5]

$$\frac{G(Q,p)}{\mu M}=\hat R\,\hat p_r\left[\left(\alpha_1\mathcal{P}^2+\beta_1\hat p_r^2+\frac{\gamma_1}{\hat R}\right)+\left(\alpha_2\mathcal{P}^4+\beta_2\mathcal{P}^2\hat p_r^2+\gamma_2\hat p_r^4+\delta_2\frac{\mathcal{P}^2}{\hat R}+\epsilon_2\frac{\hat p_r^2}{\hat R}+\frac{\eta_2}{\hat R^2}\right)+\cdots\right]\ , \tag{0.18}$$

where we have (re)introduced the dimensionless quantities

$$\mathcal{P}^2=\hat p_r^2+\frac{\hat p_\phi^2}{\hat R^2}\ ,\qquad \hat R\equiv\frac{R}{M}\ ,\ \hat p_r\equiv\frac{p_r}{\mu}\ ,\ \hat p_\phi\equiv\frac{p_\phi}{\mu M}\ ,\qquad M=m+m'\ ,\ \mu=\frac{mm'}{M}\ ,\ \nu=\frac{\mu}{M}\ .$$

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[4] A. Buonanno, T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, Phys. Rev. D59 (1999) 084006 or arXiv:gr-qc/9811091 .

[5] We already know from Newton's theory that once written in the centre of mass frame, the Newtonian two-body and effective-one-body Hamiltonians identify.

This generating function does not to depend on Φ (by isotropy) so that $P_\Phi = p_\phi$, see (0.17). Note also that for circular orbits for which $p_r = 0 \Leftrightarrow P_R = 0$, we have $\phi = \Phi$ and hence only the radial coordinates then differ (but are both constant, p_ϕ being then constant on shell).

The two-body Hamiltonian (0.6) *et seq.* is thus canonically transformed, $H(Q,p) \equiv H(Q, P(Q,p))$, using the last two relations in (0.17) (the use of G instead of F avoids having to perform inversions) and is of the form (its explicit expression is easily worked out)

$$\frac{H(Q,p) - M}{\mu} = \left(\frac{\mathcal{P}^2}{2} - \frac{h^{\rm N}}{\hat{R}}\right) + \hat{H}^{\rm 1PN} + \hat{H}^{\rm 2PN} + \cdots \tag{0.19}$$

where $h^{\rm N} = 1$, see (0.8), and where the 16 remaining coefficients appearing in $\hat{H}^{\rm 1PN}$ and $\hat{H}^{\rm 2PN}$ depend on the 14 parameters f_i introduced in (0.3) and on the 9 parameters entering the canonical transformation (0.18). Similarly, the effective Hamiltonian $H_{\rm e}(Q,p) \equiv H_{\rm e}(q(Q,p),p)$ (0.13) is transformed as, using the first two relations in (0.17) :

$$\frac{H_{\rm e}(Q,p)}{\mu} - 1 = \left(\frac{\mathcal{P}^2}{2} + \frac{a_1}{2\hat{R}}\right) + \hat{H}_{\rm e}^{\rm 1PN} + \hat{H}_{\rm e}^{\rm 2PN} + \cdots \tag{0.20}$$

where $\hat{H}_{\rm e}^{\rm 1PN}$ and $\hat{H}_{\rm e}^{\rm 2PN}$ depend on the 5 coefficients $(a_1, a_2, a_3, b_1, b_2)$ entering the effective metric coefficients at 2PN order, see (0.12), as well as on the 9 parameters of the canonical transformation (0.18), and are also easily worked out.

- *The functional relation*

Now that we have expressed the two-body and effective Hamiltonians H and $H_{\rm e}$ in the same coordinate system (Q,p), see (0.19-0.20), the EOB mapping requires in a second step to impose a functional relation between them, $H_{\rm e} = f_{\rm EOB}(H)$, which will yield, at 2PN order, the sought-for relations between the five coefficients (a_i, b_i) entering $H_{\rm e}$ and the 17 coefficients $h_i^{N{\rm PN}}$ entering H.

This functional relation can a priori be expanded, substracting the rest-mass constants, as :

$$\frac{H_{\rm e}(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p)-M}{\mu}\right)\left[1 + \frac{\bar{\nu}_1}{2}\left(\frac{H(Q,p)-M}{\mu}\right) + \bar{\nu}_2\left(\frac{H(Q,p)-M}{\mu}\right)^2 + \cdots\right] ,$$

with the (mass-shifted) Hamiltonians identifying at Newtonian order (which already yields $a_1 = -2h^{\rm N} = -2$, as can be seen from (0.19) and (0.20)). Now, as has been proven up to 4PN in general relativity and even at all orders within a post-Minkowskian scheme, the relation must be quadratic *at all orders* with $\bar{\nu}_1 = \nu = \mu/M$, $\bar{\nu}_2 = 0 \cdots$. That is, it must be :[6]

$$\frac{H_{\rm e}(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(Q,p)-M}{\mu}\right)\right] . \tag{0.21}$$

This functional relation is not guaranteed to hold *a priori*. Indeed, the generic 2PN Hamiltonians written in terms of the 17 coefficients $h_i^{N{\rm PN}}$, see (0.6), may now be considered as describing the two-body dynamics (in various coordinate systems parametrized by the 14 f_i) for an *arbitrary* theory (e.g. general relativity), which must be related to an effective Hamiltonian depending on 5 coefficients $(a_i,\, b_i)$, see (0.12), by means of a canonical transformation that depends on 9 parameters (0.18). Hence one expects $17 - 5 - 9 = 3$ conditions on the $h_i^{N{\rm PN}}$ coefficients.

At Newtonian order, no condition arise since (0.21) then reduces to the identity. At 1PN order it turns out that an effective $H_{\rm e}$ can be constructed provided that the coefficients $h_i^{\rm 1PN}$ (which do not depend on the f_i) satisfy :

$$2h_2^{\rm 1PN} + 3h_3^{\rm 1PN} = 0\,. \tag{0.22}$$

Any theory (such as general relativity) whose purely kinematical terms in their Lagrangians take the Lorentz-invariant form $m\sqrt{1-V^2} + m'\sqrt{1-V'^2}$, is such that $h_2^{\rm 1PN} = 0$ and $h_3^{\rm 1PN} = 0$. Thus this condition is not restrictive. At 2PN

[6] See T. Damour, *Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory*, Phys. Rev. D94 (2016) no.10, 104015 or arXiv:1609.00354.

Note that at 1PN order one can choose $\bar{\nu}_1 = 0$ (which then yields $a_2 = -\nu/4$ and $b_1 = (8-\nu)/4$ and effective Eddington parameters $\gamma = 1 - \nu/8$, $\beta = 1 - \nu/4$). This shows that the two-body post-Keplerian orbits, which depend on three eccentricities, see equations (57.5-6), can be transformed into the post-Keplerian orbits of a test particle in a SSS metric which depend on two eccentricities only, see eq. (51.4-5), by means of a (1PN) canonical transformation (0.17) (with $\alpha_1 = -3\nu/8$, $\beta_1 = 0$ and $\gamma_1 = (8+\nu)/8)$ where the (mass-shifted) Hamiltonians identify at 1PN-Newtonian order included.

order, the coefficients $h_i^{\rm 2PN}$ depend on the 14 parameters f_i, and the identification requires two further conditions. The first,

$$h_4^{\rm 2PN}=-\frac{2}{45}\left(12h_2^{\rm 2PN}+18h_3^{\rm 2PN}+(h_2^{\rm 1PN})^2\right)\ ,$$

is no more restrictive than (0.22), for the same reasons. However, the second condition,

$$\begin{aligned}
&h_1^{\rm 2PN}+\frac{7}{3}h_2^{\rm 2PN}+h_3^{\rm 2PN}+h_5^{\rm 2PN}+h_6^{\rm 2PN}+h_7^{\rm 2PN}=\frac{h^{\rm N}}{128}(5+2\nu+5\nu^2)+\\
&-\frac{1}{8}(1+\nu)\left((3h_1^{\rm 1PN}+h_2^{\rm 1PN})h^{\rm N}+h_4^{\rm 1PN}+h_5^{\rm 1PN}\right)+\frac{5}{2}h_1^{\rm 1PN}\left(7h_1^{\rm 1PN}h^{\rm N}+2(h_4^{\rm 1PN}+h_5^{\rm 1PN})\right)\\
&+\frac{1}{6}h_2^{\rm 1PN}\left(13h_2^{\rm 1PN}h^{\rm N}+10(h_4^{\rm 1PN}+h_5^{\rm 1PN})\right)+\frac{35}{3}h_1^{\rm 1PN}h_2^{\rm 1PN}h^{\rm N}\ ,
\end{aligned}\tag{0.23}$$

is restrictive and the mapping of the two-body motion towards an effective geodesic is possible for a subclass of theories only.

In general relativity, one checks that the coefficients $h_i^{\rm 2PN}$ (given, e.g., in (0.8-0.9) when using ADM coordinates) *do* satisfy the condition (0.23) *whatever* the values of the 14 f_i parameters, that is, independently of the coordinate system in which the two-body Hamiltonian has been written, as required by the invariance of general relativity under diffeomorphisms.[7]

- *The effective metric*

Inserting now in the left-hand side of the functional relation (0.21) the explicit expressions for the coefficients $h_i^{N\rm PN}$ of the two-body Hamiltonians H (0.19) obtained in section 69, see (0.6) *et seq.*, and, in the right-hand side, the explicit expression of the effective Hamiltonian $H_{\rm e}$ (0.20) obtained in section 70, see (0.13), the identification term by term yields a *unique* solution for the five coefficients a_i and b_i entering $H_{\rm e}$, and hence the effective SSS metric :

$$\begin{aligned}
&ds_{\rm e}^2=-A(r)\,dt_{\rm e}^2+B(r)\,dr^2+r^2d\phi^2\\
&A(r)=1+\frac{a_1}{\hat r}+\frac{a_2}{\hat r^2}+\frac{a_3}{\hat r^3}\ ,\quad B(r)=1+\frac{b_1}{\hat r}+\frac{b_2}{\hat r^2}\qquad\text{with}\quad \hat r=\frac{r}{M}\\
&\text{where}\quad a_1=-2\quad,\quad a_2=0\quad,\quad a_3=2\nu\quad,\quad b_1=2\quad,\quad b_2=2(2-3\nu)\quad\text{with}\quad \nu=\frac{mm'}{M^2}\ .
\end{aligned}\tag{0.24}$$

The simplicity of (0.24) is striking, since geodesic motion in that metric encompasses all the dynamics of the two-body problem at 2PN order. In particular, the 14 parameters f_i which parametrize the family of two-body ordinary, reduced, Lagrangians at 2PN order, see (0.2), are hidden in the canonical transformation (0.18), whose coefficients are found to be :

$$\begin{aligned}
&\alpha_1=-\frac{\nu}{2}\ ,\quad \beta_1=0\ ,\quad \gamma_1=1+\frac{\nu}{2}\ ,\quad \alpha_2=\frac{1}{8}(1-\nu)\nu\ ,\quad \beta_2=0\ ,\quad \gamma_2=\frac{\nu^2}{2}\ ,\\
&\delta_2=f_6\frac{m}{M}+f_1\frac{m'}{M}-\nu\left(f_1+f_6+(-f_3+f_5+f_6)\frac{m}{M}+(f_1+f_2-f_4)\frac{m'}{M}-\frac{3}{2}+\frac{\nu}{8}\right)\ ,\\
&\epsilon_2=-\frac{\nu^2}{8}+f_{10}\frac{m}{M}+f_7\frac{m'}{M}-\nu\left(f_7+f_{10}+(f_9+f_{10})\frac{m}{M}+(f_7+f_8)\frac{m'}{M}\right)\ ,\\
&\eta_2=\frac{\nu}{4}(-19+\nu)+f_{13}\frac{m}{M}+f_{12}\frac{m'}{M}+\nu(f_{11}-f_{12}-f_{13}+f_{14})\ .
\end{aligned}$$

SECTION 72. The EOB Hamiltonian and the resummed dynamics

Now that $H_{\rm e}$ and the associated effective metric (0.24) have been constructed, one can invert the quadratic relation (0.21) to define an exact "EOB Hamiltonian", which is a *resummed* version of the two-body Hamiltonian H and

[7] The relation (0.23) is also verified by scalar-tensor theories, see F.L. Julié and N. Deruelle, *Two-body problem in Scalar-Tensor theories as a deformation of General Relativity : an Effective-One-Body approach*, Phys. Rev. D95 (2017) 124054 or arXiv:1703.05360. In contrast, it is not satisfied by Maxwell's theory at second post-Coulombian order (see A. Buonanno, *Reduction of the two-body dynamics to a one-body description in classical electrodynamics*, Phys. Rev. D62 (2000) 104022 or arXiv:hep-th/0004042).

It is an exercise to extend the identification between generic two-body and effective Hamiltonians at 3PN or higher orders. One then sees that the identification at 3PN order (when imposing the quadratic relation (0.21) between the 2PN and effective Hamiltonians), requires a further condition which turns out *not* to be satisfied by general relativity. In consequence, the mapping of the two-body motion has then to be extended to a forced motion of a test particle in a SSS metric, see T. Damour, P. Jaranowski and G. Schäfer, *On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation*, Phys. Rev. D62 (2000) 084011 or arXiv: gr-qc/0005034 .

coincides, at 2PN, with its 2PN expansion given in (0.6) *et seq.* It reads (in the (q,p) phase space coordinates of the effective problem and returning to dimensional coordinates) :

$$H_{\rm EOB}=M\sqrt{1+2\nu\left(\frac{H_{\rm e}}{\mu}-1\right)}\quad\text{with}\quad\frac{H_{\rm e}}{\mu}=\sqrt{A\left(1+\frac{p_r^2}{\mu^2B}+\frac{p_\phi^2}{\mu^2r^2}\right)}\tag{0.25}$$

where A and B are given in (0.24) and considered now as exact. (We again recall that $M=m+m'$, $\mu=mm'/M$ and $\nu=\mu/M$).

As an illustration of the resummed dynamics defined by $H_{\rm EOB}$, let us focus on the ISCO and light-ring orbital frequency $\Omega_{\rm ISCO}=(d\Phi/dt)|_{\rm ISCO}$ and $\Omega_{\rm LR}=(d\Phi/dt)|_{\rm LR}$ of the relative motion of the two bodies.[8]

From Hamilton's equations we have

$$\Omega=\frac{\partial H_{\rm EOB}}{\partial P_\Phi}\quad\text{with}\quad\Omega=\frac{d\Phi}{dt}\,,\quad\text{that is :}\quad\Omega=\frac{1}{\sqrt{1+2\nu(E-1)}}\frac{\partial H_{\rm e}}{\partial p_\phi}$$

where E is the energy per unit mass of the effective test particle and where $P_\Phi=p_\phi$ because the generating function G does not depend on Φ ; moreover, $\Phi=\phi$ for circular orbits, see the remark below eq. (0.19). Hence

$$\Omega_{\rm circ}=\frac{\omega_{\rm circ}}{\sqrt{1+2\nu(E-1)}}\quad\text{with}\quad\omega_{\rm circ}=\frac{\partial H_{\rm e}}{\partial p_\phi}\,,\quad\text{that is}\quad M\omega_{\rm circ}=\frac{j}{E}Au^2\quad\text{where}\quad u=\frac{M}{r}\,,$$

and where the dimensionless angular momentum j and energy E of the effective test particle are given by : $E=A\sqrt{2u/(Au^2)'}$ and $j^2=-A'/(Au^2)'$, see (0.14).[9] Now, at 2PN order the metric potential A is $A=1-2u+2\nu u^3$, see (0.24), so that

$$E=\frac{1-2u+2\nu u^3}{\sqrt{1-3u+5\nu u^3}}\quad,\quad j^2=\frac{1-3\nu u^2}{u(1-3u+5\nu u^3)}\quad\text{and}\quad M\Omega_{\rm circ}=\frac{u^{3/2}\sqrt{1-3\nu u^2}}{\sqrt{1+2\nu(E-1)}}\,.\tag{0.26}$$

As for the positions, $u_{\rm ISCO}$ and $u_{\rm LR}$, of the ISCO and the light-ring they solve $A''/A'=(Au^2)''/(Au^2)'$ and $(Au^2)'=0$ respectively, see (0.15-0.16), and hence are the outermost roots of

$$\begin{aligned}\text{ISCO :}&\quad 1-6u+3\nu u^2+20\nu u^3-30\nu^2u^5=0\\ \text{light-ring :}&\quad 1-3u+5\nu u^3=0\,.\end{aligned}\tag{0.27}$$

A remarkable feature of this resummed dynamics is that it is *exact* for $\nu=0$ (since the ISCO and light-ring are those of a geodesic in the Schwarzschild metric, $r_{\rm ISCO}=6M$ and $r_{\rm LR}=3M$, see section 38). One also notes that, for all ν ($\in[0,1/4]$), $r_{\rm ISCO}<6M$ (at odds with the prediction based on the 2PN Hamiltonian H, but in agreement with the prediction based on an analysis at 3PN order). One notes too that $\Omega_{\rm LR}=0$ because the light-ring coincides with a circular timelike geodesic with infinite j (and hence p_ϕ), and that the canonical transformation (0.17-0.18) between (r,ϕ) and (R,Φ) becomes then singular.[10]

SECTION 73. EOB dynamics including the radiation reaction force

In sections 69-72, we have built a resummed EOB dynamics of the two-body problem, focussing on its conservative part. We now include radiation reaction effects which first circularize the orbit and then cause its radius to decrease. Our starting point will be the 2.5PN radiation reaction force obtained in harmonic coordinates in section 67.

• *The 2.5PN radiation reaction force*

From the 2.5PN equations of motion obtained in section 67 eq. (6-7), one derives the centre of mass, 2.5PN, radiation reaction force $\vec{\mathcal F}$ (with $\vec N=\vec R/R$, $\vec{\mathcal V}=\dot{\vec R}$ and a dot denoting derivation with respect to t) :

$$\mu\dot{\vec{\mathcal V}}=-\frac{GM\mu}{R^2}\vec N+\cdots+\vec{\mathcal F}\qquad\text{with}\quad M=m+m'\,,\ \mu=\frac{mm'}{M}$$
$$\text{and}\quad\vec{\mathcal F}=\frac{8(mm')^2}{5MR^3}\mathcal V^2\left[3\vec N(N.\mathcal V)-\vec{\mathcal V}\right]+\frac{8(mm')^2}{5R^4}\left[\frac{17}{3}\vec N(N.\mathcal V)-3\vec{\mathcal V}\right]\,.$$

---

[8] In the standard post-Newtonian approach they are given by $\Omega_{\rm PN}=\partial H/\partial P_\Phi$ where H is the 2PN Hamiltonian (0.6) *et seq.* which depends on the seventeen coefficients $h_i^{\rm NPN}$. The analysis was performed in 1993 by Kidder, Will and Wiseman (within the Lagrangian formalism and in harmonic coordinates) and by Wex and Schäfer (within the Hamiltonian formalism) and shown to lack robustness.

[9] The relation between $\Omega=d\Phi/dt$ and ω means that ω is an angular velocity with respect to the rescaled "effective" time : $\omega=d\phi/dt_{\rm e}$ with $t_{\rm e}=t/\sqrt{1+2\nu(E-1)}$.

[10] The analysis of circular orbits is as straightforward at 3PN order, see the reference to T. Damour, P. Jaranowski and G. Schäfer in Note 7 : the potentials A and B are then given by $A=1-2u+2\nu u^3+\nu a_4u^4$ with $a_4=94/3-41\pi^2/32$, and $AB=1-6\nu u^2+b_3u^3$ with $b_3=2(3\nu-26)\nu$. The motion of the effective test particle is no longer geodesic but governed by $H_{\rm e}=\sqrt{\mu^2A+Ap_\phi^2/r^2+(p_r^*)^2\left[1+z_3(p_r^*)^2\right]}$ with $p_r^*=\sqrt{A/B}\,p_r$ and $z_3=2(4-3\nu)$, but the value of z_3 does not affect the dynamics of circular orbits for which $p_r=0$.

For circular motion : $(N.\mathcal{V})=0$ and $(N\wedge\mathcal{V})_z=R\,\dot\Phi$. Hence $\mathcal{F}_R|_{\rm circ}=(N.\mathcal{F})$ vanishes, and $\mathcal{F}_\Phi|_{\rm circ}=R(N\wedge\mathcal{F})_z$ is given by

$$\mathcal{F}_\Phi|_{\rm circ}=-\frac{8(mm')^2}{5MR}\dot\Phi\left(\mathcal{V}^2+\frac{3M}{R}\right)\quad\text{with}\quad\mathcal{V}^2=(R\dot\Phi)^2\,.$$

At the lowest, Newtonian, order to which we limit ourselves here, the contact and canonical tranformations between the two body and effective coordinates performed in (0.5) *et seq.* and (0.17) *et seq.* are the identity : $\Phi=\phi$, $R=r$. Hence, for circular orbits for which $\mathcal{V}^2=M/R$ so that $R/M=(M\dot\phi)^{-2/3}$, see, e.g., section 65 eq. (18), the leading order, "quadrupolar", radiation reaction force reads :[11]

$$\mathcal{F}_r|_{\rm circ}=0\quad,\quad\mathcal{F}_\phi|^{\rm quad}_{\rm circ}=-\frac{32}{5}M\,\nu^2(M\dot\phi)^{7/3}\quad\text{with}\quad\nu=\frac{mm'}{M^2}\,.\tag{0.28}$$

- *The EOB equations of motion including the radiation reaction*

From the Lagrangian equations of motion that include the radiation reaction effects, see e.g. section 68 eq. (9), one obtains the corresponding EOB Hamilton equations of motion as (a dot denoting derivation with respect to t) :

$$\begin{gathered}\dot r=\frac{\partial H_{\rm EOB}}{\partial p_r}\quad,\quad\dot\phi=\frac{\partial H_{\rm EOB}}{\partial p_\phi}\quad,\quad\dot p_r=-\frac{\partial H_{\rm EOB}}{\partial r}+\mathcal{F}_r\quad,\quad\dot p_\phi=\mathcal{F}_\phi\\ \text{where}\quad H_{\rm EOB}=M\sqrt{1+2\nu\left(\frac{H_{\rm e}}{\mu}-1\right)}\quad\text{with}\quad\frac{H_{\rm e}}{\mu}=\sqrt{A\left(1+\frac{p_r^2}{\mu^2B}+\frac{p_\phi^2}{\mu^2r^2}\right)}\\ \text{and}\quad A=1-2u+2\nu u^3\quad,\quad B=\frac{1}{A}\left(1-6\nu u^2\right)\quad\text{with}\quad u=\frac{M}{r}\,.\end{gathered}\tag{0.29}$$

Note that B has been factored by $1/A$ in order to recover the exact Schwarzschild metric in the test mass limit $\nu=0$. As for $\mathcal{F}_i(q,\,\dot q)=\{\mathcal{F}_r(q,\dot q),\mathcal{F}_\phi(q,\dot q)\}$, they are to be considered now as functions of $q=(r,\phi)$ and $p=(p_r,p_\phi)$ through Hamilton's equations :

$$\mathcal{F}_i(q,\,\dot q(q,p))=\mathcal{F}_i(q,\,\partial H_{\rm EOB}/\partial p)\,.$$

Let us consider an initially almost circular motion (which is the case of physical interest since gravitational radiation tends to circularize orbits). The radial component of the reaction force is hence almost zero initially and will be assumed to remain negligible for the rest of the coalescence : $\mathcal{F}_r=\mathcal{F}_r|_{\rm circ}=0$. Similarly, $\mathcal{F}_\phi$ will be approximated by $\mathcal{F}_\phi=\mathcal{F}_\phi|_{\rm circ}$ (given in eq. (0.28) at the Newtonian approximation), for all r. The equations of motion hence become (a prime denoting here a derivative with respect to r) :

$$\begin{gathered}\dot r=\frac{M}{H_{\rm e}H_{\rm EOB}}\frac{A\,p_r}{B}\quad,\quad\dot p_r=-\frac{M}{2H_{\rm e}H_{\rm EOB}}\left[A'+p_r^2\left(\frac{A}{B}\right)'+p_\phi^2\left(\frac{A}{r^2}\right)'\right]\quad,\quad\dot p_\phi=\mathcal{F}_\phi|_{\rm circ}\,,\\ \dot\phi=\frac{M}{H_{\rm e}H_{\rm EOB}}\frac{A\,p_\phi}{r^2}\end{gathered}\tag{0.30}$$

where $H_{\rm e}$ and $H_{\rm EOB}$ are given in (0.29) and where $\mathcal{F}_\phi|_{\rm circ}$ is a function of $(M\dot\phi)$, that is of (r,p_r,p_ϕ) through the last (independent) Hamilton equation (0.30).

- *The initial conditions*

At $t=0$, let us set, say, $r_{\rm in}=15M$ and $\phi_{\rm in}=0$.

As for $p_r|_{\rm in}$ and $p_\phi|_{\rm in}$, they are chosen within the adiabatic, quasi-circular, approximation. We hence recall here that, in the absence of radiative effects, we have $H_{\rm e}=\mu E$ and $H_{\rm EOB}=M\sqrt{1+2\nu(E-1)}$ and, for circular orbits, see (0.26) (with $u=M/r$) :

$$p_\phi|_{\rm circ}=M^2\nu\sqrt{\frac{1-3\nu u^2}{u(1-3u+5\nu u^3)}}\ ,\ M\dot\phi|_{\rm circ}=\frac{u^{3/2}\sqrt{1-3\nu u^2}}{\sqrt{1+2\nu(E-1)}}\ ,\ E=\frac{1-2u+2\nu u^3}{\sqrt{1-3u+5\nu u^3}}\,.\tag{0.31}$$

---

[11] With the material presented in this book, one cannot derive more than the Newtonian expression (0.28) above for the radiation reaction force, which is a crude approximation. For an improved *resummed* expression of the radiative force including extra information from higher-order post-Newtonian expansion, see A. Buonnanno and T. Damour's seminal paper, *Transition from inspiral to plunge in binary black hole coalescence*, Phys. Rev. D62 (2000) 064015 or arXiv: gr-qc/0001013. For a review of the present state of the art, see T. Damour, *The general relativistic two body problem*, Brumberg Festschrift, Ed. S. M. Kopeikin, de Gruyter, Berlin, 2014 or arXiv:1312.3505.

We therefore choose : $p_\phi|_{\rm in} = p_\phi|^{\rm in}_{\rm circ}$ with $u_{\rm in} = M/r_{\rm in}$.

The initial condition $p_r|_{\rm in}$ is obtained as follows : we have, from Hamilton's equations (0.30), $\dot p_\phi = \mathcal{F}_\phi|_{\rm circ}$, that is, $(dp_\phi/dr)\dot r = \mathcal{F}_\phi|_{\rm circ}$, where $\mathcal{F}_\phi|_{\rm circ}$ can be approximated for large r by $\mathcal{F}_\phi|^{\rm quad}_{\rm circ}$ as given in (0.28) and is a function of $\dot\phi = \dot\phi|_{\rm circ}$, that is a function of $r = M/u$, see (0.31) ; similarly $p_\phi = p_\phi|_{\rm circ}$, and from eq. (0.31) $(dp_\phi|_{\rm circ}/dr)$ is hence known in terms of r ; as for $\dot r$ it is given by Hamilton's equation $\dot r = \frac{MA}{BH_{\rm e}H_{\rm EOB}}p_r$ where $H_{\rm e} = \mu E$ and $H_{\rm EOB} = M\sqrt{1+2\nu(E-1)}$ with E given in (0.31) in terms of r. Hence p_r is a known (but not very illuminating) function of r at the adiabatic, quasi-circular, approximation (at lowest order in $u_{\rm in} = M/r_{\rm in}$, it reads : $p_r|_{\rm in} = -\frac{64}{5}M\nu^2 u^3_{\rm in}$).

SECTION 74. EOB waveform of two coalescing black holes

• *The EOB dynamics of two coalescing black holes*

The EOB dynamics derived above encapsulates the inspiralling of two compact bodies, for example two black holes (or neutrons stars if the tidal effects are neglected), starting from an initially quasi-circular relative orbit.

The numerical integration of their equations of motion (0.30) –with $H_{\rm e}$, $H_{\rm EOB}$, A and B given in (0.29) and $\mathcal{F}_\phi|_{\rm circ} = \mathcal{F}_\phi|^{\rm quad}_{\rm circ}$ given in (0.28) where $\dot\phi$ is expressed in terms of (r, p_r, p_ϕ)– is straightforward, once a value of ν has been chosen, and for the adiabatic, quasi-circular, initial conditions given above. The result is given in Fig. 74a below, which shows that the inspiral of the two bodies remains quasi-circular even beyond their effective innermost stable circular orbit, and that the integration can be pushed all the way to the light-ring.

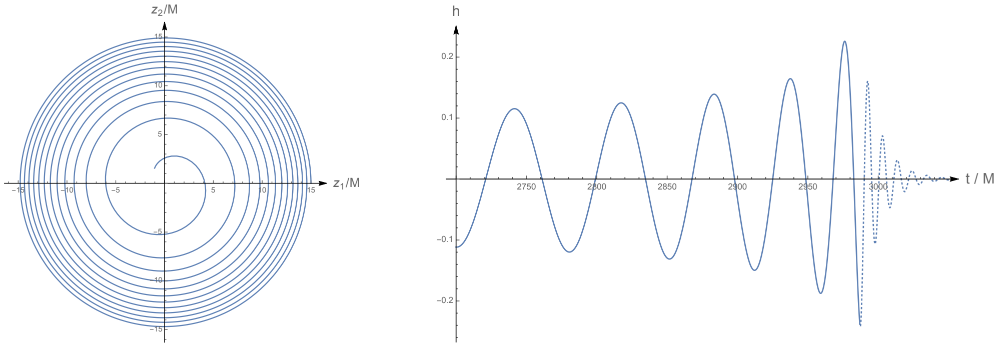

Figure 74. The left panel represents the (effective) inspiral of two equal mass ($\nu = 1/4$) black holes due to the (Newtonian) radiation reaction force (0.28), obtained by integration of the EOB equations of motion (0.30). The initial conditions at $\phi_{\rm in} = 0$, $r_{\rm in} = 15M$ are $p_\phi|_{\rm in} = 1.08\,M^2$ and $p_r|_{\rm in} = -3.61\times10^{-4}\,M$. The innermost stable circular orbit and light-ring are located at $r_{\rm ISCO} = 5.72M$ and $r_{\rm LR} = 2.85M$.

The right panel represents the EOB waveform produced during the inspiral phase of the coalescence of two equal mass black holes (solid line, see eq. (0.32), setting $\mathcal{C} = 1$) up to merger, taken to occur at the effective light-ring, where it is matched to the dominant quasi-normal mode of the final black hole (dotted line, see eq. (0.33)). The mass and angular momentum parameter of the final black hole are $M_{\rm BH} = 0.98M$ and $a_{\rm BH} = 0.79$. The quasi-normal mode pulsation and damping time are $M\omega_{\rm QNM} = 0.60$ and $M/\tau = 0.078$.

• *The gravitational waveform*

In chapter 13 (and 14) we derived the "first quadrupole formula", that is, the radiative gravitational field generated by a source at lowest (sometimes called "Newtonian") order in terms of the second time derivative of its quadrupole moment, see eq. (62.5). In the centre of mass frame (at the lowest order we restrict our attention to, calculations are made using Newton's physics) the quadrupole moment of two point-like bodies on circular orbits is $Q_{ij} = \mu(3z_iz_j - \delta_{ij}z^2)$ with $z_1 = r\cos\phi$ and $z_2 = r\sin\phi$ with the angular velocity $\dot\phi$ taken to be constant. Hence the time dependence of the amplitude of both modes of the gravitational wave is given (up to a constant phase) by

$$h = \mathcal{C}(M\dot\phi)^{2/3}\cos(2\phi) \qquad (0.32)$$

where $\mathcal{C}$ is a dimensionless factor taking into account the position of the observer with respect to the plane of the orbit and is proportional to μ/D where D is the distance to the source. More precisely, up to a constant phase, the gravitational form (0.32) reads, in a cartesian coordinate system $\mathcal{R}' = (O', z'_1, z'_2, z'_3)$ attached to the observer O', and in the TT gauge introduced in section 60, see (5.62) :

$$h'^{\,TT}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{where} \quad h_+ = \frac{4\mu}{D}\left(\frac{1+\cos^2 i}{2}\right)\left(M\dot\phi\right)^{2/3}\cos(2\phi) \quad \text{and} \quad h_\times = \frac{4\mu}{D}(\cos i)\left(M\dot\phi\right)^{2/3}\sin(2\phi)\,,$$

with i the inclination between the normal to the orbital plane and the line-of-sight OO'^i, to which the $(O'z'_3)$ axis is aligned here for simplicity.

The inspiralling motion of the binary system due to the emission of gravitational waves being given by the EOB dynamics obtained above, the amplitude and frequency of the waveform increase with time, see Fig 74b.

The merging time is taken to be the moment the effective particle reaches the light-ring.[12] The mass $M_{\rm BH}$ and angular momentum parameter $a_{\rm BH}$ of the final black hole formed after coalescence are hence estimated to be $H_{\rm EOB}$ (which is the energy of the binary system) as given in (0.29), and $p_\phi/M_{\rm BH}^2$, evaluated at the light ring.

From then on the gravitational waveform is smoothly matched to :

$$h = \mathcal{A}\, e^{-(t-t_{\rm LR})/\tau_{\rm QNM}} \cos\left(\omega_{\rm QNM}(t-t_{\rm LR})+\mathcal{B}\right) \tag{0.33}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two dimensionless constants set by the matching conditions, and where $\omega_{\rm QNM}$ and $\tau_{\rm QNM}$ characterize the dominant quasi-normal mode of the final black-hole.[13]

The whole EOB waveform is plotted on Fig. 74b.

It should be recalled that obtaining this EOB waveform relies on a number of approximations and assumptions : first, the resummed EOB dynamics is built out of the 2PN Lagrangian and the reaction force as well as the waveform are evaluated at leading, quadrupolar, order ; second, the mass and angular momentum of the final black hole are identified with the energy and angular momentum of the two-body system, evaluated on the light-right ; third, the waveform is matched at merger with the dominant quasi-normal mode of the final black hole.

In order to produce useful waveform templates to analyze the data collected by the present gravitational wave detectors, a far more elaborate EOB approach has to be, and was, developed, based on higher-order PN approximations, including the spins of the initial black holes, and defining merger by a fit to full numerical black hole coalescence simulations.[14]

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[12] Identifying the moment of merger with the moment the effective particle reaches the light-ring is supported by the fact that, in the test particle limit, the light-ring acts as a potential barrier below which the radial gravitational radiation it emits is strongly filtered, as first studied by M. Davis, R. Ruffini, W.H. Press and R.H. Price, *Gravitational Radiation from a Particle Falling Radially into a Schwarzschild Black Hole*, Phys. Rev. Lett. 27, 1466 (1971), and as conforted by present-day numerical relativity simulations.

[13] As noted in section 41, the study of the quasi-normal (or "ringing") modes which characterize a black hole was initiated by Vishveshwara in 1970. The values of $\omega_{\rm QNM}$ and $\tau_{\rm QNM}$ used here are those taken by A. Buonanno and T. Damour in their 2000 paper : $M_{\rm BH}\,\omega_{\rm QNM} = f_f[1-0.63(1-a_{\rm BH})^{3/10}]$ and $\tau\,\omega_{\rm QNM} = 4f_Q(1-a_{\rm BH})^{-9/20}$ with f_f and f_Q given in F. Echeverria, Phys. Rev. D40, 3194 (1997) (for $M_{\rm BH}=0.98M$ and $a_{\rm BH}=0.98$: $f_f=0.9587$ and $f_Q=0.9389$).

[14] For the present state of the art, see, e.g., A. Nagar, T. Damour, C. Reisswig and D. Pollney, *Energetics and phasing of nonprecessing spinning coalescing black hole binaries*, Phys. Rev. D 93, 044046 (2016) or arXiv:1506.08457, or A. Bohé *et al.* *An improved effective-one-body model of spinning, nonprecessing binary black holes for the era of gravitational-wave astrophysics with advanced detectors*, Phys. Rev. D 95, 044028 (2017) or arXiv:1611.03703 .

Chapitre 2

Généralisation de l'approche EOB aux gravités modifiées : le cas des théories scalaire-tenseur

Nous avons rappelé dans le chapitre 1 qu'en relativité générale, le problème à deux corps peut, à l'ordre 2PN, être ramené au mouvement géodésique dans la métrique effective (1.3). Cette réduction permet non seulement de simplifier la dynamique à deux corps, mais aussi d'en définir une resommation. Ceci en fait un ingrédient majeur du formalisme EOB qui vise, rappelons-le, à construire des formes d'onde gravitationnelles valables dans le régime de champ fort, près de la coalescence du système.

Afin de tester la gravité dans son régime de champ fort, on peut envisager d'obtenir, de même, des "patrons" d'onde dans le cadre de théories de gravité modifiée. Considérons par exemple les théories scalaire-tenseur, introduites en section 0.2 de l'introduction. Le Lagrangien à deux corps associé, décrivant la partie conservative du mouvement, a été calculé à l'ordre 2PK en coordonnées harmoniques par S. Mirshekari et C. Will en 2013 [96], et peut se ré-écrire selon :

$$L = -(m_A^0 + m_B^0) + \frac{1}{2} m_A^0 V_A^2 + \frac{1}{2} m_B^0 V_B^2 + \frac{G_{AB} m_A^0 m_B^0}{R} + L_{1\mathrm{PK}} + L_{2\mathrm{PK}} + \mathcal{O}(V^8)\,, \tag{2.1}$$

$$\begin{aligned}
L_{1\mathrm{PK}} = {} & \frac{1}{8} m_A^0 V_A^4 + \frac{1}{8} m_B^0 V_B^4 + \frac{G_{AB} m_A^0 m_B^0}{R} \left(\frac{3}{2}(V_A^2 + V_B^2) - \frac{7}{2} \vec{V}_A \cdot \vec{V}_B - \frac{1}{2} (\vec{N}\cdot\vec{V}_A)(\vec{N}\cdot\vec{V}_B) + \bar{\gamma}_{AB} (\vec{V}_A - \vec{V}_B)^2 \right) \\
& - \frac{G_{AB}^2 m_A^0 m_B^0}{2R^2} \left(m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A) \right)\,, \\
L_{2\mathrm{PK}} = {} & \frac{1}{16} m_A^0 V_A^6 + \frac{G_{AB} m_A^0 m_B^0}{R} \left[\frac{1}{8} (7 + 4\bar{\gamma}_{AB}) \left(V_A^4 - V_A^2 (\vec{N}\cdot\vec{V}_B)^2 \right) - (2 + \bar{\gamma}_{AB}) V_A^2 (\vec{V}_A \cdot \vec{V}_B) + \frac{1}{8} (\vec{V}_A \cdot \vec{V}_B)^2 \right. \\
& \left. + \frac{1}{16} (15 + 8\bar{\gamma}_{AB}) V_A^2 V_B^2 + \frac{3}{16} (\vec{N}\cdot\vec{V}_A)^2 (\vec{N}\cdot\vec{V}_B)^2 + \frac{1}{4} (3 + 2\bar{\gamma}_{AB}) \vec{V}_A \cdot \vec{V}_B (\vec{N}\cdot\vec{V}_A)(\vec{N}\cdot\vec{V}_B) \right] \\
& + \frac{G_{AB}^2 m_B^0 (m_A^0)^2}{R^2} \left[\frac{1}{8} \left(2 + 12\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 8\bar{\beta}_B - 4\delta_A \right) V_A^2 + \frac{1}{8} \left(14 + 20\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 4\bar{\beta}_B - 4\delta_A \right) V_B^2 \right. \\
& - \frac{1}{4} \left(7 + 16\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 4\bar{\beta}_B - 4\delta_A \right) \vec{V}_A \cdot \vec{V}_B - \frac{1}{4} \left(14 + 12\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 - 8\bar{\beta}_B + 4\delta_A \right) (\vec{V}_A \cdot \vec{N})(\vec{V}_B \cdot \vec{N}) \\
& \left. + \frac{1}{8} \left(28 + 20\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 - 8\bar{\beta}_B + 4\delta_A \right) (\vec{N}\cdot\vec{V}_A)^2 + \frac{1}{8} \left(4 + 4\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 + 4\delta_A \right) (\vec{N}\cdot\vec{V}_B)^2 \right] \\
& + \frac{G_{AB}^3 (m_A^0)^3 m_B^0}{2R^3} \left[1 + \frac{2}{3}\bar{\gamma}_{AB} + \frac{1}{6}\bar{\gamma}_{AB}^2 + 2\bar{\beta}_B + \frac{2}{3}\delta_A + \frac{1}{3}\epsilon_B \right] + \frac{G_{AB}^3 (m_A^0)^2 (m_B^0)^2}{R^3} \left[\frac{19}{8} + \bar{\gamma}_{AB} + \bar{\beta}_A + \bar{\beta}_B + \frac{1}{2}\zeta \right] \\
& - \frac{1}{8} G_{AB} m_A^0 m_B^0 \left(2(7 + 4\bar{\gamma}_{AB}) \vec{A}_A \cdot \vec{V}_B (\vec{N}\cdot\vec{V}_B) + \vec{N}\cdot\vec{A}_A (\vec{N}\cdot\vec{V}_B)^2 - (7 + 4\bar{\gamma}_{AB}) \vec{N}\cdot\vec{A}_A V_B^2 \right) + (A \leftrightarrow B)\,,
\end{aligned}$$

où $R = |\vec{Z}_A - \vec{Z}_B|$, $\vec{N} = (\vec{Z}_A - \vec{Z}_B)/R$, $\vec{V}_A = d\vec{Z}_A/dt$, $\vec{A}_A = d\vec{V}_A/dt$, et où $\vec{Z}_A$ désigne la position du corps A.

Le Lagrangien (2.1) généralise celui de la relativité générale (1.1). Il dépend maintenant de deux fonctions $m_A(\varphi)$ et $m_B(\varphi)$ caractérisant les deux corps, qui sont développées autour de la valeur φ_0 du champ scalaire à l'infini, imposée par l'environnement cosmologique du système binaire :

$$\ln m_{A/B}(\varphi) = \ln m^0_{A/B} + \alpha^0_{A/B}(\varphi - \varphi_0) + \frac{1}{2}\beta^0_{A/B}(\varphi - \varphi_0)^2 + \frac{1}{6}{\beta'}^0_{A/B}(\varphi - \varphi_0)^3 + \cdots \tag{2.2}$$

voir aussi la section 0.2.4 de l'introduction. Ainsi, le Lagrangien à deux corps dépend à l'ordre 2PK de huit paramètres ($m^0_{A/B}$, $\alpha^0_{A/B}$, $\beta^0_{A/B}$, ${\beta'}^0_{A/B}$), apparaissant sous la forme des onze combinaisons suivantes :

$$m^0_A\,, \quad G_{AB} = 1 + \alpha^0_A\alpha^0_B\,, \tag{2.3a}$$
$$\bar{\gamma}_{AB} = -\frac{2\alpha^0_A\alpha^0_B}{1+\alpha^0_A\alpha^0_B}\,, \quad \bar{\beta}_A = \frac{1}{2}\frac{(\beta_A\alpha^2_B)^0}{(1+\alpha^0_A\alpha^0_B)^2}\,, \tag{2.3b}$$
$$\delta_A = \frac{(\alpha^0_A)^2}{(1+\alpha^0_A\alpha^0_B)^2}\,, \quad \epsilon_A = \frac{(\beta'_A\alpha^3_B)^0}{(1+\alpha^0_A\alpha^0_B)^3}\,, \quad \zeta = \frac{\beta^0_A\alpha^0_A\beta^0_B\alpha^0_B}{(1+\alpha^0_A\alpha^0_B)^3}\,, \tag{2.3c}$$

ainsi que leurs contreparties ($A \leftrightarrow B$), où l'on rappelle qu'un indice 0 indique une quantité évaluée en $\varphi = \varphi_0$.[1]

À l'ordre keplerien, l'addition des interactions métrique et scalaire se manifeste par le remplacement de la constante de Newton (sans dimension, et normalisée à $G_* = 1$ dans ce manuscrit) par un couplage effectif $G_{AB} = 1 + \alpha^0_A\alpha^0_B$, dépendant des corps. À l'ordre 1PK, apparaissent deux combinaisons (2.3b) qui sont une généralisation aux corps auto-gravitants des paramètres d'Eddington [88]. Notons que les combinaisons (2.3b)-(2.3c) sont au moins quadratiques en $\alpha^0_{A/B}$. Ces deux paramètres influencent donc de façon déterminante les déviations à la relativité générale [174]. Enfin, on retrouve le Lagrangien à deux corps de la relativité générale dans la limite $m_A(\varphi) = cste$, i.e. lorsque

$$\alpha^0_{A/B} = \beta^0_{A/B} = {\beta'}^0_{A/B} = 0 \quad \Rightarrow \quad G_{AB} = 1\,,\ \bar{\gamma}_{AB} = \bar{\beta}_{A/B} = \delta_{A/B} = \epsilon_{A/B} = \zeta = 0\,. \tag{2.4}$$

À l'aide des méthodes présentées au chapitre 1, nous avons montré avec Nathalie Deruelle dans l'article [175] reproduit ci-dessous que l'on peut généraliser le Hamiltonien EOB 2PN de la relativité générale, à savoir

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}\,, \quad \text{avec} \quad H_e = \mu\sqrt{A\left(1 + \frac{p_r^2}{\mu^2 B} + \frac{p_\phi^2}{\mu^2 r^2}\right)}\,, \tag{2.5}$$

(où maintenant $M = m^0_A + m^0_B$, $\mu = m^0_A m^0_B/M$ et $\nu = \mu/M$) et réduire le problème à deux

1. Notons que connaissant la partie 1PK du Lagrangien (2.1) avec (2.3), il est aisé de calculer les paramètres β_{Edd} et γ_{Edd} d'Eddington, mentionnés en introduction, voir (14). Pour cela, il suffit de spécifier le Lagrangien (2.1) dans la limite du rapport de masses extrême, et de se placer dans le référentiel au repos du "gros" corps, e.g., en prenant $m^0_B \ll m^0_A$ et $\vec{V}_A = \vec{0}$. Lorsque de plus l'auto-gravité est négligée, i.e. $\alpha^0_{A/B} \to \alpha_0$ et $\beta^0_{A/B} \to \beta_0$, cf. (13) et au-dessus, l'identification du Lagrangien obtenu à celui d'une particule test en mouvement géodésique dans la métrique d'Eddington (donnée, e.g., dans [133]) permet de lire les paramètres β_{Edd} et γ_{Edd}.

corps des théories scalaire-tenseur, décrit par (2.1), au mouvement géodésique d'une particule test dans la métrique effective :

$$ds_e^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\phi^2\,, \quad \text{avec} \tag{2.6a}$$

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \left[2\nu + \delta a_3^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^3\,, \tag{2.6b}$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \left[2(2-3\nu) + \delta b_2^{\mathrm{ST}}\right]\left(\frac{G_{AB}M}{r}\right)^2\,, \tag{2.6c}$$

qui est une simple déformation de celle de la relativité générale.

Ce résultat appelle quelques remarques introductives. Nous avons montré au chapitre précédent que pour pouvoir être réduit à un mouvement géodésique, le Hamiltonien à deux corps doit vérifier trois contraintes sur ses coefficients à l'ordre 2PK (cf. 1.3.2). Il est remarquable qu'elles soient satisfaites en théories scalaire-tenseur, dont le Lagrangien à deux corps est nettement plus lourd que celui de la relativité générale.

De plus, la simplicité de la métrique (2.6) est frappante. La constante de Newton est remplacée à tous les ordres par G_{AB}. À l'ordre 1PK, (2.6) n'est autre que la métrique PPN d'Eddington, ainsi généralisée, grâce à l'approche EOB, au problème à deux corps (i.e. de masses comparables) ainsi qu'aux effets d'auto-gravité. Notons aussi que les corrections scalaire-tenseur d'ordre 2PK se résument à l'ajout de deux coefficients seulement, δa_3^{ST} et δb_2^{ST}, dans la métrique effective (on trouvera leurs expressions détaillées en (4.13) et (4.14) dans l'article ci-dessous), illustrant la puissance simplificatrice des transformations canoniques qui sont au cœur du formalisme EOB. Notons enfin que la métrique (2.6) ne dépend que de cinq coefficients, mettant en lumière une dégénérescence des huit paramètres fondamentaux (2.2) à l'ordre 2PK. Nous montrerons au chapitre 4 comment, en spécifiant la nature des corps, il devient possible de calculer la fonction associée $m_A(\varphi)$ explicitement, et ainsi réduire le nombre de paramètres inconnus.

Par ailleurs, H_{EOB} définit une resommation de l'information post-keplerienne. Puisque la métrique effective (2.6) a été construite comme une déformation de la relativité générale, elle est particulièrement adaptée pour estimer l'impact des corrections scalaire-tenseur sur la dynamique à deux corps, et ce, jusque dans son régime de champ fort. Dans l'article reproduit ci-dessous, on propose donc d'étudier les corrections scalaire-tenseur d'ordre 1PK et 2PK considérées comme des perturbations du Hamiltonien EOB le plus élaboré de la relativité générale, connu actuellement à l'ordre 5PN (cf. chapitre 1). On y montre par exemple que les corrections sur la position de la dernière orbite circulaire stable du système (ou "innermost stable circular orbit", ISCO) et sur la fréquence orbitale associée, typiquement atteinte par le système près de sa coalescence, deviennent significatives dès lors que $(\alpha^0_{A/B})^2 \gtrsim 10^{-2}$.

L'exemple scalaire-tenseur met en avant tout le potentiel qu'a le formalisme EOB pour rassembler les gravités modifiées au sein d'un cadre simple. À l'instar de la famille scalaire-tenseur, toute théorie satisfaisant les contraintes données en 1.3.2 peut y être incorporée via une simple déformation de la métrique effective. Rappelons d'ailleurs qu'à l'ordre 1PK, la seule contrainte restante n'est restrictive que pour les théories dont le Lagrangien à deux corps brise l'invariance de Lorentz.

Ainsi, l'article conclut par une extension de l'approche PPN d'Eddington à la coalescence des systèmes binaires, en proposant d'encoder toute déviation à la relativité générale dans une métrique générique paramétrisée ("Parametrized EOB", PEOB) de la forme :

$$A^{\mathrm{PEOB}}(u) \equiv \mathcal{P}^1_5[A^{GR}_{\mathrm{5PN}} + 2(\epsilon^0_{\mathrm{1PK}} + \nu\,\epsilon^\nu_{\mathrm{1PK}})u^2 + (\epsilon^0_{\mathrm{2PK}} + \nu\,\epsilon^\nu_{\mathrm{2PK}})u^3]\,, \tag{2.7}$$

où $\mathcal{P}^1_5$ désigne l'approximant de Padé d'indice $(1,5)$, apparaissant en relativité générale, $u = G_{AB}M/r$ en théories scalaire-tenseur, et où $\epsilon^0_{\mathrm{nPK}}$ et $\epsilon^\nu_{\mathrm{nPK}}$ sont à présent des paramètres libres. En permettant de contraindre les gravités modifiées dans leur régime de champ fort, cette approche PEOB pourrait permettre de compléter les tests limités à contraindre la relativité générale dans la phase "spiralante" des systèmes binaires (i.e., en champ faible), voir, e.g., ceux de la collaboration LIGO-Virgo [176] ; voir aussi l'approche PPE ("parametrized post-einsteinian") de N. Yunes et F. Pretorius [177].

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Two-body problem in scalar-tensor theories as a deformation of general relativity: An effective-one-body approach

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In this paper we address the two-body problem in massless scalar-tensor (ST) theories within an effective-one-body (EOB) framework. We focus on the first building block of the EOB approach, that is, mapping the conservative part of the two-body dynamics onto the geodesic motion of a test particle in an effective external metric. To this end, we first deduce the second post-Keplerian (2PK) Hamiltonian of the two-body problem from the known 2PK Lagrangian. We then build, by means of a canonical transformation, a ST deformation of the general relativistic EOB Hamiltonian that allows us to incorporate the scalar-tensor (2PK) corrections to the currently best available general relativity EOB results. This EOB-ST Hamiltonian defines a resummation of the dynamics that may provide information on the strong-field regime, in particular, the ISCO location and associated orbital frequency, and can be compared to, other, e.g., tidal, corrections.

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I. INTRODUCTION

On September 14th, 2015, the two antennas of the "Laser Interferometric Gravitational Observatory" (LIGO) detected a "chirp" signal, called GW150914, which heralded a new era in gravitational wave astronomy. Indeed, up to then, only the backreaction of the emitted gravitational waves onto the inspiralling of binary stars had been observed, through pulsar timing, and found to be in full agreement with the general relativistic predictions, see; e.g., [1–3]. That time, an actual gravitational waveform was extracted from the data by the LIGO-Virgo collaboration, which was announced on February 11th, 2016 to describe the first ever observed merger of two black holes [4].

Building libraries of accurate gravitational waveform templates is essential for the successful detection of the inspiral, merging, and "ring-down" phases of binary systems of compact objects driven by gravity. In the framework of general relativity (GR), post-Newtonian (PN) expansions of Einstein's equations are suitable to describe the weak-field inspiral phase and the associated gravitational waveforms, and numerical relativity is required to take account of the full nonlinear dynamics of the merging, whereas the settling down of the final black hole through its ringing modes can be tackled by semianalytical methods; see, e.g., the reviews, [5–7].

The effective-one-body (EOB) approach has proven to be a very powerful way to analytically match and encompass the general relativistic post-Newtonian and numerical descriptions of the inspiralling and merging (as well as ring-down) phases of the dynamics of binary systems of comparable masses. It was initiated by A. Buonanno and T. Damour in 1998 [8] who reduced the general relativistic two-body problem at 2PN order[1] to that of the geodesic motion of a test particle in an effective external metric. They did so by mapping, by means of a canonical transformation, the two-body 2PN general relativistic Hamiltonian towards a much simpler, EOB Hamiltonian related to that of a test particle in geodesic motion in an external, static and spherically symmetric metric. Taking then this EOB Hamiltonian as exact (which amounts to an implicit resummation) and including the 2.5PN radiation reaction force, they described the inspiralling phase *up to* merging, that is up to and through the last stable orbit. The gravitational waveforms thus predicted [9] turned out to be much simpler than previously argued [10], a simplicity that was confirmed later by numerical relativity [11]. This EOB approach was then extended to 3PN [12] (see [13] for a review) and, recently, to 4PN [14], and even beyond by introducing a couple of parameters that are tuned by matching to numerical relativity results. The corresponding gravitational waveforms were constructed using various resummation techniques, and were used to extract from the GW150914 data the characteristics of the coalescing black holes, that is their initial and final masses and spins; see, e.g., [15].

Matching binary system gravitational waveform templates to the present and forthcoming data from the LIGO-Virgo interferometers (and forthcoming detectors such as LISA) will allow to test gravity theories at high post-Newtonian order (if the inspiralling phase can be monitored for a sufficient number of cycles) and in the strong-field regime at merger. The template libraries being based at present on general relativistic waveforms, the tests and their

[1]That is, up to and including $(v/c)^4$ corrections to the Newtonian dynamics.

feasibility are limited to phenomenological bounds on some parametrized PN coefficients; see, e.g., [16–18].

A next step to test gravity theories in their strong-field regimes using gravitational wave detectors is to match their data with templates predicted within the framework of "modified gravities," that is, theories alternative to general relativity. Among the large "zoo" of modified gravities, scalar-tensor theories (ST) are probably those which are best motivated from a theoretical point of view and most studied. In their simplest versions, they consist in adding one massless scalar degree of freedom to gravity that, like the metric, couples universally to matter. They were introduced by Jordan, Fierz, Thiry, Brans and Dicke (see [19] for a historical review) and were put in a modern perspective by Will and Zaglauer [20], Nordvedt [21], and Damour and Esposito-Farèse [22]. However the corresponding dynamics of binary systems is known at 2.5PN order only [23][2] or numerically [24]. What was hence done in [25,26] or [27] is the computation of the gravitational waveforms in scalar-tensor theories at 2PN relative order.

What we propose to do here is a first step to go beyond what has been done up to now by extending to scalar-tensor theories the effective-one-body approach of Buonanno and Damour in their 1998 paper [8].

More precisely we start from the scalar-tensor Lagrangian of two nonspinning bodies obtained by Mirshekari and Will in [23], which is the scalar-tensor extension of the Damour-Deruelle two-body GR Lagrangian [28,29]. This "Jordan-frame" Lagrangian, written in harmonic coordinates, depends on the positions, velocities, and accelerations of the two bodies and we first reduce it to an ordinary "Einstein-frame" Lagrangian depending on positions and velocities only by means of a contact transformation similar to what is done in GR [30–31]. It is then an exercise to obtain the center-of-mass 2PK Hamiltonian.

This 2PK Hamiltonian is then mapped, after an appropriately chosen canonical transformation, to an effective one-body Hamiltonian, which reduces to the 1998 Buonanno-Damour EOB Hamiltonian [8] in the general relativity limit. Of course the (conservative) dynamics derived from this EOB-ST Hamiltonian is the same at 2PK order as the dynamics derived from the Mirshekari-Will 2PN Lagrangian but, when taken as being exact, it defines an implicit resummation and hence different dynamics in the strong-field regime which is reached near the last stable orbit.

This 2PK EOB-ST Hamiltonian, which can be seen as a scalar-tensor "deformation" of the general relativistic 2PN EOB one, is then extended to incorporate the scalar-tensor 2PK corrections into the currently best available general relativity EOB results, and is thus well suited to test scalar-tensor theories when considered as parametrized corrections to general relativity.

The paper is organized as follows: we first recall the general settings of scalar-tensor theories in Sec. II. In Sec. III we perform an order reduction of the conservative two-body Lagrangian at 2PK order. We then express the corresponding Hamiltonian in the center-of-mass frame. In Sec. IV we build the effective problem, that is, the geodesic motion in a 2PN effective external metric, and rewrite its dynamics as a "ST deformation" of general relativity at 2PN order. We then incorporate the scalar-tensor 2PK corrections that we have obtained into the currently best available EOB-NR general relativistic Hamiltonian and, after adequate "padeization," some features of the corrections to the general relativity innermost stable circular orbit (ISCO) predictions are described.

II. SCALAR-TENSOR THEORY REMINDER

In this paper we adopt the conventions of Damour and Esposito-Farèse (see, e.g., [22] or [32]) and limit ourselves to the single, massless scalar field case. In the Einstein frame, the action reads[3]

$$S_{\rm EF} = \frac{c^4}{16\pi G_*}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) + S_m[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}], \tag{2.1}$$

where R is the Ricci scalar, $g \equiv \det g_{\mu\nu}$, and Ψ generically stands for matter fields. In the following we work in units where $G_* \equiv 1$ and $c \equiv 1$ if not specified. The free dynamics of the Einstein metric $g_{\mu\nu}$, which describes the tensorial degrees of freedom of gravity, is governed by the usual Einstein-Hilbert action. The dynamics of the scalar field φ, that is, the gravitational scalar degree of freedom of gravity, arises from its coupling to the matter fields Ψ. Indeed, matter minimally couples, not to the Einstein, but to the Jordan metric

$$\tilde{g}_{\mu\nu} \equiv \mathcal{A}^2(\varphi)g_{\mu\nu}. \tag{2.2}$$

This Jordan metric $\tilde{g}_{\mu\nu}$ is often referred to as the "physical" one, since one retrieves special relativity in its locally inertial frames (i.e., frames where $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\lambda\tilde{g}_{\mu\nu} = 0$). Hence, by construction, scalar-tensor theories explicitly encompass the Einstein equivalence principle [33]. A given scalar-tensor theory is completely determined once the function $\mathcal{A}(\varphi)$ has been specified. In particular, general relativity is recovered for $\mathcal{A}(\varphi) = $ cst.

[2]or, in the terminology of [22], 2.5 post-Keplerian (2.5PK) order, to make explicit the fact that it includes the scalar field dependence of the masses due to self-gravity effects, see also footnote 5.

[3]For a comparison with the Jordan-frame parametrization of, e.g., [23], see Appendix A.

From (2.1) one derives the Einstein-frame field equations

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{2.3a}$$

$$\Box\varphi = -4\pi\alpha(\varphi)T, \tag{2.3b}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ is the Eintein-frame energy-momentum tensor, $T \equiv T^\mu{}_\mu$, and

$$\alpha(\varphi) \equiv \frac{d\ln\mathcal{A}(\varphi)}{d\varphi} \tag{2.4}$$

measures the coupling between the scalar field and matter.

When dealing with compact, self-gravitating bodies (e.g., neutron stars or black holes), we adopt the phenomenological treatment suggested by Eardley [34] and justified by Damour [35] and Damour and Esposito-Farèse [22], and "skeletonize" these extended bodies as point particles,

$$S_m = -\sum_A \int d\lambda \sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\tilde{m}_A(\varphi), \tag{2.5}$$

where λ is an affine parameter along the worldline of the particle. The Jordan-frame mass $\tilde{m}_A(\varphi)$ is not a constant but rather depends on the local value of the scalar field, on the specific theory and on body A itself (through its equation of state in particular).[4] Since $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu}$, one also has

$$S_m = -\sum_A \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}m_A(\varphi), \tag{2.6}$$

where we have defined the Einstein-frame mass of the skeletonized compact bodies as

$$m_A(\varphi) \equiv \mathcal{A}(\varphi)\tilde{m}_A(\varphi), \tag{2.7}$$

which takes into account both the universal factor $\mathcal{A}(\varphi)$ and body-dependent self-gravity effects, $\tilde{m}_A(\varphi)$. Hence the two-body problem in ST theories is fully described by *two* functions, $m_A(\varphi)$ and $m_B(\varphi)$. The trajectory of freely falling bodies is generally no longer universal, thus violating the so-called strong equivalence principle, unless their self-gravity is negligible, that is when $\tilde{m}_A$ and $\tilde{m}_B$ are constant (in which case they are geodesics of the Jordan metric). In contrast, static, spherically symmetric black holes are known to carry no massless scalar "hair" and hence reduce to Schwarzschild black holes (see, e.g., [38,22]). In that case $m_A(\varphi) = cst$, and black holes follow the geodesics of the Einstein metric. Consequently, binary black holes are usually expected to generate no deviation from general relativity. However this is not guaranteed in the strong-field, dynamical, regime of a binary coalescence; see the conclusion.

Finally, the following six, dimensionless, body-dependent functions built out of the two mass functions $m_A(\varphi)$ and $m_B(\varphi)$ are useful at the 2PK order we work at,

$$\alpha_A(\varphi) \equiv \frac{d\ln m_A}{d\varphi} = \frac{d\ln\mathcal{A}}{d\varphi} + \frac{d\ln\tilde{m}_A}{d\varphi}, \tag{2.8a}$$

$$\beta_A(\varphi) \equiv \frac{d\alpha_A}{d\varphi}, \tag{2.8b}$$

$$\beta'_A(\varphi) \equiv \frac{d\beta_A}{d\varphi}. \tag{2.8c}$$

In the negligible self-gravity limit, $\tilde{m}_A = \mathrm{cst}$, these functions become universal,

$$\alpha_A \to \alpha = \frac{d\ln\mathcal{A}}{d\varphi}, \quad \beta_A \to \beta \equiv \frac{d\alpha}{d\varphi}, \quad \beta'_A \to \beta' \equiv \frac{d\beta}{d\varphi},$$

while in the general relativity limit ($m_A(\varphi) = \mathrm{cst}$), $\alpha_A = \beta_A = \beta'_A = 0$.

III. THE TWO-BODY 2PK CONSERVATIVE HAMILTONIAN

The scalar-tensor two-body conservative Lagrangian has already been derived at second post-Keplerian order and is our starting point. In particular, its structure (derived from a Fokker action) was given by Damour and Esposito-Farèse in [32] using a diagrammatic approach, while Mirshekari and Will provided its explicit expression in [23]. Because of the harmonic coordinates in which it has been derived, this Lagrangian depends (linearly) on the accelerations of the bodies.

In this section we rewrite the Mirshekari-Will Lagrangian in the Einstein-frame conventions introduced above and in a class of coordinate systems where the Lagrangian is ordinary (i.e., only depends on positions and velocities). We then derive the associated Hamiltonian. Finally, we transform it by means of a generic canonical transformation, to prepare the mapping towards the effective problem that is performed in Sec. IV.

A. Jordan frame vs Einstein frame

From now on, any quantity that is related to the Jordan frame is denoted with a tilde superscript. The Jordan-frame two-body Lagrangian has been derived at 2PK order in harmonic coordinates in [23], using a set of Brans-Dicke-like parameters. In order to rewrite it in terms of the

[4] This scalar field dependence of the mass of pointlike objects embodies the fact that the equilibrium configuration of an extended body depends on the value of the background scalar field at its location, imposed by the other (faraway) companions. We do not discuss here how this φ dependence, or "sensitivity" is (numerically) calculated; see, e.g., [22,36,37].

Einstein-frame parametrization discussed above, one has to:

(i) Translate the parameters of [23] in terms of (2.8). The conversion is given in detail in Appendix A.

(ii) Note that the Jordan-frame Lagrangian of [23] is written in a coordinate system $\{\tilde{x}^\mu\}$ such that the Jordan metric $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$ is Minkowski at infinity, while in the Einstein frame one uses instead coordinates $\{x^\mu\}$ such that $g_{\mu\nu} \to \eta_{\mu\nu}$. Since $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi) g_{\mu\nu}$, that means the following global rescaling of coordinates has to be performed between both frames,

$$\tilde{x}^\mu = \mathcal{A}_0 x^\mu, \tag{3.1}$$

where, and from now on, a "0" index indicates a quantity evaluated at $\varphi = \varphi_0$, where φ_0 is taken to be the asymptotic constant value of the scalar field far from the system, imposed by cosmology. Therefore, in order to get the Einstein-frame Lagrangian, one has to rescale the radial variable R of [23] to $\mathcal{A}_0 R$. For the same reasons, $t \to \mathcal{A}_0 t$, i.e., the Lagrangian has to be rescaled by an overall $\mathcal{A}_0$ factor.

All that taken into account, the Mirshekari-Will two-body 2PK Lagrangian translates, in the Einstein frame and in harmonic coordinates, as

$$L = -m_A^0 - m_B^0 + L_{\mathrm{K}} + L_{1\mathrm{PK}} + L_{2\mathrm{PK}} + \cdots \tag{3.2}$$

with

$$L_{\mathrm{K}} = \frac{1}{2} m_A^0 V_A^2 + \frac{1}{2} m_B^0 V_B^2 + \frac{G_{AB} m_A^0 m_B^0}{R}, \tag{3.3}$$

$$\begin{aligned} L_{1\mathrm{PK}} = {} & \frac{1}{8} m_A^0 V_A^4 + \frac{1}{8} m_B^0 V_B^4 + \frac{G_{AB} m_A^0 m_B^0}{R} \left(\frac{3}{2}(V_A^2 + V_B^2) - \frac{7}{2} \vec{V}_A \cdot \vec{V}_B - \frac{1}{2} (\vec{N} \cdot \vec{V}_A)(\vec{N} \cdot \vec{V}_B) + \bar{\gamma}_{AB} (\vec{V}_A - \vec{V}_B)^2 \right) \\ & - \frac{G_{AB}^2 m_A^0 m_B^0}{2R^2} (m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A)), \end{aligned} \tag{3.4}$$

$$\begin{aligned} L_{2\mathrm{PK}} = {} & \frac{1}{16} m_A^0 V_A^6 \\ & + \frac{G_{AB} m_A^0 m_B^0}{R} \bigg[\frac{1}{8} (7 + 4\bar{\gamma}_{AB}) (V_A^4 - V_A^2 (\vec{N} \cdot \vec{V}_B)^2) - (2 + \bar{\gamma}_{AB}) V_A^2 (\vec{V}_A \cdot \vec{V}_B) + \frac{1}{8} (\vec{V}_A \cdot \vec{V}_B)^2 \\ & \qquad + \frac{1}{16} (15 + 8\bar{\gamma}_{AB}) V_A^2 V_B^2 + \frac{3}{16} (\vec{N} \cdot \vec{V}_A)^2 (\vec{N} \cdot \vec{V}_B)^2 + \frac{1}{4} (3 + 2\bar{\gamma}_{AB}) \vec{V}_A \cdot \vec{V}_B (\vec{N} \cdot \vec{V}_A)(\vec{N} \cdot \vec{V}_B) \bigg] \\ & + \frac{G_{AB}^2 m_B^0 (m_A^0)^2}{R^2} \bigg[\frac{1}{8} (2 + 12\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 8\bar{\beta}_B - 4\delta_A) V_A^2 + \frac{1}{8} (14 + 20\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 4\bar{\beta}_B - 4\delta_A) V_B^2 \\ & - \frac{1}{4} (7 + 16\bar{\gamma}_{AB} + 7\bar{\gamma}_{AB}^2 + 4\bar{\beta}_B - 4\delta_A) \vec{V}_A \cdot \vec{V}_B - \frac{1}{4} (14 + 12\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 - 8\bar{\beta}_B + 4\delta_A)(\vec{V}_A \cdot \vec{N})(\vec{V}_B \cdot \vec{N}) \\ & + \frac{1}{8} (28 + 20\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 - 8\bar{\beta}_B + 4\delta_A)(\vec{N} \cdot \vec{V}_A)^2 + \frac{1}{8} (4 + 4\bar{\gamma}_{AB} + \bar{\gamma}_{AB}^2 + 4\delta_A)(\vec{N} \cdot \vec{V}_B)^2 \bigg] \\ & + \frac{G_{AB}^3 (m_A^0)^3 m_B^0}{2R^3} \left[1 + \frac{2}{3} \bar{\gamma}_{AB} + \frac{1}{6} \bar{\gamma}_{AB}^2 + 2\bar{\beta}_B + \frac{2}{3} \delta_A + \frac{1}{3} \epsilon_B \right] + \frac{G_{AB}^3 (m_A^0)^2 (m_B^0)^2}{8R^3} [19 + 8\bar{\gamma}_{AB} + 8(\bar{\beta}_A + \bar{\beta}_B) + 4\zeta] \\ & - \frac{1}{8} G_{AB} m_A^0 m_B^0 (2(7 + 4\bar{\gamma}_{AB}) \vec{A}_A \cdot \vec{V}_B (\vec{N} \cdot \vec{V}_B) + \vec{N} \cdot \vec{A}_A (\vec{N} \cdot \vec{V}_B)^2 - (7 + 4\bar{\gamma}_{AB}) \vec{N} \cdot \vec{A}_A V_B^2) \\ & \quad + (A \leftrightarrow B), \end{aligned} \tag{3.5}$$

where

$$\vec{N} = \frac{\vec{Z}_A - \vec{Z}_B}{R}, \qquad R = |\vec{Z}_A - \vec{Z}_B|, \qquad \vec{V}_A = \frac{d\vec{Z}_A}{dt}, \qquad \vec{A}_A = \frac{d\vec{V}_A}{dt},$$

$\vec{Z}_A$ being the position of particle A (in our system of units the radial coordinate R has the dimension of a mass). As for the coefficients appearing in the two-body 2PK Lagrangian above, they are combinations of the following eleven constants, built out of the eight functions defined in (2.8) when evaluated at infinity (all deduced, we recall, from the mass function $m_A(\varphi)$ and its B counterpart, which define the theory and bodies under study),

$$m_A^0, \qquad G_{AB} \equiv 1 + \alpha_A^0 \alpha_B^0, \tag{3.6a}$$

$$\bar{\gamma}_{AB} \equiv -\frac{2\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0}, \qquad \bar{\beta}_A \equiv \frac{1}{2}\frac{\beta_A^0(\alpha_B^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}, \tag{3.6b}$$

$$\delta_A \equiv \frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}, \qquad \epsilon_A \equiv \frac{(\beta_A'\alpha_B^3)^0}{(1+\alpha_A^0\alpha_B^0)^3},$$
$$\zeta \equiv \frac{\beta_A^0\alpha_A^0\beta_B^0\alpha_B^0}{(1+\alpha_A^0\alpha_B^0)^3}. \tag{3.6c}$$

(Our notations are a similar, yet simplified, version of the parameters introduced in [32] in the context of the N-body, multiscalar problem and admit a diagrammatic interpretation; see [32].)[5] We note that the effective (dimensionless, since we set $G_* = 1$) gravitational constant $G_{AB} = 1 + \alpha_A^0\alpha_B^0$ *does* depend on the bodies.

Although we stick to the Einstein frame for the rest of this paper, the reader willing to rewrite any forthcoming result in terms of Jordan-frame variables should perform the replacements (we recall that tildes refer to the Jordan frame),

$$m^0_{A/B} \leftrightarrow \tilde{m}^0_{A/B}, \quad G_{AB} \leftrightarrow \tilde{G}_{AB}, \quad R \leftrightarrow \tilde{R},$$
$$\vec{N} \leftrightarrow \tilde{\vec{N}}, \quad \vec{V}_{A/B} \leftrightarrow \tilde{\vec{V}}_{A/B}, \quad \vec{A}_{A/B} \leftrightarrow \tilde{\vec{A}}_{A/B}, \tag{3.7}$$

where

$$\tilde{m}^0_{A/B} = m^0_{A/B}/\mathcal{A}_0, \qquad \tilde{G}_{AB} = G_{AB}\mathcal{A}_0^2, \qquad \tilde{R} = \mathcal{A}_0 R,$$
$$\tilde{\vec{A}}_A = \vec{A}_A/\mathcal{A}_0 \quad \text{and} \quad \tilde{\vec{N}} = \vec{N}, \qquad \tilde{\vec{V}}_{A/B} = \vec{V}_{A/B}. \tag{3.8}$$

As a final remark, the Lagrangian (3.2) generalizes the 2PN general relativity one, obtained by Damour and Deruelle in harmonic coordinates in [28], and reduces to it in the limit $m_A(\varphi) = \mathrm{cst}$, i.e.,

$$\alpha_{A/B} = \beta_{A/B} = \beta'_{A/B} = 0 \tag{3.9}$$

$$\Rightarrow G_{AB} = 1, \quad \bar{\gamma}_{AB} = \bar{\beta}_{A/B} = \delta_{A/B} = \epsilon_{A/B} = \zeta = 0. \tag{3.10}$$

B. The class of reduced Lagrangians

The Lagrangian (3.2) is expressed in harmonic coordinates [i.e., such that $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$] and depends linearly on the accelerations $\vec{A}_A$ at the 2PK level. Let us add to it a (2PK) total time derivative,

$$L \to L + \frac{df}{dt} \equiv L_f, \tag{3.11}$$

where f is a generic function,

$$\begin{aligned}\frac{f}{m_A^0 m_B^0} \equiv{}& G_{AB}[(f_1V_A^2 + f_2\vec{V}_A\cdot\vec{V}_B + f_3V_B^2)(\vec{N}\cdot\vec{V}_A) - (f_4V_A^2 + f_5\vec{V}_A\cdot\vec{V}_B + f_6V_B^2)(\vec{N}\cdot V_B)\\ &+ f_7(\vec{N}\cdot\vec{V}_A)^3 + f_8(\vec{N}\cdot\vec{V}_A)^2(\vec{N}\cdot\vec{V}_B) - f_9(\vec{N}\cdot\vec{V}_B)^2(\vec{N}\cdot\vec{V}_A) - f_{10}(\vec{N}\cdot\vec{V}_B)^3]\\ &+ G_{AB}^2\left[f_{11}\left(\frac{m_A^0}{R}\right)(\vec{N}\cdot\vec{V}_A) + f_{12}\left(\frac{m_B^0}{R}\right)(\vec{N}\cdot\vec{V}_A) - f_{13}\left(\frac{m_A^0}{R}\right)(\vec{N}\cdot\vec{V}_B) - f_{14}\left(\frac{m_B^0}{R}\right)(\vec{N}\cdot\vec{V}_B)\right],\end{aligned} \tag{3.12}$$

that depends on fourteen parameters (the G_{AB} factor appears in the definition of the f_i for dimensional convenience). This total derivative generates a boundary term and hence does not affect the equations of motion.

Now in order to deal with an ordinary Lagrangian (depending only on positions and velocities), a way to proceed is to reduce L_f by "boldly" replacing the accelerations by their leading order, that is Keplerian, on-shell expressions (as was done in [39] in general relativity),

$$L_f \to L_f\left(\vec{A}_A \to -\vec{N}\frac{G_{AB}m_B^0}{R^2}, \vec{A}_B \to \vec{N}\frac{G_{AB}m_A^0}{R^2}\right) \equiv L_f^{\mathrm{red}}. \tag{3.13}$$

This indeed amounts to making an implicit four-dimensional coordinate change, through a contact transformation, (see [30,31]). Hence the equations of motion derived from our reduced Lagrangian L_f^{red} are equivalent to those derived from [23] but written in a different coordinate system, which depends on f. The full expression of the contact transformation is given in Appendix B.

Hence we have on hand a whole class of coordinate systems (depending on the 14 parameters f_i) for which the class of Lagrangians L_f^{red} is ordinary. The harmonic coordinates do not belong to that class.

[5] In the post-Newtonian scheme, these parameters are expanded as series of the compactness $s \sim G_* m/c^2 r$ of weakly self-gravitating bodies [32]. In this paper the orbital velocity $(\frac{v}{c})^2 \sim \frac{G_* m}{c^2 R}$ is the only perturbative parameter. Hence our PK scheme is valid even for strongly self-gravitating bodies.

C. The center-of-mass two-body 2PK Hamiltonians

We now derive the ordinary Hamiltonians, corresponding to the class of coordinate systems discussed above, by a further Legendre transformation,

$$\vec{P}_A = \frac{\partial L_f^{\rm red}}{\partial \vec{V}_A}, \qquad \vec{P}_B = \frac{\partial L_f^{\rm red}}{\partial \vec{V}_B},$$
$$H = \vec{P}_A \cdot \vec{V}_A + \vec{P}_B \cdot \vec{V}_B - L_f^{\rm red}. \tag{3.14}$$

In the center-of-mass frame, $\vec{P}_A + \vec{P}_B \equiv \vec{0}$, and the conjugate variables are then easily checked to be $\vec{Z} \equiv \vec{Z}_A - \vec{Z}_B$ and $\vec{P} \equiv \vec{P}_A = -\vec{P}_B$. At 2PK order, when no spin effects come into play, the relative motion is planar. Hence, it is convenient to use polar coordinates (R, Φ), with conjugate momenta $P_R = (\vec{N} \cdot \vec{P})$, $P_\Phi = R(\vec{N} \times \vec{P})_z$, setting $\theta = \pi/2$. From now on we denote $(Q, P) \equiv (R, \Phi, P_R, P_\Phi)$.

The general structure for an isotropic, translation-invariant, center-of-mass frame, 2PK Hamiltonian $H(Q, P)$ is expected to be

$$\hat{H} \equiv \frac{H}{\mu} = \frac{M}{\mu} + \left(\frac{\hat{P}^2}{2} - \frac{h^{\rm K}}{\hat{R}}\right) + \hat{H}^{\rm 1PK} + \hat{H}^{\rm 2PK} + \cdots \tag{3.15}$$

with

$$\hat{H}^{\rm 1PK} = (h_1^{\rm 1PK}\hat{P}^4 + h_2^{\rm 1PK}\hat{P}^2\hat{P}_R^2 + h_3^{\rm 1PK}\hat{P}_R^4) + \frac{1}{\hat{R}}(h_4^{\rm 1PK}\hat{P}^2 + h_5^{\rm 1PK}\hat{P}_R^2) + \frac{h_6^{\rm 1PK}}{\hat{R}^2}, \tag{3.16a}$$

$$\hat{H}^{\rm 2PK} = (h_1^{\rm 2PK}\hat{P}^6 + h_2^{\rm 2PK}\hat{P}^4\hat{P}_R^2 + h_3^{\rm 2PK}\hat{P}^2\hat{P}_R^4 + h_4^{\rm 2PK}\hat{P}_R^6) + \frac{1}{\hat{R}}(h_5^{\rm 2PK}\hat{P}^4 + h_6^{\rm 2PK}\hat{P}_R^2\hat{P}^2 + h_7^{\rm 2PK}\hat{P}_R^4) + \frac{1}{\hat{R}^2}(h_8^{\rm 2PK}\hat{P}^2 + h_9^{\rm 2PK}\hat{P}_R^2) + \frac{h_{10}^{\rm 2PK}}{\hat{R}^3}, \tag{3.16b}$$

where we have introduced the dimensionless quantities

$$\hat{P}^2 \equiv \hat{P}_R^2 + \frac{\hat{P}_\Phi^2}{\hat{R}^2} \quad \text{with} \quad \hat{P}_R \equiv \frac{P_R}{\mu},$$
$$\hat{P}_\Phi \equiv \frac{P_\Phi}{\mu M}, \qquad \hat{R} \equiv \frac{R}{M}, \tag{3.17}$$

together with the reduced mass, total mass, and symmetric mass ratio,

$$\mu \equiv \frac{m_A^0 m_B^0}{M}, \qquad M \equiv m_A^0 + m_B^0, \qquad \nu \equiv \frac{\mu}{M}. \tag{3.18}$$

The scalar-tensor Hamiltonians derived from the reduced Lagrangians (3.2) and (3.11)–(3.13) fall into the class (3.15) and (3.16), and their seventeen coefficients $h_i^{N\rm PK}$ are computed to be (written here when $f = 0$ for simplicity)

$$h^{\rm K} = G_{AB}, \tag{3.19}$$

at Keplerian order,

$$h_1^{\rm 1PK} = -\frac{1}{8}(1 - 3\nu),$$
$$h_2^{\rm 1PK} = h_3^{\rm 1PK} = 0,$$
$$h_4^{\rm 1PK} = -\frac{G_{AB}}{2}(3 + \nu + 2\bar{\gamma}_{AB}),$$
$$h_5^{\rm 1PK} = -\frac{G_{AB}}{2}\nu,$$
$$h_6^{\rm 1PK} = \frac{G_{AB}^2}{2M}(m_A^0(1 + 2\bar{\beta}_B) + m_B^0(1 + 2\bar{\beta}_A)), \tag{3.20}$$

at 1PK order, and

$$h_1^{\rm 2PK} = \frac{1}{16}(5\nu^2 - 5\nu + 1), \qquad h_2^{\rm 2PK} = h_3^{\rm 2PK} = h_4^{\rm 2PK} = 0,$$
$$h_5^{\rm 2PK} = \frac{G_{AB}}{8}[5 + 4\bar{\gamma}_{AB} - (22 + 16\bar{\gamma}_{AB})\nu - 3\nu^2], \qquad h_6^{\rm 2PK} = -\frac{G_{AB}}{4}\nu(\nu - 1), \qquad h_7^{\rm 2PK} = -\frac{3G_{AB}}{8}\nu^2,$$
$$h_8^{\rm 2PK} = \frac{G_{AB}^2}{8}\left[22 - 4\frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{M} + 4\frac{m_A^0\delta_A + m_B^0\delta_B}{M} + 28\bar{\gamma}_{AB} + 9\bar{\gamma}_{AB}^2 + \nu\left(58 - 4\frac{m_A^0\bar{\beta}_A + m_B^0\bar{\beta}_B}{M} + 36\bar{\gamma}_{AB}\right)\right],$$
$$h_9^{\rm 2PK} = G_{AB}^2\left[-\frac{1}{2} - \frac{1}{2}\frac{m_A^0\delta_A + m_B^0\delta_B}{M} - \frac{\bar{\gamma}_{AB}}{2} - \frac{\bar{\gamma}_{AB}^2}{8} + \nu\left(-4 + (\bar{\beta}_A + \bar{\beta}_B) - 3\bar{\gamma}_{AB} + \frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{M}\right)\right],$$
$$h_{10}^{\rm 2PK} = G_{AB}^3\left[-\frac{1}{2} - \frac{m_B^0\bar{\beta}_A + m_A^0\bar{\beta}_B}{M} - \frac{1}{6}\frac{m_A^0\epsilon_B + m_B^0\epsilon_A}{M} - \frac{1}{3}\frac{m_A^0\delta_A + m_B^0\delta_B}{M} - \frac{\bar{\gamma}_{AB}}{3} - \frac{\bar{\gamma}_{AB}^2}{12}, + \nu\left(-\frac{15}{4} - \zeta + \frac{\bar{\gamma}_{AB}^2}{6} - \frac{4}{3}\bar{\gamma}_{AB} + \frac{\delta_A + \delta_B}{3} + \frac{\epsilon_A + \epsilon_B}{6} - (\bar{\beta}_A + \bar{\beta}_B)\right)\right], \tag{3.21}$$

at 2PK order.

The $f = 0$ scalar-tensor two-body Hamiltonian given above is written in terms of the 17 coefficients $\mathrm{h}_i^{N\mathrm{PK}}$ which are in turn expressed in terms of the 11 constants (3.6) [which are themselves functions of the eight parameters m_A^0, α_A^0, β_A^0, and $\beta_A^{\prime 0}$ characterizing at 2PK order the functions $m_A(\varphi)$ and $m_B(\varphi)$].[6] In the other coordinate systems discussed in Sec. III B, the 14 coefficients of the function f at 2PK order modify the ten 2PK coefficients, which can be found in Appendix C. Each $\{f_i\}$ setting implicitly corresponds to a distinct coordinate system.

D. The canonically transformed class of real Hamiltonians

As discussed in the introduction, the EOB mapping requires imposing a functional relation between the real two-body Hamiltonian $H(Q,P)$ (that is, the ST two-body 2PK class of Hamiltonians obtained in the previous subsection), and an effective Hamiltonian H_e, $H_e = f_{\mathrm{EOB}}(H)$, by means of a canonical transformation.

We thus perform a further general canonical transformation on the real two-body Hamiltonians $H(Q,P)$,

$$(Q,P) \to (q,p), \tag{3.22}$$

where, for the moment, $(q,p) \equiv (r,\phi,p_r,p_\phi)$ is a distinct set of canonical variables with no particular interpretation. The canonical transformation is generated by a time-independent function[7] $F(q,Q)$ such that the Lagrangian is shifted by a total derivative $L_f^{\mathrm{red}}(Q,\dot{Q}) = L'(q,\dot{q}) + dF/dt$ and the Hamiltonian is a scalar $H(Q,P) = H'(q,p)$, so that, see, e.g, (3.14),

$$\begin{aligned} S &\equiv \int (L'dt + dF) = \int (p_r dr + p_\phi d\phi - Hdt + dF) \\ &= \int (P_R dR + P_\Phi d\Phi - Hdt) \quad \text{and thus} \end{aligned}$$

$$dF = P_R dR + P_\Phi d\Phi - (p_r dr + p_\phi d\phi).$$

For practical reasons, we rather consider the generating function $G(Q,p)$ such that

$$\begin{aligned} G &\equiv F + (p_r r + p_\phi \phi) - (p_r R + p_\phi \Phi), \\ \Rightarrow dG(Q,p) &= dR(P_R - p_r) + d\Phi(P_\Phi - p_\phi) \\ &\quad + dp_r(r - R) + dp_\phi(\phi - \Phi), \end{aligned} \tag{3.23}$$

which leads to the canonical transformation

$$\begin{aligned} r(Q,p) &= R + \frac{\partial G}{\partial p_r}, \qquad \phi(Q,p) = \Phi + \frac{\partial G}{\partial p_\phi}, \\ P_R(Q,p) &= p_r + \frac{\partial G}{\partial R}, \qquad P_\Phi(Q,p) = p_\phi + \frac{\partial G}{\partial \Phi}. \end{aligned} \tag{3.24}$$

We now consider a generic ansatz for G that generates 1PK and higher order coordinate changes,[8] which depends on nine parameters,

$$\begin{aligned} \frac{G(Q,p)}{\mu M} = \hat{R}\hat{p}_r \bigg(&\alpha_1 \mathcal{P}^2 + \beta_1 \hat{p}_r^2 + \frac{\gamma_1}{\hat{R}} + \alpha_2 \mathcal{P}^4 \\ &+ \beta_2 \mathcal{P}^2 \hat{p}_r^2 + \gamma_2 \hat{p}_r^4 + \delta_2 \frac{\mathcal{P}^2}{\hat{R}} \\ &+ \epsilon_2 \frac{\hat{p}_r^2}{\hat{R}} + \frac{\eta_2}{\hat{R}^2} + \cdots \bigg), \end{aligned} \tag{3.25}$$

where we have introduced the dimensionless quantities

$$\mathcal{P}^2 \equiv \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}, \quad \hat{R} \equiv \frac{R}{M}, \quad \hat{p}_r \equiv \frac{p_r}{\mu}, \quad \hat{p}_\phi \equiv \frac{p_\phi}{\mu M}. \tag{3.26}$$

We chose this generating function not to depend on Φ so that $P_\Phi = p_\phi$. Also, for circular orbits for which $p_r = 0 \Leftrightarrow P_R = 0$, we note that $\phi = \Phi$ and hence only the radial coordinates differ, $r \neq R$.

Rather than inverting iteratively both first relations of (3.24), the real and effective Hamiltonians are expressed in the following in the intermediate coordinate system (Q,p) for computational convenience. The two-body Hamiltonian (3.15) and (3.16), together with its coefficients (3.19)–(3.21), is transformed to the intermediate coordinate system $H'(Q,p) = H(Q,P)$ using the last two relations in (3.24) and is computed to be (dropping the prime)

$$\hat{H} = \frac{M}{\mu} + \left(\frac{\mathcal{P}^2}{2} - \frac{h^{\mathrm{K}}}{\hat{R}} \right) + \hat{H}^{1\mathrm{PK}} + \hat{H}^{2\mathrm{PK}} + \cdots, \tag{3.27}$$

where $h^{\mathrm{K}} = G_{AB}$, and where the explicit expressions for $\hat{H}^{1\mathrm{PK}}$ and $\hat{H}^{2\mathrm{PK}}$ for a generic function f are given in Appendix D. It depends on the eight fundamental parameters characterizing the theory and the two bodies at 2PK order, that is, m_A^0, α_A^0, β_A^0, ${\beta'_A}^0$ and their B counterparts, on the 14 parameters f_i characterizing the coordinate system used, and the nine parameters of the canonical transformation.

[6]The fact that $h_2^{1\mathrm{PK}}$ and $h_3^{1\mathrm{PK}}$, as well as $h_3^{2\mathrm{PK}}$, $h_3^{2\mathrm{PK}}$, and $h_4^{2\mathrm{PK}}$, vanish is due to the structure of the kinetic term, as is seen in more detail below.

[7]The generating function cannot depend on time since it has to relate two conservative problems.

[8]We know from Newton's theory that once written in the center-of-mass frame, the Keplerian two-body Hamiltonian does not necessitate any further canonical transformation and is the effective-one-body Hamiltonian. From (3.24), one checks that the Keplerian order coordinate change is indeed the identity.

E. The functional relation between the real and effective Hamiltonians

We have obtained a class of ordinary 2PK Hamiltonians that implicitly correspond to different coordinate systems, $H(Q,P)$. By means of a canonical transformation we have transformed them into an even larger class $H(Q,p)$. Our aim in the next section is to find the canonical transformations that relate them to the Hamiltonian H_e of an effective-one-body problem by means of a functional relation, $H_e = f_{\rm EOB}(H)$.

At 2PK order, this functional relation can *a priori* be expanded as, subtracting the rest-mass constants,

$$\frac{H_e(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p)-M}{\mu}\right)\left[1 + \frac{\bar{\nu}_1}{2}\left(\frac{H(Q,p)-M}{\mu}\right) + \bar{\nu}_2\left(\frac{H(Q,p)-M}{\mu}\right)^2 + \cdots\right], \tag{3.28}$$

with the Hamiltonians identifying at Keplerian order. Now, as justified in detail in, e.g., [8,12], and [14] up to at least 4PN in general relativity, and as proven to be true at all orders in GR *as well as* in ST theories in [40] within a post-Minkowskian scheme, the relation must be quadratic *at all orders*, with $\bar{\nu}_1 = \nu = \mu/M$, $\bar{\nu}_2 = 0\cdots$, that is

$$\frac{H_e(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(Q,p)-M}{\mu}\right)\right]. \tag{3.29}$$

As we shall see, H_e will be *uniquely* determined. Inverting (3.29) hence defines the unique, "resummed" EOB Hamiltonian,

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}. \tag{3.30}$$

The dynamics deduced from $H_{\rm EOB}$ and the real Hamiltonians H are, by construction, equivalent up to 2PK order.

The topic of Sec. IV is to propose a scalar-tensor effective-one-body Hamiltonian $H_{\rm EOB}$ that reduces, in the limit where the scalar interaction is switched off, to the EOB Hamiltonian of general relativity obtained in [8].

IV. ST DEFORMATION OF THE GENERAL RELATIVISTIC EOB HAMILTONIAN

In this section, which is the core of the paper, we first recall the structure of the Hamiltonian H_e for geodesic motion in an (effective) static, spherically symmetric metric (in Schwarzschild-Droste coordinates). We then proceed to the EOB mapping and show that the resulting effective metric is unique and can be considered as a scalar-tensor-deformed version of the 2PN results of [8].

A. The 2PN geodesic dynamics in an effective external one-body metric

Let us consider a static, spherically symmetric metric, written in Schwarzschild-Droste coordinates (for $\theta = \pi/2$),

$$ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2. \tag{4.1}$$

The geodesic dynamics of a test particle coupled to this external metric, with mass μ [which is identified to the real two-body reduced mass defined in (3.18)], is described by the Lagrangian

$$L_e = -\mu\sqrt{-g^e_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -\mu\sqrt{A - B\dot{r}^2 - r^2\dot{\phi}^2} \tag{4.2}$$

where $\dot{r} \equiv dr/dt$, $\dot{\phi} \equiv d\phi/dt$. The (dimensionless) effective Hamiltonian is

$$\hat{H}_e \equiv \frac{H_e}{\mu} = \sqrt{A\left(1 + \frac{\hat{p}_r^2}{B} + \frac{\hat{p}_\phi^2}{\hat{r}^2}\right)} \quad \text{with} \quad p_r \equiv \frac{\partial L_e}{\partial \dot{r}}, \qquad p_\phi \equiv \frac{\partial L_e}{\partial \dot{\phi}}, \tag{4.3}$$

and where

$$\hat{r} \equiv \frac{r}{M}, \quad \hat{p}_r \equiv \frac{p_r}{\mu}, \quad \hat{p}_\phi \equiv \frac{p_\phi}{\mu M}, \quad \hat{p}^2 \equiv \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{r}^2}, \tag{4.4}$$

M being an effective mass, identified with the real two-body total mass.

Now, the A and B metric functions are generically expanded (at the required 2PN order) as

$$A(r) = 1 + \frac{a_1}{\hat{r}} + \frac{a_2}{\hat{r}^2} + \frac{a_3}{\hat{r}^3} + \cdots, \quad B(r) = 1 + \frac{b_1}{\hat{r}} + \frac{b_2}{\hat{r}^2} + \cdots, \tag{4.5}$$

where a_1, a_2, a_3, b_1, and b_2 are the five (dimensionless) effective parameters to be determined. The 2PN effective Hamiltonian then becomes

$$\hat{H}_e = 1 + \hat{H}_e^{\rm N} + \hat{H}_e^{\rm 1PN} + \hat{H}_e^{\rm 2PN} + \cdots, \tag{4.6}$$

where

$$\hat{H}_e^{\rm N} = \frac{\hat{p}^2}{2} + \frac{a_1}{2\hat{r}},$$
$$\hat{H}_e^{\rm 1PN} = -\frac{\hat{p}^4}{8} - \hat{p}_r^2\frac{b_1}{2\hat{r}} + \frac{1}{4}\frac{a_1}{\hat{r}}\hat{p}^2 + \frac{a_2 - a_1^2/4}{2\hat{r}^2},$$
$$\hat{H}_e^{\rm 2PN} = \frac{\hat{p}^6}{16} - \frac{\hat{p}^2(a_1\hat{p}^2 - 4b_1\hat{p}_r^2)}{16\hat{r}} + \frac{(4a_2 - a_1^2)\hat{p}^2 + 4(2b_1^2 - 2b_2 - a_1b_1)\hat{p}_r^2}{16\hat{r}^2} + \frac{a_1^3 - 4a_1a_2 + 8a_3}{16\hat{r}^3}.$$

In the previous section we performed a generic canonical transformation $(Q,P)\to(q,p)$ and wrote the real Hamiltonians $H(Q,P)$ in terms of the intermediate coordinates (Q,p). In order to be in a position to relate them to the effective Hamiltonian $H_e(q,p)$ considered here, we have to express the latter in the same variables (Q,p) by means of the generic canonical transformation (3.24) and (3.25). We thus turn it into the class of Hamiltonians (recalling the notation $\mathcal{P}^2 \equiv \hat{p}_r^2 + \hat{p}_\phi^2/\hat{R}^2$),

$$\hat{H}_e = 1 + \left(\frac{\mathcal{P}^2}{2} - \frac{a_1}{2\hat{R}}\right) + \hat{H}_e^{\rm 1PK} + \hat{H}_e^{\rm 2PK} + \cdots, \tag{4.7}$$

where the explicit expressions for $\hat{H}_e^{\rm 1PK}$ and $\hat{H}_e^{\rm 2PK}$ are given in Appendix E. These Hamiltonians depend on the five parameters a_i and b_i entering the effective metric coefficients at 2PN order, see (4.5), and on the nine parameters entering the canonical transformation (3.24) and (3.25).

B. The scalar-tensor effective-one-body metric at 2PK order

As we saw in Sec. III E the effective Hamiltonian H_e and the two-body 2PK Hamiltonians H obtained in Sec. III must be related through the quadratic relation (3.29), that is

$$\frac{H_e(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(Q,p)-M}{\mu}\right)\right].$$

Consider now the generic (theory-agnostic) two-body Hamiltonian written in terms of the 17 coefficients $h_i^{N{\rm PK}}$; see (3.15) and (3.16). It turns out that an effective H_e can be constructed at 1PK level provided that

$$2h_2^{\rm 1PK} + 3h_3^{\rm 1PK} = 0. \tag{4.8}$$

Any theory (such as scalar-tensor) whose purely kinetical terms take the form $m_A^0\sqrt{1-V_A^2} + m_B^0\sqrt{1-V_B^2}$ at the Lagrangian level is such that $h_2^{\rm 1PK} = 0$ and $h_3^{\rm 1PK} = 0$ (as anticipated in footnote 6). Thus this condition is not restrictive. At 2PK level, the identification requires two further conditions; the first one

$$h_4^{\rm 2PK} = -\frac{2}{45}(12h_2^{\rm 2PK} + 18h_3^{\rm 2PK} + (h_2^{\rm 1PK})^2) \tag{4.9}$$

is no more restrictive than (4.8), for the same reasons. The second condition however,

$$h_1^{\rm 2PK} + \frac{7}{3}h_2^{\rm 2PK} + h_3^{\rm 2PK} + h_5^{\rm 2PK} + h_6^{\rm 2PK} + h_7^{\rm 2PK} = -\frac{h^{\rm K}}{128}(5+2\nu+5\nu^2) + \frac{1}{8}(1+\nu)((3h_1^{\rm 1PK} + h_2^{\rm 1PK})h^{\rm K} + h_4^{\rm 1PK} + h_5^{\rm 1PK}) + \frac{5}{2}h_1^{\rm 1PK}(7h_1^{\rm 1PK}h^{\rm K} + 2(h_4^{\rm 1PK} + h_5^{\rm 1PK})) + \frac{1}{6}h_2^{\rm 1PK}(13h_2^{\rm 1PK}h^{\rm K} + 10(h_4^{\rm 1PK} + h_5^{\rm 1PK})) + \frac{35}{3}h_1^{\rm 1PK}h_2^{\rm 1PK}h^{\rm K}, \tag{4.10}$$

is restrictive and the mapping of the two-body problem towards an effective geodesic is only possible for a subclass of theories.

In the scalar-tensor case, one checks that the coefficients (3.19)–(3.21) (see also Appendix C) *do* satisfy the condition (4.10) *whatever* the values of the 14 f_i parameters, that is, independently of the coordinate system in which the two-body Hamiltonian has been written, as they should.[9]

Inserting now in the functional relation (3.29) recalled above the explicit expressions for the ST coefficients $h_i^{N{\rm PK}}$ of the real two-body Hamiltonians H obtained in the previous section, the identification term by term is then seen to yield a *unique* solution for H_e and hence for the effective one-body metric, which can be written as

[9]The relation (4.10) is thus also verified by general relativity. In contrast, it is not satisfied by Electrodynamics at second post-Coulombian order (see [41]). It is an exercise to extend this computation to 3PK. The identification then requires a further condition at 2PK, which first is not natural and, second, is not satisfied by the ADM Hamiltonian of general relativity [8]. In consequence, this condition will not be satisfied in scalar-tensor theories either (since they include GR as a limit) and, as in GR [12], it will no longer be possible to map the two body problem towards a geodesic.

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}]\left(\frac{G_{AB}M}{r}\right)^2 + [2\nu + \delta a_3^{\rm ST}]\left(\frac{G_{AB}M}{r}\right)^3 + \cdots, \tag{4.11}$$

$$B(r) = 1 + 2[1 + \bar{\gamma}_{AB}]\left(\frac{G_{AB}M}{r}\right) + [2(2-3\nu) + \delta b_2^{\rm ST}]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots, \tag{4.12}$$

where

$$\begin{aligned}\delta a_3^{\rm ST} \equiv \frac{1}{12}[&-20\bar{\gamma}_{AB} - 35\bar{\gamma}_{AB}^2 - 24\langle\bar{\beta}\rangle(1-2\bar{\gamma}_{AB}) + 4(\langle\delta\rangle - \langle\epsilon\rangle)\\ &+ \nu(-36(\bar{\beta}_A + \bar{\beta}_B) + 4\bar{\gamma}_{AB}(10 + \bar{\gamma}_{AB}) + 4(\epsilon_A + \epsilon_B) + 8(\delta_A + \delta_B) - 24\zeta)],\end{aligned} \tag{4.13}$$

$$\delta b_2^{\rm ST} \equiv \left[4\langle\bar{\beta}\rangle - \langle\delta\rangle + \bar{\gamma}_{AB}\left(9 + \frac{19}{4}\bar{\gamma}_{AB}\right) + \nu(2\langle\bar{\beta}\rangle - 4\bar{\gamma}_{AB})\right], \tag{4.14}$$

where we introduced the "mean" quantities

$$\langle\bar{\beta}\rangle \equiv \frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{M}, \qquad \langle\delta\rangle \equiv \frac{m_A^0\delta_A + m_B^0\delta_B}{M}, \qquad \langle\epsilon\rangle \equiv \frac{m_A^0\epsilon_B + m_B^0\epsilon_A}{M}. \tag{4.15}$$

That is the main result of this paper, which shows that one can interpret 2PK scalar-tensor theories as a deformation of the 2PN general relativity results [8] [which are retrieved in the limit (3.10)].

A few comments are in order.

(i) one sees that the bare (dimensionless) gravitational constant is replaced by the effective one $G_* \to G_{AB}$ at every order[10]

(ii) one recognizes, at 1PK level, a parametrized post-Newtonian Eddington metric written in Droste coordinates, with

$$\beta^{\rm Edd} = 1 + \langle\bar{\beta}\rangle, \qquad \gamma^{\rm Edd} = 1 + \bar{\gamma}_{AB}$$

being the Eddington parameters, such that $\beta^{\rm Edd} = \gamma^{\rm Edd} = 1$ in general relativity. Interestingly, these effective Eddington parameters encompass the self-gravity of both real bodies through the simple mean quantities (4.15), extending the results of [40].[11] The reader should note that the two-body parameters (3.6) were initially defined in [22] consistently with the parametrized-post-Newtonian (PPN) approach, where the N-body problem is to be interpreted as point particles following geodesics of a PPN *metric*. Hence it is not surprising that properties (i) and (ii) emerge in the context of a metric effective problem.

It must also be noted that the effective metric does not depend on the function f introduced in Sec. III B, i.e., on the coordinate system (R, Φ) in which the two-body Hamiltonian is written. Indeed, as it should be, f is absorbed by the canonical transformation (3.24) and (3.25), whose parameters are found to be

$$\begin{aligned}&\alpha_1 = -\frac{\nu}{2}, \qquad \beta_1 = 0, \qquad \gamma_1 = G_{AB}\left(1 + \bar{\gamma}_{AB} + \frac{\nu}{2}\right), \qquad \alpha_2 = \frac{1}{8}(1-\nu)\nu, \qquad \beta_2 = 0, \qquad \gamma_2 = \frac{\nu^2}{2},\\ &\delta_2 = G_{AB}\left[f_6\frac{m_A^0}{M} + f_1\frac{m_B^0}{M} - \nu\left(f_1 + f_6 + (-f_3 + f_5 + f_6)\frac{m_A^0}{M} + (f_1 + f_2 - f_4)\frac{m_B^0}{M} - \frac{3}{2}(1 + \bar{\gamma}_{AB}) + \frac{\nu}{8}\right)\right],\\ &\epsilon_2 = G_{AB}\left[-\frac{\nu^2}{8} + f_{10}\frac{m_A^0}{M} + f_7\frac{m_B^0}{M} - \nu\left(f_7 + f_{10} + (f_9 + f_{10})\frac{m_A^0}{M} + (f_7 + f_8)\frac{m_B^0}{M}\right)\right],\\ &\eta_2 = \frac{1}{8}G_{AB}^2[8\langle\bar{\beta}\rangle - 4\langle\delta\rangle + 4\bar{\gamma}_{AB} + 3\bar{\gamma}_{AB}^2 + \nu(-38 + 4(\bar{\beta}_A + \bar{\beta}_B) - 24\bar{\gamma}_{AB} + 2\nu)]\\ &\qquad + G_{AB}^2\left(f_{13}\frac{m_A^0}{M} + f_{12}\frac{m_B^0}{M} + \nu(f_{11} - f_{12} - f_{13} + f_{14})\right),\end{aligned} \tag{4.16}$$

and reduce to the general relativity (ADM) values of [8] in the limit (3.10), and (C1).

[10]As was done up to 2PK, the parameters (3.6) can always be defined so as to factorize out the appropriate $(G_{AB})^n$ factor corresponding to any R^n term at the Lagrangian level (3.2). One may then anticipate this property to hold at higher PK orders, so that the coordinate R always comes in the form R/G_{AB}.

[11]We thank Thibault Damour for having pointed out to us this important feature.

In this section, the scalar-tensor two-body problem has thus been mapped towards the geodesic of an effective external metric (4.1), written in Droste coordinates, which is well suited when scalar-tensor effects are to be considered as perturbative with respect to general relativity. We turn in the next subsection to the study of some aspects of the dynamics this EOB problem defines.

C. The 2PK effective problem as a ST deformation of general relativity at 2PN order

Solar system and binary pulsar observations have put stringent constraints on scalar-tensor theories. In particular, the decay rate of the orbital period of binary pulsars (excluding dipolar radiation) has led to the constraint (see [42,43])

$$(\alpha_A^0)^2 < 4 \times 10^{-6}, \tag{4.17}$$

for any body A, regardless of its self-gravity or equation of state.[12] Now, the two-body Lagrangian parameters (3.6) are all driven by a factor $(\alpha_{A/B}^0)^i$, where $i \geq 2$ (as can be understood from the diagrammatic approach of [32]) and can be conjectured to be all of the same order. This overall factor is also seen to appear at the level of the scalar-tensor corrections to $A(r)$ and $B(r)$ at any PK order; see (4.11)–(4.14).

Hence, the dynamics defined by the effective metric (4.1), that is

$$ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2, \tag{4.18}$$

with A and B given in (4.11)–(4.14), is particularly well suited to regard scalar-tensor effects as perturbations to general relativity. Remarkably, and as we recall below, when studying the conservative dynamics of circular orbits in Droste coordinates, only the g_{00}^e component of the metric intervenes and can be written as

$$A(r) = A_{2\mathrm{PN}}^{\mathrm{GR}}\left(\frac{G_{AB}M}{r};\nu\right) + \delta A^{\mathrm{ST}}\left(\frac{G_{AB}M}{r};\nu\right), \tag{4.19}$$

where $A_{2\mathrm{PN}}^{\mathrm{GR}}(\frac{G_{AB}M}{r};\nu)$ is the 2PN GR limit obtained by Buonanno-Damour (with $M \to G_{AB}M$) and where, as can be seen from (4.11)–(4.13),

$$\delta A^{\mathrm{ST}} = \delta A_{1\mathrm{PK}}^{\mathrm{ST}} + \delta A_{2\mathrm{PK}}^{\mathrm{ST}}, \tag{4.20a}$$

$$\delta A_{1\mathrm{PK}}^{\mathrm{ST}}(r) = 2\left(\frac{G_{AB}M}{r}\right)^2 [\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}],$$

$$\delta A_{2\mathrm{PK}}^{\mathrm{ST}}(r) = \left(\frac{G_{AB}M}{r}\right)^3 \delta a_3^{\mathrm{ST}}(\nu). \tag{4.20b}$$

[12] In particular, this bound constrains certain classes of scalar-tensor theories, which predict that strongly self-gravitating bodies such as neutron stars can develop significant scalar "charges," i.e., a significant α_A^0 parameter, even when $\alpha^0 = \frac{d\ln A}{d\varphi}|_{\varphi_0}$ is vanishingly small; see [36].

Introducing finally the notations

$$\frac{G_{AB}M}{r} \equiv u, \qquad \langle\bar{\beta}\rangle - \bar{\gamma}_{AB} \equiv \epsilon_{1\mathrm{PK}},$$
$$\delta a_3^{\mathrm{ST}}(\nu) \equiv \epsilon_{2\mathrm{PK}}^0 + \nu\epsilon_{2\mathrm{PK}}^\nu,$$

$$\text{where } \epsilon_{2\mathrm{PK}}^0 \equiv \frac{1}{12}[-20\bar{\gamma}_{AB} - 35\bar{\gamma}_{AB}^2 - 24\langle\bar{\beta}\rangle(1-2\bar{\gamma}_{AB}) + 4(\langle\delta\rangle - \langle\epsilon\rangle)],$$

$$\epsilon_{2\mathrm{PK}}^\nu \equiv -3(\bar{\beta}_A + \bar{\beta}_B) + \frac{1}{3}\bar{\gamma}_{AB}(10+\bar{\gamma}_{AB}) + \frac{1}{3}(\epsilon_A + \epsilon_B) + \frac{2}{3}(\delta_A + \delta_B) - 2\zeta, \tag{4.21}$$

(4.19) simply reads

$$A(u) = A_{2\mathrm{PN}}^{\mathrm{GR}}(u;\nu) + 2\epsilon_{1\mathrm{PK}}u^2 + (\epsilon_{2\mathrm{PK}}^0 + \nu\epsilon_{2\mathrm{PK}}^\nu)u^3. \tag{4.22}$$

Therefore, the scalar-tensor 2PK corrections to the g_{00}^e component of the effective general relativistic 2PN metric are completely described by three parameters $(\epsilon_{1\mathrm{PK}}, \epsilon_{2\mathrm{PK}}^0, \epsilon_{2\mathrm{PK}}^\nu)$ that are numerically of the same order of magnitude [since they are driven by $(\alpha_{A/B}^0)^2$].

When scalar-tensor effects are to be considered as perturbative, our result (4.22) can be refined by replacing $A_{2\mathrm{PN}}^{\mathrm{GR}}(u;\nu)$ by the currently best available general relativity EOB results, to which we turn now.

D. ST-parametrized EOB dynamics

We now propose to evaluate the effect of these ST post-Keplerian corrections to the general relativistic predictions for the innermost stable circular orbit (ISCO) frequency. To do so, we do not restrict ourselves to the 2PN GR expression for $A_{2\mathrm{PN}}^{\mathrm{GR}}(u;\nu)$ but use instead the best available EOB-NR function $A^{\mathrm{GR}}(u;\nu)$ (in the nonspinning case)

$$A^{\mathrm{GR}}(u;\nu) = \mathcal{P}_5^1[A_{5\mathrm{PN}}^{\mathrm{Taylor}}], \tag{4.23}$$

i.e., the (1,5) Padé approximant of the truncated 5PN expansion

$$A_{5\mathrm{PN}}^{\mathrm{Taylor}} = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + (a_5^c + a_5^{\ln}\ln u)u^5 + \nu(a_6^c + a_6^{\ln}\ln u)u^6, \tag{4.24}$$

where $a_6^c(\nu)$ has been obtained by calibration with numerical relativity results, the other coefficients being known analytically; see [44–46] for their explicit expressions. Comparing (4.22) to (4.24), scalar-tensor effects are clearly seen to induce a quadratic $\mathcal{O}(u^2)$ term that does not exist in general relativity, and a (ν-dependent) correction to the cubic $\mathcal{O}(u^3)$ coefficient.

One could, in principle, phenomenologically anticipate ST corrections coming from higher PK orders. However, first, it is known from general relativity that from 3PN order on, the effective dynamics cannot be that of a pure geodesic anymore (as mentioned in footnote 9). Second, the two-body 3PK Lagrangian is not known in scalar-tensor theories. We hence leave these questions to future work and, for the time being, content ourselves with the study of the ST 2PK corrections only.

The study of the dynamics is now straightforward. By staticity and spherical symmetry of the metric (4.18),

$$u_t = -A\frac{dt}{d\lambda} \equiv -E, \qquad u_\phi = r^2\frac{d\phi}{d\lambda} \equiv L \tag{4.25}$$

are the conserved energy and angular momentum of the orbit, per unit mass μ. One also normalizes the four-velocity $u^\alpha u_\alpha = -\epsilon$, where $\epsilon = 1$ for $\mu \neq 0$, $\epsilon = 0$ for null geodesics ($\mu = 0$). The radial motion in the metric (4.18) is hence determined by

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{AB}F(u), \tag{4.26}$$

$$\text{where } F(u) \equiv E^2 - A(u)(\epsilon + j^2u^2),$$
$$j \equiv \frac{L}{G_{AB}M}, \qquad u \equiv \frac{G_{AB}M}{r}. \tag{4.27}$$

In the following we focus on circular orbits, assuming that gravitational radiation has suppressed any eccentricity during the early inspiral.[13]

When $\epsilon = 1$, the radial velocity vanishes when $F(u) = 0$, while the circularity of the orbit also requires $F'(u) = 0$; hence j^2 and E are related to u by

$$j^2(u) = -\frac{A'}{(Au^2)'}, \qquad E(u) = A\sqrt{\frac{2u}{(Au^2)'}}. \tag{4.28}$$

The ISCO requires the third (inflection point) condition $F''(u) = 0$; i.e., $u_{\rm ISCO}$ is the root of the equation,

$$F'(u_{\rm ISCO}) = F''(u_{\rm ISCO}) = 0 \Rightarrow \frac{A''}{A'} = \frac{(Au^2)''}{(Au^2)'}. \tag{4.29}$$

[As anticipated in the previous subsection the circular orbits are determined by the function $A(u)$ only.]

Let us now turn to the real two-body dynamics. The quadratic relation between the real and effective Hamiltonians H and H_e (3.28) can be inverted to yield the EOB Hamiltonian; see (3.30),

$$H_{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_e}{\mu}-1\right)}, \quad \text{where}$$
$$\frac{H_e}{\mu} = \sqrt{A\left(1+\frac{\hat{p}_r^2}{B}+\frac{\hat{p}_\phi^2}{\hat{r}^2}\right)}, \tag{4.30}$$

which defines a resummed two-body dynamics. Since $H_{\rm EOB}$ and H_e are conservative, we have on shell

$$\left(\frac{\partial H_{\rm EOB}}{\partial H_e}\right) = \frac{1}{\sqrt{1+2\nu(E-1)}} \quad \text{since}$$
$$H_e = \mu E \quad \text{on shell}. \tag{4.31}$$

Hence the real (two-body problem) equations of motion have been drastically simplified, since they now read

$$\frac{dr}{dt} = \frac{\partial H_{\rm EOB}}{\partial p_r}, \qquad \frac{d\phi}{dt} = \frac{\partial H_{\rm EOB}}{\partial p_\phi},$$
$$\frac{dp_r}{dt} = -\frac{\partial H_{\rm EOB}}{\partial r}, \qquad \frac{dp_\phi}{dt} = -\frac{\partial H_{\rm EOB}}{\partial \phi} = 0, \tag{4.32}$$

and are identical to the effective equations of motion, to within the *constant* multiplicative factor (4.31), i.e., a simple time rescaling $t \to t\sqrt{1+2\nu(E-1)}$. Consequently, the effective orbital frequency [deduced from Hamilton's equations, or equivalently from (4.25)] being given by

$$\omega(u) \equiv \frac{d\phi}{dt} = \frac{\partial H_e}{\partial p_\phi} = \frac{ju^2A}{G_{AB}ME},$$

the real frequency, deduced from $H_{\rm EOB}$, is

$$\Omega(u) = \frac{\partial H_{\rm EOB}}{\partial H_e}\frac{\partial H_e}{\partial p_\phi} = \frac{ju^2A}{G_{AB}ME\sqrt{1+2\nu(E-1)}}, \tag{4.33}$$

where $E(u)$ and $j(u)$ are given for circular orbits in (4.28).[14]

Figure 1 below shows the ISCO location in Droste coordinates and associated frequency when $\nu = 1/4$ in the following cases:

(i) considering only 1PK corrections, i.e., keeping only the $\mathcal{O}(u^2)$, $\epsilon_{\rm 1PK}$ term in (4.22). Note that $\epsilon_{\rm 1PK}$ can be negative. For instance, for identical bodies, $\epsilon_{\rm 1PK} = (\alpha_A^0)^2(\frac{1}{2}\beta_A^0+2)+(\mathcal{O}(\alpha_A^0)^4)$ is driven by an overall $(\alpha_A^0)^2$ factor but is negative when $\beta_A^0 < -4+\mathcal{O}((\alpha_A^0)^2)$;

(ii) adding 2PK, $\mathcal{O}(u^3)$ corrections. As discussed above, in scalar-tensor theories the 2PK coefficients are expected to be of the same order; we hence incorporate them and, for simplicity, limit ourselves to the specific example,

$$\epsilon_{\rm 2PK}^0 + \nu\epsilon_{\rm 2PK}^\nu \equiv \epsilon_{\rm 1PK}$$

in the equal-mass case ($\nu = 1/4$).

[13]Note that for circular orbits, motion can still be considered as geodesic at 3 and higher PN orders in general relativity; see [12].

[14]The orbital frequency has been derived in the "effective," Droste, coordinate system, (q,p). The real coordinates, (Q,P), are linked to (q,p) through the canonical transformation (3.24) and (3.25) so that $\Phi \neq \phi$ in general. However, $\Phi = \phi$ for circular orbits ($p_r = P_R = 0$) and hence (4.33) is the real, observed, orbital frequency. Indeed, only the radial coordinates differ $r \neq R$, but are not observables. See Sec. III D.

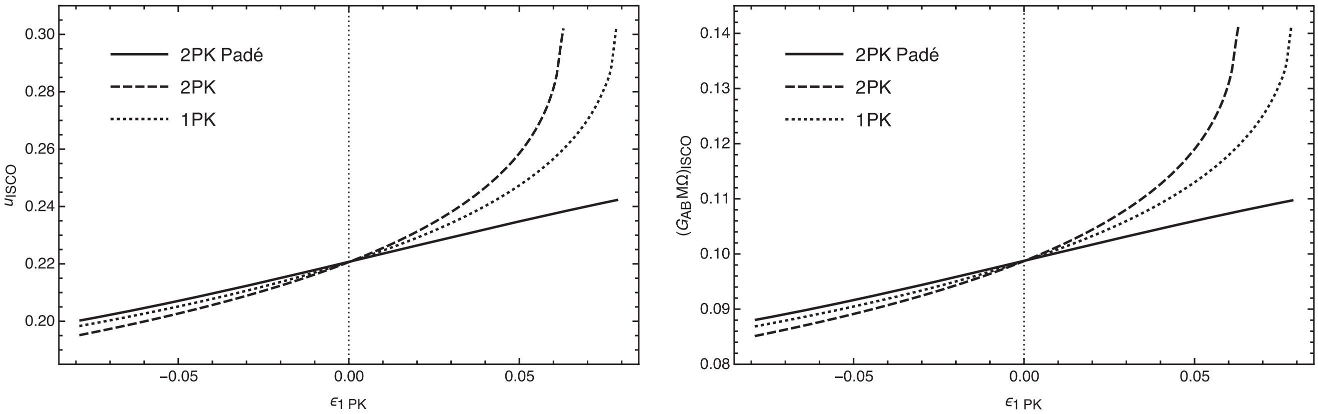

FIG. 1. Scalar-tensor corrections to the ISCO location in Droste coordinates (left panel) and associated frequency (right panel) vs ϵ_{1PK} for $\nu = 0.25$ and for $\epsilon^0_{2PK} + \nu\epsilon^\nu_{2PK} = \epsilon_{1PK}$. General relativity is recovered when $\epsilon_{1PK} = 0$. The first (dotted lines) and second (dashed lines) PK corrections quickly lead to divergences. The overall Padé resummation (solid line) cures them efficiently. The table gathers a few numerical values in the 2PK Padé resummed case; $x \equiv G_{AB}M\Omega_{ISCO}/(G_{AB}M\Omega_{ISCO})_{GR}$.

In both cases, the ISCO location and frequency are seen to increase dramatically as soon as ϵ_{1PK} approaches $\sim 10^{-1}$. What is happening here is similar to what was discussed at 3PN order in general relativity in [12]: when ϵ_{1PK} becomes too large and positive, the function $A(u)$ is no longer a good representation of the scalar-tensor deformations, since, in particular, $A(u)$ has no 0 anymore (in particular, it does not exhibit a horizon). This phenomenon is another reason to recall that this effective geodesic should be taken seriously only when scalar-tensor corrections are to be considered as perturbative (here, $\epsilon_{1PK} \ll 1$).

(iii) For that reason, we follow the suggestion of [12] and further resum $A(u)$ through an overall Padé approximant, by continuity with the general relativity ($\epsilon_{1PK} = \epsilon^0_{2PK} = \epsilon^\nu_{2PK} = 0$) limit,

$$A^{2PK}(u) \equiv \mathcal{P}^1_5[A^{\mathrm{Taylor}}_{5PN} + 2\epsilon_{1PK}u^2 + (\epsilon^0_{2PK} + \nu\epsilon^\nu_{2PK})u^3], \tag{4.34}$$

ensuring also that $A(u)$ has a simple 0. As one can see from Fig. 1, the divergences are then efficiently cured.[15]

The ISCO frequency is roughly linear in ϵ_{1PK}. The slope, or sensitivity of the ISCO frequency to scalar-tensor corrections, is

$$\left.\frac{d(G_{AB}M\Omega)_{ISCO}}{d\epsilon_{1PK}}\right|_{\nu=1/4} \simeq 0.13, \qquad \left.\frac{d(G_{AB}M\Omega)_{ISCO}}{d\epsilon_{1PK}}\right|_{\nu=0} \simeq 0.048. \tag{4.35}$$

Finally, the relative correction reaches a few percent when $\epsilon_{1PK} \sim 10^{-2}$; see the x column in the table below. It seems thus unlikely that measurements of this specific effect lead to improvements to the current (binary pulsar) constraints on scalar-tensor theories (4.17).

$\epsilon > 0$	u_{ISCO}	$G_{AB}M\Omega_{ISCO}$	x	$\epsilon < 0$	u_{ISCO}	$G_{AB}M\Omega_{ISCO}$	x
10^{-3}	0.2209	0.09886	1.002	-10^{-3}	0.2203	0.09857	0.9985
2.5×10^{-3}	0.2213	0.09908	1.004	-2.5×10^{-3}	0.2199	0.09835	0.9963
5×10^{-3}	0.2221	0.09945	1.008	-5×10^{-3}	0.2192	0.09798	0.9926
7.5×10^{-3}	0.2228	0.09982	1.011	-7.5×10^{-3}	0.2185	0.09761	0.9889
10^{-2}	0.2235	0.1002	1.012	-10^{-2}	0.2178	0.09725	0.9852
2.5×10^{-2}	0.2278	0.1024	1.038	-2.5×10^{-2}	0.2137	0.09510	0.9634
5×10^{-2}	0.2349	0.1060	1.074	-5×10^{-2}	0.2072	0.09168	0.9287
7.5×10^{-2}	0.2414	0.1093	1.107	-7.5×10^{-2}	0.2011	0.08851	0.8966

[15] See [47] for a different resummation method.

The study of circular geodesics in the metric (4.18) has allowed us to describe the impact of the 2PK ST deviations to general relativity (4.34) on the ISCO frequency. In fact, as discussed in Sec. IV B, any theory whose two-body Lagrangian verifies the constraints (4.8)–(4.10) may also be mapped towards an effective geodesic. This suggests, by extension of the ST results, that (4.34) takes the generic parametrized form

$$A^{\mathrm{PEOB}}(u) \equiv \mathcal{P}^1_5[A^{\mathrm{Taylor}}_{5\mathrm{PN}} + 2(\epsilon^0_{1\mathrm{PK}} + \nu\epsilon^\nu_{1\mathrm{PK}})u^2 + (\epsilon^0_{2\mathrm{PK}} + \nu\epsilon^\nu_{2\mathrm{PK}})u^3], \quad (4.36)$$

where $\epsilon^0_{1\mathrm{PK}}$, $\epsilon^\nu_{1\mathrm{PK}}$, $\epsilon^0_{2\mathrm{PK}}$, and $\epsilon^\nu_{2\mathrm{PK}}$ are now to be regarded as theory-agnostic parametrized EOB (PEOB) coefficients, and is suitable to encompass the (conservative) dynamics of a generic deviation to general relativity at 2PK order. We note that no Keplerian parameter is needed since it can always be absorbed by a redefinition of the total mass (see, for example, G_{AB} in the ST case). For scalar-tensor theories, $\epsilon^\nu_{1\mathrm{PK}} = 0$ and $\epsilon^0_{1\mathrm{PK}} \sim \epsilon^0_{2\mathrm{PK}} \sim \epsilon^\nu_{2\mathrm{PK}}$.

V. CONCLUDING REMARKS

It is a remarkable fact that the EOB approach can be extended beyond the framework of general relativity: the two-body (2PK) problem has indeed been mapped here towards the geodesic of an effective metric in Schwarzschild-Droste coordinates. This paper is (to our knowledge) the first EOB description of a modified gravity, in the simplest example of massless scalar-tensor theories.

This mapping has led to a much simpler and compact (still, canonically equivalent) description of the two-body conservative dynamics in the 2PK regime, partly hiding some of the irrelevant information of its Hamiltonian in an appropriate canonical transformation. The effective problem also defines a resummation of the two-body dynamics that may capture some of its strong-field features, in particular, concerning the ISCO frequencies. In a second paper (in preparation), we shall build *another* EOB Hamiltonian that maps the two-body problem to a ν-deformed version of the scalar-tensor one-body problem.

The general relativity EOB approach has been extended in [48] to the case of binary neutron stars. There, tidal effects were phenomenologically included by adding corrections to the $-g^e_{00} = A(u)$ part of the effective (Droste) metric, starting at 5PN order, i.e., $\mathcal{O}(u^6)$ (tidal EOB, or TEOB). In contrast, our work should be regarded as a different extension, towards parametrized scalar-tensor theories (PEOB), and modifies the effective metric at 1PK order already, i.e., $\mathcal{O}(u^2)$ in $A(u)$. When applied to neutron stars, the scalar-tensor corrections must be compared to tidal effects. Our model shows that the PEOB $\mathcal{O}(u^2)$ corrections are generically numerically much smaller than the TEOB $\mathcal{O}(u^6)$ correction close to the merger, assuming the constraint $(\alpha^0_{A/B})^2 < 4\times10^{-6}$ discussed in Sec. IV C. However, systems that are subject to *dynamical* scalarization [49] may develop nonperturbative scalar charges in the strong-field regime, and hence escape this constraint. In that case, the ISCO frequency can be significantly modified as soon as $(\alpha^0_{A/B})^2 \gtrsim 10^{-2}$; see Fig. 1.

When it comes to the question of binary black holes, it is well known that static black holes in the scalar-tensor theories we are considering here cannot carry scalar hair and reduce to the Schwarzschild solution. However, this may no longer be true in the strong-field, dynamical regime (i.e., near merger), which is *precisely* explored by the EOB approach. Moreover, scalar hair can be induced by means of a potential $V(\varphi)$ or massless gauge fields. We leave the investigation of such effects to further work.

It should also be noted that while Solar System and binary pulsar observations have put stringent constraints on scalar-tensor theories, gravitational wave detectors are designed to detect highly redshifted sources, that is, at cosmological epochs when scalar-tensor effects may have been more manifest (see, e.g., [50] or [51]). Therefore gravitational wave astronomy should be regarded as an opportunity to constrain also the cosmological history of scalar-tensor theories.

Finally, we restricted ourselves in this paper to the conservative part of the dynamics of the scalar-tensor two-body problem. The corresponding EOB radiation reaction force and gravitational waveforms still remain to be investigated and are the topic of further work.

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APPENDIX A: EINSTEIN VS JORDAN FRAME—CONVERSION OF THE TWO-BODY PARAMETERS

In this appendix we convert the parameters appearing in the two-body (harmonic) Lagrangian of [23] using the conventions introduced in [22]. The scalar-tensor action reads, in the Einstein frame (see Sec. I),

$$S_{\mathrm{EF}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) + S_m[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}], \quad (A1)$$

while in the conventions of [23], the action is written in the Jordan frame as

$$S_{\mathrm{JF}} = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left(\phi\tilde{R} - \frac{\omega(\phi)}{\phi}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) + S_m[\Psi, \tilde{g}_{\mu\nu}]. \tag{A2}$$

Hence, for a given function $\mathcal{A}(\varphi)$ characterizing the ST theory in the Einstein frame, the Jordan metric and function $\omega(\phi)$ characterizing it in the Jordan frame are given by

$$\tilde{g}_{\mu\nu} = \mathcal{A}^2 g_{\mu\nu}, \quad \alpha = \frac{d\ln\mathcal{A}(\varphi)}{d\varphi}, \quad 3 + 2\omega(\phi) = \alpha(\varphi)^{-2}, \tag{A3}$$

where $\varphi(\phi)$ is obtained by inverting $\mathcal{A}(\varphi) = 1/\sqrt{\phi}$. The parameters defined in Table I of [23] are translated using (2.7) and (A3) and are gathered in Table I. In particular, we note that $\varphi(\phi_0) = \varphi_0$ are the background cosmological values of the scalar fields.

The notations of this paper are given in the third column. Some of them are a slight simplification of the Damour-Esposito Farèse parameters. Our table of correspondence agrees with [27], except for λ_1, λ_2, s'_A, s''_A. However this has no consequence on the two-body Lagrangian parameters, which we found to be in full agreement.

TABLE I. Conversion of the two-body Lagrangian parameters.

MW [23]	DEF [22,32]	This paper
Scalar-tensor parameters		
G	$\mathcal{A}_0^2(1+\alpha_0^2)$	…
ζ	$\frac{\alpha_0^2}{1+\alpha_0^2}$	…
λ_1	$\frac{1}{2}\frac{\beta_0}{1+\alpha_0^2}$	…
λ_2	$-\frac{1}{2(1+\alpha_0^2)^2}(\frac{\beta'_0\alpha_0}{2} + \beta_0\alpha_0^2 - 2\beta_0^2)$	…
Self-gravity (sensitivity) parameters		
s_A	$\frac{1}{2} - \frac{\alpha_A^0}{2\alpha_0}$	…
s'_A	$\frac{\beta_A^0}{4\alpha_0^2} - \frac{\alpha_A^0\beta_0}{4\alpha_0^3}$	…
s''_A	$-\frac{1}{2\alpha_0}(\frac{\beta'^0_A}{4\alpha_0^2} - \frac{3\beta_A^0\beta_0+\alpha_A^0\beta'_0}{4\alpha_0^3} + \frac{3\beta_0^2\alpha_A^0}{4\alpha_0^4})$	…
Two-body Lagrangian parameters		
Kepler		
m_1	$m_A^0/\mathcal{A}_0$	$m_A^0/\mathcal{A}_0 \equiv \tilde{m}_A^0$
m_2	$m_B^0/\mathcal{A}_0$	$m_B^0/\mathcal{A}_0 \equiv \tilde{m}_B^0$
$G\alpha$	$(1+\alpha_A^0\alpha_B^0)\mathcal{A}_0^2 \equiv G_{AB}\mathcal{A}_0^2$	$G_{AB}A_0^2 \equiv \tilde{G}_{AB}$
1PK		
$\bar{\gamma}$	$-2\frac{\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0} \equiv \bar{\gamma}_{AB}$	$\bar{\gamma}_{AB}$
$\bar{\beta}_1$	$\frac{1}{2}\frac{(\beta_A\alpha_B^2)_0}{(1+\alpha_A^0\alpha_B^0)^2} \equiv \bar{\beta}^A_{BB}$	$\bar{\beta}_A$
$\bar{\beta}_2$	$\frac{1}{2}\frac{(\beta_B\alpha_A^2)_0}{(1+\alpha_A^0\alpha_B^0)^2} \equiv \bar{\beta}^B_{AA}$	$\bar{\beta}_B$
2PK		
$\bar{\delta}_1$	$\frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$	δ_A
$\bar{\delta}_2$	$\frac{(\alpha_B^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}$	δ_B
$\bar{\chi}_1$	$-\frac{1}{4}\frac{(\beta'_A\alpha_B^3)_0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{4}\epsilon^A_{BBB}$	$-\frac{1}{4}\epsilon_A$
$\bar{\chi}_2$	$-\frac{1}{4}\frac{(\beta'_B\alpha_A^3)_0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{4}\epsilon^B_{AAA}$	$-\frac{1}{4}\epsilon_B$
$\bar{\beta}_1\bar{\beta}_2/\bar{\gamma}$	$-\frac{1}{8}\frac{\beta_A^0\alpha_A^0\beta_B^0\alpha_B^0}{(1+\alpha_A^0\alpha_B^0)^3} \equiv -\frac{1}{8}\zeta_{ABAB}$	$-\frac{1}{8}\zeta$

APPENDIX B: THE CONTACT TRANSFORMATIONS DEFINING THE CLASS OF REDUCED LAGRANGIANS

In Sec. III B, we performed a 2PK position redefinition (through a contact transformation) depending on the 14 parameters f_i of the function f introduced in (3.12). Its full expression is $\vec{Z'}_A = \vec{Z}_A + \delta\vec{Z}_A$ and $\vec{Z'}_B = \vec{Z}_B + \delta\vec{Z}_B$, where

$$\begin{aligned}\delta\vec{Z}_A =& \frac{G_{AB}m_B^0}{8}[2(7+4\bar{\gamma}_{AB})\vec{V}_B(\vec{N}\cdot\vec{V}_B) - \vec{N}((7+4\bar{\gamma}_{AB})V_B^2 - (\vec{N}\cdot\vec{V}_B)^2)]\\ &- G_{AB}m_B^0\bigg[\vec{V}_A(2f_1(N\cdot V_A) - 2f_4(N\cdot V_B)) + \vec{V}_B(f_2(N\cdot V_A) - f_5(N\cdot V_B))\\ &+\vec{N}\bigg(f_1V_A^2 + f_2V_A\cdot V_B + f_3V_B^2 + 3f_7(N\cdot V_A)^2 + 2f_8(N\cdot V_A)(N\cdot V_B) - f_9(N\cdot V_B)^2 + f_{11}\frac{G_{AB}m_A^0}{R} + f_{12}\frac{G_{AB}m_B^0}{R}\bigg)\bigg],\\ \delta\vec{Z}_B =& \frac{G_{AB}m_A^0}{8}[-2(7+4\bar{\gamma}_{AB})\vec{V}_A(\vec{N}\cdot\vec{V}_A) + \vec{N}((7+4\bar{\gamma}_{AB})V_A^2 - (\vec{N}\cdot\vec{V}_A)^2)]\\ &- G_{AB}m_A^0\bigg[\vec{V}_A(f_2(N\cdot V_A) - f_5(N\cdot V_B)) + \vec{V}_B(2f_3(N\cdot V_A) - 2f_6(N\cdot V_B))\\ &+\vec{N}\bigg(-f_4V_A^2 - f_5V_A\cdot V_B - f_6V_B^2 + f_8(N\cdot V_A)^2 - 2f_9(N\cdot V_A)(N\cdot V_B) - 3f_{10}(N\cdot V_B)^2 - f_{13}\frac{G_{AB}m_A^0}{R} - f_{14}\frac{G_{AB}m_B^0}{R}\bigg)\bigg].\end{aligned}$$

APPENDIX C: THE TWO-BODY 2PK HAMILTONIANS FOR $f \neq 0$

When $f \neq 0$, we have on hands a whole class of ordinary Hamiltonians, corresponding implicitly to different coordinate systems. Only the 2PK coefficients differ from (3.21) (because of the 2PK order contact transformations, see Appendix B) and read (see Sec. III C)

$$\begin{aligned}h_1^{2\mathrm{PK}} =& \frac{1}{16}(5\nu^2 - 5\nu + 1), \qquad h_2^{2\mathrm{PK}} = h_3^{2\mathrm{PK}} = h_4^{2\mathrm{PK}} = 0,\\ h_5^{2\mathrm{PK}} =& \frac{G_{AB}}{8}[5 + 4\bar{\gamma}_{AB} - (22+16\bar{\gamma}_{AB})\nu - 3\nu^2]\\ &+ G_{AB}\bigg[-f_6\frac{m_A^0}{M} - f_1\frac{m_B^0}{M} + \nu\bigg(f_1 + f_6 + (f_1+f_2-f_4)\frac{m_B^0}{M} + (-f_3+f_5+f_6)\frac{m_A^0}{M}\bigg)\bigg],\\ h_6^{2\mathrm{PK}} =& G_{AB}\bigg[-\frac{\nu(\nu-1)}{4} + (f_1 - 3f_7)\frac{m_B^0}{M} + (f_6 - 3f_{10})\frac{m_A^0}{M}\\ &-\nu\bigg(f_1 + f_6 - 3f_7 - 3f_{10} + \frac{m_B^0}{M}(f_1+f_2-f_4-3f_7-3f_8) + \frac{m_A^0}{M}(-f_3+f_5+f_6-3f_9-3f_{10})\bigg)\bigg],\\ h_7^{2\mathrm{PK}} =& G_{AB}\bigg[-\frac{3}{8}\nu^2 + 3\bigg(f_7\frac{m_B^0}{M} + f_{10}\frac{m_A^0}{M}\bigg) - 3\nu\bigg(f_7 + f_{10} + (f_7+f_8)\frac{m_B^0}{M} + (f_9+f_{10})\frac{m_A^0}{M}\bigg)\bigg],\\ h_8^{2\mathrm{PK}} =& \frac{G_{AB}^2}{8}\bigg[22 - 4\frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{M} + 4\frac{m_A^0\delta_A + m_B^0\delta_B}{M} + 28\bar{\gamma}_{AB} + 9\bar{\gamma}_{AB}^2 + \nu\bigg(58 - 4\frac{m_A^0\bar{\beta}_A + m_B^0\bar{\beta}_B}{M} + 36\bar{\gamma}_{AB}\bigg)\bigg]\\ &+ G_{AB}^2\bigg[(f_1 - f_{12})\frac{m_B^0}{M} + (f_6 - f_{13})\frac{m_A^0}{M}\\ &+\nu\bigg(-f_1 - f_6 - f_{11} + f_{12} + f_{13} - f_{14} - (f_1+f_2-f_4)\frac{m_B^0}{M} + (f_3 - f_5 - f_6)\frac{m_A^0}{M}\bigg)\bigg],\end{aligned}$$

$$h_9^{2\mathrm{PK}} = G_{AB}^2\left[-\frac{1}{2}-\frac{1}{2}\frac{m_A^0\delta_A+m_B^0\delta_B}{M}-\frac{\bar\gamma_{AB}}{2}-\frac{\bar\gamma_{AB}^2}{8}+\nu\left(-4+(\bar\beta_A+\bar\beta_B)-3\bar\gamma_{AB}+\frac{m_A^0\bar\beta_B+m_B^0\bar\beta_A}{M}\right)\right]$$
$$+G_{AB}^2\left[(2f_1+3f_7+2f_{12})\frac{m_B^0}{M}+(2f_6+3f_{10}+2f_{13})\frac{m_A^0}{M}-\nu\left(2f_1+2f_6+3f_7+3f_{10}-2f_{11}+2f_{12}+2f_{13}-2f_{14}\right.\right.$$
$$\left.\left.+(2f_1+2f_2-2f_4+3f_7+3f_8)\frac{m_B^0}{M}+(-2f_3+2f_5+2f_6+3f_9+3f_{10})\frac{m_A^0}{M}\right)\right],$$
$$h_{10}^{2\mathrm{PK}} = G_{AB}^3\left[-\frac{1}{2}-\frac{m_B^0\bar\beta_A+m_A^0\bar\beta_B}{M}-\frac{1}{6}\frac{m_A^0\epsilon_B+m_B^0\epsilon_A}{M}-\frac{1}{3}\frac{m_A^0\delta_A+m_B^0\delta_B}{M}-\frac{\bar\gamma_{AB}}{3}-\frac{\bar\gamma_{AB}^2}{12}+f_{12}\frac{m_B^0}{M}+f_{13}\frac{m_A^0}{M}\right.$$
$$\left.+\nu\left(-\frac{15}{4}-\zeta+\frac{\bar\gamma_{AB}^2}{6}-\frac{4}{3}\bar\gamma_{AB}+\frac{\delta_A+\delta_B}{3}+\frac{\epsilon_A+\epsilon_B}{6}-(\bar\beta_A+\bar\beta_B)+f_{11}-f_{12}-f_{13}+f_{14}\right)\right].$$

They reduce to II.22 for $f=0$. As a consistency check, one retrieves the general relativistic ADM coordinates Hamiltonian (given, e.g., in [8]), that is, in the limit (3.10), setting

$$f_3 = f_4 = -\frac{1}{4},\qquad f_{12}=f_{13}=\frac{1}{4},\qquad f_{11}=f_{14}=\frac{7}{4}, \tag{C1}$$

the other f_i coefficients being 0.

APPENDIX D: CANONICALLY TRANSFORMED TWO-BODY HAMILTONIAN

By means of a generic canonical transformation (3.24) and (3.25), the two-body (2PK) Hamiltonian (see Sec. III C) is rewritten in the intermediate coordinate system $(Q,P)\to(Q,p)$ (recalling the notation $\mathcal{P}^2\equiv\hat p_r^2+\frac{\hat p_\phi^2}{\hat R^2}$),

$$\hat H = \frac{M}{\mu}+\left(\frac{\mathcal{P}^2}{2}-\frac{h^{\mathrm{K}}}{\hat R}\right)+\hat H^{1\mathrm{PK}}+\hat H^{2\mathrm{PK}}+\cdots, \tag{D1}$$

where

$$\hat H^{1\mathrm{PK}} = h_1^{1\mathrm{PK}}\mathcal{P}^4+\mathcal{P}^2\hat p_r^2(h_2^{1\mathrm{PK}}-\alpha_1)+\hat p_r^4(2\alpha_1+\beta_1+h_3^{1\mathrm{PK}})+\frac{h_4^{1\mathrm{PK}}\mathcal{P}^2+h_5^{1\mathrm{PK}}\hat p_r^2}{\hat R}+\frac{h_6^{1\mathrm{PK}}}{\hat R^2},$$
$$\hat H^{2\mathrm{PK}} = \mathcal{P}^2\hat p_r^4(-2\alpha_1^2+4\alpha_2-\beta_2+4\beta_1h_1^{1\mathrm{PK}}+\alpha_1(-\beta_1+8h_1^{1\mathrm{PK}}+2h_2^{1\mathrm{PK}}-4h_3^{1\mathrm{PK}})+2\beta_1h_2^{1\mathrm{PK}}+h_3^{2\mathrm{PK}})$$
$$+\frac{1}{2}\mathcal{P}^4\hat p_r^2(\alpha_1^2-4\alpha_1(2h_1^{1\mathrm{PK}}+h_2^{1\mathrm{PK}})+2(h_2^{2\mathrm{PK}}-3\alpha_2)),$$
$$+\hat p_r^6\left(2\alpha_1^2+\frac{\beta_1^2}{2}+2\beta_2+\gamma_2+2\alpha_1(\beta_1+2h_2^{1\mathrm{PK}}+4h_3^{1\mathrm{PK}})+2\beta_1(h_2^{1\mathrm{PK}}+2h_3^{1\mathrm{PK}})+h_4^{2\mathrm{PK}}\right)+h_1^{2\mathrm{PK}}\mathcal{P}^6$$
$$+\frac{1}{\hat R}[\mathcal{P}^2\hat p_r^2(-2\delta_2-2\alpha_1(h_4^{1\mathrm{PK}}+h_5^{1\mathrm{PK}})+h_6^{2\mathrm{PK}})+\hat p_r^4(2\delta_2+4\alpha_1(h_4^{1\mathrm{PK}}+h_5^{1\mathrm{PK}})+2\beta_1(h_4^{1\mathrm{PK}}+h_5^{1\mathrm{PK}})+h_7^{2\mathrm{PK}})+h_5^{2\mathrm{PK}}\mathcal{P}^4]$$
$$+\frac{1}{\hat R^2}[h_8^{2\mathrm{PK}}\mathcal{P}^2+\hat p_r^2(h_9^{2\mathrm{PK}}-\eta_2)]+\frac{h_{10}^{2\mathrm{PK}}}{\hat R^3},$$

and where the 17 coefficients $h_i^{N\mathrm{PK}}$, which depend on the 14 parameters f_i, are given in Appendix C.

APPENDIX E: CANONICALLY TRANSFORMED EFFECTIVE HAMILTONIANS

Performing the canonical transformation (3.24) and (3.25), the effective (2PK expanded) Hamiltonians are rewritten in the intermediate coordinate system $(q,p)\to(Q,p)$. The Keplerian order is unaffected by the canonical transformation, as discussed below (3.24).

The Hamiltonians presented in Sec. IVA read (recalling the notation $\mathcal{P}^2 \equiv \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{R}^2}$)

$$\begin{aligned}
\hat{H}_e^{\rm 1PN} &= -\left(\alpha_1 + \frac{1}{8}\right)\mathcal{P}^4 - \hat{p}_r^2\mathcal{P}^2(\alpha_1 + 3\beta_1) + \hat{p}_r^4(2\alpha_1 + 3\beta_1)\\
&\quad + \frac{1}{4\hat{R}}[\mathcal{P}^2(a_1(1-2\alpha_1) - 4\gamma_1) - 2\hat{p}_r^2(a_1(2\alpha_1 + 3\beta_1) + b_1 - 2\gamma_1)] + \frac{4a_2 - a_1(a_1 + 4\gamma_1)}{8\hat{R}^2},\\
\hat{H}_e^{\rm 2PN} &= \frac{1}{16}(24\alpha_1^2 + 8\alpha_1 - 16\alpha_2 + 1)\mathcal{P}^6 + \frac{1}{2}\hat{p}_r^2\mathcal{P}^4(\alpha_1(18\beta_1 + 1) + 9\alpha_1^2 - 6\alpha_2 + 3\beta_1 - 6\beta_2)\\
&\quad + \frac{1}{2}\hat{p}_r^4\mathcal{P}^2(2\alpha_1(9\beta_1 - 1) + 8\alpha_2 + 27\beta_1^2 - 3\beta_1 + 2\beta_2 - 10\gamma_2) - \frac{1}{2}\hat{p}_r^6(36\alpha_1\beta_1 + 12\alpha_1^2 + 27\beta_1^2 - 4\beta_2 - 10\gamma_2)\\
&\quad + \frac{1}{16\hat{R}}[8\hat{p}_r^4(a_1(2\alpha_1(6\beta_1 + 1) + 4\alpha_1^2 + 9\beta_1^2 + 3\beta_1 - 2\beta_2 - 5\gamma_2) - 12\alpha_1\gamma_1 + b_1(2\alpha_1 + 3\beta_1) - 18\beta_1\gamma_1 + 4\delta_2 + 6\epsilon_2)\\
&\quad + 4\hat{p}_r^2\mathcal{P}^2(a_1(4\alpha_1(3\beta_1 - 1) + 8\alpha_1^2 - 8\alpha_2 - 9\beta_1 - 6\beta_2) + 2(\gamma_1(6\alpha_1 + 18\beta_1 - 1) - 2(\delta_2 + 3\epsilon_2)) + (2\alpha_1 + 1)b_1)\\
&\quad + \mathcal{P}^4(a_1(8\alpha_1^2 - 12\alpha_1 - 8\alpha_2 - 1) + 8((6\alpha_1 + 1)\gamma_1 - 2\delta_2))]\\
&\quad + \frac{1}{16\hat{R}^2}[4\hat{p}_r^2(a_1^2(2\alpha_1 + 3\beta_1) - a_1(b_1 - 2(\gamma_1(4\alpha_1 + 6\beta_1 + 1) - 2\delta_2 - 3\epsilon_2)) + 2(-4a_2\alpha_1 - 6a_2\beta_1 + b_1\gamma_1 + b_1^2 - b_2\\
&\quad - 3\gamma_1^2 + 2\eta_2)) + \mathcal{P}^2(4a_1((4\alpha_1 - 3)\gamma_1 - 2\delta_2) + 4(a_2(1 - 4\alpha_1) + 6\gamma_1^2 - 4\eta_2) + (4\alpha_1 - 1)a_1^2)]\\
&\quad + \frac{1}{16\hat{R}^3}[-4a_1(a_2 - 2\gamma_1^2 + 2\eta_2) + 4a_1^2\gamma_1 + 8(a_3 - 2a_2\gamma_1) + a_1^3].
\end{aligned}$$

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Chapitre 3

Problème à deux corps en théories scalaire-tenseur : un formalisme EOB "sur mesure"

Dans leurs travaux de 1998, A. Buonanno et T. Damour ont réduit la dynamique 2PN de la relativité générale au mouvement géodésique dans une ν-déformation de la métrique de Schwarzschild [68], où $\nu = m_A m_B/(m_A + m_B)^2$ reste petit, $\nu < 1/4$, comme nous l'avons vu au chapitre 1. Cette construction, centrée sur le "problème à un corps" exact de la relativité générale, auquel elle se ramène dans la limite $\nu = 0$, renforce ainsi la pertinence des prédictions EOB jusqu'aux dernières orbites précédant la coalescence du système.

Dans le chapitre précédent, nous avons aussi réduit le problème à deux corps des théories scalaire-tenseur au mouvement géodésique dans une métrique effective. Une telle approche, calquée sur celle développée en relativité générale, est bien adaptée, mais aussi restreinte, à l'étude des effets du champ scalaire lorsqu'ils sont perturbatifs.

L'objet de ce chapitre est de développer une autre approche EOB, spécifique aux théories scalaire-tenseur.

En théories scalaire-tenseur, la généralisation de l'espace-temps de Schwarzschild, solution statique et à symétrie sphérique des équations du champ dans le vide, s'écrit en coordonnées de Just [178] :

$$ds_*^2 = -\left(1-\frac{\mathfrak{a}_*}{\rho}\right)^{\frac{\mathfrak{b}_*}{\mathfrak{a}_*}} dt^2 + \left(1-\frac{\mathfrak{a}_*}{\rho}\right)^{-\frac{\mathfrak{b}_*}{\mathfrak{a}_*}} d\rho^2 + \left(1-\frac{\mathfrak{a}_*}{\rho}\right)^{1-\frac{\mathfrak{b}_*}{\mathfrak{a}_*}} \rho^2\left(d\theta^2+\sin^2\!\theta\, d\phi^2\right) , \tag{3.1a}$$

$$\varphi_*(\rho) = \varphi_0 + \frac{\mathfrak{q}_*}{\mathfrak{a}_*}\ln\left(1-\frac{\mathfrak{a}_*}{\rho}\right) , \tag{3.1b}$$

où φ_0 est la valeur asymptotique du champ scalaire, imposée, e.g., par l'environnement cosmologique, et où les constantes d'intégration $\mathfrak{a}_*$, $\mathfrak{b}_*$, et $\mathfrak{q}_*$ satisfont la relation $\mathfrak{a}_*^2 = \mathfrak{b}_*^2 + 4\mathfrak{q}_*^2$. Notons que la solution de Schwarzschild est retrouvée dans la limite $\mathfrak{q}_* = 0$.

Ainsi, le "problème à un corps" en théories scalaire-tenseur est celui du mouvement d'une particule test orbitant dans les champs (3.1), i.e. décrit par l'action

$$S_*[x^\mu] = -\int m_*(\varphi_*)\, ds_* , \tag{3.2}$$

où $ds_* = \sqrt{-g^*_{\mu\nu}dx^\mu dx^\nu}$ et où $x^\mu[s_*]$ désigne la ligne d'univers de la particule test, décrite par une fonction $m_*(\varphi_*)$.

Dans l'article qui suit [179], on se propose de transposer les fondements de [68] aux théories scalaire-tenseur, c'est-à-dire de bâtir un Hamiltonien EOB,

$$H_{\text{EOB}} = M\sqrt{1+2\nu\left(\frac{H_e}{\mu}-1\right)}\,, \tag{3.3}$$

avec $M = m_A^0 + m_B^0$, $\mu = m_A^0 m_B^0/M$ et $\nu = \mu/M$, à partir du Lagrangien 2PK (2.1) des théories scalaire-tenseur et des transformations canoniques présentées en chapitre 2. La nouveauté de la construction proposée est que le Hamiltonien effectif H_e (voir (4.5-4.8) et (4.15-4.17) ci-dessous pour son expression détaillée), est maintenant une ν-déformation du Hamiltonien d'une particule test $m_*(\varphi)$, orbitant dans la métrique et le champ scalaire (3.1), générés par un corps central $M_*(\varphi)$, dont les sensibilités se développent selon :

$$\ln m_*(\varphi) = \ln m_*^0 + \alpha_*^0(\varphi-\varphi_0) + \frac{1}{2}\beta_*^0(\varphi-\varphi_0)^2 + \frac{1}{6}\beta'^0_*(\varphi-\varphi_0)^3 + \cdots \tag{3.4a}$$

$$\ln M_*(\varphi) = \ln M_*^0 + A_*^0(\varphi-\varphi_0) + \cdots \tag{3.4b}$$

avec $\mathfrak{b}_* = 2M_*^0$, $\mathfrak{q}_* = M_*^0 A_*^0$, et où les paramètres (3.4) sont reliés à l'ordre 2PK à ceux des corps réels A et B par :

$$m_*^0 = m_A^0 m_B^0/(m_A^0+m_B^0) \qquad \text{et} \qquad M_*^0 = m_A^0 + m_B^0\,, \tag{3.5}$$

ainsi que

$$(A_*^0)^2 = \frac{m_A^0(\alpha_A^0)^2 + m_B^0(\alpha_B^0)^2}{m_A^0+m_B^0}\,, \tag{3.6a}$$

$$\alpha_*^0 = \frac{\alpha_A^0\alpha_B^0}{A_*^0}\,, \tag{3.6b}$$

$$\beta_*^0 = \frac{(m_A\alpha_A^2)^0\beta_B^0 + (m_B\alpha_B^2)^0\beta_A^0}{(m_A\alpha_A^2)^0 + (m_B\alpha_B^2)^0}\,, \tag{3.6c}$$

$$\beta'^0_* = \frac{(m_A\alpha_A^3)^0\beta'^0_B + (m_B\alpha_B^3)^0\beta'^0_A}{(m_A^0+m_B^0)(A_*^0)^3}\,. \tag{3.6d}$$

On rappelle qu'un indice 0 indique une quantité évaluée à l'infini, $\varphi = \varphi_0$. Notons qu'aux masses réduite et totale μ et M retrouvées en (3.5), s'ajoutent les nouvelles combinaisons (3.6), spécifiques aux théories scalaire-tenseur.

Bien sûr, les dynamiques à deux corps dérivées de ce Hamiltonien EOB et de celui du chapitre précédent sont canoniquement équivalentes à l'ordre 2PK ; mais, de par sa construction, le nouvel H_{EOB} en définit une resommation différente près de la coalescence du système.

Ainsi, en calculant par exemple la position de l'ISCO et la fréquence orbitale associée dans le cas simple des théories Jordan-Fierz-Brans-Dicke, on montre que notre nouvelle dynamique resommée est cohérente avec celle du chapitre précédent dans leur domaine de validité commun (i.e., lorsque les effets du champ scalaire perturbent la relativité générale), mais qu'elle permet, au delà, de décrire les régimes qui diffèrent grandement de la relativité générale puisque, entre autres, la fréquence orbitale à l'ISCO se réduit à sa valeur exacte dans la limite $\nu = 0$, qui peut être très différente de celle de l'ISCO de la métrique de Schwarzschild.

Cet article clôt la première partie de cette thèse, dans laquelle nous avons construit, à partir d'un même Lagrangien 2PK, deux resommations complémentaires de la partie conservative du mouvement à deux corps :

La premiere, présentée au chapitre précédent, est centrée sur le mouvement géodésique dans une métrique effective. Elle est adaptée à la description des théories scalaire-tenseur lorsque les effets du champ scalaire sont perturbatifs, et permet de s'appuyer sur les résultats EOB les plus récents en relativité générale (dont la dynamique est actuellement décrite à l'ordre 5PN, cf. chapitre 1).

La seconde est bâtie sur le mouvement d'une particule test dans une métrique et un champ scalaire effectifs. Elle permet de considérer les théories scalaire-tenseur sur le même pied que la relativité générale, et donc, d'en décrire le problème à deux corps dans des régimes différant grandement de ceux de la relativité générale. Elle permet donc, par exemple, de décrire les systèmes binaires constitués d'étoiles ayant "scalarisé" [180], voir aussi la section 0.2.5.

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Reducing the two-body problem in scalar-tensor theories to the motion of a test particle: A scalar-tensor effective-one-body approach

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Starting from the second post-Keplerian (2PK) Hamiltonian describing the conservative part of the two-body dynamics in massless scalar-tensor (ST) theories, we build an effective-one-body (EOB) Hamiltonian which is a ν deformation (where $\nu = 0$ is the test mass limit) of the analytically known ST Hamiltonian of a test particle. This ST-EOB Hamiltonian leads to a simple (yet canonically equivalent) formulation of the conservative 2PK two-body problem, but also defines a resummation of the dynamics which is well-suited to ST regimes that depart strongly from general relativity (GR) and which may provide information on the strong field dynamics; in particular, the ST innermost stable circular orbit location and associated orbital frequency. Results will be compared and contrasted with those deduced from the ST-deformation of the (5PN) GR-EOB Hamiltonian previously obtained in [Phys. Rev. D **95**, 124054 (2017)].

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I. INTRODUCTION

Building libraries of accurate gravitational waveform templates is essential for detecting the coalescence of compact binary systems. To this aim, the effective-one-body (EOB) approach has proven to be a very powerful framework to analytically encompass and combine the post-Newtonian (PN) and numerical descriptions of the inspiral and merger, as well as "ring-down" phases of the dynamics of binary systems of comparable masses in general relativity; see, e.g., [1].

Matching and comparing gravitational wave templates to the present and future data from the LIGO-Virgo and forthcoming interferometers will bring the opportunity to test general relativity (GR) at high PN order *and* in the strong field regime of a merger. A next step to test gravity in this regime is to match gravitational wave data with templates predicted in the framework of modified gravities. In this context, scalar-tensor (ST) theories with a single massless scalar field have been the most thoroughly studied. For instance, the corresponding dynamics of binary systems is known at 2.5PN order [2], or, adopting the terminology of [3], 2.5 post-Keplerian (PK) order, to highlight the fact that (strong) self-gravity effects are encompassed in the body-dependent "Eardley-type" mass functions $m_A(\varphi)$ assigned to each compact body A (see, e.g., Sec. II B). What was hence done in [4–6] is the computation of ST waveforms at 2PK relative order (although part of this computation requires information on the ST 3PK dynamics, which are, for now, unknown).

In that context, the aim of [7] (henceforth Paper 1) was to go beyond the (as yet poorly known) PK dynamics of modified gravities by extending the EOB approach to scalar-tensor theories. More precisely, we started from the ST two-body 2PK Lagrangian obtained by Mirshekari and Will [2] (no spins, nor finite-size, "tidal" effects) and deduced from it the corresponding centre-of-mass frame 2PK Hamiltonian. That two-body 2PK Hamiltonian was then mapped to that of geodesic motion in an effective, "ST-deformed" metric, which has the important property of reducing to the 1998 Buonanno-Damour EOB metric [8] in the general relativity limit. When extended to encompass the currently best available (5PN) GR-EOB results, the corresponding ST-EOB Hamiltonian of Paper 1 is therefore well-suited to test scalar-tensor theories when considered as parametrized corrections to GR. However, the scope of this GR-centered EOB Hamiltonian is, by construction, restricted to a regime where the scalar field effects are *perturbative* with respect to general relativity.

In their 1998 paper [8], Buonanno and Damour successfully reduced the general relativistic two-body problem to an effective geodesic motion in a static, spherically symmetric (SSS) metric. In their approach, they ensured that the effective-one-body dynamics is centered on a particular *one-body problem* in general relativity, namely, the geodesic motion of the reduced mass of the system $\mu = m_A m_B/M$ in the Schwarzschild metric produced by a central body, $M = m_A + m_B$, to which it indeed reduces to in the test-mass limit (i.e., $\nu = 0$ with $\nu = \mu/M$). Consequently, the associated predictions were smoothly connected to those of the motion of a test mass in the Schwarzschild metric (which is known exactly), ensuring an accurate resummation of the two-body dynamics that could be pushed up to the strong field regime of the last few orbits before plunge.

With the same motivation this paper proposes a mapping where the ST-EOB Hamiltonian reduces, in contrast with what was done in Paper 1, to the *scalar-tensor* one-body Hamiltonian in the test mass limit, which describes the motion of a test particle in the metric *and scalar field* generated by a central SSS body. Although the conservative dynamics derived from this Hamiltonian and that proposed in Paper 1 (and from the Mirshekari-Will Lagrangian) are the same at 2PK order, when taken as being exact, they define different resummations and hence, *a priori* different dynamics in the strong field regime which is reached near the last stable orbit. In particular, we shall highlight the fact that our new, *ST-centered*, EOB Hamiltonian is well-suited to investigate ST regimes that depart strongly from general relativity.

This paper is organized as follows: In Sec. II we present the Hamiltonian describing the motion of a test particle orbiting in the metric and scalar field generated by a central body (when written in Just coordinates) in scalar-tensor theories, henceforth referred to as the real one-body Hamiltonian. In order for the paper to be self-contained, in Sec. III we recall the expression of the two-body Hamiltonian in the centre-of-mass frame obtained in Paper 1 at 2PK order. In Sec. IV we then reduce the two-body problem to an EOB ν-deformed version of the ST one-body problem, by means of a canonical transformation and imposing the EOB mapping relation between their Hamiltonians. We finally study the resummed dynamics it defines; in particular, we compute the innermost stable circular orbit (ISCO) location and associated orbital frequency in the case of Jordan-Fierz-Brans-Dicke theory. Corrections to general relativity ISCO predictions are compared to the results obtained in Paper 1.

II. THE SCALAR-TENSOR REAL ONE-BODY PROBLEM

A. The metric and scalar field outside a static, spherically symmetric body

In this paper we limit ourselves to the single, massless scalar field case. Adopting the conventions of Damour and Esposito-Farèse (DEF, see, e.g., [3] or [9]), the Einstein-frame action reads in vacuum, that is, outside the sources (setting $G_* = c = 1$),

$$S_{\mathrm{EF}}^{\mathrm{vac}}[g_{\mu\nu},\varphi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi), \qquad (2.1)$$

where R is the Ricci scalar and $g = \det g_{\mu\nu}$. The vacuum field equations follow:

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi, \qquad (2.2a)$$

$$\Box\varphi = 0, \qquad (2.2b)$$

where $R_{\mu\nu}$ is the Ricci tensor and $\Box\varphi = \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi)$.

The vacuum, static and spherically symmetric (SSS) solutions to the Einstein-frame field equations (2.2), henceforth, real one-body metric $g^*_{\mu\nu}$ and scalar field φ_*, have a simple analytical expression in Just coordinates (see, e.g., [10]) as follows[1]:

$$ds_*^2 = -D_* dt^2 + \frac{d\rho^2}{D_*} + C_*\rho^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.3a)$$

$$\text{with} \quad D_*(\rho) = \left(1 - \frac{\mathfrak{a}_*}{\rho}\right)^{\frac{\mathfrak{b}_*}{\mathfrak{a}_*}},$$
$$C_*(\rho) = \left(1 - \frac{\mathfrak{a}_*}{\rho}\right)^{1-\frac{\mathfrak{b}_*}{\mathfrak{a}_*}}, \qquad (2.3b)$$

and

$$\varphi_*(\rho) = \varphi_0 + \frac{\mathfrak{q}_*}{\mathfrak{a}_*}\ln\left(1 - \frac{\mathfrak{a}_*}{\rho}\right), \qquad (2.4)$$

where φ_0 is a constant scalar background that must not be considered as an arbitrary integration constant, but rather as imposed, say, by the cosmological environment [11,12], while the other integration constants $\mathfrak{a}_*$, $\mathfrak{b}_*$ and $\mathfrak{q}_*$ have the dimension of a mass and satisfy the following constraint:

$$\mathfrak{a}_*^2 = \mathfrak{b}_*^2 + 4\mathfrak{q}_*^2. \qquad (2.5)$$

We note that when $\mathfrak{q}_* = 0$, i.e., $\mathfrak{a}_* = \mathfrak{b}_*$, the scalar field is a constant, the metric (2.3) reduces to Schwarzschild's, and Droste and Just coordinates coincide. Note also that pure vacuum (black hole) solutions exhibit singular scalar field and curvature invariants at $\rho = \mathfrak{a}_*$. For that reason, SSS black holes cannot carry massless scalar "hair" (thus $\mathfrak{q}_* = 0$) and hence do not differ from Schwarzschild's; see, e.g. [3,13].

One easily checks that expanding (2.3)–(2.4) at infinity and in isotropic coordinates ($\rho = \bar{\rho} + \frac{\mathfrak{a}_*}{2} + \cdots$), the metric and scalar field behave as

$$\bar{g}^*_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{\mathfrak{b}_*}{\bar{\rho}}\right) + \mathcal{O}\left(\frac{1}{\bar{\rho}^2}\right), \qquad (2.6a)$$

$$\varphi_* = \varphi_0 - \left(\frac{\mathfrak{a}_*}{\bar{\rho}}\right) + \mathcal{O}\left(\frac{1}{\bar{\rho}^2}\right), \qquad (2.6b)$$

where $\delta_{\mu\nu}$ is the Kronecker symbol.

In order to relate the constants of the vacuum solution to the structure of the body generating the fields, we need the Einstein-frame action inside the source,

$$S_{\mathrm{EF}}[g_{\mu\nu},\varphi,\Psi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) + S_m[\Psi,\mathcal{A}^2(\varphi)g_{\mu\nu}], \qquad (2.7)$$

[1]In the following, a star $(*)$ shall stand for quantities that refer to the real one-body problem.

where $\mathcal{A}(\varphi)$ characterizes the ST theory and Ψ generically stands for matter fields that are minimally coupled to the Jordan metric, $\tilde{g}_{\mu\nu} \equiv \mathcal{A}^2(\varphi) g_{\mu\nu}$. The field equations read

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{2.8a}$$

$$\Box\varphi = -4\pi\alpha(\varphi)T, \tag{2.8b}$$

where $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ is the Einstein-frame energy-momentum tensor of the source, $T \equiv T^\mu{}_\mu$ and where

$$\alpha(\varphi) \equiv \frac{d\ln\mathcal{A}(\varphi)}{d\varphi} \tag{2.9}$$

measures the universal coupling strength between the scalar field and matter.

The constants $\mathfrak{b}_*$ and $\mathfrak{q}_*$ can then be matched to the internal structure of the central body through integration of (2.8a) and (2.8b) as

$$\mathfrak{b}_* = 2\int_0^{\rho_0} d^3x\sqrt{-g}(-T^0{}_0 + T^i{}_i),$$
$$\mathfrak{q}_* = -\int_0^{\rho_0} d^3x\sqrt{-g}\alpha(\varphi)T, \tag{2.10}$$

where ρ_0 denotes the radius of the central body.[2] The numerical values of these integrals generically depend on the asymptotic value of the scalar field at infinity φ_0. Indeed, one can, for example, model a star as a perfect fluid, together with its equation of state. Given some central density and value for the scalar field $\varphi_c \equiv \varphi(\rho = 0)$, one integrates (2.8) and the matter equations of motion from the regular center of the body up to ρ_0 where the pressure vanishes. The metric and scalar field are then matched to the exterior solution (2.3)–(2.4), fixing uniquely $\mathfrak{b}_*$, $\mathfrak{q}_*$, and φ_0 in terms of the central density and φ_c. When the equation of state and the baryonic number of the star are held fixed, the exterior fields (i.e., $\mathfrak{b}_*$ and $\mathfrak{q}_*$) are completely known as functions of φ_c only, or, equivalently, of the scalar field value at infinity, φ_0, see, e.g., [14] for an explicit computation.

B. Skeletonizing the source of the gravity field

In order to clarify the analysis to come in the forthcoming sections, we now "skeletonize" the body creating the gravity field; that is, we phenomenologically replace S_m in (2.7) with a point particle action, as was suggested by Eardley in [15],

$$S_m^{\rm skel}[X^\mu, g_{\mu\nu}, \varphi] = -\int M_*(\varphi)dS, \tag{2.11}$$

where $dS = \sqrt{-g_{\mu\nu}dX^\mu dX^\nu}$ and where $X^\mu(S)$ denotes the location of the skeletonized body. The Einstein-frame mass $M_*(\varphi)$ depends on the value of the scalar field at $X^\mu(S)$ (subtracting divergent self contributions), on the specific theory and on the body itself [contrarily to (2.7) where the coupling to the scalar field was universal], hence encompassing the effects of the background scalar field on its equilibrium configuration.[3] For a discussion on the validity of the skeletonization procedure, see [3,17].

The question addressed now is to relate the function $M_*(\varphi)$ to the parameters describing the exterior solutions, that is $\mathfrak{b}_*$ and $\mathfrak{q}_*$, given a scalar field value at infinity φ_0. The field equations are given by

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

$$\text{with}\quad T^{\mu\nu} = \int dSM_*(\varphi)\frac{\delta^{(4)}(x-X)}{\sqrt{-g}}\frac{dX^\mu}{dS}\frac{dX^\nu}{dS}, \tag{2.12a}$$

$$\text{and}\quad \Box\varphi = 4\pi\int dSM_*(\varphi)A_*(\varphi)\frac{\delta^{(4)}(x-X)}{\sqrt{-g}}, \tag{2.12b}$$

where we introduced the body-dependent function ("capital alpha")

$$A_*(\varphi) \equiv \frac{d\ln M_*(\varphi)}{d\varphi}, \tag{2.13}$$

which measures the coupling between the skeletonized body and the scalar field. Note that because of the body-dependent function $M_*(\varphi)$, the effective scalar field equation is different from (2.8b) with $T^{\mu\nu}$ given in (2.12a), because (2.8b) was derived from the universally coupled action (2.7). Note also that since black holes cannot carry scalar hair, A_* must vanish in that case, i.e., M_* must then reduce to a constant, and one recovers general relativity.

We now solve these equations in the rest-frame of the skeletonized body, setting $\vec{X} = \vec{0}$. Outside it, the metric and scalar field are of the form (2.3) and (2.4). Moreover, solving the field equations (2.12) perturbatively around the metric and scalar field backgrounds, i.e., $\bar{g}^*_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\varphi_* = \varphi_0 + \delta\varphi$, in harmonic coordinates $\partial_\mu(\sqrt{-\bar{g}}\bar{g}^{\mu\nu}) = 0$, easily yields, at linear order

[2]For example, one rewrites (2.8b) as $(\sqrt{-g}g^{\rho\rho}\varphi')' = -4\pi\sqrt{-g}\alpha(\varphi)T$ (where $a' \equiv da/d\rho$) and integrates both sides between the center of the star, where the fields are supposed to be regular, and its radius ρ_0. The left-hand side hence reads $\int_0^{\rho_0}(\sqrt{-g}g^{\rho\rho}\varphi')'d\rho = \sqrt{-g}g^{\rho\rho}\varphi'|_{\rho=\rho_0} = \mathfrak{q}_*\sin\theta$, using the vacuum expressions (2.3)–(2.4), by continuity at $\rho = \rho_0$. Hence, one has $\mathfrak{q}_* = -(4\pi/\sin\theta)\int_0^{\rho_0}d\rho\sqrt{-g}\alpha T = -\int_0^{\rho_0}d\rho d\theta d\phi\sqrt{-g}\alpha T$, i.e., (2.10). One similarly obtains $\mathfrak{b}_*$ through integration of the $t-t$ component of Einstein's equation (2.8a), see [10] for the details.

[3]Note that Eardley-type actions do not depend on the local gradients of $g_{\mu\nu}$ and φ and hence cannot account for finite-size, "tidal" effects, nor out-of-equilibrium effects, see, e.g., [16], that will hence be neglected in the present paper.

$$\bar{g}^*_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{2M_*(\varphi_0)}{\bar{\rho}}\right) + \mathcal{O}\left(\frac{1}{\bar{\rho}^2}\right), \tag{2.14a}$$

$$\varphi_* = \varphi_0 - \frac{M_*(\varphi_0)A_*(\varphi_0)}{\bar{\rho}} + \mathcal{O}\left(\frac{1}{\bar{\rho}^2}\right), \tag{2.14b}$$

where the φ_0 dependence of the fields recalls the fact that the skeletonized body is "sensitive" to the background value of the scalar field in which it is immersed, that is, φ_0, as already discussed below (2.10).[4]

Moreover, by comparing (2.14) to (2.6), one obtains the following relations (knowing that the harmonic and isotropic coordinates identify at linear order):

$$\mathfrak{b}_* = 2M^0_*, \qquad \mathfrak{q}_* = M^0_* A^0_*, \qquad \mathfrak{a}_* = 2M^0_*\sqrt{1+(A^0_*)^2}, \tag{2.15}$$

see (2.5), where and from now on, a zero index denotes a quantity evaluated for $\varphi = \varphi_0$. Hence, by means of the matching conditions (2.15), we have traded the integration constants of the vacuum solution $\mathfrak{b}_*$ and $\mathfrak{q}_*$, which are related to the source stress-energy tensor by (2.10), for their "skeleton" counterparts, M^0_* and A^0_*, which are the values of the function $M_*(\varphi)$ and its logarithmic derivative evaluated at the background φ_0.

C. The real one-body problem: The motion of a test particle in the fields of a skeletonized body in ST theories

We now turn to the motion of a self-gravitating test particle $m_*(\varphi)$, coupled to the fields obtained above, i.e., generated by the central body only. The dynamics is described again by an Eardley-type action,

$$S_*[x^\mu] = -\int m_*(\varphi_*) ds_*, \tag{2.16}$$

where $ds_* = \sqrt{-g^*_{\mu\nu}dx^\mu dx^\nu}$ and where φ_* and $g^*_{\mu\nu}$ are the real one-body metric and scalar field, given explicitly in Just coordinates in (2.3), (2.4) together with (2.15). Note that the function $m_*(\varphi_*)$ characterizing the particle can also be related to the properties of an extended test body following the steps presented above, but where the scalar environment is not φ_0 anymore, and is replaced by the value of the scalar field generated by the central body φ_*, at the location of the test particle, $\varphi_*(x^\mu(s_*))$.

To simplify notations it is convenient to replace $m_*(\varphi_*)$ with the rescaled function

$$V_*(\varphi_*) \equiv \left(\frac{m_*(\varphi_*)}{m^0_*}\right)^2,$$

$$\text{such that}\quad S_*[x^\mu] = -m^0_*\int\sqrt{V_*}ds_*, \tag{2.17}$$

where we recall that $m^0_* = m_*(\varphi_0)$ is the value of $m_*(\varphi_*)$ when the test particle is infinitely far away from the central body. Therefore, the scalar-tensor Lagrangian for our test particle, defined as $S_* \equiv \int dt L_*$, reads (restricting the motion to the equatorial plane, $\theta = \pi/2$)

$$\begin{aligned} L_* &= -m^0_*\sqrt{-(V_* g^*_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \\ &= -m^0_*\sqrt{V_*\left(D_* - \frac{\dot{\rho}^2}{D_*} - C_*\rho^2\dot{\phi}^2\right)}, \\ \dot{\rho} &\equiv \frac{d\rho}{dt}, \qquad \dot{\phi} \equiv \frac{d\phi}{dt}, \end{aligned} \tag{2.18}$$

with

$$D_*(\rho) = \left(1 - \frac{a_*}{\hat{\rho}}\right)^{\frac{b_*}{a_*}}, \qquad C_*(\rho) = \left(1 - \frac{a_*}{\hat{\rho}}\right)^{1-\frac{b_*}{a_*}}, \tag{2.19}$$

where we have introduced the dimensionless radial coordinate

$$\hat{\rho} \equiv \rho/M^0_*, \tag{2.20}$$

and where the rescaled constants b_* and a_* follow from (2.15),

$$b_* = 2, \qquad a_* = 2\sqrt{1+(A^0_*)^2}. \tag{2.21}$$

In contrast, the expression of $V_*(\varphi_*(\rho))$ (or, equivalently, m_*) as an explicit function of ρ depends on the specific ST theory and on the internal structure of the test particle. At 2PK order, to which we restrict ourselves in this paper, it will prove sufficient to replace it with its Taylor expansion around φ_0. To do so, let us introduce the three quantities

$$\alpha_*(\varphi) \equiv \frac{d\ln m_*}{d\varphi}, \qquad \beta_*(\varphi) \equiv \frac{d\alpha_*}{d\varphi}, \qquad \beta'_*(\varphi) \equiv \frac{d\beta_*}{d\varphi}, \tag{2.22}$$

such that, expanding $m_*(\varphi)$ around φ_0 (where we recall that φ_0 is the value at infinity of the scalar field imposed by cosmology) yields

[4]Meanwhile, as in GR, the asymptotic (constant) metric at infinity can always be "gauged away" to Minkowski by means of an appropriate coordinate change.

$$m_*(\varphi) = m_*^0\left[1+\alpha_*^0(\varphi-\varphi_0)+\frac{1}{2}(\alpha_*^{02}+\beta_*^0)(\varphi-\varphi_0)^2 + \frac{1}{6}(3\beta_*^0\alpha_*^0+\alpha_*^{03}+\beta'^0_*)(\varphi-\varphi_0)^3+\cdots\right]. \tag{2.23}$$

Now, the scalar field generated by the central body is given in (2.4) together with (2.15). Hence, V_* reads, at 2PK order,

$$V_*(\hat\rho) = \left(\frac{m_*(\varphi_*(\hat\rho))}{m_*^0}\right)^2 = 1+\frac{v_1^*}{\hat\rho}+\frac{v_2^*}{\hat\rho^2}+\frac{v_3^*}{\hat\rho^3}+O\left(\frac{1}{\hat\rho^4}\right), \tag{2.24}$$

where the dimensionless constants v_1^*, v_2^*, and v_3^* depend on the functions $M_*(\varphi)$ and $m_*(\varphi)$ characterizing the central body and the test particle and are given by

$$v_1^* = -2\alpha_*^0 A_*^0, \tag{2.25a}$$

$$v_2^* = (2(\alpha_*^0)^2+\beta_*^0)(A_*^0)^2 - 2\alpha_*^0A_*^0\sqrt{1+(A_*^0)^2}, \tag{2.25b}$$

$$v_3^* = -\left(\frac{4}{3}(\alpha_*^0)^3+\frac{1}{3}\beta'^0_*+2\alpha_*^0\beta_*^0\right)(A_*^0)^3 + (4(\alpha_*^0)^2+2\beta_*^0)(A_*^0)^2\sqrt{1+(A_*^0)^2} - \frac{8}{3}\alpha_*^0A_*^0(1+(A_*^0)^2). \tag{2.25c}$$

To summarize, we have obtained in this section the Lagrangian that describes the dynamics of a test particle orbiting around a central (skeletonized) body in scalar-tensor theories of gravity. At 2PK order, it is entirely described by five coefficients, a_*, b_*, v_1^*, v_2^*, and v_3^*, which are in turn expressed in terms of the five fundamental parameters: M_*^0, A_*^0 describing the central body, and α_*^0, β_*^0, β'^0_*, describing the orbiting particle.[5]

III. THE REAL TWO-BODY DYNAMICS AT 2PK ORDER, A REMINDER

In this section, we recall the results from Paper 1 [7] that will be needed in the forthcoming sections.

A. The two-body 2PK Hamiltonians in scalar-tensor theories

The two-body dynamics is conveniently described in the Einstein-frame (following DEF), by means of an Eardley-type action

$$S_{\rm EF}[x_A^\mu, g_{\mu\nu}, \varphi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi) - \sum_A\int ds_A m_A(\varphi), \tag{3.1}$$

where $ds_A = \sqrt{-g_{\mu\nu}dx_A^\mu dx_A^\nu}$, and where $x_A^\mu(s_A)$ denotes the position of body A. The masses $m_A(\varphi)$ depend on the (regularized) local value of the scalar field and are related to their Jordan-frame counterparts through $m_A(\varphi) \equiv \mathcal{A}(\varphi)\tilde m_A(\varphi)$. In the negligible self-gravity limit, the "Jordan masses" reduce to constants, $\tilde m_A(\varphi) = cst$, so that the motion is a geodesic of the Jordan metric $\tilde g_{\mu\nu} = \mathcal{A}^2 g_{\mu\nu}$. In contrast, general relativity is recovered when the "Einstein masses" are constants, $m_A(\varphi) = cst$.

We now define a set of body-dependent quantities, consistently with (2.13) and (2.22),

$$\alpha_A(\varphi) \equiv \frac{d\ln m_A}{d\varphi}\left(=\frac{d\ln\mathcal{A}}{d\varphi}+\frac{d\ln\tilde m_A}{d\varphi}\right), \tag{3.2a}$$

$$\beta_A(\varphi) \equiv \frac{d\alpha_A}{d\varphi}, \tag{3.2b}$$

$$\beta'_A(\varphi) \equiv \frac{d\beta_A}{d\varphi}, \tag{3.2c}$$

that appear in the 2PK two-body Lagrangian. In the negligible self-gravity limit, $\tilde m_A = cst$, and hence

$$\alpha_A \to \alpha \equiv \frac{d\ln\mathcal{A}}{d\varphi}, \qquad \beta_A \to \beta \equiv \frac{d\alpha}{d\varphi}, \qquad \beta'_A \to \beta' \equiv \frac{d\beta}{d\varphi}, \tag{3.3}$$

become universal, while in the general relativity limit, $m_A = cst$, implying $\alpha_A = \beta_A = \beta'_A = 0$.

The conservative part of the scalar-tensor two-body problem has been studied at 1PK order by Damour and Esposito-Farèse (DEF) in [3] and at 2PK order by DEF in [9] and Mirshekari and Will (MW) in [2], performing a small orbital velocities, weak field expansion ($V^2 \sim m/R$) around $\eta_{\mu\nu}$ and a constant cosmological background φ_0. Because of the harmonic coordinates in which it has been computed, the two-body Lagrangian depends linearly on the accelerations of the bodies at 2PK level.

In Paper 1, we started from this MW Lagrangian, $L(\vec Z_{A/B}, \dot{\vec Z}_{A/B}, \ddot{\vec Z}_{A/B})$. Once translated in terms of the DEF conventions presented above (see also Paper 1, Appendix A), we eliminated the dependence in the accelerations $\ddot{\vec Z}_{A/B}$ by means of suitable contact transformations of the form

$$\vec Z'_A(t) = \vec Z_A(t) + \delta\vec Z_A(\vec Z_{A/B}, \dot{\vec Z}_{A/B}), \tag{3.4}$$

[5]Note that $b_* = \mathfrak{b}_*/M_*^0$ (with $\mathfrak{b}_* = 2M_*^0$) is a parameter since M_*^0 has been factorized out in the definition of $\hat\rho = \rho/M_*^0$.

that is, four-dimensional 2PK coordinate changes. We found a whole class of coordinate systems, labeled by fourteen parameters f_i, in which the Lagrangian is ordinary (see Paper 1 Appendix B and below). By means of a further Legendre transformation, we obtained the associated Hamitonians $H(Q,P)$ in the center-of-mass frame, the conjugate variables being $\vec{Z}=\vec{Z}_A-\vec{Z}_B$ and $\vec{P}=\vec{P}_A=-\vec{P}_B$, and in polar coordinates: $(Q,P)\equiv(R,\Phi,P_R,P_\Phi)$ where $P_R=\vec{N}\cdot\vec{P}$ and $P_\Phi=R(\vec{N}\times\vec{P})_z$. The resulting isotropic, translation-invariant, ordinary Hamiltonians are given at 2PK order in Paper 1, Sec. III C,

$$\hat{H}\equiv\frac{H}{\mu}=\frac{M}{\mu}+\left(\frac{\hat{P}^2}{2}-\frac{G_{AB}}{\hat{R}}\right)+\hat{H}^{1\mathrm{PK}}+\hat{H}^{2\mathrm{PK}}+\cdots, \tag{3.5}$$

where we have introduced the rescaled quantities

$$\hat{P}^2\equiv\hat{P}_R^2+\frac{\hat{P}_\Phi^2}{\hat{R}^2}\quad\text{with}\quad\hat{P}_R\equiv\frac{P_R}{\mu},$$
$$\hat{P}_\Phi\equiv\frac{P_\Phi}{\mu M},\qquad\hat{R}\equiv\frac{R}{M}, \tag{3.6}$$

and the reduced mass, total mass, and symmetric mass ratio

$$\mu\equiv\frac{m_A^0m_B^0}{M},\qquad M\equiv m_A^0+m_B^0,\qquad \nu\equiv\frac{\mu}{M}, \tag{3.7}$$

where m_A^0 and m_B^0 are the values of the functions $m_A(\varphi)$ and $m_B(\varphi)$ at $\varphi=\varphi_0$.

At 2PK order, the two-body Hamiltonians depend on seventeen coefficients $(h_i^{n\,\mathrm{PK}})$ (which are very lengthy and are given explicitly in Appendix C of Paper 1), which in turn depend on the fourteen f_i parameters and on the eleven following combinations of the eight fundamental mass parameters (3.2) [m_A^0, α_A^0, β_A^0, and $\beta_A'^0$ and B counterparts, characterizing at 2PK order the functions $m_{A/B}(\varphi)$]:

$$m_A^0,\qquad G_{AB}\equiv1+\alpha_A^0\alpha_B^0, \tag{3.8a}$$

$$\bar{\gamma}_{AB}\equiv-\frac{2\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0},\qquad\bar{\beta}_A\equiv\frac{1}{2}\frac{\beta_A^0(\alpha_B^0)^2}{(1+\alpha_A^0\alpha_B^0)^2}, \tag{3.8b}$$

$$\delta_A\equiv\frac{(\alpha_A^0)^2}{(1+\alpha_A^0\alpha_B^0)^2},\quad\epsilon_A\equiv\frac{(\beta_A'\alpha_B^3)^0}{(1+\alpha_A^0\alpha_B^0)^3},\quad\zeta\equiv\frac{\beta_A^0\alpha_A^0\beta_B^0\alpha_B^0}{(1+\alpha_A^0\alpha_B^0)^3}, \tag{3.8c}$$

and $(A\leftrightarrow B)$ counterparts, where we recall that a zero index indicates a quantity evaluated at infinity, $\varphi=\varphi_0$. In the general relativity limit, $m_A=cst$, the Hamiltonian considerably simplifies since these combinations are reduced to

$$G_{AB}=1,\quad\text{and}\quad\bar{\gamma}_{AB}=\bar{\beta}_A=\delta_A=\epsilon_A=\zeta=0. \tag{3.9}$$

B. The canonical transformation

The EOB mapping consists in imposing a functional relation between the two-body Hamiltonian $H(Q,P)$, and an effective Hamiltonian H_e (that we shall build in the next section), by means of a canonical transformation,

$$(Q,P)\to(q,p), \tag{3.10}$$

where $(q,p)\equiv(\rho,\phi,p_\rho,p_\phi)$. The canonical transformation is generated by the (time-independent and isotropic) generic function $G(Q,p)$ introduced in [7], Sec. III D, which depends on nine parameters at 2PK order,

$$\begin{aligned}\frac{G(Q,p)}{\mu M}=\hat{R}\hat{p}_\rho\Bigg[&\left(\alpha_1\mathcal{P}^2+\beta_1\hat{p}_\rho^2+\frac{\gamma_1}{\hat{R}}\right)\\&+\Bigg(\alpha_2\mathcal{P}^4+\beta_2\mathcal{P}^2\hat{p}_\rho^2+\gamma_2\hat{p}_\rho^4+\delta_2\frac{\mathcal{P}^2}{\hat{R}}\\&+\epsilon_2\frac{\hat{p}_\rho^2}{\hat{R}}+\frac{\eta_2}{\hat{R}^2}\Bigg)+\cdots\Bigg],\end{aligned} \tag{3.11}$$

where we introduced the reduced quantities

$$\mathcal{P}^2\equiv\hat{p}_\rho^2+\frac{\hat{p}_\phi^2}{\hat{R}^2},\quad\hat{R}\equiv\frac{R}{M},\quad\hat{p}_\rho\equiv\frac{p_\rho}{\mu},\quad\hat{p}_\phi\equiv\frac{p_\phi}{\mu M}. \tag{3.12}$$

The associated canonical transformation reads

$$\begin{aligned}\rho(Q,p)&=R+\frac{\partial G}{\partial p_\rho},\\\phi(Q,p)&=\Phi+\frac{\partial G}{\partial p_\phi},\\P_R(Q,p)&=p_\rho+\frac{\partial G}{\partial R},\\P_\Phi(Q,p)&=p_\phi+\frac{\partial G}{\partial\Phi},\end{aligned} \tag{3.13}$$

and leads to 1PK and higher order coordinate changes. Note that the Φ independence of $G(Q,p)$ yields $P_\Phi=p_\phi$. Moreover, for circular orbits, $p_\rho=0\Leftrightarrow P_R=0$, we note that $\phi=\Phi$ and hence only the radial coordinates differ $\rho\neq R$.

The two-body Hamiltonian (3.5) is thus rewritten in the intermediate coordinate system $H'(Q,p)=H(Q,P(Q,p))$ using the last two equations in (3.13) which yield (dropping the prime)

$$\hat{H}=\frac{M}{\mu}+\left(\frac{\mathcal{P}^2}{2}-\frac{G_{AB}}{\hat{R}}\right)+\hat{H}^{1\mathrm{PK}}+\hat{H}^{2\mathrm{PK}}+\cdots, \tag{3.14}$$

where the explicit expressions for $\hat{H}^{1\mathrm{PK}}$ and $\hat{H}^{2\mathrm{PK}}$ are given in Appendix D of Paper 1. It depends on the eight fundamental parameters (3.2), on the fourteen parameters

f_i characterizing the coordinate system in which the two-body Hamiltonian $H(Q,P)$ was written, and on the nine parameters of the canonical transformation (3.11).

IV. THE SCALAR-TENSOR EOB HAMILTONIAN

In this section we relate the canonically transformed, two-body Hamiltonians $H(Q,p)$ to the Hamiltonian H_e of an effective test-particle in the fields of an effective central body.

To this aim, we shall propose a ST-centered Hamiltonian H_e that contrasts with what was done in Paper 1, where H_e was centered on the GR limit.

A. The effective Hamiltonian

In view of reducing the two-body dynamics to that of an effective test particle coupled to the generic SSS fields of an effective single body, and taking inspiration from (2.16), let us consider the action (setting again $\theta = \pi/2$)

$$S_e[x^\mu] = -\int m_e(\varphi_e)ds_e \tag{4.1}$$

where $ds_e = \sqrt{-g^e_{\mu\nu}dx^\mu dx^\nu}$ and where $x^\mu[s_e]$ is the world line of the effective particle characterized by the function $m_e(\varphi_e)$. As in (2.18), we write the effective metric in Just coordinates,

$$ds_e^2 = -D_e dt^2 + \frac{d\rho^2}{D_e} + C_e\rho^2 d\phi^2, \tag{4.2}$$

where D_e and C_e are effective functions to be determined later.

We now replace, for notational convenience, $m_e(\varphi_e)$ with the function

$$V_e \equiv \left(\frac{m_e(\varphi_e)}{\mu}\right)^2, \quad \text{such that} \quad S_e[x^\mu] = -\mu\int \sqrt{V_e}ds_e, \tag{4.3}$$

which is the third effective function to be determined, and where μ is identified to the real two-body reduced mass, defined in (3.7). The associated Lagrangian, defined as $S_e \equiv \int dt L_e$, therefore reads

$$\begin{aligned} L_e &= -\mu\sqrt{-(V_e g^e_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \\ &= -\mu\sqrt{V_e\left(D_e - \frac{\dot\rho^2}{D_e} - C_e\rho^2\dot\phi^2\right)}, \\ &\text{where } \dot\rho \equiv d\rho/dt, \qquad \dot\phi \equiv d\phi/dt. \end{aligned} \tag{4.4}$$

Note that L_e identifies to the Lagrangian of a geodesic in the *body-dependent* conformal metric, $(V_e g^e_{\mu\nu})$.

One easily deduces the effective momenta and Hamiltonian,

$$p_\rho \equiv \frac{\partial L_e}{\partial\dot\rho}, \qquad p_\phi \equiv \frac{\partial L_e}{\partial\dot\phi}, \qquad H_e \equiv p_\rho\dot\rho + p_\phi\dot\phi - L_e,$$

that is

$$\hat H_e \equiv \frac{H_e}{\mu} = \sqrt{V_e D_e + D_e^2\hat p_\rho^2 + \frac{D_e}{C_e}\frac{\hat p_\phi^2}{\hat\rho^2}}, \tag{4.5}$$

where we used the reduced (dimensionless) variables

$$\hat\rho \equiv \frac{\rho}{M}, \qquad \hat p_\rho \equiv \frac{p_\rho}{\mu}, \qquad \hat p_\phi \equiv \frac{p_\phi}{\mu M}, \qquad \hat p^2 \equiv \hat p_\rho^2 + \frac{\hat p_\phi^2}{\hat\rho^2}, \tag{4.6}$$

M being identified to the real total mass; see (3.7).

In order to relate the effective Hamiltonian H_e to the two-body (perturbative) Hamiltonian H, we now restrict H_e to 2PK order also. To this end, one could in principle expand V_e, D_e, and C_e in the form of $1/\hat\rho$ series. However, our aim being to build an effective dynamics as close as possible to the *scalar-tensor* test-body problem, we shall rather introduce the nonperturbative, "resummed" ansatz for the metric functions D_e and C_e,

$$D_e(\rho) \equiv \left(1 - \frac{a}{\hat\rho}\right)^{\frac{b}{a}}, \qquad C_e(\rho) \equiv \left(1 - \frac{a}{\hat\rho}\right)^{1-\frac{b}{a}}, \tag{4.7}$$

as suggested by (2.19), and where a and b are two effective parameters that we shall determine in the following. As already remarked below Eq. (4.4), the effective dynamics is equivalent to the geodesic motion in the conformal metric $(V_e g^e_{\mu\nu})$. The ansatz (4.7) that we shall use rather than a simple $1/\hat\rho$ expansion of D_e and C_e is hence crucial, since the latter would be equivalent, to within a mere coordinate change ($r^2 = C_e V_e \rho^2$), to the GR-centered approach of Paper 1.

In contrast, a specific ansatz for the function V_e can be proposed in the framework of a specific ST theory and when the internal structure of the two real bodies is known; see discussion below (2.21). (For an example, see Subsection IV D.) For the moment, we hence expand V_e a 2PK order, similarly to what was done in (2.24),

$$V_e(\rho) = 1 + \frac{v_1}{\hat\rho} + \frac{v_2}{\hat\rho^2} + \frac{v_3}{\hat\rho^3} + \cdots, \tag{4.8}$$

where v_1, v_2, and v_3 are three further effective parameters to determine later.

Expanding the effective Hamiltonian (4.5) and (4.7)–(4.8) hence reads

$$\hat{H}_e = 1 + \hat{H}_e^{\mathrm{K}} + \hat{H}_e^{1\mathrm{PK}} + \hat{H}_e^{2\mathrm{PK}} + \cdots \tag{4.9}$$

with, at 1PK,

$$\begin{aligned}
\hat{H}_e^{\mathrm{K}} &= \frac{\hat{p}^2}{2} + \frac{v_1 - b}{2\hat{\rho}}, \\
H_e^{1\mathrm{PK}} &= -\frac{\hat{p}^4}{8} + \frac{1}{4\hat{\rho}}[\hat{p}^2(2a - 3b - v_1) - 2a\hat{p}_\rho^2] \\
&\quad + \frac{1}{8\hat{\rho}^2}[-2ab + b^2 - 2bv_1 - v_1^2 + 4v_2],
\end{aligned} \tag{4.10}$$

and, at 2PK,

$$\begin{aligned}
H_e^{2\mathrm{PK}} &= \frac{\hat{p}^6}{16} + \frac{1}{16\hat{\rho}}[\hat{p}^4(5b + 3v_1 - 4a) + 4a\hat{p}^2\hat{p}_\rho^2] \\
&\quad + \frac{1}{16\hat{\rho}^2}[4a\hat{p}_\rho^2(-2a + 3b + v_1) \\
&\quad + (8a^2 + 9b^2 + 6bv_1 + 3v_1^2 \\
&\quad - 2a(9b + 2v_1) - 4v_2)\hat{p}^2] \\
&\quad + \frac{1}{48\hat{\rho}^3}[-8a^2b - b^3 + 6ab(b - v_1) \\
&\quad + 3b^2v_1 + 3b(v_1^2 - 4v_2) \\
&\quad + 3(v_1^3 - 4v_1v_2 + 8v_3)].
\end{aligned} \tag{4.11}$$

In order to relate the two-body Hamiltonians of the previous Sec. III B and the present effective Hamiltonian $H_e(q,p)$, we finally express the latter in the same coordinate system $H_e'(Q,p) = H_e(q(Q,p),p)$ using the first two relations in (3.13). The resulting effective Hamiltonian reads (dropping again the prime)

$$\hat{H}_e = 1 + \left(\frac{\mathcal{P}^2}{2} + \frac{v_1 - b}{2\hat{R}}\right) + \hat{H}_e^{1\mathrm{PK}} + \hat{H}_e^{2\mathrm{PK}} + \cdots \tag{4.12}$$

where we recall that $\mathcal{P}^2 \equiv \hat{p}_\rho^2 + \hat{p}_\phi^2/\hat{R}^2$ and where $H_e^{1\mathrm{PK}}$ and $H_e^{2\mathrm{PK}}$ are explicitly given in Appendix A of this paper.

B. The EOB mapping

By means of the generic canonical transformation (3.13) to (3.11), the real and (*a priori* independent) effective Hamiltonians $H(Q,p)$ and $H_e(Q,p)$ have been written in a common coordinate system, (Q,p); see (3.14) and (4.12). Now, as discussed in, e.g., [8,18,19], and as proven to be indeed necessary at all orders in GR as well as in ST theories in [20], both Hamiltonians shall be related by means of the quadratic functional relation (we recall that $\nu = \mu/M$):

$$\frac{H_e(Q,p)}{\mu} - 1 = \left(\frac{H(Q,p) - M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(Q,p) - M}{\mu}\right)\right]. \tag{4.13}$$

The identification (4.13) proceeds order by order and term by term to yield a *unique* solution for H_e, that is for the functions introduced in the previous subsection,

$$\begin{aligned}
D_e(\rho) &\equiv \left(1 - \frac{a}{\hat{\rho}}\right)^{\frac{b}{a}}, \\
C_e(\rho) &\equiv \left(1 - \frac{a}{\hat{\rho}}\right)^{1-\frac{b}{a}}, \\
V_e(\rho) &= 1 + \frac{v_1}{\hat{\rho}} + \frac{v_2}{\hat{\rho}^2} + \frac{v_3}{\hat{\rho}^3} + \cdots,
\end{aligned} \tag{4.14}$$

whose effective parameters now depend on the combinations (3.8) and are the main technical result of this paper,

$$b = 2, \quad v_1 = -2\alpha_A^0\alpha_B^0, \tag{4.15a}$$

$$a = 2\mathcal{R}, \quad v_2 = 2 - 4G_{AB} + 2(1 + \langle\bar{\beta}\rangle)G_{AB}^2 - 2\alpha_A^0\alpha_B^0\mathcal{R}, \tag{4.15b}$$

$$\begin{aligned}
\frac{v_3}{4} &= 1 - \frac{5}{3}G_{AB} + \left(1 + \langle\bar{\beta}\rangle + \frac{2}{3}\langle\delta\rangle\right)G_{AB}^2 \\
&\quad - \frac{1}{3}\left(1 + 3\langle\bar{\beta}\rangle + \frac{1}{4}\langle\epsilon\rangle + 2\langle\delta\rangle\right)G_{AB}^3 \\
&\quad + (1 - 2G_{AB} + (1 + \langle\bar{\beta}\rangle)G_{AB}^2)\mathcal{R} \\
&\quad + \nu\left[\frac{17}{3}G_{AB} - \frac{1}{3}(19 + 4\langle\bar{\beta}\rangle + 6\zeta)G_{AB}^2\right. \\
&\quad + \left(\frac{2}{3} - \frac{3}{4}(\bar{\beta}_A + \bar{\beta}_B) + \frac{1}{12}(\epsilon_A + \epsilon_B)\right. \\
&\quad \left.\left. + \frac{1}{6}(\delta_A + \delta_B) + \frac{3}{2}\langle\bar{\beta}\rangle\right)G_{AB}^3\right],
\end{aligned} \tag{4.15c}$$

where we have introduced

$$\mathcal{R} \equiv \sqrt{1 + \langle\delta\rangle G_{AB}^2 + \nu[8G_{AB} - 2(1 + \langle\bar{\beta}\rangle)G_{AB}^2]}, \tag{4.16}$$

and the "mean" quantities

$$\begin{aligned}
\langle\bar{\beta}\rangle &\equiv \frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{M}, \\
\langle\delta\rangle &\equiv \frac{m_A^0\delta_A + m_B^0\delta_B}{M}, \\
\langle\epsilon\rangle &\equiv \frac{m_A^0\epsilon_B + m_B^0\epsilon_A}{M}.
\end{aligned} \tag{4.17}$$

We note that as they should, these parameters (4.15) can alternatively be deduced from the effective metric found in

Paper 1, using the 2PK-expanded coordinate change $r^2 = C_e V_e \rho^2$, where r is the Schwarschild-Droste coordinate used there.[6]

As a first consistency check, we note that the effective coefficients (4.15) do not depend on the f_i parameters introduced in Sec. III A, i.e., on the coordinate system (R, Φ) in which the two-body Hamiltonian has been initially written, as expected by covariance of the theory. Indeed, the f_i parameters are absorbed in the 2PK part of the canonical transformation (3.11), whose parameters read

$$\begin{aligned}
&\alpha_1 = -\frac{\nu}{2}, \qquad \beta_1 = 0, \qquad \gamma_1 = G_{AB}\left[\frac{1}{2}\nu + \left(1+\frac{1}{2}\bar{\gamma}_{AB}\right)\mathcal{R}\right],\\
&\alpha_2 = \frac{1}{8}(1-\nu)\nu, \qquad \beta_2 = 0, \qquad \gamma_2 = \frac{\nu^2}{2},\\
&\delta_2 = G_{AB}\left[f_6\frac{m_A^0}{M} + f_1\frac{m_B^0}{M} - \nu\left(f_1+f_6+(-f_3+f_5+f_6)\frac{m_A^0}{M} + (f_1+f_2-f_4)\frac{m_B^0}{M} - \frac{3}{2} - \bar{\gamma}_{AB} + \frac{\nu}{8}\right)\right],\\
&\epsilon_2 = G_{AB}\left[-\frac{\nu^2}{8} + f_{10}\frac{m_A^0}{M} + f_7\frac{m_B^0}{M} - \nu\left(f_7+f_{10}+(f_9+f_{10})\frac{m_A^0}{M} + (f_7+f_8)\frac{m_B^0}{M}\right)\right],\\
&\eta_2 = G_{AB}^2\left[f_{13}\frac{m_A^0}{M} + f_{12}\frac{m_B^0}{M} + \nu(f_{11}-f_{12}-f_{13}+f_{14}) + \nu\left(-\frac{7}{4} - \bar{\gamma}_{AB} - \langle\bar{\beta}\rangle + \frac{\bar{\beta}_A+\bar{\beta}_B}{2} + \frac{\nu}{4}\right)\right].
\end{aligned} \tag{4.18}$$

The real two-body Hamiltonian (3.5), whose full expression is relegated to Sec. III C and Appendix C of Paper 1, has hence been reduced to a compact effective Hamiltonian, where most of the two-body Hamiltonian complexity is hidden in the canonical transformation (3.11), (4.18) (e.g., information regarding the initial coordinate system) and in the mapping relation (4.13).

1. The $\nu = 0$ limit

Setting formally $\nu = 0$ in (4.15)–(4.16), the parameters reduce to, when written in terms of the fundamental quantities (3.2),

$$\begin{aligned}
&b = 2,\\
&v_1 = -2\alpha_A^0\alpha_B^0,
\end{aligned} \tag{4.19a}$$

$$\begin{aligned}
&a = 2\mathcal{R},\\
&v_2 = 2(\alpha_A^0\alpha_B^0)^2 + \frac{(m_A\alpha_A^2)^0\beta_B^0 + (m_B\alpha_B^2)^0\beta_A^0}{M} - 2\alpha_A^0\alpha_B^0\mathcal{R},
\end{aligned} \tag{4.19b}$$

$$\begin{aligned}
v_3 = &-\frac{4}{3}(\alpha_A^0\alpha_B^0)^3 - \frac{1}{3}\frac{(m_A\alpha_A^3)^0\beta'^0_B + (m_B\alpha_B^3)^0\beta'^0_A}{M}\\
&-2\alpha_A^0\alpha_B^0\frac{(m_A\alpha_A^2)^0\beta_B^0 + (m_B\alpha_B^2)^0\beta_A^0}{M}\\
&-\frac{8}{3}\left(1+\frac{(m_A\alpha_A^2)^0 + (m_B\alpha_B^2)^0}{M}\right)\alpha_A^0\alpha_B^0\\
&+\left(4(\alpha_A^2\alpha_B^2)^0 + 2\frac{(m_A\alpha_A^2)^0\beta_B^0 + (m_B\alpha_B^2)^0\beta'^0_A}{M}\right)\mathcal{R},
\end{aligned}$$

$$\text{with } \mathcal{R} = \sqrt{1+\frac{(m_A\alpha_A^2)^0 + (m_B\alpha_B^2)^0}{M}}. \tag{4.19c}$$

Identifying now (4.19) to the parameters (2.21) and (2.25) of the real one-body problem presented in Sec. II C does yield a unique solution,

$$(A_*^0)^2 = \frac{m_A^0(\alpha_A^0)^2 + m_B^0(\alpha_B^0)^2}{m_A^0 + m_B^0}, \tag{4.20a}$$

$$\alpha_*^0 = \frac{\alpha_A^0\alpha_B^0}{A_*^0}, \tag{4.20b}$$

$$\beta_*^0 = \frac{(m_A\alpha_A^2)^0\beta_B^0 + (m_B\alpha_B^2)^0\beta_A^0}{(m_A\alpha_A^2)^0 + (m_B\alpha_B^2)^0}, \tag{4.20c}$$

$$\beta'^0_* = \frac{(m_A\alpha_A^3)^0\beta'^0_B + (m_B\alpha_B^3)^0\beta'^0_A}{(m_A^0 + m_B^0)(A_*^0)^3}, \tag{4.20d}$$

together with $m_*^0 = \mu$, $M_*^0 = M$.

We hence conclude that the dynamics described by H_e is a ν deformation of a scalar-tensor test-body problem, describing an effective test particle characterized by

[6] Note also that the present results (4.15)–(4.17) have been simplified using the relation $\bar{\gamma}_{AB} = -2 + 2/G_{AB}$, relating $\bar{\gamma}_{AB}$ to the *dimensionless* combination G_{AB}; see (3.8). The reader wishing to establish G_* (i.e., Newton's constant) again should note that it only appears through $\hat{\rho} \equiv \rho/(G_* M)$.

$$\ln m_*(\varphi) = \ln m_*^0 + \alpha_*^0(\varphi - \varphi_0) + \frac{1}{2}\beta_*^0(\varphi - \varphi_0)^2 + \frac{1}{6}\beta'^0_*(\varphi - \varphi_0)^3 + \cdots, \quad (4.21)$$

orbiting around an effective central body characterized by

$$\ln M_*(\varphi) = \ln M_*^0 + A_*^0(\varphi - \varphi_0) + \cdots, \quad (4.22)$$

whose fundamental parameters [M_*^0, A_*^0, m_*^0, α_*^0, β_*^0, and β'^0_*] are related to the real, two-body ones through (4.20). Since $\nu \to 0$ means, say, $m_B^0 \gg m_A^0$, one retrieves consistently

$$M_*^0 \to m_B^0, \qquad A_*^0 \to \alpha_B^0, \qquad m_*^0 \to m_A^0,$$
$$\alpha_*^0 \to \alpha_A^0, \qquad \beta_*^0 \to \beta_A^0, \qquad \beta'^0_* \to \beta'^0_B,$$

that is, A becomes a test body orbiting around the central body B.

We note also that ν deformations do not enter the coefficients b and v_1 in the generic $\nu \neq 0$ case, see (4.15a), which are hence particularly simple; we hence recover a feature of the *linearized* effective dynamics which is common with that of the general relativity case (see Buonanno and Damour in [8]), and which is related to the very specific form of the quadratic functional relation (4.13).[7]

2. General relativity

Finally, in the general relativity limit (3.9), (4.21), and (4.22) become the well-known reduced and total masses $m_*(\varphi) = \mu$ and $M_*(\varphi) = M$, and the effective coefficients (4.15) reduce to

$$a = 2\sqrt{1+6\nu}, \qquad b = 2, \quad (4.23a)$$

$$v_1 = v_2 = v_3 = 0. \quad (4.23b)$$

In other words, $V_e = 1$, i.e., the effective scalar field effects disappear. The (nonperturbative) metric sector is now written in Just coordinates and differs from the results of Buonanno and Damour [8], who worked out their analysis in Schwarzschild-Droste coordinates. In the present paper, we hence have on hand a resummation of the 2PN general relativity dynamics that differs from the one explored in [8]. The comparison and consistency of the two shall be commented upon in Subsection IV D. When, moreover, $\nu = 0$, $a = b$ and the metric consistently reduces to Schwarzschild's, see the comment below (2.5).

[7] We also recall that the gravitational coupling $G_{AB} = 1 + \alpha_A^0\alpha_B^0$, appearing in the two-body Hamiltonian [see (3.5)], Subsection III A, and Paper 1 Sec. III C), encompasses the linear addition of the metric and scalar interations at linear level [9]. The present mapping has consistently split it again, between the effective metric and scalar sectors, i.e., b and v_1, see (4.15a), contrarily to the GR-centered, fully *metric* mapping of Paper 1, where G_{AB} appeared at each post-Keplerian order in the form $(G_{AB}M)/r$, r being the Schwarzschild-Droste coordinate used there.

C. ST-EOB dynamics

Inverting the EOB mapping relation (4.13) yields the "EOB Hamiltonian,"

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)},$$
$$\text{where } \frac{H_e}{\mu} = \sqrt{D_e V_e + D_e^2 \hat{p}_\rho^2 + \frac{D_e}{C_e}\left(\frac{\hat{p}_\phi}{\hat{\rho}}\right)^2}, \quad (4.24)$$

[where D_e, C_e, and V_e are given in (4.14) and (4.15)] which defines a resummation of the two-body 2PK Hamiltonian, H. In the following we focus on some features of the resultant resummed dynamics, in the strong field regime. Henceforth, the 2PK-truncated function V_e is to be considered as exact, along with D_e and C_e.

1. Effective dynamics

As we shall see, the ST-EOB dynamics will follow straightforwardly from that derived from the effective Hamiltonian H_e. This can be obtained from Hamilton's equations ($\dot{q} = \partial H_e/\partial p$, $\dot{p} = -\partial H_e/\partial q$), or, as already remarked below (4.4), can be equivalently interpreted as a geodesic of the conformal metric $\tilde{g}_{\mu\nu} = V_e g^e_{\mu\nu}$,

$$d\tilde{s}_e^2 \equiv -D_e V_e dt^2 + \frac{V_e}{D_e}d\rho^2 + C_e V_e \rho^2 d\phi^2. \quad (4.25)$$

The staticity and spherical symmetry of this metric imply the conservation of the energy and angular momentum of the orbit (per unit mass μ),

$$u_t = -D_e V_e \frac{dt}{d\lambda} \equiv -E, \qquad u_\phi = C_e V_e \rho^2 \frac{d\phi}{d\lambda} \equiv L, \quad (4.26)$$

λ being an affine parameter along the trajectory. When, moreover, the 4-velocity is normalized as $u^\mu u_\mu = -\epsilon$ (where $\epsilon = 1$ for $\mu \neq 0$, $\epsilon = 0$ for null geodesics), the radial motion is driven by an effective potential F_ϵ,

$$\left(\frac{d\rho}{d\lambda}\right)^2 = \frac{1}{V_e^2}F_\epsilon(u), \quad (4.27)$$

where

$$F_\epsilon(u) \equiv E^2 - D_e V_e\left(\epsilon + \frac{j^2 u^2}{C_e V_e}\right),$$
$$j \equiv \frac{L}{M}, \qquad u \equiv \frac{1}{\hat{\rho}} = \frac{M}{\rho},$$
and $D_e(u) = (1-au)^{b/a}, \qquad C_e(u) = (1-au)^{1-b/a},$
$$V_e(u) = 1 + v_1 u + v_2 u^2 + v_3 u^3. \tag{4.28}$$

2. ISCO location

We now focus on circular orbits when $\epsilon = 1$, i.e., $F_{\epsilon=1}(u) = F'_{\epsilon=1}(u) = 0$; j^2 and E are therefore related to u through

$$j^2(u) = -\frac{(D_e V_e)'}{(u^2 D_e/C_e)'},$$
$$E(u) = \sqrt{D_e V_e\left(1 + \frac{j^2(u)u^2}{C_e V_e}\right)}. \tag{4.29}$$

A characteristic feature of the strong field regime is the innermost stable circular orbit, which is reached when the third (inflection point) condition is satisfied $F''_{\epsilon=1}(u) = 0$, i.e., when $u_{\rm ISCO}$ is the root, if any, of the equation,

$$F'_{\epsilon=1}(u_{\rm ISCO}) = F''_{\epsilon=1}(u_{\rm ISCO}) = 0$$
$$\Rightarrow \frac{(D_e V_e)''}{(D_e V_e)'} = \frac{(u^2 D_e/C_e)''}{(u^2 D_e/C_e)'}. \tag{4.30}$$

3. Light-ring location

When $\epsilon = 0$, $F_{\epsilon=0}(u) = E^2 - j^2 u^2 \frac{D_e}{C_e}$ and one can define a light-ring (LR), i.e., the radius of null circular orbits, through $F'_{\epsilon=0}(u_{\rm LR}) = 0$,

$$u_{LR} = \frac{1}{b + \frac{a}{2}} \Leftrightarrow \rho_{LR} = M(2 + \mathcal{R}), \tag{4.31}$$

where $\mathcal{R}$ is given in (4.15). In particular, one retrieves $\mathcal{R} = 1$, i.e., $\rho_{LR} = 3M$ (Schwarzschild's LR location) in the test-mass ($\nu \to 0$), general relativity limit (3.9).

4. ST-EOB orbital frequency

We now turn to the resummed two-body dynamics defined by the EOB Hamiltonian (4.24). Since $H_{\rm EOB}$ and H_e are conservative, we have

$$\left(\frac{\partial H_{\rm EOB}}{\partial H_e}\right) = \frac{1}{\sqrt{1 + 2\nu(E-1)}} \tag{4.32}$$

since $H_e = \mu E$ is a constant on shell. Therefore, the resummed equations of motion

$$\frac{d\rho}{dt} = \frac{\partial H_{\rm EOB}}{\partial p_\rho}, \qquad \frac{d\phi}{dt} = \frac{\partial H_{\rm EOB}}{\partial p_\phi},$$
$$\frac{dp_\rho}{dt} = -\frac{\partial H_{\rm EOB}}{\partial \rho}, \qquad \frac{dp_\phi}{dt} = -\frac{\partial H_{\rm EOB}}{\partial \phi} = 0, \tag{4.33}$$

are identical to the effective ones, i.e., derived from the effective Hamiltonian, $H_e(q, p)$, to within the (constant) time rescaling $t \to t\sqrt{1 + 2\nu(E-1)}$. In particular, for circular orbits, the orbital frequency reads

$$\Omega(u) \equiv \frac{d\phi}{dt} = \frac{\partial H_{\rm EOB}}{\partial H_e}\frac{\partial H_e}{\partial p_\phi} = \frac{D_e}{C_e}\frac{j u^2}{ME\sqrt{1 + 2\nu(E-1)}}, \tag{4.34}$$

where $E(u)$ and $j(u)$ are given for circular orbits in (4.29). Its ISCO value is reached when $u = u_{\rm ISCO}$, as defined in (4.30).

Note that the orbital frequency has been derived in the Just coordinate system, (q, p), which is related to the real one, (Q, P), through the canonical transformation presented in Subsection III B. Moreover, for circular orbits ($p_\rho = P_R = 0$), $\Phi = \phi$, and hence (4.34) is the *observed* orbital frequency. See also Subsections III B and IV D.

D. An example: The Jordan-Fierz-Brans-Dicke theory

1. A simple one-parameter model

We now illustrate the previous results through the example of the Jordan-Fierz-Brans-Dicke theory [21,22], which depends on a unique parameter α, such that[8]

$$S_{\rm JFBD}[g_{\mu\nu}, \varphi, \Psi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi)$$
$$+ S_m[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}],$$
where $\mathcal{A}(\varphi) = e^{\alpha\varphi}$,
$$\alpha = \frac{d\ln\mathcal{A}}{d\varphi} = cst, \tag{4.35}$$

while general relativity is retrieved when $\alpha = 0$.

The two-body dynamics is then described by replacing S_m with its "skeleton" version,

$$S_m^{\rm skel}[x_A^\mu, g_{\mu\nu}, \varphi] = -\sum_A \int m_A(\varphi) ds_A, \tag{4.36}$$

where, for the sake of simplicity, we shall neglect self-gravity effects, i.e., $m_A(\varphi) = \mathcal{A}(\varphi)\tilde{m}_A$, where $\tilde{m}_A$ are constants; see the discussion above (3.2). In that case, since $\mathcal{A}(\varphi)$ is known and the Jordan masses $\tilde{m}_A$ are

[8]For a comparison with the Jordan-frame parameter ω, such that $3 + 2\omega = \alpha^{-2}$, see [7] Appendix A.

constants, there is no need to expand $m_A(\varphi)$ as in (2.23) since it is entirely determined as

$$m_A(\varphi) = m_A^0 e^{\alpha(\varphi-\varphi_0)}, \qquad m_A^0 = cst. \tag{4.37}$$

Therefore, the fundamental parameters (3.2) become universal (3.3) and reduce to

$$\alpha_A = \frac{d\ln m_A}{d\varphi} = \alpha, \qquad \beta_A = 0, \qquad \beta'_A = 0, \tag{4.38}$$

and the post-Keplerian (two-body) parameters (3.8) greatly simplify as well to

$$G_{AB} = 1+\alpha^2, \quad \bar\gamma_{AB} = -\frac{2\alpha^2}{1+\alpha^2}, \quad \delta_A = \delta_B = \frac{\alpha^2}{(1+\alpha^2)^2},$$
$$\bar\beta_A = \bar\beta_B = 0, \quad \epsilon_A = \epsilon_B = 0, \quad \zeta = 0. \tag{4.39}$$

Hence, the coefficients (4.15) of the functions

$$D_e = \left(1-\frac{a}{\hat\rho}\right)^{\frac{b}{a}},$$
$$C_e = \left(1-\frac{a}{\hat\rho}\right)^{1-\frac{b}{a}},$$
$$V_e = 1+\frac{v_1}{\hat\rho}+\frac{v_2}{\hat\rho^2}+\frac{v_3}{\hat\rho^3}+\cdots, \tag{4.40}$$

depend only on α and $\nu = \mu/M$ and reduce to

$$b = 2, \qquad v_1 = -2\alpha^2, \tag{4.41a}$$

$$a = 2\mathcal{R}, \qquad v_2 = 2\alpha^4 - 2\alpha^2\mathcal{R}, \tag{4.41b}$$

$$v_3 = \frac{4}{3}\alpha^2(3\alpha^2\mathcal{R} - (2+2\alpha^2+\alpha^4) - \nu(14+12\alpha^2-2\alpha^4)),$$
$$\text{with} \quad \mathcal{R}(\nu) = \sqrt{(1+\alpha^2)(1+2(3-\alpha^2)\nu)}. \tag{4.41c}$$

2. An improved V_e function

As discussed in Subsection IV B, the effective dynamics is a ν deformation of a ST test-body problem, which, in the present case, describes a test particle $m_*(\varphi) = \mu e^{\alpha(\varphi-\varphi_0)}$ orbiting around a central body $M_*(\varphi) = M e^{\alpha(\varphi-\varphi_0)}$, where $\mu = m_A^0 m_B^0/M$ and $M = m_A^0 + m_B^0$; see (4.20) and below.

Therefore, in keeping with our approach consisting of centering as much as possible the effective dynamics on the test-body problem, we can "improve" V_e by factorizing out its exact, $\nu = 0$ expression,

$$V_e = V_{\text{exact}}^{\nu=0} P(\nu),$$
$$P(\nu) = 1+\frac{p_1}{\hat\rho}+\frac{p_2}{\hat\rho^2}+\frac{p_3}{\hat\rho^3}+\cdots, \tag{4.42}$$

where, by definition [see (2.17)],

$$V_{\text{exact}}^{\nu=0} \equiv \left(\frac{m_*(\varphi_e)}{m_*^0}\right)^2 = e^{2\alpha(\varphi_e-\varphi_0)}, \tag{4.43}$$

and where φ_e is the scalar field generated by the central body [see (2.21)],

$$\varphi_e = \varphi_0 + \frac{\alpha}{2\sqrt{1+\alpha^2}}\ln\left(1-\frac{2\sqrt{1+\alpha^2}}{\hat\rho}\right), \qquad \hat\rho = \rho/M. \tag{4.44}$$

The 2PK identification of (4.42)–(4.44) with (4.40)–(4.41) then gives

$$V_e = \left(1-\frac{2\sqrt{1+\alpha^2}}{\hat\rho}\right)^{\frac{\alpha^2}{\sqrt{1+\alpha^2}}} P(\nu),$$
$$P(\nu) = 1+\frac{p_1}{\hat\rho}+\frac{p_2}{\hat\rho^2}+\frac{p_3}{\hat\rho^3},$$
$$\text{with} \quad p_1 = 0,$$
$$p_2 = 2\alpha^2[\mathcal{R}(0)-\mathcal{R}(\nu)],$$
$$p_3 = -\frac{8}{3}\alpha^2(7+6\alpha^2-\alpha^4)\nu, \tag{4.45}$$

where $P(\nu = 0) = 1$. In doing so, in the test-mass limit, D_e and C_e, *as well as* V_e, reduce to their *exact*, nonperturbative expressions, to which they are smoothly connected.

3. The ST-EOB orbital frequency at the ISCO

We now have on hand all the necessary material to study the ISCO location, $u_{\text{ISCO}} \equiv M/\rho_{\text{ISCO}}$, and associated orbital frequency, $M\Omega_{\text{ISCO}}$, as defined in the previous subsection, using (4.29), (4.30), and (4.34). The results are even in α, as expected from (4.41) and (4.45), and are gathered in Fig. 1 for $0 < \alpha^2 < 1$.

The limit $\alpha = 0$ reduces to general relativity. When, moreover, $\nu = 0$, one recovers the well-known Schwarzschild values $u_{\text{ISCO}} = 1/6$, $M\Omega_{\text{ISCO}} = 0.06804$ [since then the Just and Droste-Schwarzschild coordinates coincide, see the comment below (4.23)]. Note that when $\alpha = 0$ but $\nu \neq 0$, u_{ISCO} is *less* than $1/6$. This does not contradict the general relativity results of Buonanno and Damour [8], who worked in Droste coordinates rather than Just's; rather, this illustrates the fact that the effective radii are physically irrelevant, contrary to the orbital frequency $M\Omega_{\text{ISCO}}$ which *is* an observable: for $\alpha = 0$ and for all $\nu \neq 0$, the ISCO frequency turns out to be always larger than the Schwarzschild one (see right panel of Fig. 1), as in [8]. For

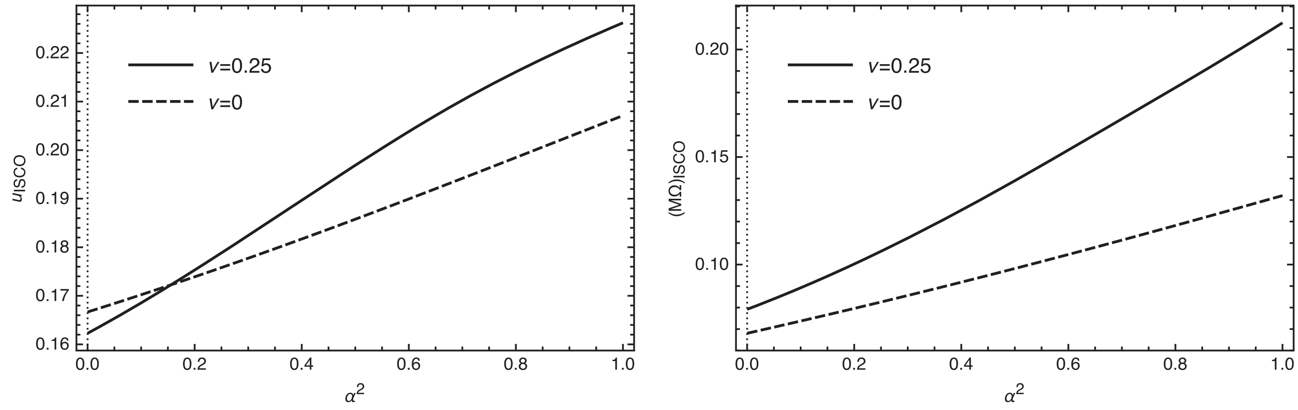

FIG. 1. ISCO location (left panel) in Just coordinates and ISCO frequency (right panel) versus the (squared) Jordan-Fierz-Brans-Dicke parameter α^2, when $\nu = 0$ (dashed lines) and $\nu = 0.25$ (solid lines).

instance, when $\nu = 1/4$, we find $M\Omega_{\text{ISCO}} = 0.07919$, i.e., slightly higher than the value 0.07340 quoted in [8]. The ~7% difference in the numerical values is reasonable considering that the two resummations [see (4.23)] are different and built on 2PK information only.

Now, when $\alpha \neq 0$, i.e., when the scalar field is switched on, the ISCO frequency increases roughly linearly in α^2, as can be seen from the right panel of Fig. 1, with a slope

$$\left.\frac{d(M\Omega_{\text{ISCO}})}{d(\alpha^2)}\right|_{\nu=1/4} \simeq 0.13 \quad \text{and} \quad \left.\frac{d(M\Omega_{\text{ISCO}})}{d(\alpha^2)}\right|_{\nu=0} \simeq 0.063. \tag{4.46}$$

Interestingly, when restricted to a perturbative regime $\alpha \ll 1$, these results are qualitatively consistent with the ones obtained from the *distinct* GR-centered resummation of [7], where ST effects were considered as perturbations of general relativity. There, we started from the best available EOB-NR metric, known in GR at 5PN order; see [23–25]. We then perturbed this effective metric by scalar-tensor 2PK corrections and studied their impact on the strong field dynamics. The ISCO frequency was also found there to increase linearly with the "PPN," Eddington parameter

$$\epsilon_{\text{1PK}} \equiv \langle\bar{\beta}\rangle - \bar{\gamma}_{AB} \tag{4.47}$$

[which reduces to $\epsilon_{\text{1PK}} \sim 2\alpha^2$ in the present case, see (3.8b), (4.17), and (4.39)], the slope being numerically of the same order of magnitude, hence illustrating the robustness of the EOB description of the strong field regime.[9]

More importantly, we have developed, throughout this paper, a ST-centered EOB Hamiltonian that reduces to the *exact* test-body Hamiltonian in the test-mass limit. Consequently, the ISCO predictions are well-defined even when $|\alpha| \sim 1$, that is, can be pushed to a regime that strongly departs from general relativity: there, the estimated ISCO location and frequency significantly deviate from the GR ones *and* remain smoothly connected to the test-mass ($\nu = 0$) limit (see Fig. 1), which we know exactly even in the strong field regime.[10]

We hence have illustrated, in the simple case of the Jordan-Fierz-Brans-Dicke theory, the *complementarity* of two EOB resummations of the scalar-tensor dynamics:

(i) The first one, introduced in Paper 1, which is built on rich (5PN) general relativity information, is oriented towards regimes where ST effects are considered as perturbations of GR [while the dynamics is ill-defined in nonperturbative regimes; this necessitates, e.g., the use of appropriate Padé resummations of the ST perturbations as soon as $\epsilon_{1PK} \gtrsim 10^{-1}$, see [7] for details].

(ii) The second, ST-centered one that we have developed throughout this paper, which has been shown to be well-suited to describe regimes that may depart strongly from general relativity; the price to pay being that it is based on 2PK information only.

An exhaustive study of generic ST theories [that depend on five parameters (4.4)] is left to future work.

V. CONCLUDING REMARKS

The reduction to a simple, effective-one-body motion has been a key element in the treatment of the two-body

[9] In particular, we found $d(G_{AB}M\Omega)/d\epsilon_{\text{1PK}} \simeq 0.13$ in the equal-mass case. In the present paper we will not proceed to any detailed, quantitative comparison of the two resummations since the present ST-centered approach is limited in this section to the JFBD case and since Paper 1 included some extra 5PN GR information.

[10] It must be noted that when $\alpha > \alpha_{\text{crit}} \simeq 1.6$, the exact test-body problem (which is reached when $\nu = 0$) does not feature any ISCO anymore, since then (4.30) has no root. This phenomenon is encompassed by our mapping; when ν is nonzero and increases, the value of α_{crit} smoothly decreases to reach $\alpha_{\text{crit}}(\nu = 1/4) \simeq 1.03$.

problem in general relativity. In the pioneering 1998 paper [8] of Buonanno and Damour, the 2PN effective dynamics was found to be a ν deformation of the test-body problem in GR, namely, the geodesic motion of a test particle μ in the Schwarzschild metric generated by a central body M. Remarkably, the fruitfulness of the EOB approach spreads beyond the scope of general relativity: indeed, by means of a canonical transformation and the same EOB quadratic relation (4.13), we reduced the 2PK two-body dynamics in scalar-tensor theories to a ν-deformed version of the ST test-body problem; namely, the motion of a test particle [μ, α^0_*, β^0_*, β'^0_*] orbiting in the fields of a central body [M, A^0_*].

The present mapping has led, just like that of Paper 1 [7], to a much simpler and compact description of the two-body dynamics in the 2PK regime, "gauging away" the irrelevant information in a canonical transformation. The (conservative) dynamics derived from the two ST-EOB Hamiltonians presented in [7] and in the present paper are, by construction, canonically equivalent at 2PK order but, when taken as being exact, they define two distinct resummations of the dynamics in the strong field regime. The fact that both lead to consistent ISCO predictions (in their overlapping ST regimes) is a hint that they may have captured accurately some of the strong field features of binary coalescence in ST theories.

To summarize, we have on hands two complementary EOB dynamics: (i) the geodesic motion in an effective metric in Schwarschild-Droste coordinates, encompassing the most accurate (5PN) GR information, which is particularly well-suited to test scalar-tensor theories when considered as parametrized corrections to general relativity [7]; and (ii) a ST effective test-body problem, in Just coordinates, that allows the investigation of regimes that depart strongly from GR. For example, the coupling α^0_A between the scalar field and stars that are subject to *spontaneous* scalarization can reach the order of unity [14]; binary systems involving such stars are hence encompassed in the present work.

Note that one cannot perform the 2PK Droste-Just coordinate change $r^2 = C_e V_e \rho^2$ without spoiling either the resummation towards the ST test-body problem of (ii) or the 5PN accurate GR information of (i).

Now, Solar System and binary pulsar experiments have already put stringent constraints on ST theories, namely, $(\alpha^0_A)^2 < 4\times 10^{-6}$ for any body A, and $\alpha^2 < 2\times 10^{-5}$ in (non–self-gravitating) JFBD theory (see, e.g., [26,27]). Since the parameters (4.19) contain terms that are all driven by at least $(\alpha^0_{A/B})^i$, $i \geq 2$, these constraints seem to imply that scalar-tensor effects are negligible. However, gravitational wave astronomy allows the observation of new regimes of gravity that might escape these constraints.

For example, from the cosmological point of view, GR is indeed an attractor of ST theories [11,12], and hence, the gravitational wave detectors LIGO-Virgo (and forthcoming LISA), which are designed to observe highly redshifted sources, can probe epochs when ST effects may have been stronger.

Also, stars that are subject to *dynamical* scalarization [28,29] can develop nonperturbative α_A couplings to the scalar field during the last few orbits before plunge. It must be noted that although the present paper aims at exploring the strong field regime near merger, it is based on PK information only, and hence, cannot cover *dynamical* scalarization phenomena as it is. Their implementation within the present ST-EOB framework is left to future work; note that analytical approaches to dynamical scalarization can be found in, e.g., [30,31].

Hence, the tools developed in the present paper, which goes beyond the scope of [7], could turn out to become useful in practice.

As for now, we restricted ourselves to the conservative part of the ST two-body problem. The incorporation of the EOB radiation reaction force will be the topic of further work. In particular, the comparison of the resulting gravitational waveforms to their numerical relativity counterparts, as investigated in [28] (at least at prescalarization stages), will allow us to estimate the accuracy of our ST-EOB approach to comparable mass ($\nu \simeq 1/4$) binary systems. Other information such as, e.g., the binding energy of a binary system predicted in [32], could also serve as a reference.

Finally, we recall that static, spherically symmetric black holes cannot carry scalar hair in the class of ST theories we are considering here (provided that the no hair theorems hold in the highly dynamical regime of a merger), see, e.g., the comments below Eq. (2.5) and references quoted there. An interesting alternative would be to induce hair by means of a massless gauge vector field, as for, e.g., Einstein-Maxwell-dilaton theories [33,34], which will be the subject of future works.

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APPENDIX: CANONICALLY TRANSFORMED EFFECTIVE HAMILTONIANS

Performing the canonical transformation (3.13)–(3.11), the effective 2PK Hamiltonian (4.9) is rewritten in the intermediate coordinate system $(q,p) \to (Q,p)$ as follows:

$$\hat{H}_e = 1 + \left(\frac{\mathcal{P}^2}{2} + \frac{v_1 - b}{2\hat{R}}\right) + \hat{H}_e^{1\mathrm{PK}} + \hat{H}_e^{2\mathrm{PK}} + \cdots \tag{A1}$$

where

$$\begin{aligned}
\hat{H}_e^{1\mathrm{PK}} &= \hat{p}_r^4(2\alpha_1+3\beta_1)-\hat{p}_r^2\mathcal{P}^2(\alpha_1+3\beta_1)+\mathcal{P}^4\left(-\alpha_1-\frac{1}{8}\right)\\
&\quad+\frac{1}{4\hat{R}}[\mathcal{P}^2(2a+2\alpha_1 b-3b-4\gamma_1-(2\alpha_1+1)v_1)-2\hat{p}_r^2(a-2\alpha_1(b-v_1)-3\beta_1(b-v_1)-2\gamma_1)]\\
&\quad+\frac{1}{8\hat{R}^2}[b(-2a+b+4\gamma_1)-2v_1(b+2\gamma_1)-v_1^2+4v_2],\\
\hat{H}_e^{2\mathrm{PK}} &= -\frac{1}{2}\hat{p}_r^6(36\alpha_1\beta_1+12\alpha_1^2+27\beta_1^2-4\beta_2-10\gamma_2)+\frac{1}{2}\hat{p}_r^4\mathcal{P}^2(2\alpha_1(9\beta_1-1)+8\alpha_2+27\beta_1^2-3\beta_1+2\beta_2-10\gamma_2)\\
&\quad+\frac{1}{2}\hat{p}_r^2\mathcal{P}^4(\alpha_1(18\beta_1+1)+9\alpha_1^2-6\alpha_2+3\beta_1-6\beta_2)+\frac{1}{16}(24\alpha_1^2+8\alpha_1-16\alpha_2+1)\mathcal{P}^6\\
&\quad+\frac{1}{16\hat{R}}[8\hat{p}_r^4(2\alpha_1(3(a-2b\beta_1-b-2\gamma_1)+(6\beta_1-1)v_1)+3\beta_1(3a-3b-6\gamma_1-v_1)+2b\beta_2+5b\gamma_2-4\alpha_1^2(b-v_1)\\
&\quad-9\beta_1^2(b-v_1)+4\delta_2-2\beta_2 v_1-5\gamma_2 v_1+6\epsilon_2)\\
&\quad-4\hat{p}_r^2\mathcal{P}^2(-2\alpha_1(-3a-6b\beta_1+6b+6\gamma_1+(6\beta_1+2)v_1)+18a\beta_1-a-27b\beta_1-6b\beta_2-36\beta_1\gamma_1\\
&\quad+8\alpha_1^2(b-v_1)-8\alpha_2(b-v_1)+2\gamma_1+4\delta_2-9\beta_1 v_1+6\beta_2 v_1+12\epsilon_2)\\
&\quad+\mathcal{P}^4(\alpha_1(-24a+36b+48\gamma_1)-4a-8\alpha_1^2 b+8\alpha_2 b+5b+8\gamma_1-16\delta_2+(8\alpha_1^2+12\alpha_1-8\alpha_2+3)v_1)]\\
&\quad+\frac{1}{16\hat{R}^2}[\mathcal{P}^2(8a^2+8\alpha_1 ab+v_1(-4a+8\alpha_1(b+2\gamma_1)+6b+12\gamma_1-8\delta_2)-18ab-24a\gamma_1-4\alpha_1 b^2+9b^2\\
&\quad-16\alpha_1 b\gamma_1+36b\gamma_1+8b\delta_2+24\gamma_1^2-16\eta_2-4(4\alpha_1+1)v_2+(4\alpha_1+3)v_1^2)\\
&\quad-4\hat{p}_r^2(2a^2-6ab\beta_1-v_1(a+4\alpha_1(b+2\gamma_1)+6\beta_1(b+2\gamma_1)-2\gamma_1-4\delta_2-6\epsilon_2)+\alpha_1(2b(-2a+b+4\gamma_1)+8v_2)\\
&\quad-3ab-6a\gamma_1+3b^2\beta_1+12b\beta_1\gamma_1+6b\gamma_1-4b\delta_2-6b\epsilon_2+6\gamma_1^2-4\eta_2-v_1^2(2\alpha_1+3\beta_1)+12\beta_1 v_2)]\\
&\quad+\frac{1}{48\hat{R}^3}[-3v_1(2ab-b^2-8\gamma_1(b+\gamma_1)+8\eta_2+4v_2)-12(b\gamma_1(-2a+b+2\gamma_1)-2b\eta_2+v_2(b+4\gamma_1))\\
&\quad-b(b-4a)(b-2a)+3v_1^2(b+4\gamma_1)+3v_1^3+24v_3].
\end{aligned}$$

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Deuxième partie

Trous noirs chevelus et rayonnement gravitationnel : l'exemple des théories Einstein-Maxwell-dilaton

Chapitre 4

Systèmes binaires de trous noirs en théories Einstein-Maxwell-dilaton

Dans la première partie de cette thèse, nous avons présenté le formalisme EOB, qui permet de décrire la dynamique d'un système binaire compact jusqu'au plus près de sa coalescence, et est donc très utile à la description de trous noirs binaires, qui sont pour l'heure les principales sources d'ondes gravitationnelles observées par les détecteurs LIGO-Virgo.

Nous avons ensuite montré au chapitre 2 qu'il est possible de généraliser ce formalisme à toute une classe de gravités modifiées, dont les théories scalaire-tenseur, à l'aide d'une métrique paramétrisée de la forme (2.7). Cependant, si l'on s'intéresse aux signatures d'éventuels "cheveux" dans la dynamique d'un système binaire de trous noirs, il est nécessaire d'étendre nos travaux au-delà des théories scalaire-tenseur puisque, dans ces dernières, les trous noirs (statiques et à symétrie sphérique tout du moins) se réduisent à ceux de la relativité générale.

Considérons les théories Eintein-Maxwell-dilaton (EMD), présentées en section 0.3.2 de l'introduction, qui consistent à élargir les théories scalaire-tenseur par l'ajout d'un champ de jauge vectoriel (ou "graviphoton"), couplé non minimalement au champ scalaire. Ces théories prédisent, en particulier, un trou noir caractérisé par des "cheveux" scalaire et vectoriel, solution statique et à symétrie sphérique de leurs équations du champ dans le vide, introduit par G.W. Gibbons et K.I. Maeda [123] :

$$\begin{aligned}
&ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} d\Omega^2\,,\\
&A_t = -\frac{Q\,e^{2a\varphi_\infty}}{r}\,,\quad A_i = 0\,,\qquad \text{avec}\qquad Q^2 = \frac{r_+ r_-}{1+a^2}\,e^{-2a\varphi_\infty}\,,\\
&\varphi = \varphi_\infty + \frac{a}{1+a^2}\ln\left(1-\frac{r_-}{r}\right)\,,
\end{aligned}\tag{4.1}$$

avec $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ et où a est le paramètre de couplage du "graviphoton" A^μ au champ scalaire φ.

La solution (4.1) dépend de trois constantes d'intégration : le rayon r_+ de son horizon, la position r_- de la singularité de courbure, et la valeur asymptotique φ_∞ du champ scalaire qui, en présence d'un compagnon lointain, s'identifie à la valeur locale du champ scalaire générée par ce dernier. On note que (4.1) se réduit au trou noir de Schwarzschild lorsque $Q = 0 \Leftrightarrow r_- = 0$, et à celui de Reissner-Nordström dans la limite de découplage vecteur-scalaire, $a = 0$.

Rappelons aussi que l'action EMD du vide, i.e. 0.3 avec $S_\mathrm{m} = 0$, est symétrique sous la transformation

$$\varphi \to \varphi + \Delta\varphi\,, \qquad A^\mu \to e^{a\Delta\varphi} A^\mu\,, \tag{4.2}$$

où $\Delta\varphi$ est une constante, se traduisant dans la solution (4.1) par $\varphi_\infty \to \varphi_\infty + \Delta\varphi$ (et donc $Q \to e^{-a\Delta\varphi}Q$). Cette symétrie est souvent utilisée pour poser $\varphi_\infty = 0$ dans (4.1), ce qui en allège l'écriture [181, 182], mais risque de faire perdre de vue que la solution dépend non pas de deux mais de trois paramètres, la valeur de φ à l'infini pouvant, dans une situation dynamique, varier.

L'objectif de ce chapitre est dans un premier temps d'obtenir le Lagrangien à deux corps en théorie EMD et de l'"EOBiser" (ce qui sera possible, à l'ordre 1PK tout du moins, vue son invariance de Lorentz), réduisant ainsi la dynamique à deux corps, y compris celle de trous noirs tels que (4.1), au mouvement géodésique dans la métrique EOB paramétrisée du chapitre 2.

À cette fin, on propose dans l'article reproduit ci-dessous [183] de généraliser le procédé de "skeletonisation" aux théories EMD, en montrant que, tant que les effets de "taille finie" (e.g., de marée) restent négligeables, les corps compacts (sans spins) peuvent être réduits à des particules ponctuelles décrites par l'action générale :

$$S_{\rm m}^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x_A^\mu\}] = -\sum_A \int m_A(\varphi)\, ds_A + \sum_A q_A \int A_\mu\, dx_A^\mu\,, \tag{4.3}$$

où $ds_A = \sqrt{-g_{\mu\nu}dx_A^\mu dx_A^\nu}$, et où apparaissent les sensibilités $m_A(\varphi)$ des corps d'Eardley [83], ainsi que les constantes q_A paramétrisant le couplage linéaire des lignes d'univers $x_A^\mu[s_A]$ au champ vectoriel A^μ.

Les corps ainsi "skeletonisés", il devient possible de généraliser les techniques de calcul du Lagrangien à deux corps en théorie scalaire-tenseur (voir, e.g., [88]) pour y inclure les contributions du champ vectoriel, en introduisant les paramètres m_A^0, α_A^0 et β_A^0 issus du développement à l'ordre 1PK des fonctions $m_A(\varphi)$ autour de la valeur $\varphi = \varphi_0$ imposée par l'environnement cosmologique du système,

$$\ln m_A(\varphi) = \ln m_A^0 + \alpha_A^0(\varphi - \varphi_0) + \frac{1}{2}\beta_A^0(\varphi-\varphi_0)^2 + \cdots\,, \tag{4.4a}$$

$$\text{ainsi que} \quad e_A = (q_A/m_A^0)e^{a\varphi_0}\,, \tag{4.4b}$$

où (4.4b) joue le rôle de couplage du corps A au "graviphoton".

Tous calculs faits, on trouve le fait remarquable que la structure du Lagrangien à deux corps, écrit en coordonnées harmoniques et dans la jauge de Lorenz, est identique à celle de son homologue scalaire-tenseur à l'ordre 1PK (2.1). De fait, les contributions, nouvelles, du champ vectoriel, sont entièrement encodées dans la redéfinition des combinaisons G_{AB}, $\bar\gamma_{AB}$ et $\bar\beta_{A/B}$ selon :

$$G_{AB} \equiv 1 + \alpha_A^0\alpha_B^0 - e_A e_B\,, \tag{4.5a}$$

$$\bar\gamma_{AB} \equiv \frac{-4\,\alpha_A^0\alpha_B^0 + 3\,e_A e_B}{2(1+\alpha_A^0\alpha_B^0 - e_A e_B)}\,, \tag{4.5b}$$

$$\bar\beta_A \equiv \frac{1}{2}\frac{\beta_A^0{\alpha_B^0}^2 - 2\,e_A e_B(a\,\alpha_B^0 - \alpha_A^0\alpha_B^0) + e_B^2(1 + a\,\alpha_A^0 - e_A^2)}{(1+\alpha_A^0\alpha_B^0 - e_A e_B)^2}\,, \quad (A \leftrightarrow B) \tag{4.5c}$$

où le couplage gravitationnel effectif G_{AB} inclut à présent la contribution "répulsive" de l'interaction vectorielle, et où $\bar\gamma_{AB}$ et $\bar\beta_{A/B}$ contiennent maintenant des termes quadratiques en $e_{A/B}$. Notons que nous venons, au passage, de calculer le Lagrangien à deux corps en théorie Einstein-Maxwell, correspondant au cas particulier $m_{A/B}(\varphi) = cste$, i.e. $\alpha^0_{A/B} = \beta^0_{A/B} = 0$; on retrouve celui des théories scalaire-tenseur lorsque $q_{A/B} = 0$, i.e. $e_{A/B} = 0$.

Par conséquent, l'inclusion des théories EMD au sein du formalisme EOB est triviale à l'ordre 1PK : il suffit en effet pour cela de remplacer G_{AB}, $\bar{\gamma}_{AB}$ et $\bar{\beta}_{A/B}$, apparaissant dans tous les résultats du chapitre 2, par leurs nouvelles expressions (4.5). Pour cette même raison, on conclut qu'à l'ordre 1PK, la dynamique à deux corps en théorie scalaire-tenseur et en théorie EMD sont indiscernables ; à ceci près, rappelons-le, qu'en théorie scalaire-tenseur sans "graviphoton", les trous noirs n'ont pas de cheveux...

Dans un second temps, on se propose donc de particulariser le résultat (4.5) au cas où le système binaire implique un trou noir (4.1). Pour cela, on montre, par identification à l'infini de la solution (4.1) avec les champs générés par une particule (4.3) dans son référentiel propre, à l'ordre $\mathcal{O}(1/r)$, que :

(i) le paramètre q_A s'identifie à la charge électrique conservée Q du trou noir ;

(ii) sa sensibilité $m_A(\varphi)$ doit satisfaire, $\forall a \neq 0$, l'équation différentielle suivante [1] :

$$\left(\frac{dm_A}{d\varphi}\right)\left(m_A(\varphi) - \frac{1-a^2}{2a}\frac{dm_A}{d\varphi}\right) = \frac{a}{2}\, q_A^2\, e^{2a\varphi}\,, \tag{4.6}$$

dont la résolution nécessite l'introduction d'une unique constante d'intégration, notée μ_A dans l'article, qu'il faut identifier à la masse irréductible du trou noir, $\mu_A = \sqrt{A_H/16\pi}$, où A_H est l'aire de son horizon. À titre d'exemple, on trouve pour la théorie $a = 1$:

$$m_A(\varphi) = \sqrt{\mu_A^2 + q_A^2\,\frac{e^{2\varphi}}{2}}\,. \tag{4.7}$$

On vérifie aisément que la symétrie discutée en (4.2) et dessous est préservée par notre action "skeleton" (4.3) ainsi que l'équation (4.6) déterminant $m_A(\varphi)$: en effet, ces deux dernières sont invariantes sous les transformations simultanées (4.2) et $q_A \to e^{-a\Delta\varphi} q_A$.

Le résultat (4.7) généralise sur un exemple précis, et pour la première fois, la notion de sensibilité aux trous noirs. Sa simplicité remarquable contraste avec le calcul (numérique) de la sensibilité d'une étoile à neutrons (e.g., en théorie scalaire-tenseur), faisant intervenir l'équation d'état de l'étoile, ainsi que le couplage de la matière à la métrique de Jordan, voir, e.g., [103, 104, 106, 184].

Ainsi, la "skeletonisation" (4.3) décrit un trou noir (4.1) qui, lorsqu'il est soumis à une variation de son environnement scalaire φ_∞ due au mouvement de son compagnon lointain, réajuste adiabatiquement la valeur de ses paramètres r_+ et r_-, à charge Q et aire A_H fixées. L'interprétation d'un tel résultat, à la lumière de la thermodynamique à l'équilibre des trous noirs, fera l'objet du chapitre 7.

Notons enfin que la résolution de l'équation (4.6) permet d'obtenir l'expression analytique exacte de la sensibilité $m_A(\varphi)$ d'un trou noir (4.1), valable à tout ordre post-keplerien [2]. Ainsi, une fois spécifié l'environnement cosmologique φ_0 d'un système binaire, sa dynamique est entièrement déterminée par la donnée de deux paramètres (q_A, μ_A) par trou noir seulement.

1. Dans le cas $a = 0$, on trouve $m_A = (r_+ + r_-)/2 = cste$ et $q_A = Q$; la particule décrit alors un trou noir de Reissner-Nordström, cf. (4.1). De même, $\forall a$, mais lorsque $q_A = Q = 0$, on trouve $m_A = r_+/2 = cste$, décrivant un trou noir de Schwarzschild.

2. La raison en est que pour construire la solution (4.1) en résolvant itérativement les équations du champ du vide, il suffit de se donner son développement asymptotique à l'ordre $\mathcal{O}(1/r)$ pour fixer uniquement les ordres suivants. L'équation (4.6) étant construite avec cette information, la résolution des équations de champ avec membre de droite distributionnel où $q_A = Q$ et où $m_A(\varphi)$ satisfait (4.6) redonnera donc nécessairement, dans son référentiel propre, la solution exacte à un corps (4.1). Notons qu'à l'ordre 1PK considéré dans ce chapitre, il suffit et il est facile de vérifier l'identification, à l'ordre $\mathcal{O}(1/r^2)$, des champs.

Connaissant explicitement la fonction $m_A(\varphi)$, on calcule alors les paramètres $\alpha^0_{A/B}$, $\beta^0_{A/B}$ et $e_{A/B}$, définis en (4.4), apparaissant dans le Lagrangien à deux corps via les combinaisons (4.5) (et on peut, plus généralement, calculer les $n+3$ paramètres par corps à l'ordre nPK apparaissant dans (4.4)).

La figure 4.1 reproduit le tracé, obtenu dans l'article ci-dessous, du couplage $\alpha^0_A = (d\ln m_A/d\varphi)(\varphi_0)$ en fonction de φ_0, pour un trou noir tel que $|\mu_A/q_A| = 10^3$, et pour les théories $a \in [\![0, 10]\!]$. Cette figure met en évidence l'influence capitale de l'environnement cosmologique φ_0 sur la dynamique d'un système binaire. En effet, on voit que selon la valeur de φ_0, un trou noir peut exhiber deux régimes dynamiques antipodaux :

(i) un régime découplé des champs scalaire et vectoriel, où $\alpha^0_A \to 0$, $\beta^0_A = (d\alpha_A/d\varphi)^0 \to 0$, et $e_A \to 0$, pour lequel la dynamique du trou noir est indiscernable de son homologue schwarzschildienne en relativité générale ;

(ii) un régime "quasi-extrémal", i.e. $r_- \to r_+$ (avec $r_- < r_+$), tel que le trou noir est fortement couplé aux champs scalaire et vectoriel, $\alpha^0_A \to a$, $\beta^0_A \to 0$, et $(e_A)^2 \to 1 + a^2$, impliquant de grandes déviations à la dynamique à deux corps de la relativité générale.

Ainsi, les valeurs spécifiques des paramètres (q_A, μ_A) d'un trou noir A, i.e. sa charge et son aire, n'ont d'influence (pour un a donné) que sur la valeur critique φ^{crit}_A de φ_0 sur laquelle est centrée la transition abrupte entre les régimes asymptotiques (i) et (ii). On montre d'ailleurs que φ^{crit}_A dépend uniquement du logarithme de $|\mu_A/q_A|$ (cf. (4.5) dans l'article ci-dessous), de sorte que tout trou noir, même de charge "graviphotonique" Q arbitrairement petite (mais non nulle) peut atteindre le régime universel (ii).

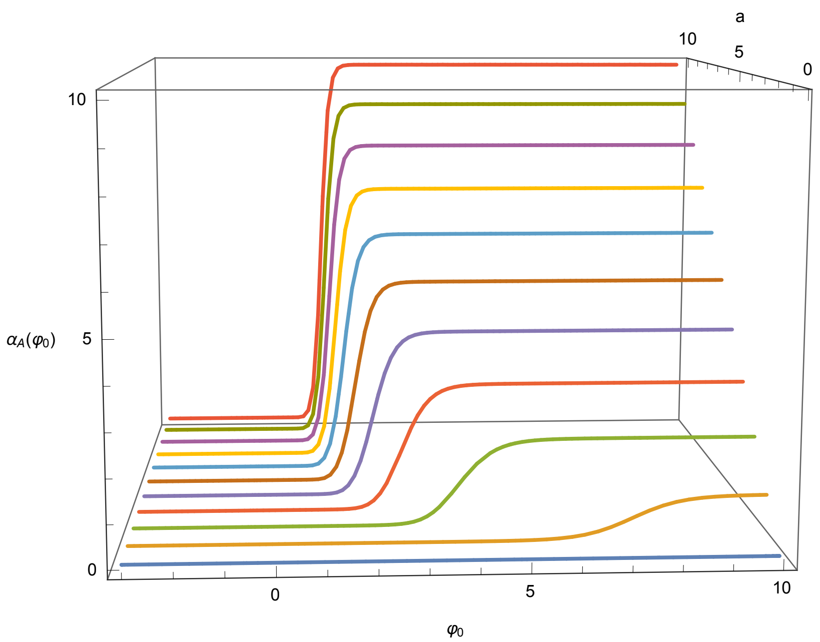

FIGURE 4.1 – Couplage au champ scalaire $\alpha_A(\varphi_0)$ d'un trou noir A, tel que $|\mu_A/q_A| = 10^3$, en fonction de l'environnement cosmologique φ_0, pour les théories $a \in [\![0, 10]\!]$. Lorsque φ_0 augmente, le trou noir "scalarise" d'un régime schwarzschildien ($\alpha^0_A \to 0$) à un régime fortement couplé aux champs scalaire et vectoriel ($\alpha^0_A \to a$).

Le fait que, selon la valeur de φ_0, la dynamique d'un système binaire de trous noirs puisse être indiscernable de celle qu'il aurait en relativité générale, ou bien, au contraire, en différer grandement, illustre tout à fait l'utilité d'observer, grâce à des détecteurs tels que

LISA, des sources de configurations variées, et situées à des redshifts supérieurs à ceux atteints jusqu'à présent : rappelons en effet qu'aux redshifts d'où rayonne un système binaire, la valeur numérique de φ_0 n'est pas nécessairement la même que dans le système solaire, à cause de son évolution cosmologique ; voir, e.g., [102] et la section 0.2.5.

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On the motion of hairy black holes in Einstein-Maxwell-dilaton theories

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Abstract. Starting from the static, spherically symmetric black hole solutions in massless Einstein-Maxwell-dilaton (EMD) theories, we build a "skeleton" action, that is, we phenomenologically replace black holes by an appropriate effective point particle action, which is well suited to the formal treatment of the many-body problem in EMD theories. We find that, depending crucially on the value of their scalar cosmological environment, black holes can undergo steep "scalarization" transitions, inducing large deviations to the general relativistic two-body dynamics, as shown, for example, when computing the first post-Keplerian Lagrangian of EMD theories.

Keywords: gravitational waves / sources, gravitational waves / theory, modified gravity

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Contents

1 Introduction

The observations of gravitational waves emitted by the coalescence of binary black holes by the LIGO-Virgo collaboration [1–4], together with the recent joint detection of the coalsecence of a binary neutron star system with electromagnetic counterparts [5], have opened a new era in gravitational wave astronomy. In particular, these detections provide the opportunity to challenge general relativity (GR), as well as modified gravities (i.e., gravity theories beyond GR), in the strong field regime of the coalescence of two compact objects, in diversified configurations, and at various redshifts.

In GR, post-Newtonian (PN) expansions of Einstein's equations (in the small orbital velocity, weak field limit) are suitable to describe analytically the inspiral phase of binary systems and associated gravitational waveforms. They often rely on the very convenient "skeletonization" of compact bodies, that is, on their phenomenological reduction to effective point particles, that follow the geodesics of the metric they produce (although necessitating regularization schemes due to the artificial introduction of Dirac distributions in Einstein's equations), see, e.g., [6] and [7].

The generalization of skeletonization to modified gravities was introduced by Eardley in the simplest case of (massless) scalar-tensor theories in [8]. There, he proposed to endow each point particle with an effective "mass function", $m_A(\varphi)$, that depends on the (regularized) value of the scalar field evaluated at its location, which in turn identifies to the scalar environment of the skeletonized body. The resultant simplification of the formal treatment of the two-body problem enabled to address it up to 2.5 PN order [9–11], or, adopting the terminology of [9], 2.5 post-Keplerian (PK) order, to highlight the fact that (strong) self-gravity effects are encompassed in the mass function $m_A(\varphi)$. The explicit computation of $m_A(\varphi)$ for a given body A is well-known in the case of neutron stars. It is based on the (numerical) integration of the field equations inside the star, depending on its internal structure (e.g.,

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its equation of state), up to the exterior of the star, where the fields are matched to the near-worldine fields sourced by the effective point particle, see, e.g., [12–14]; see also [15]. On the other hand, in that class of theories, its is known that black holes cannot carry scalar "hair", and hence, are reduced to Schwarzschild's [16].

In the present paper, we propose to go beyond what was done so far in scalar-tensor theories (ST), and consider the class of Einstein-Maxwell-dilaton theories (EMD). They consist in supplementing ST theories with a (massless) vector gauge field that is non-minimally coupled to the scalar field. Note that this vector does not necessarily correspond to the Maxwell field of electrodynamics, but must rather be considered as a new "graviphotonic" degree of freedom of gravity. These EMD theories will be at the center of our attention, since, contrarily to ST theories, they predict the existence of black holes that differ from the general relativistic ones through the presence of vector and induced scalar "hairs" [17–20]. Our aim will therefore be to address the problem of motion of two compact bodies, including these "hairy" black holes, by generalizing the Eardley-type point particle action to EMD theories.

The dynamics of binary black holes in EMD theories has been studied numerically in [21]. There, the authors found that for very small values of their scalar cosmological environment φ_0, the influence of the scalar interaction can be largely neglected in understanding their dynamics, which is therefore hardly distinguishable from its general relativistic counterpart. Now, although these conclusions will coincide with the (analytical) results obtained in the present paper, we shall hint that, when φ_0 is increased (which, as we shall argue, is physically reasonnable), black holes can undergo steep "scalarization" transitions, i.e. become strongly coupled to the scalar and vector fields, inducing large deviations to the general relativistic two-body dynamics.

The paper is organized as follows: in section 2 we introduce EMD theories and build an appropriate point particle action, endowing it with a mass function $m_A(\varphi)$ ("à la" Eardley) and a constant charge q_A. We then compute the resultant two-body Lagrangian at post-Keplerian order (1PK), which describes the conservative part of the dynamics of arbitrary compact bodies in interaction. In section 3, we present a class of EMD black hole solutions with an "electric" charge and dilatonic "hair". By means of an appropriate matching condition, we then show how to skeletonize these black holes, that is, how to compute explicitly the corresponding mass function and charge. Finally, in section 4, we specify the values of the post-Keplerian couplings that enter the two-body Lagrangian for the black holes mentioned above, and highlight that, depending on their cosmological scalar environment φ_0, black holes can transition between two extremal regimes: a "Schwarzschild"-like behaviour, which makes them undistinguishable from their general relativistic counterparts, and a "scalarized" regime, where their couplings to the scalar and vector fields reach the order of unity, inducing large deviations from general relativity.

2 The two-body problem in Einstein-Maxwell-dilaton theories

2.1 The Einstein-Maxwell-dilaton skeleton action

The Einstein-Maxwell-dilaton (EMD) theories describe general relativity supplemented by a massless scalar field and a massless vector gauge field. In vacuum, the EMD action is conveniently written in the Einstein frame as (setting $G \equiv c \equiv 1$)

$$S_{\text{vac}}[g_{\mu\nu}, A_\mu, \varphi] = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi} F^{\mu\nu} F_{\mu\nu} \right), \tag{2.1}$$

where R is the Ricci scalar, $g = det\, g_{\mu\nu}$, and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The fundamental parameter a induces a coupling between the vector and scalar fields.[1] In consequence, vacuum EMD theories encompass the usual U(1) gauge symmetry, $A_\mu \to A_\mu + \partial_\mu \chi$, χ being an arbitrary scalar function, while breaking the scalar "shift symmetry", $\varphi \to \varphi + cst$. As we shall see in the following, this will be crucial in endowing EMD black holes, that is, vacuum solutions of the theory, with scalar "hair".

In this paper, we shall restrict ourselves to $a > 0$ only, since (2.1) is symmetric under the simultaneous redefinitions $a \to -a$ and $\varphi \to -\varphi$.

The vacuum field equations follow from the variation of (2.1),

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right), \tag{2.2a}$$

$$\nabla_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 0\,, \tag{2.2b}$$

$$\Box\,\varphi = -\frac{a}{2}e^{-2a\varphi}F^2\,, \tag{2.2c}$$

where $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, where ∇_μ denotes the covariant derivative associated to $g_{\mu\nu}$ and where $\Box \equiv \nabla_\mu\nabla^\mu$.

In presence of matter, the Einstein-frame action becomes

$$S = S_{\rm vac} + S_{\rm m}[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}, A_\mu]\,, \tag{2.3}$$

where Ψ generically denotes matter fields, that are minimally coupled to the gauge vector A_μ and to the Jordan metric, $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu}$, $\mathcal{A}(\varphi)$ being a scalar function of φ that specifies the theory, together with a.

However, our aim being to address the problem of motion of two compact bodies in interaction, it will prove convenient to "skeletonize" them, that is to replace $S_{\rm m}$ in (2.3) by a phenomenological point particle action, $S_{\rm m}^{\rm pp}$. Now, extending the treatment of Eardley in scalar-tensor theories [8] to incorporate the presence of the vector field, we shall consider the most generic covariant ansatz

$$S_{\rm m}^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x_A^\mu\}] = -\sum_A \int m_A(\varphi)\, ds_A + \sum_A q_A \int A_\mu\, dx_A^\mu\,, \tag{2.4}$$

where $ds_A = \sqrt{-g_{\mu\nu}dx_A^\mu dx_A^\nu}$, $x_A^\mu[s_A]$ being the worldline of particle A, and where $m_A(\varphi)$ is a scalar function to be related to the skeletonized body A later, which depends on the value of the scalar field evaluated at its location $x_A^\mu(s_A)$ (substracting divergent self contributions). In contrast, q_A will be considered as a constant parameter that does not depend on φ. Indeed, as one easily sees from (2.4), the linear coupling of the worldlines to the vector field A_μ preserves the U(1) gauge symmetry provided that the following current, j^μ, is conserved:

$$\partial_\mu j^\mu = 0\,, \qquad \text{where} \qquad j^\mu(y) = \sum_A q_A\, \delta^{(3)}(\vec{y} - \vec{x}_A(t))\frac{dx_A^\mu}{dt}\,. \tag{2.5}$$

This in turn implies, as usual, that the charge q_A of each body is conserved (provided that it remains separated from its companion), and cannot depend upon the variation of $\varphi(\vec{x}_A(t))$.

[1] Note that $a = 0$ reduces to Einstein-Maxwell theory minimally coupled to a scalar field, while $a \neq 0$ is motivated by low-energy limits of string theory, see, e.g., [18]; $a = \sqrt{3}$ is equivalent to the dimensional reduction of the Kaluza-Klein theory [19, 20].

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Note that our ansatz (2.4) does not depend on the four-gradients $\partial_\mu = (\partial_t, \partial_i)$ of the fields and, therefore, cannot take into account any finite size (e.g., tidal) nor out of equilibrium effect, that we will neglect in the present paper; see, e.g., [22]. The functions $m_A(\varphi)$ must therefore be understood as depending on the homogeneous, adiabatically varying, scalar field environment of the body A in equilibrium, created by the faraway and slowly orbiting companion.

Finally, for weakly self-gravitating bodies, the mass functions are reduced to $m_A(\varphi) = \tilde{m}_A\mathcal{A}(\varphi)$, where $\tilde{m}_A = cst$ is the Jordan-frame mass, such that bodies with $q_A = 0$ follow the geodesics of the Jordan metric, see (2.3) and below. In contrast, general relativity is recovered when $m_A(\varphi) = cst$, together with $q_A = 0$.

2.2 The post-Keplerian Lagrangian

We now have the necessary material to address the dynamics of compact binary systems in EMD theories: our starting point is the "skeleton" action

$$\begin{aligned} S[g_{\mu\nu}, A_\mu, \varphi, \{x^\mu_A\}] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi}F^2\Big) \\ - \sum_A \int m_A(\varphi)\, ds_A + \sum_A q_A \int A_\mu\, dx^\mu_A\,, \end{aligned} \tag{2.6}$$

with $ds_A = \sqrt{-g_{\mu\nu}dx^\mu_A dx^\nu_A}$, that depends on the fondamental parameter a, together with two mass functions $m_A(\varphi)$ and two constant charges q_A. The field equations are now sourced by the effective particles:

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right) + 8\pi\sum_A\left(T^A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^A\right), \tag{2.7a}$$

$$\nabla_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi\sum_A q_A\int ds_A\,\frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}\frac{dx^\mu_A}{ds_A}, \tag{2.7b}$$

$$\Box\varphi = -\frac{a}{2}e^{-2a\varphi}F^2 + 4\pi\sum_A\int ds_A\,\frac{dm_A}{d\varphi}\frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}, \tag{2.7c}$$

where $\delta^{(4)}\left(x-y\right)$ is the 4-dimensional Dirac distribution, and where $T^A_{\mu\nu}$ is the stress-energy tensor associated to the skeletonized body A,

$$T^{\mu\nu}_A = \int ds_A\, m_A(\varphi)\frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}\frac{dx^\mu_A}{ds_A}\frac{dx^\nu_A}{ds_A}\,. \tag{2.8}$$

In this paper, we shall focus on the conservative part of the dynamics, neglecting all radiation reaction forces. Moreover, for bound orbits, it is convenient to implement relativistic corrections to the Keplerian dynamics in the weak field, slow orbital velocity limit.

In appendix A, we derive the first post-Keplerian (1PK), $\mathcal{O}(m/R) \sim \mathcal{O}(V^2)$ corrections to the two-body Lagrangian (where R is the distance separating the bodies, and V denotes their relative orbital velocity), extending to EMD theories the standard methods presented in, e.g., [9] in massless scalar-tensor theories (ST). There, the field equations (2.7) are solved perturbatively around the Minkowski metric $\eta_{\mu\nu}$ and the constant value φ_0 of the background scalar field which, being associated to no gauge symmetry, see (2.1) and below, cannot be set equal to zero, and is imposed by the cosmological environment of the binary system.

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At 1PK order, it is also useful to define the following body-dependent quantities

$$\alpha_A(\varphi) \equiv \frac{d\ln m_A}{d\varphi}\,, \qquad \beta_A(\varphi) \equiv \frac{d\alpha_A}{d\varphi}\,, \tag{2.9}$$

$$\text{such that} \qquad \ln m_A(\varphi) = \ln m_A^0 + \alpha_A^0(\varphi-\varphi_0) + \frac{1}{2}\beta_A^0(\varphi-\varphi_0)^2 + \cdots\,,$$

where and from now on, a zero superscript indicates a quantity evaluated at infinity, $\varphi = \varphi_0$, as in $m_A^0 = m_A(\varphi_0)$. Note that the parameter α_A^0 measures the effective coupling between the scalar field and the skeletonized body A, and will play an essential role in the following. Note also that for a weakly self-gravitating body ($m_A(\varphi) = \tilde{m}_A\mathcal{A}(\varphi)$, see discussion below (2.5)), the quantities (2.9) become universal: $\alpha = d\ln\mathcal{A}/d\varphi$, and $\beta = d^2\ln\mathcal{A}/d^2\varphi$.

With the definitions given above, the post-Keplerian Lagrangian reads, in harmonic coordinates $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$, and in the Lorenz gauge $\nabla_\mu A^\mu = 0$ (see, again, appendix A for details):

$$\begin{aligned} L_{AB} = &-m_A^0 - m_B^0 + \frac{1}{2}m_A^0V_A^2 + \frac{1}{2}m_B^0V_B^2 + \frac{G_{AB}m_A^0m_B^0}{R} + \frac{1}{8}m_A^0V_A^4 + \frac{1}{8}m_B^0V_B^4 \\ &+ \frac{G_{AB}m_A^0m_B^0}{R}\left[\frac{3}{2}(V_A^2+V_B^2) - \frac{7}{2}(V_A.V_B) - \frac{1}{2}(N.V_A)(N.V_B) + \bar{\gamma}_{AB}(\vec{V}_A-\vec{V}_B)^2\right] \\ &- \frac{G_{AB}^2m_A^0m_B^0}{2R^2}\left[m_A^0(1+2\bar{\beta}_B) + m_B^0(1+2\bar{\beta}_A)\right] + \mathcal{O}(V^6)\,, \end{aligned} \tag{2.10}$$

where $R \equiv |\vec{x}_A - \vec{x}_B|$, $\vec{N} \equiv (\vec{x}_A - \vec{x}_B)/R$, and $\vec{V}_A \equiv d\vec{x}_A/dt$. We have also introduced the combinations

$$G_{AB} \equiv 1 + \alpha_A^0\alpha_B^0 - e_Ae_B\,, \tag{2.11a}$$

$$\bar{\gamma}_{AB} \equiv \frac{-4\,\alpha_A^0\alpha_B^0 + 3\,e_Ae_B}{2(1+\alpha_A^0\alpha_B^0 - e_Ae_B)}\,, \tag{2.11b}$$

$$\bar{\beta}_A \equiv \frac{1}{2}\frac{\beta_A^0{\alpha_B^0}^2 - 2\,e_Ae_B(a\,\alpha_B^0 - \alpha_A^0\alpha_B^0) + e_B^2(1 + a\,\alpha_A^0 - e_A^2)}{(1+\alpha_A^0\alpha_B^0 - e_Ae_B)^2}\,, \tag{2.11c}$$

and $A \leftrightarrow B$ counterpart, together with the convenient notations

$$e_A \equiv \frac{q_A}{m_A^0}e^{a\varphi_0} \quad \text{and} \quad e_B \equiv \frac{q_B}{m_B^0}e^{a\varphi_0}\,. \tag{2.12}$$

The expression (2.10)–(2.12) of the post-Keplerian Lagrangian in EMD theories, which results from our generic "skeleton" ansatz (2.4), is the first new technical result of this paper.

Remarkably, (2.10) has exactly the same structure than the 1PK Lagrangian in ST theories, see, e.g., [9]. Indeed, the effects of the scalar *and* vector fields have been entirely gathered in the four body-dependent combinations (2.11), which generalize those introduced by Damour and Esposito-Farèse in ST theories [9], that we retreive when the charges vanish, $q_{A/B} = 0$. The effective (dimensionless, since we set $G = 1$) gravitational coupling G_{AB} reflects the addition of the metric, scalar and (repulsive) vector interactions at the linear level, and is reduced to unity in the general relativity limit, $q_{A/B} = 0$ and $\alpha_{A/B}^0 = \beta_{A/B}^0 = 0$. The combinations $\bar{\gamma}_{AB}$ and $\bar{\beta}_{A/B}$ depend quadratically on the couplings to the scalar field $\alpha_{A/B}^0$ and to the vector field $e_{A/B}$, and vanish in the GR limit.

Finally, we note that our Lagrangian also encompasses Einstein-Maxwell's theory, which is retrieved when the bodies decouple from the scalar field, $m_{A/B} = cst$, i.e. $\alpha^0_{A/B} = \beta^0_{A/B} = 0$.

In this section, we proposed a generic "skeleton" ansatz (2.4) that is well-suited to address the problem of motion of two arbitrary compact bodies in EMD theories (e.g., neutron stars, or black holes). At 1PK order, we showed that the resultant two-body Lagrangian (2.10) only differs from the ST one through the redefinition of G_{AB}, $\bar{\gamma}_{AB}$ and $\bar{\beta}_{A/B}$ given in (2.11), which makes their associated two-body dynamics *a priori* undistinguishable at this order.

However, EMD theories deserve special attention since, contrarily to ST theories [16], they predict the existence of "hairy" black holes (i.e. black holes that differ from the general relativistic ones), that are encompassed in the generic approach developped above. Consequently, our aim in the next sections will be to specify our results for systems involving such "hairy" black holes, that is, to compute explcitely the values for their post-Keplerian parameters α^0_A, β^0_A and e_A entering our Lagrangian (2.10).

3 Reducing a hairy black hole to a point particle

In the following, we will restrict our attention to a specific subclass of "hairy" black holes with an electric charge, presented in subsection 3.1. In order to address their motion in presence of a companion, we shall, in a second step, "skeletonize" them, that is, compute explicitly the effective "mass function" $m_A(\varphi)$ and parameter q_A that enter the effective point particle action describing them,

$$S^{\rm pp}_{\rm m,A} = -\int m_A(\varphi)\, ds_A + q_A \int A_\mu\, dx^\mu_A\,, \tag{3.1}$$

see (2.4) and the discussion below.

To this extent, we will "zoom in" to the near-wordline region of particle A, which will take on the role of the black hole, and match the effective fields it generates to the real black hole solution. In particular, in keeping with neglecting all finite-size effects (see, again, section 2.1), the only *gauge-invariant* influence of the faraway companion B will be to impose a value for the adiabatically varying scalar environment φ_∞ of our black hole.

3.1 Electrically charged dilatonic black holes

Isolated black holes, that is, solutions of the vacuum field equations (2.2), have been thoroughly studied in the literature. In particular, static, spherically symmetric black holes with an electric and (or) magnetic charge were introduced in [17] and [18], while their axisymmetric counterparts were found in [19] and [20] in Kaluza-Klein theory (i.e. when $a=\sqrt{3}$ only).

In this paper, we shall restrict our attention to the class of electrically charged, non-spinning black hole solutions that read, in Just coordinates,

$$ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} d\Omega^2\,, \tag{3.2a}$$

$$A_t = -\frac{Q\, e^{2a\varphi_\infty}}{r}\,, \quad A_i = 0\,, \tag{3.2b}$$

$$\varphi = \varphi_\infty + \frac{a}{1+a^2}\ln\left(1-\frac{r_-}{r}\right), \tag{3.2c}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and where Q is not an independent integration constant but rather satisfies

$$Q^2 = \frac{r_+ r_-}{1+a^2}\, e^{-2a\varphi_\infty}\,. \tag{3.3}$$

Note that Q is the conserved U(1) charge of the black hole, as one easily sees from a direct integration of (2.2b). Note also that we have "gauged away" the asymptotic value of A_μ to zero for convenience.

The solution (3.2)–(3.3) hence depends on three integration constants: the radius r_+ of the horizon, the location r_- of the curvature singularity, and the asymptotic value φ_∞ of the scalar field which, in the presence of the faraway companion, identifies to the local, adiabatically varying value of the scalar field it produces.

We note that the electric charge is crucial to induce our black hole with a scalar "secondary hair". Indeed, when $r_- = 0$, (3.2) is reduced to Schwarzschild's black hole. Another important limit is the scalar-vector decoupling, $a = 0$, see (2.1). In that case, $\varphi = \varphi_\infty$, $Q^2 = r_+ r_-$, and (3.2) is reduced to Reissner-Nordström's black hole.

Finally, as part of the "skeletonization" to come below, it will prove sufficient to expand the solution (3.2) at infinity and in isotropic coordinates ($r = \tilde r + [r_+ + r_-]/2 + \cdots$) as

$$\tilde g_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{r_+ + \frac{1-a^2}{1+a^2}\, r_-}{\tilde r}\right) + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\,, \tag{3.4a}$$

$$A_t = -\frac{Q\, e^{2a\varphi_\infty}}{\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\,, \tag{3.4b}$$

$$\varphi = \varphi_\infty - \frac{a}{1+a^2}\frac{r_-}{\tilde r} + \mathcal{O}\left(\frac{1}{\tilde r^2}\right)\,. \tag{3.4c}$$

Indeed, we shall see in the next subsection that the asymptotic expansion (3.4), which depends on the three integration constants r_+, r_- and φ_∞, encodes all the necessary information to skeletonize the black hole, i.e. to fix uniquely the function $m_A(\varphi)$ and constant q_A of the effective particle that will take on the role of the black hole.

3.2 The matching conditions

In the near-wordline region of the effective particle A, and at leading order in the large separation limit, the effective field equations (2.7) are reduced to

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right) + 8\pi\left(T^A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^A\right)\,, \tag{3.5a}$$

$$\nabla_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi q_A \int ds_A\, \frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}\,\frac{dx^\mu_A}{ds_A}\,, \tag{3.5b}$$

$$\Box\varphi = -\frac{a}{2}e^{-2a\varphi}F^2 + 4\pi\int ds_A\, \frac{dm_A}{d\varphi}\frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}\,, \tag{3.5c}$$

where we recall that $ds_A = \sqrt{-g_{\mu\nu}dx^\mu_A dx^\nu_A}$ and that

$$T^{\mu\nu}_A = \int ds_A\, m_A(\varphi)\frac{\delta^{(4)}\left(y - x_A(s_A)\right)}{\sqrt{-g}}\frac{dx^\mu_A}{ds_A}\frac{dx^\nu_A}{ds_A}\,. \tag{3.6}$$

Let us solve these field equations perturbatively around Minkowski's metric $\eta_{\mu\nu}$, a vector field that can also be "gauged away" to zero, and a scalar background, φ_∞, which is the

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remaining influence of body B, as in (3.4). In harmonic coordinates (that identify to isotropic coordinates at this order), $\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}) = 0$, and in the rest-frame of the particle A, i.e. setting $\tilde{\vec{x}}_A = \vec{0}$, this yields, at linear order,

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{2m_A(\varphi_\infty)}{\tilde{r}}\right) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right), \tag{3.7a}$$

$$A_t = -\frac{q_A\, e^{2a\varphi_\infty}}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right), \tag{3.7b}$$

$$\varphi = \varphi_\infty - \frac{1}{\tilde{r}}\frac{dm_A}{d\varphi}(\varphi_\infty) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right), \tag{3.7c}$$

which depends on φ_∞ and on the three effective parameters q_A, $m_A(\varphi_\infty)$ and $m'_A(\varphi_\infty)$.

Now, by comparing (3.4) to (3.7), one obtains the following matching conditions:

$$m_A(\varphi_\infty) = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}\, r_-\right), \tag{3.8a}$$

$$q_A = Q\,, \tag{3.8b}$$

$$\frac{dm_A}{d\varphi}(\varphi_\infty) = \frac{a}{1+a^2}\, r_-\,, \tag{3.8c}$$

where we recall that the electric charge Q satisfies

$$Q^2 = \frac{r_+ r_-}{1+a^2}\, e^{-2a\varphi_\infty}\,, \tag{3.9}$$

see (3.3). Therefore, one "skeletonizes" the real black hole (3.2) as an effective point particle, provided that q_A, m_A and its derivative evaluated at φ_∞ satisfy the three matching conditions (3.8).

Note that as expected, q_A identifies to the electric charge Q of the black hole, see (3.8b), while $m_A(\varphi_\infty)$ coincides with its standard, say, ADM mass [23], $M_{\rm ADM} = \left(r_+ + \frac{1-a^2}{1+a^2}\, r_-\right)/2$, see (3.8a), which will therefore not be conserved when the scalar environment φ_∞ of the black hole A, as created by its companion B, varies along its orbit (see (3.8c)).

Moreover, the system (3.8) is integrable. Indeed, inverting (3.8) to substitute m_A, $dm_A/d\varphi$ and q_A to r_+, r_- and Q in (3.9), yields the first order differential equation to be satisfied by the function $m_A(\varphi)$:

$$\left(\frac{dm_A}{d\varphi}\right)\left(m_A(\varphi) - \frac{1-a^2}{2a}\frac{dm_A}{d\varphi}\right) = \frac{a}{2}\, q_A^2\, e^{2a\varphi}\,. \tag{3.10}$$

For a given theory a, the solution of (3.10) depends on the charge q_A of the black hole, together with a unique integration constant, that we shall denote μ_A. The obtention of the matching conditions (3.8), together with the differential equation (3.10) is the second new technical result of this paper, which shows that the dynamics of the black hole (3.2) is described by two constant parameters only, q_A and μ_A.

4 The sensitivity of a hairy black hole

In this section, we solve the differential equation (3.10), that is, we compute explicitly the function $m_A(\varphi)$ that describes the dynamics of our "hairy" black hole (3.2) through the point particle action

$$S^{\rm pp}_{\rm m,A} = -\int m_A(\varphi)\, ds_A + q_A \int A_\mu\, dx^\mu_A\,. \tag{4.1}$$

This will then allow us to come back to the two-body Lagrangian (2.10) obtained in section 2.2, and compute the values of the fundamental post-Keplerian parameters α_A^0, β_A^0 and e_A that enter it. In particular, we will find that the dynamics of our black hole might depart significantly from that of black holes in general relativity.

4.1 The theory $a = 1$

In the particular case $a = 1$, (3.10) is easily integrated as

$$m_A(\varphi) = \sqrt{\mu_A^2 + q_A^2 \frac{e^{2\varphi}}{2}}\,, \tag{4.2}$$

where μ_A is a positive integration constant. We hence have an explicit expression for the function $m_A(\varphi)$, which characterizes our black hole by means of two constant parameters:

(i) Its electric charge $q_A = Q$ (see (3.8b)), given in (3.9), which is essential for our black hole to carry a scalar "hair". Indeed, when $q_A = 0$, (4.2) is reduced to $m_A(\varphi) = \mu_A$ and the black hole decouples both from the vector and the scalar fields;

(ii) The constant μ_A, whose physical interpretation can be obtained thus: expressing it in terms of r_+, r_-, φ_∞, using the matching conditions (3.8), yields

$$\mu_A^2 = \frac{r_+(r_+ - r_-)}{4} = \frac{A_H}{16\pi}\,. \tag{4.3}$$

Therefore, μ_A^2 is proportional to the area A_H of the horizon of the black hole (and hence, μ_A may be referred to as its irreducible mass [24]).

In other words, the skeletonization describes a black hole that, when submitted to an adiabatic variation of its scalar field environment φ_∞ (created by the slowly orbiting, faraway companion), readjusts its equilibrium configuration (that is, r_+ and r_-) in order for both its electric charge and horizon area to remain constant (while we recall that its ADM mass is not conserved, see (3.8a) and comments below).

Moreover, given a specific black hole (i.e. specific values for q_A and μ_A), one can come back to the two-body problem, and compute the fundamental parameters that enter the 1PK Lagrangian (2.10), see (2.9) and (2.12),

$$\alpha_A^0 = \frac{d\ln m_A}{d\varphi}(\varphi_0)\,, \qquad \beta_A^0 = \frac{d\alpha_A}{d\varphi}(\varphi_0)\,, \qquad e_A = \frac{q_A}{m_A^0} e^{a\varphi_0}\,, \tag{4.4}$$

in terms of q_A, μ_A, and φ_0 which is the value of the scalar field far from the binary system, imposed by cosmology. Injecting the explicit mass function (4.2) in (4.4), we find

$$\alpha_A(\varphi_0) = \frac{1}{1 + e^{2\left(\ln\left|\frac{\mu_A\sqrt{2}}{q_A}\right| - \varphi_0\right)}}\,, \tag{4.5}$$

$$\text{together with} \qquad \beta_A^0 = 2\alpha_A^0(1 - \alpha_A^0) \qquad \text{and} \qquad (e_A)^2 = 2\alpha_A^0\,. \tag{4.6}$$

The simplicity of (4.5) is striking: the coupling $\alpha_A(\varphi_0)$ between the black hole and the scalar field is given by a "Fermi-Dirac" distribution, as shown in figure 1, which highlights the crucial influence of the scalar cosmological background φ_0 on the dynamics of a black

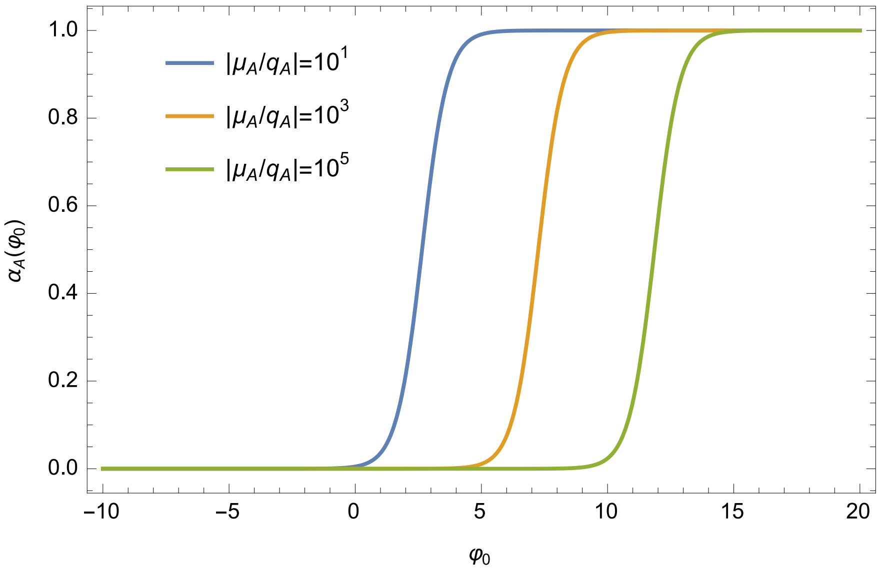

Figure 1. The "Fermi-Dirac" coupling $\alpha_A(\varphi_0)$ between the black hole A and the scalar field, as a function of its cosmological environment φ_0, when $a = 1$. The "scalarization" is centered on $\varphi_A^{\rm crit} = \ln\left|\mu_A\sqrt{2}/q_A\right|$. Three distinct black holes are represented, $|\mu_A/q_A| = \{10, 10^3, 10^5\}$, q_A and $16\pi\mu_A^2$ being respectively their conserved charges and areas.

hole. Indeed, when φ_0 is increased through $\varphi_A^{\rm crit} = \ln\left|\mu_A\sqrt{2}/q_A\right|$, the black hole A transitions steeply between two extremal regimes:

(a) A decoupled regime, $\alpha_A^0 \to 0$, where it is moreover undistinguishable from the general relativistic Schwarzschild black hole, since α_A^0 *as well as* β_A^0 (and higher order derivatives of α_A) and e_A vanish, see (4.6);

(b) A regime where it is strongly coupled both to the scalar and to the vector fields, $\alpha_A^0 \to 1$, together with $\beta_A^0 \to 0$ and $(e_A)^2 \to 2$, which by definition, induces large deviations to the general relativistic two-body Lagrangian through the combinations G_{AB}, $\bar{\gamma}_{AB}$ and $\bar{\beta}_{A/B}$ that enter it, see (2.11).

We note that the specific black hole considered (i.e., q_A and μ_A) only influences the location $\varphi_A^{\rm crit} = \ln\left|\mu_A\sqrt{2}/q_A\right|$ of the transition, while the values reached by α_A^0, β_A^0 and $|e_A|$ in the extremal regimes (a) and (b) are *universal*.

From now on, this transition phenomenon will be referred to as "scalarization", by analogy with the terminology introduced in the context of scalar-tensor theories to describe the spontaneous [12] and dynamical [14] scalarization of neutron stars (altough it must be noted that the phenomenon we highlight here is not a "phase transition", and that the scalar coupling α_A^0 transitions together with the vector coupling, e_A).

4.2 Generic EMD theories

The remarkable features presented above are now easily transposed to generic EMD theories, that is, to an arbitrary value for the fundamental parameter a. First, one solves numerically the differential equation (3.10), holding fixed the values for q_A and for the integration constant μ_A, in keeping with considering one specific black hole.[2] Then, one deduces the resultant coupling to the scalar field, $\alpha_A^0 = d\ln m_A/d\varphi(\varphi_0)$, together with β_A^0 and e_A through the relations[3]

$$\beta_A^0 = \alpha_A^0\,(a-\alpha_A^0)\left[\frac{(1-a^2)\alpha_A^0-2a}{(1-a^2)\alpha_A^0-a}\right], \qquad (e_A)^2 = \alpha_A^0\left[\frac{2a-(1-a^2)\alpha_A^0}{a^2}\right]. \tag{4.7}$$

The resulting parameter α_A^0 is shown in figure 2, for $a \in [\![0,10]\!]$, setting $|\mu_A/q_A| = 10^3$. Again, for all a, the black hole transitions between two extremal regimes:

(a) A decoupling, "Schwarzschild-like" regime $\alpha_A^0 \to 0$, together with $\beta_A^0 \to 0$ and $e_A \to 0$, see (4.7);

(b) A *universally* scalarized regime, $\alpha_A^0 \to a$, with $\beta_A^0 \to 0$ and $(e_A)^2 \to 1 + a^2$.

Note that the curves $\alpha_A(\varphi_0)$ shown in figure 2 are slightly deformed "Fermi-Dirac" distributions (with an exact identification when $a = 1$ only, see discussion above), the steepness of the transition increasing with a. Note also that, for a fixed theory a, the value of q_A and μ_A only influences the location $\varphi_A^{\rm crit}$ of the transition (which is shifted to greater φ_0 values when $|\mu_A/q_A|$ is increased) and the sign of e_A.

A numerical study of the dynamics of binary black holes in EMD theories has been performed recently in [21]. There, the authors found that when $|\varphi_0| = 10^{-10}$ and $a = \{1, 10^3, 3\times 10^3\}$, the influence of the scalar field on binary black holes can be neglected in understanding their dynamics, which in turn becomes hardly distinguishable from its general relativistic counterpart. These features are consistent with the approach developped throughout the present paper: for example, when $|\mu_A/q_A| = 10^3$, $|\varphi_0| = 10^{-10}$ and $a = 3\times 10^3$, we find vanishingly small values for α_A^0, β_A^0 and e_A, see, again, figure 2.

However, our results above hint that this Schwarzschild-like behaviour might be considerably revised when the value of φ_0 is increased, as shown by the "scalarization" phenomenon discussed above. Now, gravitational wave detectors such as LIGO-Virgo (or the forthcoming LISA detector) are designed to detect highly redshifted sources, where, indeed, the cosmological environment φ_0 might have had a non-negligible value (and that differs from today). In such situations, one could expect large deviations to the general relativistic black hole dynamics.

The obtention of the explicit mass function $m_A(\varphi)$ and charge q_A that describe EMD black holes, together with the identification of the transition phenomenon regarding their coupling to the scalar and vector fields, or "scalarization", is the third, and main result of this paper.

[2] Note that although we solve (3.10) numerically, this differential equation admits an exact analytical solution in the form of a parametric equation on $m_A(\varphi)$, which is not enlightening to show here. It is however easy to find that the asymptotic behaviour of the solution is $m_A(\varphi) \underset{-\infty}{\sim} \mu_A$ and $m_A(\varphi) \underset{+\infty}{\sim} [q_A^2 e^{2a\varphi}/(1+a^2)]^{1/2}$, which is sufficient for the purpose of this paper.

[3] The obtention of (4.7) is straightforward if one injects, by definition of α_A^0 and β_A^0 (4.4), the expansion $m_A(\varphi) = m_A^0[1 + \alpha_A^0(\varphi-\varphi_0) + \frac{1}{2}({\alpha_A^0}^2 + \beta_A^0)(\varphi-\varphi_0)^2 + \cdots]$ into the differential equation (3.10), and solves it order by order to get

$$\alpha_A^0 = \frac{a}{1-a^2}\left(1-\sqrt{1-e_A^2(1-a^2)}\right), \qquad \beta_A^0 = \frac{a^2e_A^2}{1-a^2}\left(1-a^2/\sqrt{1-e_A^2(1-a^2)}\right),$$

which is easily inverted to give (4.7).

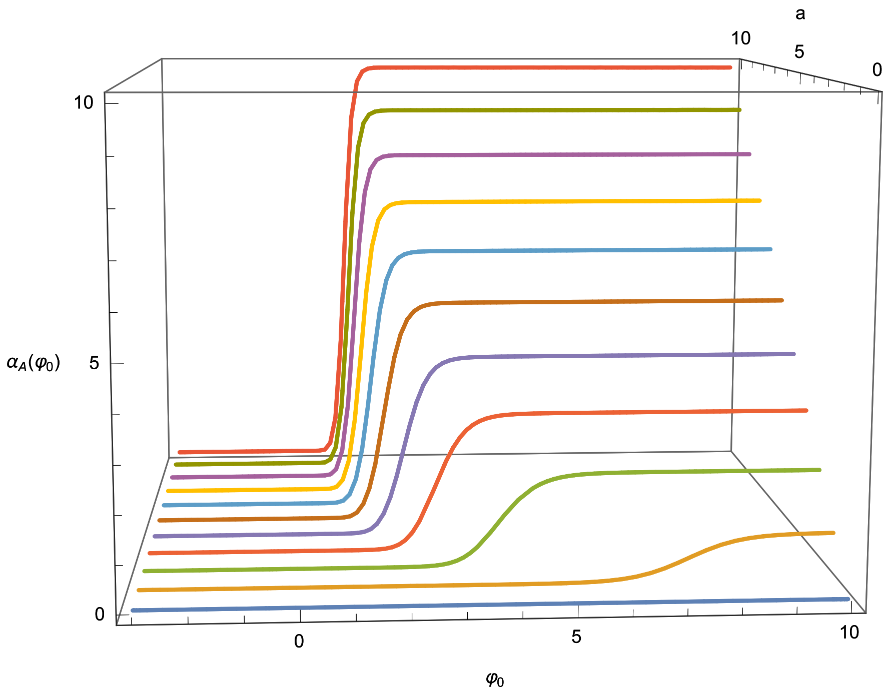

Figure 2. The coupling $\alpha_A(\varphi_0)$ between the black hole and the scalar field, as a function of its cosmological environment φ_0, for the theories $a \in [\![0, 10]\!]$. The specific class of black holes such that $|\mu_A/q_A| = 10^3$ is represented. When $a = 0$, $\alpha_A = 0$ and the black hole is Reissner-Nordstöm's. For non-zero values of a, $\alpha_A(\varphi_0)$ becomes a function of φ_0 that transitions between 0 and a, the steepness of the "scalarization" increasing with a.

4.3 Scalarized binary black holes

As discussed above, in the presence of one (or two) black holes, the dynamics of binary systems might depart significantly from its general relativistic counterpart. For the sake of illustration, let us consider a binary system composed of two "scalarized" black holes , i.e. in the *idealistic* limit when both are described by the regime (b) presented above, *and* with electric charges q_A and q_B of the same sign:

$$\alpha^0_{A/B} \to a\,, \quad \beta^0_{A/B} \to 0\,, \quad \text{and} \quad e_{A/B} \to \sqrt{1+a^2}\,, \tag{4.8}$$

the value of the parameters $(q_A,\, \mu_A)$ and $(q_B,\, \mu_B)$ influencing only the location $\varphi^{\rm crit}_{A/B}$ of their transition, see the discussions above, and where an irrelevant sign was chosen in the definition of $e_{A/B}$.

Injecting (4.8) into the combinations G_{AB}, $\bar{\gamma}_{AB}$ and $\bar{\beta}_{A/B}$, given in (2.11), that enter the post-Keplerian Lagrangian (2.10) presented in section 2, yields $G_{AB} \to 0$, $G_{AB}\,\bar{\gamma}_{AB} \to (3-a^2)/2$, and $G^2_{AB}\,\bar{\beta}_{A/B} \to 0$.

Therefore, all in all, the final two-body Lagrangian is reduced, at 1PK order, to

$$L_{AB} \to -m^0_A\sqrt{1-V^2_A} - m^0_B\sqrt{1-V^2_B} + \left(\frac{3-a^2}{2}\right)\frac{m^0_A m^0_B}{R}(\vec{V}_A - \vec{V}_B)^2 + \mathcal{O}(V^6)\,. \tag{4.9}$$

The simplification (4.9) is spectacular: among all the interaction terms present in the generic Lagrangian (2.10), only the $\bar{\gamma}_{AB}$-driven one remains. Moreover, this last term vanishes as well when the bodies are at relative rest ($\vec{V}_A - \vec{V}_B = \vec{0}$). In other words, EMD binary black holes can transition to a universally "scalarized" regime where their (attractive) metric, scalar and (repulsive) vector interactions compensate to allow for configurations at rest, at 1PK level at least.

This result is consistent with the Majumdar-Papapetrou spacetime,[4] which is a solution of the Einstein-Maxwell field equations that describes two *extremal* Reissner-Nordström black holes at rest, see [25–28], and their extensions to EMD theories [29–32]. For a "scalarizing" black hole, e_A behaves as

$$(e_A)^2 = \left(\frac{q_A}{m_A^0}\right)^2 e^{2a\varphi_0} \to 1 + a^2\,, \tag{4.10}$$

where $q_A = Q$ is the electric charge of the black hole, and $m_A^0 = M_{\rm ADM}$ its ADM mass, see (3.8) and the discussion below, which generalizes the notion of an extremal black hole to generic EMD theories, $a \neq 0$: indeed, (4.10) is equivalent to $r_- \to r_+$ (as proven using again the matching conditions (3.8)).

In other words, our "scalarizing" black holes are in fact transitioning from a Schwarzschild regime towards a quasi-extremal regime (but never reach an exact extremal, $r_- = r_+$, "naked singularity" configuration, since φ_0 is always finite).

Another striking consequence of (4.9) is that in the context of Kaluza-Klein theory, i.e., $a = \sqrt{3}$, the last interaction term in (4.9) vanishes as well. Therefore, each component of the scalarized black hole binary behaves as a free particle at 1PK order. The generalization of this phenomenon to higher PK order remains to be investigated.

4.4 General relativity and Einstein-Maxwell theory

Let us conclude this section by mentioning that the approach developped above is consistent with the well-known general relativity and Einstein-Maxwell theory limits, as it should.

(i) Firstly, for any EMD theory, i.e. any value for a, but when the charge of the black hole vanishes, $q_A = 0$, the three matching conditions (3.8) obtained in section 3.2 are reduced to

$$m_A = \frac{r_+}{2} = cst.\,, \qquad q_A = Q = 0\,. \tag{4.11}$$

Therefore, the mass function m_A identifies to the constant "Schwarzschild" mass, our skeletonized black hole decouples both from the vector and scalar fields, and describes a Schwarzschild black hole, as expected from (3.2) and the discussion below. This corresponds to the general relativity limit.

(ii) Secondly, for any charge q_A, but in the scalar-vector decoupling limit $a = 0$, see (2.1), the matching conditions (3.8) become

$$m_A = \frac{r_+ + r_-}{2} = cst.\,, \qquad q_A = Q\,. \tag{4.12}$$

This time, our effective particle describes a Reissner-Nordström black hole, as predicted in Einstein-Maxwell theory, see again discussion below (3.2).

[4] I am grateful to Thibault Damour for mentioning and sharing his expertise on the Majumdar-Papapetrou binary black hole solutions.

The fact that in both cases (i) and (ii), the "mass function" $m_A(\varphi)$ must be reduced to a constant can be understood thus: the vacuum action (2.1) which describes the black hole (with, respectively, $F_{\mu\nu}=0$ or $a=0$) then encompasses a further "shift symmetry", $\varphi\to\varphi+cst$ (see section 2.1), which is preserved by our effective point particle action (2.4) provided that $m_A(\varphi)$ does not depend on φ. In consequence, the scalar "hair" is switched off.

5 Concluding remarks

Reducing compact bodies to effective point particles has proven to be a very efficient tool to address analytically the many-body problem in general relativity and scalar-tensor theories. In this paper, we generalized this "skeletonization" to Einstein-Maxwell-dilaton (EMD) theories, and showed that the corresponding most generic skeleton ansatz was the combination of two well-known point particle actions: that introduced by Eardley in scalar-tensor theories, which consists in endowing point particles with a specific "mass function" $m_A(\varphi)$, and that of a charged particle in electrodynamics, endowed with a constant charge q_A.

More importantly, we computed, for the first time, the mass functions $m_A(\varphi)$ to be assigned to black holes, and shed light on two major properties regarding their dynamics: firstly, that it is encoded into two parameters (per black hole) only, q_A and μ_A, which are related to their charge and area;[5] secondly, that, in keeping their charge and area fixed, black holes can undergo a new type of "scalarization" phenomenon, leading to a transition between a regime where their dynamics is undistinguishable from that of Schwarzschild's black holes, and a "quasi-extremal" regime where they strongly and universally couple to the scalar and vector fields (to within the sign of their electric charge). We note that the simplicity of our results, as illustrated by, e.g., figure 1, is inherent to black holes, and contrasts with the rather involved study of neutron stars and their scalarization, see [12–14] or [36] (which, for example, depends on their equation of state, and on their coupling to the Jordan metric, $\tilde{g}_{\mu\nu}=\mathcal{A}^2(\varphi)g_{\mu\nu}$, see (2.3)).

As already discussed in section 4, the "scalarization" of a given black hole depends drastically on the value φ_0 of its cosmological environment, which, in turn, is expected to vary in the large range of redshifts that will be reachable with the gravitational wave detectors LIGO-Virgo (and forthcoming LISA); see, for example, [37] for the cosmological evolution of φ_0 in scalar-tensor theories (to which EMD theories can be fairly supposed to reduce to on cosmological scales). The fact that, depending on the redshift from which, say, a binary black hole system emitted, its dynamics can be undistinguishable from the general relativistic one, or, rather, significantly depart from it, supports the importance to investigate sources emitting from the broadest possible range of redshifts in the future.

In another paper [38], we extended the effective-one-body (EOB) approach (see, e.g., [39] and [40]) to massless scalar-tensor theories (ST); that is, we reduced the ST two-body dynamics, which is known at second post-Keplerian order [11], to the geodesic motion of a test particle in an effective static, spherically symmetric metric. In doing so, we implemented the impact of ST theories on the coalescence of compact binary systems as parametrized corrections to the best available general relativistic (5PN) EOB results. Now, in the present paper, we computed the two-body Lagrangian at 1PK level and showed that it only differs from the 1PK ST one in redefining the specific combinations G_{AB}, $\bar{\gamma}_{AB}$, and $\bar{\beta}_{A/B}$, see (2.11) and discussions below. In consequence, the ST-EOB results, presented in [38], are trivially

[5]Note that we proved that μ_A is related to the area of the black hole for the specific theory $a=1$, see (4.3). Its generalization to arbitrary EMD theories is the topic of [33].

extended to EMD theories (including black holes). For example, the EMD correction to the orbital frequency of a compact binary system at the innermost stable circular orbit (ISCO), presented in [38] figure 1 in ST theories, can be obtained replacing simply G_{AB}, $\bar{\gamma}_{AB}$, and $\bar{\beta}_{A/B}$ by their EMD values.

We note that although the point particle action (2.4) from which we started, together with the matching conditions (3.8), have fixed uniquely the mass function $m_A(\varphi)$ and charge q_A to be assigned to a specific EMD black hole, their validity remains to be justified. Now, as discussed in section 2.1, our point particle action (2.4) does not depend on the four-gradients of the fields; this presupposes in turn that, for a well-separated binary system, each body is adiabatically readjusting its equilibrium configuration when submitted to the influence of its slowly orbiting, faraway companion. Consequently, in [33], the thermodynamics of EMD black holes *in equilibrium* provides a convenient framework to interpret and justify their skeletonization.

Finally, the inclusion of the radiation reaction force to the two-body dynamics in EMD theories is left to further work. Another interesting extension of the present paper could be to apply our Eardley-type approach to binary black holes in the extreme mass ratio limit, extending, e.g., [34], and study its implications on merger estimates, as initiated in [35].

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A The two-body Lagrangian at 1PK order

In this appendix, we derive the two-body Lagrangian in EMD theories at post-Keplerian (1PK) order (2.10). The method presented below is an adaptation of the standard one given in [9] in the scalar-tensor case. For bound orbits, we shall implement the relativistic corrections to the Keplerian Lagrangian in the weak field, slow orbital velocity approximation, $\mathcal{O}(m/R) \sim \mathcal{O}(V^2)$, R being the distance separating the bodies, and V denoting their orbital velocity.

The first step is to solve the covariant field equations (2.7) sourced by the two skeletonized bodies at 1PK order. To do so, let us rewrite them as

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\right) + 8\pi\sum_A\left(T^A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^A\right), \tag{A.1a}$$

$$\sqrt{-g}\,\nabla_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi\sum_A q_A\,\delta^{(3)}\left(\vec{y} - \vec{x}_A(t)\right)\frac{dx^\mu_A}{dt}, \tag{A.1b}$$

$$\sqrt{-g}\,\Box\varphi = -\frac{a}{2}\sqrt{-g}\,e^{-2a\varphi}F^2 + 4\pi\sum_A\frac{ds_A}{dt}\frac{dm_A}{d\varphi}\delta^{(3)}\left(\vec{x} - \vec{x}_A(t)\right), \tag{A.1c}$$

where we recall that $\Box \equiv \nabla^\mu\nabla_\mu$, where

$$T^{\mu\nu}_A = m_A(\varphi)\frac{\delta^{(3)}\left(\vec{y} - \vec{x}_A(t)\right)}{\sqrt{g g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}\frac{dx^\mu_A}{dt}\frac{dx^\nu_A}{dt} \tag{A.2}$$

is the stress-energy tensor of body A, and where $x^\mu_A = (t, \vec{x}_A)$ denotes its location.

In cartesian coordinates, the metric is now conveniently expanded around Minkowski's as

$$g_{00} = -e^{-2U} + \mathcal{O}(V^6)\,, \tag{A.3a}$$
$$g_{0i} = -4\,g_i + \mathcal{O}(V^5)\,, \tag{A.3b}$$
$$g_{ij} = \delta_{ij}\,e^{2U} + \mathcal{O}(V^4)\,, \tag{A.3c}$$

(this ansatz allows to bypass unnecessary computational complications, see [41] and [42]) where, as we shall consistently check below, $U = \mathcal{O}(V^2)$ and $g_i = \mathcal{O}(V^3)$.
Similarly, the vector field is expanded around $A_\mu(r \to \infty) = 0$, that is

$$A_\mu = \left(\delta A_t + \mathcal{O}(V^6)\,,\ \delta A_i + \mathcal{O}(V^5)\right)\,, \tag{A.4}$$

where $\delta A_t = \mathcal{O}(V^2)$ and $\delta A_i = \mathcal{O}(V^3)$.

Finally, the scalar field is expanded around the local value of the cosmological scalar background, φ_0, as

$$\varphi = \varphi_0 + \delta\varphi + \mathcal{O}(V^6)\,, \tag{A.5}$$

where $\delta\varphi = \mathcal{O}(V^2)$.

Therefore, the fonctions $m_A(\varphi)$ are expanded around $\varphi = \varphi_0$ at 1PK order, following the definitions (2.9), as:

$$m_A(\varphi) = m_A^0\left[1 + \alpha_A^0(\varphi-\varphi_0) + \frac{1}{2}({\alpha_A^0}^2 + \beta_A^0)(\varphi-\varphi_0)^2 + \mathcal{O}(V^6)\right] \tag{A.6}$$

where we recall that a zero superscript indicates a quantity evaluated at φ_0, which is the value of the scalar field at spatial infinity.

In the harmonic gauge $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$, the tt and ti components of Einstein's field equations (A.1a) then read[6]

$$\begin{aligned}\Box_\eta U = {}& -4\pi\sum_A m_A^0\left[1 + \frac{3}{2}V_A^2 - U + \alpha_A^0(\varphi-\varphi_0)\right]\delta^{(3)}(\vec{x}-\vec{x}_A(t))\\ & - (\partial_i A_t)(\partial_i A_t)e^{-2a\varphi_0} + \mathcal{O}(V^6)\,, \end{aligned}\tag{A.7a}$$
$$\Delta g_i = -4\pi\sum_A m_A^0 V_A^i\delta^{(3)}(\vec{x}-\vec{x}_A(t)) + \mathcal{O}(V^5)\,, \tag{A.7b}$$

where $\Box_\eta \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ denotes the flat Dalembertian. The Maxwell field equations (A.1b) read, in the Lorenz gauge $\nabla_\mu A^\mu = 0$ (such that $\nabla_\nu F^{\mu\nu} = R^\mu{}_\nu A^\nu - \Box A^\mu$):

$$\begin{aligned}\Box_\eta A_t = {}& 4\pi\sum_A q_A e^{2a\varphi_0}\Big[1 - 2U + 2a(\varphi-\varphi_0)\Big]\delta^{(3)}(\vec{x}-\vec{x}_A(t))\\ & + 2a(\partial_i A_t)(\partial_i\varphi) - 2(\partial_i A_t)(\partial_i U) + \mathcal{O}(V^6)\,, \end{aligned}\tag{A.8a}$$
$$\Delta A_i = -4\pi\sum_A q_A e^{2a\varphi_0}V_A^i\delta^{(3)}(\vec{x}-\vec{x}_A(t)) + \mathcal{O}(V^5)\,. \tag{A.8b}$$

[6]Note that the harmonic condition $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$ implies $\partial_t U + \partial_i g_i = 0$. Other useful intermediate results are $\sqrt{-g} = e^{2U} + \mathcal{O}(V^4)$, $R^{tt} = -\Box_\eta U + \mathcal{O}(V^6)$, $R^{ti} = -2\Delta g_i + \mathcal{O}(V^5)$, as well as $g_{\mu\nu}F^{t\mu}F^{t\nu} = (\partial_i A_t)(\partial_i A_t) + \mathcal{O}(V^6)$ and $F^2 = -2(\partial_i A_t)(\partial_i A_t) + \mathcal{O}(V^6)$.

Finally, the scalar field equation (A.1c) is

$$\Box_\eta \varphi = 4\pi \sum_A m_A^0 \alpha_A^0 \left[1 - \frac{1}{2}V_A^2 - U + \left(\alpha_A^0 + \frac{\beta_A^0}{\alpha_A^0}\right)(\varphi - \varphi_0)\right] \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$
$$+ a(\partial_i A_t)(\partial_i A_t) e^{-2a\varphi_0} + \mathcal{O}(V^6)\,, \tag{A.9}$$

where $\Delta = \delta^{ij}\partial_i\partial_j$ denotes the flat Laplacian.

We now solve these equations, working iteratively. In particular, restricting ourselves to the conservative part of the dynamics, we shall use the half-retarded, half-advanced Green's function,

$$\Box_\eta G(x - x') = -4\pi\delta^{(3)}(\vec{x} - \vec{x}')\delta(t - t') \tag{A.10}$$

where

$$G(x - x') = \frac{1}{2}\left[\frac{\delta(t - t' - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} + \frac{\delta(t - t' + |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}\right] \tag{A.11}$$
$$= \delta(t - t')\left[\frac{1}{|\vec{x} - \vec{x}'|} + \frac{1}{2}\partial_t^2|\vec{x} - \vec{x}'| + \cdots\right]. \tag{A.12}$$

The right hand side of the above field equations also contains extended source terms that are easily integrated when using the identity [43]

$$f(\vec{x}, \vec{y}_1, \vec{y}_2) \equiv \frac{1}{2|\vec{x}-\vec{y}_1||\vec{x}-\vec{y}_2|} - \frac{1}{2|\vec{y}_1-\vec{y}_2|}\left(\frac{1}{|\vec{x}-\vec{y}_1|} + \frac{1}{|\vec{x}-\vec{y}_2|}\right) \quad \Rightarrow \quad \Delta f = \partial_i \frac{1}{|\vec{x}-\vec{y}_1|}\partial_i \frac{1}{|\vec{x}-\vec{y}_2|}\,, \tag{A.13}$$

which is easily shown using Leibniz' rule.

Hence, all in all, the fields evaluated at any point $x^\mu = (t, \vec{x})$ are found to be

$$\mathrm{U}(x) = \sum_A \frac{m_A^0}{\rho_A}\left[1 + \frac{3}{2}V_A^2 - \sum_{B\neq A}\frac{m_B^0}{R}(1 + \alpha_A^0\alpha_B^0)\right]$$
$$- e^{2a\varphi_0}\sum_{A,B} q_A q_B\, f\left(\vec{x},\, \vec{x}_A(t),\, \vec{x}_B(t)\right) + \mathcal{O}(V^6)\,, \tag{A.14a}$$

$$g_i(x) = \sum_A \frac{m_A^0}{|\vec{x} - \vec{x}_A(t)|}V_A^i + \mathcal{O}(V^5)\,, \tag{A.14b}$$

$$A_t(x) = -e^{2a\varphi_0}\sum_A \frac{q_A}{\rho_A}\left[1 - 2\sum_{B\neq A}\frac{m_B^0}{R}(1 + a\,\alpha_B^0)\right]$$
$$+ 2e^{2a\varphi_0}\sum_{A,B} q_A m_B^0(1 + a\,\alpha_B^0) f\left(\vec{x},\, \vec{x}_A(t),\, \vec{x}_B(t)\right) + \mathcal{O}(V^6)\,, \tag{A.14c}$$

$$A_i(x) = e^{2a\varphi_0}\sum_A \frac{q_A}{|\vec{x} - \vec{x}_A(t)|}V_A^i + \mathcal{O}(V^5)\,, \tag{A.14d}$$

and, finally,

$$\varphi(x) = \varphi_0 - \sum_A \frac{m_A^0\alpha_A^0}{\rho_A}\left[1 - \frac{1}{2}V_A^2 - \sum_{B\neq A}\frac{m_B^0}{R}\left(1 + \alpha_A^0\alpha_B^0 - \frac{\beta_A^0\alpha_B^0}{\alpha_A^0}\right)\right]$$
$$+ a\, e^{2a\varphi_0}\sum_{A,B} q_A q_B f\left(\vec{x},\, \vec{x}_A(t),\, \vec{x}_B(t)\right) + \mathcal{O}(V^6)\,, \tag{A.15}$$

where

$$\frac{1}{\rho_A} \equiv \frac{1}{|\vec{x}-\vec{x}_A(t)|} + \frac{1}{2}\partial_t^2|\vec{x}-\vec{x}_A(t)| \tag{A.16}$$
$$= \frac{1}{|\vec{x}-\vec{x}_A(t)|}\left[1+\frac{1}{2}\vec{V}_A^2 - \frac{1}{2}(N_A.V_A)^2\right] + \frac{1}{2}(N_A.A_A)\,,$$

with $\vec{N}_A \equiv (\vec{x}_A - \vec{x})/|\vec{x}_A - \vec{x}|$ and $\vec{A}_A \equiv d\vec{V}_A/dt$.

The obtention of the two-body Lagrangian is now straightforward. First, one writes the Lagrangian of, say, body B when considered as a test particle in the fields generated by body A, and defined as $S^{\rm pp}_{\rm m,B} \equiv \int dt\, L_B$, see, e.g., (2.4):

$$L_B = -m_B(\varphi)\,\frac{ds_B}{dt} + q_B\, A_\mu\,\frac{dx^\mu_B}{dt} \tag{A.17}$$
$$= -m_B(\varphi)\sqrt{e^{-2U} + 8g_iV_B^i - e^{2U}V_B^2} + q_B(A_t + A_iV_B^i) + \mathcal{O}(V^6)\,,$$

setting formally $m_B^0 = 0$, $q_B = 0$ and $\vec{x} = \vec{x}_B$ in (A.14a)–(A.15). In particular, (A.16) is easily rewritten as

$$\frac{1}{\rho_A} = \frac{1}{R}\left[1+\frac{1}{2}(V_A.V_B) - \frac{1}{2}(N_A.V_A)(N.V_B)\right] + \frac{1}{2}\frac{d}{dt}(N.V_A)\,, \tag{A.18}$$

where $R = |\vec{x}_A - \vec{x}_B|$, $\vec{N} = (\vec{x}_A - \vec{x}_B)/R$, and where the last term is a total derivative that can be neglected in the Lagrangian (A.17).

In a last step, one symmetrizes L_B with respect to $A \leftrightarrow B$ to obtain the total two-body Lagrangian:

$$\begin{aligned}
L_{AB} = {}& -m_A^0 - m_B^0 + \tfrac{1}{2}m_A^0V_A^2 + \tfrac{1}{2}m_B^0V_B^2 + \frac{m_A^0m_B^0}{R}(1+\alpha_A^0\alpha_B^0) - \frac{\tilde{q}_A\tilde{q}_B}{R} \\
&+ \tfrac{1}{8}m_A^0V_A^4 + \tfrac{1}{8}m_B^0V_B^4 \\
&+ \frac{m_A^0m_B^0}{R}\left[\left(\frac{V_AV_B}{2}\right)(-7+\alpha_A^0\alpha_B^0) + \left(\frac{V_A^2+V_B^2}{2}\right)(3-\alpha_A^0\alpha_B^0) - \left(\frac{(N.V_A)(N.V_B)}{2}\right)(1+\alpha_A^0\alpha_B^0)\right] \\
&+ \frac{\tilde{q}_A\tilde{q}_B}{R}\left[\left(\frac{V_AV_B}{2}\right) + \left(\frac{(N.V_A)(N.V_B)}{2}\right)\right] \\
&- \frac{m_A^0m_B^0}{2R^2}\left[m_A^0\left((1+\alpha_A^0\alpha_B^0)^2 + \beta_B^0{\alpha_A^0}^2\right) + m_B^0\left((1+\alpha_A^0\alpha_B^0)^2 + \beta_A^0{\alpha_B^0}^2\right)\right] \\
&+ \frac{\tilde{q}_A\tilde{q}_B}{R^2}\left[m_A^0(1+a\,\alpha_A^0) + m_B^0(1+a\,\alpha_B^0)\right] - \frac{\tilde{q}_B^2}{2R^2}\left[m_A^0(1+a\,\alpha_A^0)\right] - \frac{\tilde{q}_A^2}{2R^2}\left[m_B^0(1+a\,\alpha_B^0)\right],
\end{aligned} \tag{A.19}$$

where we introduced the convenient notation $\tilde{q}_A \equiv q_A e^{a\varphi_0}$. Finally, introducing $e_A \equiv \tilde{q}_A/m_A^0 = (q_A/m_A^0)e^{a\varphi_0}$, (A.19) is straightforwardly rewritten as (2.10).

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Chapitre 5

Rayonnement gravitationnel en théories EMD

Dans les chapitres précédents, nous avons uniquement discuté des aspects conservatifs du problème à deux corps en théories scalaire-tenseur et Einstein-Maxwell-dilaton. Afin de décrire le rétrécissement de l'orbite d'un système binaire ainsi que la forme de l'onde émise jusqu'à sa coalescence, il reste à en compléter la dynamique par les effets dissipatifs.

Dans l'article ci-dessous, nous étudions le rayonnement gravitationnel d'un système binaire en orbite quasi-circulaire en théorie EMD (le cas scalaire-tenseur étant obtenu dans la limite des charges "graviphotoniques" nulles, $q_{A/B} = 0$).

Cette étude repose sur l'obtention des flux d'énergie rayonnée à l'infini, que l'on peut obtenir à l'aide de lois de conservation à la Landau-Lifshitz [185].

On y montre que la partie due au champ métrique se réduit, à l'ordre le plus bas (ou 0PK) et dans le référentiel du centre de masse, à la deuxième formule du quadrupôle d'Einstein habillée par les contributions scalaire et "graviphotonique" :

$$\mathcal{F}_g = \frac{32}{5} \frac{\nu^2 \left(G_{AB} M \dot{\phi}\right)^{10/3}}{G_* \left(1 + \alpha_A^0 \alpha_B^0 - e_A e_B\right)^2} + \cdots , \tag{5.1}$$

où $G_{AB} = G_*(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)$ (G_* est la constante de Newton dans le "Einstein frame", $c = 1$ et les coefficients $\alpha_{A/B}$ et $e_{A/B}$ ont été introduits dans le chapitre précédent et sont rappelés dans l'article qui suit), et où ϕ est la phase orbitale, avec $\dot{\phi} = d\phi/dt$.

Le flux "graviphotonique", quand à lui, est d'ordre dipolaire (-1PK), et le calcul de sa contribution 0PK nécessite d'itérer les "équations de Maxwell" pour tenir compte du couplage à la métrique et au champ scalaire. On obtient :

$$\begin{aligned}\mathcal{F}_A = {} & \frac{\nu^2 \left(G_{AB} M \dot{\phi}\right)^{8/3}}{G_* \left(1 + \alpha_A^0 \alpha_B^0 - e_A e_B\right)^2} \left\{ \frac{2}{3}(e_A - e_B)^2 \right. \\ & + \left(G_{AB} M \dot{\phi}\right)^{2/3} \left[\frac{8}{5} \left(\frac{m_A^0 e_B + m_B^0 e_A}{M} \right)^2 + \frac{4}{9}(e_A - e_B)^2 \Big(\nu - 3 - \bar{\gamma}_{AB} - 2\langle \bar{\beta} \rangle \Big) \right. \\ & \left. \left. + 4(e_A - e_B) \left(\frac{(m_A^0)^2 e_B - (m_B^0)^2 e_A}{15 M^2} - \frac{e_A(1 + a\,\alpha_B^0) - e_B(1 + a\,\alpha_A^0)}{3M(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)} \right) \right] \right\} + \cdots ,\end{aligned} \tag{5.2}$$

où $\bar{\gamma}_{AB}$ et $\langle \bar{\beta} \rangle$ ont également été introduits dans le chapitre précédent. Notons qu'en absence de champ scalaire, cette formule donne le flux électrique rayonné, e.g., par des trous noirs de Reissner-Nordström (ce qui est aussi un résultat neuf à cet ordre). Par ailleurs lorsque $e_A = e_B$, la composante dipolaire disparait et $\mathcal{F}_A$ s'identifie alors au flux quadrupolaire maxwellien, $\mathcal{F}_A \propto (d^3 \mathcal{Q}_A^{ij}/dt^3)^2$ où $\mathcal{Q}_A^{ij}$ est le quadrupole électrique et où les charges sont habillées par le champ scalaire.

Finalement, le flux scalaire est a priori monopolaire (-2PK). Cependant le calcul montre que pour les orbites circulaires considérées ici, ce flux est en fait dipolaire (-1PK), et sa partie 0PK s'obtient par itération de "l'équation de Klein-Gordon" pour tenir compte du couplage à la métrique et au "graviphoton". On obtient :

$$\begin{aligned}\mathcal{F}_\varphi = & \frac{\nu^2 \left(G_{AB} M \dot{\phi}\right)^{8/3}}{G_* \left(1 + \alpha_A^0 \alpha_B^0 - e_A e_B\right)^2} \left\{ \frac{1}{3} (\alpha_A^0 - \alpha_B^0)^2 \right. \\ & + \left(G_{AB} M \dot{\phi}\right)^{2/3} \left[\frac{16}{15} \left(\frac{m_A^0 \alpha_B^0 + m_B^0 \alpha_A^0}{M} \right)^2 + \frac{2}{9} (\alpha_A^0 - \alpha_B^0)^2 \left(\nu - 3 - \bar{\gamma}_{AB} - 2 \langle \bar{\beta} \rangle \right) \right. \\ & \left. \left. + 2(\alpha_A^0 - \alpha_B^0) \left(\frac{(m_A^0)^2 \alpha_B^0 - (m_B^0)^2 \alpha_A^0}{5M^2} + \frac{m_A^0 \left[\alpha_B^0 + \alpha_A^0 \, (\alpha_B^0)^2 + \beta_B^0 \alpha_A^0 - a \, e_A e_B \right] - (A \leftrightarrow B)}{3M(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)} \right) \right] \right\} + \cdots\end{aligned} \tag{5.3}$$

Lorsque $\alpha_A^0 = \alpha_B^0$, le flux se réduit à sa composante quadrupolaire. On note aussi que l'on retrouve le résultat de T. Damour et G. Esposito-Farèse dans la limite scalaire-tenseur, $e_{A/B} = 0$ [88], en remarquant toutefois que le flux EMD ne s'en déduit pas par simple généralisation de G_{AB} et des charges, à cause de la présence de a dans le dernier terme de (5.3).

Ces flux permettent dans un premier temps de définir le "chirp", c.-à-d. l'évolution $\ddot{\phi}$ de la vitesse orbitale $\dot{\phi}$ à l'ordre considéré. Pour ce faire, on part de l'énergie mécanique du système conservatif à 1PK, déduite du Lagrangien à deux corps obtenu au chapitre précédent, qui s'écrit, dans le référentiel du centre de masse et pour des orbites quasi-circulaires :

$$E = -\frac{1}{2} \mu \left(G_{AB} M \dot{\phi}\right)^{2/3} \left[1 - \left(G_{AB} M \dot{\phi}\right)^{2/3} \left(\frac{3}{4} + \frac{\nu}{12} + \frac{2}{3} \left(\bar{\gamma}_{AB} - \langle \bar{\beta} \rangle \right) \right) + \cdots \right] \tag{5.4}$$

Par un bilan que l'on admet entre énergie mécanique perdue et énergie rayonnée,

$$-\frac{dE}{dt} = \mathcal{F}_g + \mathcal{F}_A + \mathcal{F}_\varphi \,, \tag{5.5}$$

on obtient ainsi la relation recherchée :

$$\begin{aligned}\ddot{\phi} = & G_* \mu \, \dot{\phi}^3 \left\{ \left[2(e_A - e_B)^2 + (\alpha_A^0 - \alpha_B^0)^2 \right] \right. \\ & + \left(G_{AB} M \dot{\phi}\right)^{2/3} \left[\frac{96}{5} + \frac{24}{5} \left(\frac{m_A^0 e_B + m_B^0 e_A}{M} \right)^2 + \frac{16}{5} \left(\frac{m_A^0 \alpha_B^0 + m_B^0 \alpha_A^0}{M} \right)^2 \right. \\ & \qquad + \left(2(e_A - e_B)^2 + (\alpha_A^0 - \alpha_B^0)^2 \right) \left(\frac{5\nu}{6} - \frac{1}{2} + \frac{2}{3} \bar{\gamma}_{AB} - \frac{8}{3} \langle \bar{\beta} \rangle \right) \\ & \qquad + 4(e_A - e_B) \left(\frac{(m_A^0)^2 e_B - (m_B^0)^2 e_A}{5M^2} - \frac{e_A (1 + a \, \alpha_B^0) - e_B (1 + a \, \alpha_A^0)}{M(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)} \right) \\ & \left. \left. + 2(\alpha_A^0 - \alpha_B^0) \left(\frac{3}{5} \frac{(m_A^0)^2 \alpha_B^0 - (m_B^0)^2 \alpha_A^0}{M^2} + \frac{m_A^0 \left[\alpha_B^0 + \alpha_A^0 \, (\alpha_B^0)^2 + \beta_B^0 \alpha_A^0 - a \, e_A e_B \right] - (A \leftrightarrow B)}{M(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)} \right) \right] + \cdots \right\} .\end{aligned} \tag{5.6}$$

Cette formule redonne dans la limite de la relativité générale l'expression bien connue :

$$\ddot{\phi} = \frac{96}{5} (G_* \mathcal{M})^{5/3} \dot{\phi}^{11/3} \quad \text{où} \quad \mathcal{M} = \nu^{3/5} M \tag{5.7}$$

est la "chirp mass". En revanche, en présence de rayonnement dipolaire, le premier terme de (5.6) domine, et la formule devient

$$\ddot{\phi} = (G_* \mathcal{M}_{\mathcal{D}})\dot{\phi}^3 \quad \text{avec} \quad \mathcal{M}_{\mathcal{D}} = \nu \left[2(e_A - e_B)^2 + (\alpha_A^0 - \alpha_B^0)^2 \right] M \,, \tag{5.8}$$

formule qui régit le mouvement lorsque, par exemple, l'un des deux corps est un trou noir EMD "scalarisé", tel qu'étudié au chapitre précédent. Enfin, lorsque le rayonnement dipolaire est négligeable ($\alpha_A^0 \approx \alpha_B^0$ et $e_A \approx e_B$), la "chirp mass" de la relativité générale est habillée par les contributions quadrupolaires des champs vectoriel et scalaire :

$$\ddot{\phi} = \frac{96}{5}(G_* \mathcal{M}_{\mathcal{Q}})^{5/3}\dot{\phi}^{11/3} \quad \text{avec} \tag{5.9}$$

$$\mathcal{M}_{\mathcal{Q}} = \nu^{3/5}(1 + \alpha_A^0\alpha_B^0 - e_A e_B)^{2/5} \left[1 + \frac{1}{4}\left(\frac{m_A^0 e_B + m_B^0 e_A}{M} \right)^2 + \frac{1}{6}\left(\frac{m_A^0 \alpha_B^0 + m_B^0 \alpha_A^0}{M} \right)^2 \right]^{3/5} M \,.$$

Comme nous le verrons plus bas, la vitesse angulaire $\dot{\phi}$ est proportionnelle à la fréquence f de l'onde gravitationnelle observée à l'infini, ce qui fait de la "chirp mass" une combinaison mesurable. Les formules ci-dessus montrent donc qu'en présence de rayonnement dipolaire, cf. (5.8), l'évolution typique de la fréquence f du signal permettrait facilement de détecter une violation de la relativité générale. En revanche, en l'absence de rayonnement dipolaire, le "chirp" se réduit à (5.9), ce qui le rend indistinguable à cet ordre, par la seule connaissance de la valeur numérique de sa "chirp mass", de celui de la relativité générale.

Connaissant les flux (5.1-5.3), il est aussi possible de calculer la force de réaction de rayonnement en égalant sa puissance au flux d'énergie rayonnée à l'infini. Pour les orbites quasi-circulaires considérées, cette force se réduit à une composante tangentielle à la trajectoire,

$$F_\phi = -\frac{\mathcal{F}_g + \mathcal{F}_A + \mathcal{F}_\varphi}{\dot{\phi}} \,. \tag{5.10}$$

Ayant en main cette force, les équations du mouvement peuvent alors s'obtenir à partir du Lagrangien 1PK donné au chapitre précédent (R étant la distance "harmonique" entre les corps) :

$$\frac{d}{dt}\left(\frac{\partial L_{1PK}}{\partial \dot{R}} \right) - \frac{\partial L_{1PK}}{\partial R} = 0 \,, \quad \frac{d}{dt}\left(\frac{\partial L_{1PK}}{\partial \dot{\phi}} \right) - \frac{\partial L_{1PK}}{\partial \phi} = F_\phi \,, \tag{5.11}$$

qui ne fait que redonner l'équation du "chirp" (5.6).

Cette formule peut cependant être resommée, en partant non pas du Lagrangien 1PK, mais du Hamiltonien EOB scalaire-tenseur présenté au chapitre 2, dont nous avons montré que la généralisation au cas EMD à 1PK est triviale au chapitre précédent. Ainsi dans le système de coordonnées effectives (r, ϕ), où l'angle ϕ s'identifie, dans l'approximation des orbites circulaires, à la phase orbitale observée, les équations du mouvement deviennent :

$$\dot{r} = \frac{\partial H_{\text{EOB}}}{\partial p_r} \quad , \quad \dot{p}_r = -\frac{\partial H_{\text{EOB}}}{\partial r} \quad , \quad \dot{\phi} = \frac{\partial H_{\text{EOB}}}{\partial p_\phi} \quad , \quad \dot{p}_\phi = -\frac{\partial H_{\text{EOB}}}{\partial \phi} + F_\phi \,, \tag{5.12}$$

$$\text{avec} \quad H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1 \right)} \,, \quad \text{où} \quad H_e = \mu\sqrt{A\left(1 + \frac{p_r^2}{\mu^2 B} + \frac{p_\phi^2}{\mu^2 r^2} \right)} \,, \tag{5.13}$$

$$\text{et} \quad A(r) = \mathcal{P}_1^1\left[1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2\right], \tag{5.14a}$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right), \tag{5.14b}$$

où $\mathcal{P}_1^1$ désigne l'approximant de Padé de type $(1,1)$ sur la variable $(G_{AB}M/r)$.

Ces équations sont intégrées pour donner l'évolution d'un système binaire jusqu'à sa dernière orbite stable (ISCO). La figure 5.1 ci-dessous montre l'exemple particulièrement illustratif de deux trous noirs EMD, dont l'un est "scalarisé" et l'autre est schwarzschildien. On y voit que les rayonnements dipolaires (scalaire et "graviphotonique") impliquent un rétrécissement de l'orbite relative nettement plus rapide qu'en relativité générale, comme on pouvait s'y attendre.

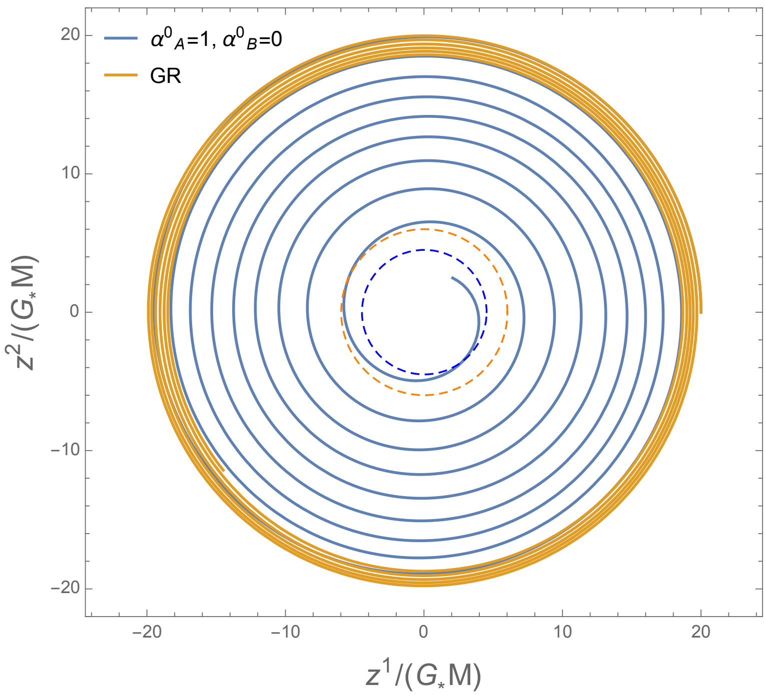


FIGURE 5.1 – Trajectoires effectives ($z^1 = r\cos\phi, z^2 = r\sin\phi$) pour un système binaire de trous noirs EMD de même masse ($\nu = 1/4$), en théorie $a = 1$. En bleu, le cas où l'un des trous noirs est "scalarisé" ($\alpha_A^0 = 1$, $e_A = \sqrt{2}$), l'autre étant schwarzschildien ($\alpha_B^0 = 0$, $e_B = 0$) ; en orange, le cas de deux trous noirs de Schwarzschild, se réduisant à la relativité générale. Les ISCO respectives sont représentées en lignes pointillées. En partant de $r = 20\,G_*M$, le système "scalarisé" atteint son ISCO en un temps $\Delta t \simeq 3000\,G_*M$, contre $\Delta t = 10\,000\,G_*M$ en relativité générale.

Une fois les trajectoires connues et poussées jusqu'à l'ISCO, il est alors aisé de prédire, à l'ordre le plus bas, les formes d'onde mesurables.

Il est possible de calculer la partie vectorielle de la forme d'onde, mais cela n'est pas nécessaire si l'on suppose que les détecteurs sont "graviphotoniquement" neutres. En revanche, les miroirs des détecteurs interférométriques suivent des géodésiques de la métrique de Jordan, qui s'écrit, dans le système solaire :

$$\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu} = \mathcal{A}_\odot^2\left[\eta_{\mu\nu}(1 + 2\alpha_\odot\delta\varphi) + h_{\mu\nu}^{TT}\right] + \mathcal{O}(1/x^2), \tag{5.15}$$

où x est la distance à la source, où $\mathcal{A}_\odot = \mathcal{A}(\varphi_\odot)$ est la valeur du couplage du champ scalaire à la matière dans le système solaire, et $\alpha_\odot = (d \ln \mathcal{A}/d\varphi)(\varphi_\odot)$. Par ailleurs, $\delta\varphi = \varphi - \varphi_\odot$ est l'onde scalaire, donnée à l'ordre dominant par

$$\delta\varphi = -G_* \frac{n_i \dot{\mathcal{D}_S}^i}{x} \quad \text{où} \quad \mathcal{D}_S^i = \sum_A m_A^0 \alpha_A^0 z_A^i \,, \tag{5.16}$$

$n_i \dot{\mathcal{D}_S}^i$ étant la projection de la dérivée temporelle du dipôle (scalaire) $\mathcal{D}_S{}^i$ de la source sur la direction d'observation n_i, les coordonnées harmoniques $z^i_{A/B}$ repérant les corps dans le référentiel de leur centre de masse. Enfin, les composantes utiles de $h^{TT}_{\mu\nu}$ sont

$$h^{TT}_{ij} = \frac{2G_*}{3} \frac{\mathcal{P}_{ij}{}^{kl} \ddot{\mathcal{Q}}_{kl}}{x} \quad \text{où} \quad \mathcal{Q}^{ij} = \sum_A m_A^0 \left(3 z_A^i z_A^j - \delta^{ij} z_A^2\right) \,, \tag{5.17}$$

$\mathcal{P}_{ij}{}^{kl} \ddot{\mathcal{Q}}_{kl}$ étant la projection transverse et sans trace de la dérivée temporelle seconde du quadrupôle (de masse) du système.

On constate que $\delta\varphi$ est d'ordre -0.5PK, tandis que h^{TT}_{ij} est, comme il se doit, d'ordre 0PK. Cependant, la contribution de $\delta\varphi$ est numériquement atténuée par la présence du facteur $\alpha_\odot$, qui est contraint à etre inférieur à 10^{-2} par les observations dans le système solaire, voir, e.g., [186], ce qui permet de s'épargner le calcul de $\delta\varphi$ à l'ordre 0PK. Rappelons qu'en revanche les $\alpha^0_{A/B}$ apparaissant dans (5.16) sont évalués sur la valeur de l'environnement scalaire des sources, φ_0, et, dans le cas de trous noirs "scalarisés", peuvent être de l'ordre de l'unité.

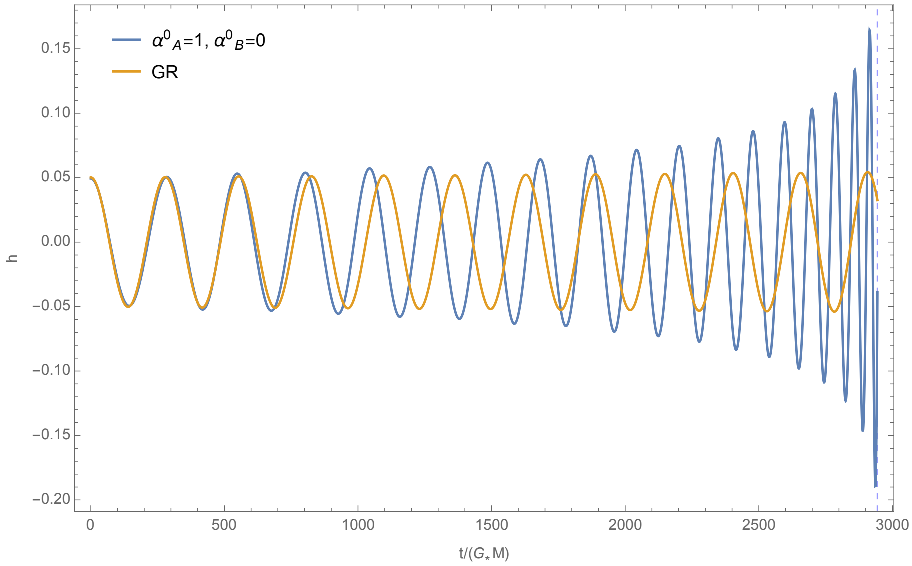
À l'ordre considéré, $\delta\varphi$ et h^{TT}_{ij} se calculent à l'aide des lois de Kepler, pour des orbites circulaires, et s'expriment en fonction de la seule vitesse angulaire $\dot{\phi}$ pour donner :

$$\delta\varphi = (\alpha_A^0 - \alpha_B^0) \frac{G_* M \nu}{x} (\sin i) \left(G_{AB} M \dot{\phi}\right)^{1/3} \cos\phi \,, \tag{5.18a}$$

$$h_+ = \frac{4 G_* M \nu}{x} \left(\frac{1 + \cos^2 i}{2}\right) \left(G_{AB} M \dot{\phi}\right)^{2/3} \cos(2\phi) \,, \tag{5.18b}$$

$$h_\times = \frac{4 G_* M \nu}{x} (\cos i) \left(G_{AB} M \dot{\phi}\right)^{2/3} \sin(2\phi) \,, \tag{5.18c}$$

où i désigne l'angle entre la normale au plan de l'orbite et la direction d'observation. La dernière étape consiste à injecter la trajectoire $\phi(t)$ obtenue par intégration des équations du mouvement (5.12). On obtient ainsi les formes d'ondes $\delta\varphi$ et h^{TT}_{ij} jusqu'à l'ISCO. La figure ci-dessous les donne dans le cas des orbites de la figure 5.1.



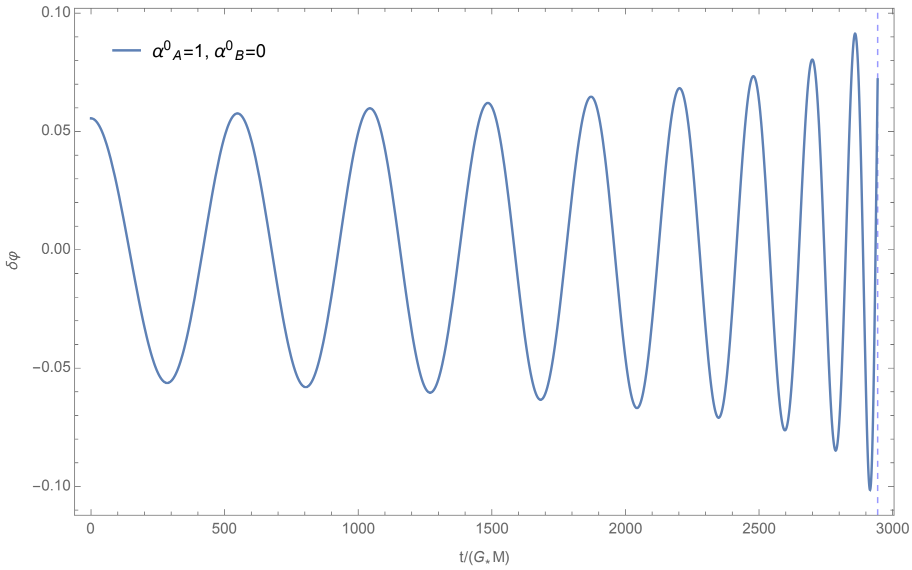


FIGURE 5.2 – Formes d'ondes gravitationnelles correspondant aux trajectoires représentées en figure 5.1, en théorie EMD avec $a = 1$. En haut, $h = (G_{AB} M \dot{\phi})^{2/3} \cos(2\phi)$; en bas, $\delta\varphi = (1/4)(\alpha_A^0 - \alpha_B^0)\,(G_{AB} M \dot{\phi})^{1/3} \cos(\phi)$. L'instant de l'ISCO est représenté en lignes pointillées verticales.

Enfin, notons que pour que la métrique de Jordan (5.15) se réduise à celle de Minkowski en l'absence d'onde gravitationnelle, il est nécessaire d'effectuer le changement de coordonnées local $d\tilde{t} = \mathcal{A}_{\odot} dt$, $d\tilde{x}^i = \mathcal{A}_{\odot} dx^i$. Ainsi, la fréquence observée du signal (5.18b,5.18c) dans le système solaire, définie par $2\pi f \equiv 2 d\phi / d\tilde{t}$, est donnée par $f = \dot{\phi} / (\pi \mathcal{A}_{\odot})$.

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Gravitational radiation from compact binary systems in Einstein-Maxwell-dilaton theories

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Abstract. We derive the energy fluxes radiated by compact binary systems, including "hairy" black holes, in Einstein-Maxwell-dilaton theories, for circular orbits and at quadrupolar order. This enables to include in their resummed effective-one-body (EOB) dynamics the effect of the radiation reaction force at the origin of their inspiral and merger. We also exhibit typical examples of the resulting tensor and scalar waveforms.

Keywords: gravitational waves / sources, gravitational waves / theory, modified gravity

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The Einstein-Maxwell-dilaton theories consist in supplementing general relativity with massless scalar and vector fields, and are described by the following Einstein frame action:

$$S[g_{\mu\nu}, A_\mu, \varphi] = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi}F^{\mu\nu}F_{\mu\nu}\right) + S_{\rm m}[\Psi, \mathcal{A}^2(\varphi)g_{\mu\nu}, A_\mu]\,, \tag{1}$$

where R is the Ricci scalar associated to $g_{\mu\nu}$, where $g = \det g_{\mu\nu}$, and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As for matter fields Ψ, they are minimally coupled to A_μ and to the Jordan metric $\tilde{g}_{\mu\nu} = \mathcal{A}^2(\varphi)g_{\mu\nu}$, $\mathcal{A}(\varphi)$ being a scalar function that specifies the theory, together with the coupling parameter a.

In paper [1], we studied the conservative sector of the dynamics of compact binary systems in EMD theories. To do so, we phenomenologically replaced $S_{\rm m}$ in (1) by a point particle action which generalizes that of Eardley in scalar-tensor theories [2],

$$S_{\rm m} \to S_{\rm m}^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x_A^\mu\}] = -\sum_A \int m_A(\varphi)\, ds_A + \sum_A q_A \int A_\mu\, dx_A^\mu\,, \tag{2}$$

where $ds_A = \sqrt{-g_{\mu\nu}dx_A^\mu dx_A^\nu}$, $x_A^\mu[s_A]$ being the worldline of body A, which is characterized by a constant charge q_A and a "sensitivity" $m_A(\varphi)$ that depends on its internal structure and on the value of the scalar field at its location.

The field equations derived from the "skeleton" action (1), (2) read:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi\left(T^{\mu\nu}_{(\varphi)} + T^{\mu\nu}_{(A)} + T^{\mu\nu}_{(\rm m)}\right)\,, \tag{3a}$$

$$\nabla_\nu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 4\pi\sum_A q_A\int ds_A\,\frac{\delta^{(4)}\left(x - x_A(s_A)\right)}{\sqrt{-g}}\frac{dx_A^\mu}{ds_A}\,, \tag{3b}$$

$$\Box\,\varphi = -\frac{a}{2}e^{-2a\varphi}F^2 + 4\pi\sum_A\int ds_A\,\frac{dm_A}{d\varphi}\frac{\delta^{(4)}\left(x - x_A(s_A)\right)}{\sqrt{-g}}\,, \tag{3c}$$

where ∇_μ denotes the covariant derivative, where $\delta^{(4)}\left(x - y\right)$ is the 4-dimensional Dirac distribution, and where

$$T^{\mu\nu}_{(\varphi)} = \frac{1}{4\pi}\left(\partial^\mu\varphi\,\partial^\nu\varphi - \frac{1}{2}g^{\mu\nu}(\partial\varphi)^2\right)\,, \quad T^{\mu\nu}_{(A)} = \frac{1}{4\pi}e^{-2a\varphi}\left(F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^2\right)\,, \tag{4}$$

$$\text{and} \quad T^{\mu\nu}_{(\rm m)} = \sum_A\int ds_A\, m_A(\varphi)\frac{\delta^{(4)}\left(x - x_A(s_A)\right)}{\sqrt{-g}}\frac{dx_A^\mu}{ds_A}\frac{dx_A^\nu}{ds_A}\,.$$

In [1], we solved the field equations (3), (4) perturbatively around a flat, Minkowski background $\eta_{\mu\nu}$ and the constant value φ_0 of the scalar field at infinity, which is imposed by the cosmological environment of the binary system. We then derived the two-body Lagrangian, at post-keplerian order (1PK) and in harmonic coordinates, which generalizes that of Einstein, Hilbert and Hoffman in general relativity. To do so, we proceeded à la Droste-Fichtenholz, which, at this order, is strictly equivalent to computing, e.g., a Fokker Lagrangian. The resulting Lagrangian depends on the quantities:

$$\alpha_A^0 = \frac{d\ln m_A}{d\varphi}(\varphi_0)\,, \quad \beta_A^0 = \frac{d\alpha_A}{d\varphi}(\varphi_0)\,, \quad \text{and} \quad e_A = \frac{q_A}{m_A^0}e^{a\varphi_0}\,, \tag{5}$$

and their B counterparts, where and from now on, a 0 index indicates a quantity evaluated at infinity, $\varphi = \varphi_0$. We recall here its expression, introducing $R = |\vec{x}_A - \vec{x}_B|$, $\vec{N} = (\vec{x}_A - \vec{x}_B)/R$, and $\vec{V}_A = d\vec{x}_A/dt$:

$$\begin{aligned} L = & -m_A^0 - m_B^0 + \frac{1}{2} m_A^0 V_A^2 + \frac{1}{2} m_B^0 V_B^2 + \frac{G_{AB} m_A^0 m_B^0}{R} + \frac{1}{8} m_A^0 V_A^4 + \frac{1}{8} m_B^0 V_B^4 \\ & + \frac{G_{AB} m_A^0 m_B^0}{R} \left[\frac{3}{2}(V_A^2 + V_B^2) - \frac{7}{2}(\vec{V}_A . \vec{V}_B) - \frac{1}{2}(\vec{N}.\vec{V}_A)(\vec{N}.\vec{V}_B) + \bar{\gamma}_{AB}(\vec{V}_A - \vec{V}_B)^2 \right] \\ & - \frac{G_{AB}^2 m_A^0 m_B^0}{2R^2} \left[m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A) \right] + \mathcal{O}(V^6) , \end{aligned} \tag{6}$$

where G_{AB}, $\bar{\gamma}_{AB}$ and $\bar{\beta}_{A/B}$ are the following combinations of the body-dependent quantities (5):

$$G_{AB} = G_* \left(1 + \alpha_A^0 \alpha_B^0 - e_A e_B \right) , \tag{7a}$$

$$\bar{\gamma}_{AB} = \frac{-4\, \alpha_A^0 \alpha_B^0 + 3\, e_A e_B}{2(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)} , \tag{7b}$$

$$\bar{\beta}_A = \frac{1}{2} \frac{\beta_A^0 {\alpha_B^0}^2 - 2\, e_A e_B (a\, \alpha_B^0 - \alpha_A^0 \alpha_B^0) + e_B^2 (1 + a\, \alpha_A^0 - e_A^2)}{(1 + \alpha_A^0 \alpha_B^0 - e_A e_B)^2} \quad \text{and} \quad A \leftrightarrow B , \tag{7c}$$

G_* being Newton's constant in the Einstein frame, which we shall keep track of in the following for clarity.

From this Lagrangian, one can derive the linear momenta $P_A^i = \partial L / \partial V_A^i$ and the associated mechanical energy $E = \vec{P}_A . \vec{V}_A + \vec{P}_B . \vec{V}_B - L$. For circular orbits, which is the case of interest, and in the center-of-mass frame (such that $P_A^i + P_B^i = 0$), the 1PK mechanical energy E can be expressed in terms of the orbital angular velocity $\dot{\phi}$ alone, using the equations of motion deduced from (6), and reads:

$$E = -\frac{1}{2} \mu \left(G_{AB} M \dot{\phi} \right)^{2/3} \left[1 - \left(G_{AB} M \dot{\phi} \right)^{2/3} \left(\frac{3}{4} + \frac{\nu}{12} + \frac{2}{3} (\bar{\gamma}_{AB} - \langle \bar{\beta} \rangle) \right) + \mathcal{O}(V^4) \right] , \tag{8}$$

where $M = m_A^0 + m_B^0$, $\mu = m_A^0 m_B^0 / M$, and $\nu = \mu / M$, and where $\langle \bar{\beta} \rangle = \left(m_A^0 \bar{\beta}_B + m_B^0 \bar{\beta}_A \right) / M$.

In [1], we then particularized our results to binary systems composed of two charged, non-spinning black holes with vector and scalar "hair", see Gibbons and Maeda [3, 4]. The sensitivity $m_A(\varphi)$ characterizing them was obtained analytically in [1] for all a; in the simple case $a = 1$ that we will consider here, it is given by:

$$m_A(\varphi) = \sqrt{\mu_A^2 + q_A^2 \frac{e^{2\varphi}}{2}} . \tag{9}$$

Here q_A is the constant U(1) charge of the black hole appearing in the action (2). As for the constant μ_A, it is its irreducible mass: $\mu_A = M_{\rm irr}$ (and not its ADM mass which is not conserved when orbiting aroud a companion). Since $M_{\rm irr} = \sqrt{S/4\pi}$, the constancy of μ_A implies that of the black hole entropy. This fact derives from the "skeletonization" approximation, hence showing its limitations [5].

The "sensitivities" being known, all the body-dependent quantities (5) are known for a given black hole A, that is, for given values of (q_A, μ_A), as functions of φ_0. In particular, as highlighted in [1], we have that $\alpha_A^0 \equiv \alpha_A(\varphi_0)$ (which is an exact "Fermi-Dirac distribution" when $a = 1$) transitions from zero (Schwarzschild limit, with $\beta^0_A \to 0$ and $e^2_A \to 0$) to a (fully scalarized black hole, with $\beta^0_A \to 0$ and $e^2_A \to 1 + a^2$) when the scalar cosmological background φ_0 increases.

These previously obtained results being recalled, we now proceed to calculate the gravitational waves emitted by EMD compact binary systems. In order to describe the shrinking of the orbit due to gravitational radiation, we first compute the energy flux radiated away by the system:

$$-\frac{d\mathcal{E}}{dt} = \mathcal{F}_g + \mathcal{F}_A + \mathcal{F}_\varphi\,, \tag{10}$$

$$\text{where}\quad \mathcal{E} = \int d^3x\,|g|\left(T^{00}_{(\varphi)} + T^{00}_{(A)} + T^{00}_{(\mathrm{m})} + t^{00}_{\mathrm{LL}}\right), \tag{11a}$$

$$\mathcal{F}_g = \int_{x\to\infty} t^{0i}_{\mathrm{LL}}\, n_i x^2 d\Omega, \quad \mathcal{F}_A = \int_{x\to\infty} T^{0i}_{(A)}\, n_i x^2 d\Omega, \quad \mathcal{F}_\varphi = \int_{x\to\infty} T^{0i}_{(\varphi)}\, n_i x^2 d\Omega\,, \tag{11b}$$

with $n^i = x^i/x$, x^i being the distance of the observation point from the source, and $d\Omega = \sin\theta\, d\theta d\phi$. $\mathcal{F}_g$ is the well-known flux in general relativity, $t^{\mu\nu}_{\mathrm{LL}}$ being the Landau-Lifshitz pseudo-tensor [6], while $\mathcal{F}_A$ and F_φ are the extra "graviphotonic" and scalar fluxes.

The calculation of the fluxes (11b) is standard but a bit heavy, and will be detailed elsewhere (it is an extension of the general relativistic text-book calculation, see [7]). The result, which is presented in [8], can be decomposed as follows:

The metric flux reduces, at leading order (that is 0PK), to that of Einstein's second quadrupole formula, but dressed up by the scalar and "graviphotonic" contributions; that is, for circular orbits and in the center-of-mass frame:

$$\mathcal{F}_g = \frac{32}{5}\frac{\nu^2\left(G_{AB}M\dot\phi\right)^{10/3}}{G_*\left(1+\alpha^0_A\alpha^0_B - e_Ae_B\right)^2} + \cdots\,, \tag{12}$$

where we recall that $\dot\phi = d\phi/dt$ is the (gauge invariant) orbital angular velocity, and that $(G_{AB}M\dot\phi)^{2/3} = \mathcal{O}(V^2)$.

The "graviphotonic" flux is dominated by a dipolar (-1PK) term. In order to determine its next-to-leading (0PK) contributions, one has to iterate (3b) to take into account the couplings to the metric and scalar field. We found:

$$\begin{aligned}\mathcal{F}_A = \frac{\nu^2\left(G_{AB}M\dot\phi\right)^{8/3}}{G_*\left(1+\alpha^0_A\alpha^0_B - e_Ae_B\right)^2}&\left\{\frac{2}{3}(e_A-e_B)^2\right.\\ &+\left(G_{AB}M\dot\phi\right)^{2/3}\left[\frac{8}{5}\left(\frac{m^0_Ae_B + m^0_Be_A}{M}\right)^2 + \frac{4}{9}(e_A-e_B)^2\left(\nu - 3 - \bar\gamma_{AB} - 2\langle\bar\beta\rangle\right)\right.\\ &\left.\left.+\,4(e_A-e_B)\left(\frac{(m^0_A)^2e_B - (m^0_B)^2e_A}{15M^2} - \frac{e_A(1+a\,\alpha^0_B) - e_B(1+a\,\alpha^0_A)}{3M(1+\alpha^0_A\alpha^0_B - e_Ae_B)}\right)\right] + \cdots\right\},\end{aligned} \tag{13}$$

where $\bar\gamma_{AB}$ and $\langle\bar\beta\rangle$ are given in (7) and seq. Note that when the scalar field is switched off, this formula gives the electric flux emitted by charged systems such as Reissner-Nordström binary black holes (which, in itself, is also a new result at this order). Note also that when

$e_A = e_B$, the dipolar contribution disappears and $\mathcal{F}_A$ identifies to the maxwellian quadrupolar flux, the charges being dressed up by the value of the background scalar field.

Finally, the scalar flux is *a priori* monopolar (-2PK), see, e.g., [9]. However, for circular orbits to which we restrict ourselves here, it is dipolar (-1PK), and its 0PK part is obtained by iterating (3c), to take into account the coupling to the metric and to the "graviphoton". It is given by:

$$\begin{aligned}
\mathcal{F}_\varphi = \; & \frac{\nu^2 \left(G_{AB} M \dot\phi\right)^{8/3}}{G_* \left(1+\alpha_A^0\alpha_B^0 - e_A e_B\right)^2} \left\{ \frac{1}{3}(\alpha_A^0-\alpha_B^0)^2 \right. \\
& + \left(G_{AB} M \dot\phi\right)^{2/3} \left[\frac{16}{15}\left(\frac{m_A^0\alpha_B^0 + m_B^0\alpha_A^0}{M}\right)^2 + \frac{2}{9}(\alpha_A^0-\alpha_B^0)^2\left(\nu - 3 - \bar\gamma_{AB} - 2\langle\bar\beta\rangle\right) \right. \\
& \left.\left. + 2(\alpha_A^0-\alpha_B^0)\left(\frac{(m_A^0)^2\alpha_B^0 - (m_B^0)^2\alpha_A^0}{5M^2} + \frac{m_A^0\left[\alpha_B^0 + \alpha_A^0(\alpha_B^0)^2 + \beta_B^0\alpha_A^0 - a\, e_A e_B\right] - (A\leftrightarrow B)}{3M(1+\alpha_A^0\alpha_B^0 - e_A e_B)}\right)\right] + \cdots \right\}.
\end{aligned} \tag{14}$$

When $\alpha_A^0 = \alpha_B^0$, this flux is reduced to its purely (scalar) quadrupolar term. This expression generalizes that of Damour and Esposito-Farèse in scalar-tensor theories [9], which is recovered when $e_{A/B} = 0$; note however that the EMD flux cannot be deduced from [9] by a mere generalization of G_{AB} to include $e_{A/B}$, due to the presence of a in the last term of (14).

From these fluxes, we can then determine the characteristics of the "chirp", i.e., of the evolution $\ddot\phi$ of the orbital velocity $\dot\phi$ at 0PK order. To do so, one assumes that the energy $\mathcal{E}$, given in (11a) and whose radiative decay is given in (10), is equal to the mechanical energy of the system, E, given in (8). This yields:

$$\begin{aligned}
\ddot\phi = \; & G_* \mu\, \dot\phi^3 \left\{ \left[2(e_A-e_B)^2 + (\alpha_A^0-\alpha_B^0)^2 \right] \right. \\
& + \left(G_{AB} M \dot\phi\right)^{2/3} \left[\frac{96}{5} + \frac{24}{5}\left(\frac{m_A^0 e_B + m_B^0 e_A}{M}\right)^2 + \frac{16}{5}\left(\frac{m_A^0\alpha_B^0 + m_B^0\alpha_A^0}{M}\right)^2 \right. \\
& + \left(2(e_A-e_B)^2 + (\alpha_A^0-\alpha_B^0)^2\right)\left(\frac{5\nu}{6} - \frac{1}{2} + \frac{2}{3}\bar\gamma_{AB} - \frac{8}{3}\langle\bar\beta\rangle\right) \\
& + 4(e_A-e_B)\left(\frac{(m_A^0)^2 e_B - (m_B^0)^2 e_A}{5M^2} - \frac{e_A(1+a\,\alpha_B^0) - e_B(1+a\,\alpha_A^0)}{M(1+\alpha_A^0\alpha_B^0 - e_A e_B)}\right) \\
& \left.\left. + 2(\alpha_A^0-\alpha_B^0)\left(\frac{3}{5}\frac{(m_A^0)^2\alpha_B^0 - (m_B^0)^2\alpha_A^0}{M^2} + \frac{m_A^0\left[\alpha_B^0 + \alpha_A^0(\alpha_B^0)^2 + \beta_B^0\alpha_A^0 - a\, e_A e_B\right] - (A\leftrightarrow B)}{M(1+\alpha_A^0\alpha_B^0 - e_A e_B)}\right)\right] + \cdots \right\}.
\end{aligned} \tag{15}$$

In the general relativistic limit ($e_{A/B} = 0$, $m_{A/B}(\varphi) = \text{cst}$), equation (15) is reduced to the well-known expression $\ddot\phi \propto \dot\phi^{11/3}$:

$$\ddot\phi = \frac{96}{5}(G_*\mathcal{M})^{5/3}\dot\phi^{11/3}\,, \quad \text{where} \quad \mathcal{M} = \nu^{3/5} M \tag{16}$$

is the "chirp mass". By contrast, in presence of dipolar radiation, the right-hand side of (15) is dominated, during early inspiral, by its first term, $\ddot\phi \propto \dot\phi^3$:

$$\ddot\phi = (G_*\mathcal{M}_\mathcal{D})\dot\phi^3 \quad \text{with} \quad \mathcal{M}_\mathcal{D} = \nu\left[2(e_A-e_B)^2 + (\alpha_A^0-\alpha_B^0)^2\right]M\,, \tag{17}$$

which governs the motion of a binary system when, for example, one of its components is a scalarized EMD black hole described above. Finally, when the dipoles are negligible

($\alpha_A^0 \simeq \alpha_B^0$ and $e_A \simeq e_B$), we have again $\ddot{\phi} \propto \dot{\phi}^{11/3}$, but the general relativistic "chirp mass" is now dressed up by the vector and scalar quadrupoles:

$$\ddot{\phi} = \frac{96}{5}(G_* \mathcal{M}_\mathcal{Q})^{5/3}\dot{\phi}^{11/3} \quad \text{with} \tag{18}$$
$$\mathcal{M}_\mathcal{Q} = \nu^{3/5}(1+\alpha_A^0\alpha_B^0 - e_Ae_B)^{2/5}\left[1+\frac{1}{4}\left(\frac{m_A^0 e_B + m_B^0 e_A}{M}\right)^2 + \frac{1}{6}\left(\frac{m_A^0\alpha_B^0 + m_B^0\alpha_A^0}{M}\right)^2\right]^{3/5} M\,.$$

As we shall see below, the angular velocity $\dot{\phi}$ is proportional to the frequency of the gravitational wave at infinity, and is hence an observable. The results above therefore show that in presence of significant dipolar radiation, cf. (17), the evolution of the frequency f will deviate from the general relativistic predictions. In contrast, when dipolar radiation is absent, the "chirp" is reduced to (18), and the deviations from general relativity can, at this order, be absorbed in a redefinition of the masses.

Having in hand the fluxes (12)–(14), one can also deduce the radiation reaction force exerted on the system by equating its power to the energy fluxes at infinity [10]. For the quasi-circular orbits considered here, this force is tangent to the trajectory and reads

$$F_\phi = -\frac{\mathcal{F}_g + \mathcal{F}_A + \mathcal{F}_\varphi}{\dot{\phi}}\,. \tag{19}$$

The equations of motion, deduced from the 1PK Lagrangian computed in [1] and recalled above, can hence be generalized to encompass radiation reaction effects as:

$$\frac{d}{dt}\left(\frac{\partial L_{1PK}}{\partial \dot{R}}\right) - \frac{\partial L_{1PK}}{\partial R} = 0\,, \quad \frac{d}{dt}\left(\frac{\partial L_{1PK}}{\partial \dot{\phi}}\right) - \frac{\partial L_{1PK}}{\partial \phi} = F_\phi\,. \tag{20}$$

Now, the range of validity of these equations of motion can be extended, hopefully up to merger, by resumming them. To do so, we shall start, not from the 1PK Lagrangian, but rather, from the scalar-tensor effective-one-body (EOB) Hamiltonian presented in [11]: indeed, it is trivially generalized to EMD theories at 1PK order, as we showed in [1], since the Lagrangian (6) has exactly the same structure as that of scalar-tensor theories at this order. In this EOB approach, the equations of motion (20) are replaced by

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r} \quad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r}\,, \quad \dot{\phi} = \frac{\partial H_{\rm EOB}}{\partial p_\phi} \quad \dot{p}_\phi = -\frac{\partial H_{\rm EOB}}{\partial \phi} + F_\phi\,, \tag{21}$$

$$\text{with} \quad H_{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_e}{\mu}-1\right)}\,, \quad \text{where} \quad H_e = \mu\sqrt{A\left(1+\frac{p_r^2}{\mu^2 B}+\frac{p_\phi^2}{\mu^2 r^2}\right)}\,, \tag{22}$$

$$\text{and} \quad A(r) = \mathcal{P}_1^1\left[1-2\left(\frac{G_{AB}M}{r}\right)+2\Big[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\Big]\left(\frac{G_{AB}M}{r}\right)^2\right]\,, \tag{23a}$$
$$B(r) = 1+2\Big[1+\bar{\gamma}_{AB}\Big]\left(\frac{G_{AB}M}{r}\right)\,, \tag{23b}$$

where $\mathcal{P}_1^1$ denotes the Padé approximant of order $(1,1)$, with respect to the variable $u = (G_{AB}M/r)$, and where $(r,\phi;p_r,p_\phi)$ are the effective phase space coordinates introduced

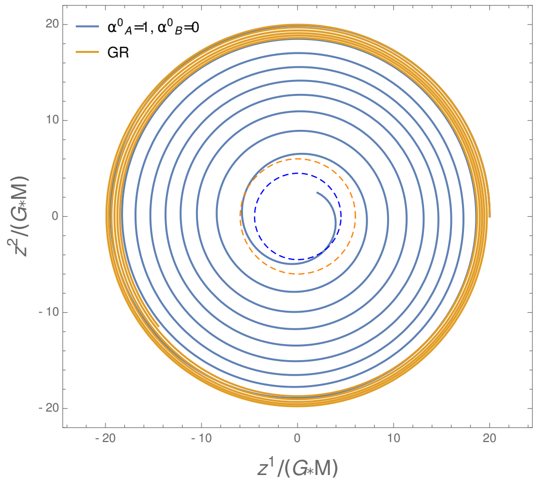

Figure 1. Effective trajectories ($z^1 = r\cos\phi$, $z^2 = r\sin\phi$) for a binary system composed of two EMD black holes with identical masses ($\nu = 1/4$), in the theory $a = 1$. In blue, one of the black holes is scalarized ($\alpha^0_A = 1$, $e_A = \sqrt{2}$), while the second one is Schwarzschild's ($\alpha^0_B = 0$, $e_B = 0$); in orange, the general relativistic case of two Schwarzschild black holes. The corresponding ISCOs are also shown (dashed lines). Starting from $r = 20\,G_*M$, the scalarized system reaches its ISCO within $\Delta t \simeq 3000\,G_*M$, to be compared to $\Delta t \simeq 10\,000\,G_*M$ in general relativity. Note that we have $(\dot r/r\dot\phi)^2_{\rm ISCO} = 0.019$, hence justifying the validity of our quasi-circular approximation, at least up to the ISCO.

in [11], such that ϕ identifies, for circular orbits, to the (observable) orbital phase used above. When considered as exact, these equations can be numerically integrated to yield the evolution of the binary system, e.g., up to the innermost stable circular orbit $u_{\rm ISCO}$, which is defined as the (outermost) solution of $A''/A' = (Au^2)''/(Au^2)'$, a prime denoting a derivative with respect to u. For a detailed study of the ISCO and its characteristics, including, e.g., its location and the associated orbital frequency in scalar-tensor and EMD theories, see, again, [11]. As for the initial conditions, they are determined exactly as in general relativity, see [10].

Figure 1 above shows the illustrative example of two EMD black holes for the theory $a = 1$, the first one being scalarized, while the second one is a Schwarzschild black hole. As expected, a strong dipolar (i.e. "graviphotonic" and scalar) radiation is driving the radius of the orbit to decrease at a much greater rate than in general relativity.

The trajectories being known and pushed up to the ISCO, one easily predicts, at leading order, the associated waveforms.

Although we can compute the vector component of the waveform, this is not necessary, supposing that the detectors are "gravielectrically" neutral. Rather, the mirrors of the interferometers follow the geodesics of the Jordan metric, see (1) and seq., which reads, in the solar system:

$$\tilde g_{\mu\nu} = \mathcal{A}^2(\varphi) g_{\mu\nu} = \mathcal{A}^2_\odot \left[\eta_{\mu\nu}(1 + 2\alpha_\odot\delta\varphi) + h^{TT}_{\mu\nu}\right] + \mathcal{O}\left(\frac{1}{x^2}\right) , \tag{24}$$

where x is the distance to the source, $\mathcal{A}_\odot = \mathcal{A}(\varphi_\odot)$ is the value of the coupling of matter to the scalar field in the solar system, and where $\alpha_\odot = (d\ln\mathcal{A}/d\varphi)(\varphi_\odot)$. As for $\delta\varphi = \varphi - \varphi_\odot$, it is the scalar wave, given at leading order by

$$\delta\varphi = -G_* \frac{n_i \dot{\mathcal{D}}^i_S}{x} \quad \text{with} \quad \mathcal{D}^i_S = \sum_A m^0_A \alpha^0_A x^i_A \,, \tag{25}$$

$n_i \dot{\mathcal{D}}^i_S$ being the projection of the time derivative of the (scalar) dipole $\mathcal{D}^i_S$ of the source on the line of sight $n^i = x^i/x$, and where $x^i_{A/B}$ denote the (harmonic) position of the bodies in their center-of-mass frame. Finally, the useful components of $h^{TT}_{\mu\nu}$ are

$$h^{TT}_{ij} = \frac{2G_*}{3} \frac{\mathcal{P}_{ij}{}^{kl} \ddot{\mathcal{Q}}_{kl}}{x} \quad \text{where} \quad \mathcal{Q}^{ij} = \sum_A m^0_A \left(3x^i_A x^j_A - \delta^{ij} x^2_A\right) , \tag{26}$$

$\mathcal{P}_{ij}{}^{kl} \ddot{\mathcal{Q}}_{kl}$ being the transverse (to n^i) and traceless part of the second time derivative of the (mass) quadrupole of the system.

When the distance between the mirrors (initially at rest) of the detector is small compared to the wavelenght of the gravitational wave, their separation ξ^i obeys [9]:

$$\frac{d^2\xi^i}{dt^2} \simeq -\tilde{R}^i{}_{0j0}\, \xi^j \quad \text{with} \quad \tilde{R}^i{}_{0j0} = -\mathcal{A}^2_\odot \left[\frac{1}{2}\ddot{h}^{TT}_{ij} + \alpha_\odot \delta\ddot{\varphi}\,(\delta_{ij} - n_i n_j)\right] + \mathcal{O}\left(\frac{1}{x^2}\right) , \tag{27}$$

where $\tilde{R}^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor associated to (24). Note that although the new scalar, conformal but transverse ("breathing") polarization in (27) cannot be disentangled from the tensor ones for the time being, it will when a third non-aligned detector (e.g., KAGRA) joins the LIGO-Virgo network, see, e.g., [12].

We note that $\delta\varphi$ is of -0.5 PK order relative to h^{TT}_{ij}. However, the contribution of $\delta\varphi$ to the wave (24) is numerically lowered by the factor $\alpha_\odot$, which is already constrained by $\alpha_\odot \lesssim 10^{-2}$ from solar system observations, see, e.g., [13]. It is hence not necessary to compute the 0PK corrections to $\delta\varphi$. Let us however emphasize that the quantities $\alpha^0_{A/B}$ appearing in (25) are evaluated on the cosmological environment of the sources, φ_0; and, for scalarized black holes, they can numerically reach the order of unity.

At the order considered here, $\delta\varphi$ and h^{TT}_{ij} are computed using Kepler's laws for circular orbits, and can be expressed as functions of the angular velocity $\dot{\phi}$ only to yield:

$$\delta\varphi = (\alpha^0_A - \alpha^0_B) \frac{G_* M\nu}{x} (\sin i) \left(G_{AB} M \dot{\phi}\right)^{1/3} \cos\phi \,, \tag{28a}$$

$$h_+ = \frac{4G_* M\nu}{x} \left(\frac{1+\cos^2 i}{2}\right) \left(G_{AB} M \dot{\phi}\right)^{2/3} \cos(2\phi) \,, \tag{28b}$$

$$h_\times = \frac{4G_* M\nu}{x} (\cos i) \left(G_{AB} M \dot{\phi}\right)^{2/3} \sin(2\phi) \,, \tag{28c}$$

where i denotes the angle between the normal to the orbital plane and the line of sight. The last step consists in injecting in (28) the trajectory $\phi(t)$ obtained when integrating (21). The waveforms $\delta\varphi$ and h^{TT}_{ij} are thereby known up to the ISCO. Figure 2 gives those associated to the trajectories shown in figure 1.

Finally, we note that for the Jordan metric (24) to reduce to that of Minkowski in the absence of gravitational waves, one has to perform the local coordinate change $d\tilde{t} = \mathcal{A}_\odot dt$, $d\tilde{x}^i = \mathcal{A}_\odot dx^i$. Therefore, the observed frequency of the signal (28b), (28c) in the solar system, defined by $2\pi f \equiv 2d\phi/d\tilde{t}$, is given by $f = \dot{\phi}/(\pi\mathcal{A}_\odot)$.

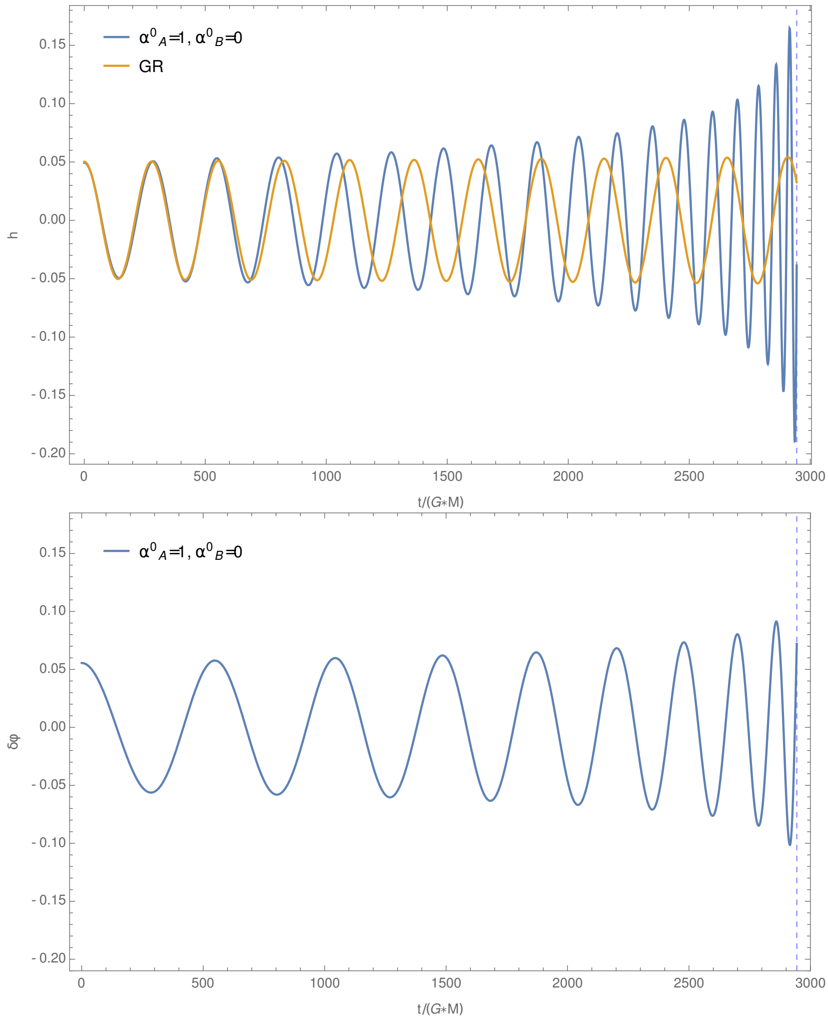

Figure 2. Gravitational waveforms associated to the trajectories ploted in figure 1, for the theory $a = 1$. On the top panel, $h = (G_{AB}M\dot{\phi})^{2/3}\cos(2\phi)$; on the bottom panel, $\delta\varphi = (1/4)(\alpha_A^0 - \alpha_B^0)(G_{AB}M\dot{\phi})^{1/3}\cos(\phi)$. The scalar waveform amplitude is, in this example, numerically comparable to the tensor one; however its contribution to (24) is numerically lowered by the solar system factor $\alpha_\odot$. The moment at which the system crosses the ISCO is represented by vertical dashed lines.

Beyond the ISCO, the binary system plunges to form a final black hole. The question of its characteristics and of the waveforms associated to its quasi-normal modes, to which our EOB waveforms will be matched, is left to future work.

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Note added in proof. The new results concerning the fluxes and waveforms presented in this paper represent part of my Ph.D. thesis [8]. While I was about to complete this paper, Khalil et al. submitted partly similar results in [14].

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Troisième partie

Charges nœtheriennes et thermodynamique des trous noirs

Chapitre 6

Charges globales et thermodynamique des trous noirs chevelus

Dans le chapitre 4, nous avons étudié le problème de deux trous noirs en interaction en théorie EMD, en décrivant leurs trajectoires par les lignes d'univers de particules munies de sensibilités $m_A(\varphi)$. Nous avons ensuite montré que l'expression spécifique de ces sensibilités apporte des indications sur la façon dont un trou noir se réajuste lorsque, en présence d'un compagnon lointain orbitant autour de lui, son environnement scalaire varie.

Dans cette troisième partie de thèse, nous allons apporter un nouvel éclairage sur la description des trous noirs issue de cette "skeletonisation", à l'aide des lois de leur thermodynamique.

Pour ce faire, notre premier objectif, auquel est dédié ce chapitre, est de montrer comment établir lesdites lois en théories EMD. En effet, si la prise en compte d'un "graviphoton", e.g., dans la première loi de la thermodynamique, est un exercice aisé, elle est plus subtile pour un champ scalaire qui, n'étant pas un champ de jauge, ne permet pas de définir de "charge scalaire" associée.

Nous reproduisons ci-dessous le premier article produit durant cette thèse, et réalisé en collaboration avec A. Anabalón et N. Deruelle [187]. On y propose d'étendre la "technologie" des superpotentiels de Katz, présentée en introduction 0.4, pour inclure la contribution d'un champ scalaire dans la définition des charges nœtheriennes d'un espace-temps (en l'occurrence, la masse).

Pour ce faire, on considère une classe spécifique de trous noirs statiques, à symétrie sphérique, et asymptotiquement anti de Sitter (AdS). Si un tel comportement asymptotique est, bien entendu, d'un intérêt premier dans le contexte de la correspondance AdS-CFT [144, 145] que nous ne discutons pas ici, il sera très instructif dans le cadre de cette thèse : un trou noir-AdS constitue un exemple techniquement complet, permettant de démontrer, comme nous le verrons plus bas, les vertus régularisatrices de la "technologie" katzienne, et sera aisément transposé au chapitre suivant à nos trous noirs asymptotiquement plats (4.1), plus simples.

Plus précisément, on considère l'action Einstein-Maxwell-dilaton-Katz définie par [1] :

$$16\pi\, I = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{U(\phi)}{\ell^2} - \frac{1}{4}A(\phi)F^{\mu\nu}F_{\mu\nu}\right) \qquad (6.1)$$
$$- \int d^4x\sqrt{-\bar{g}}\left(\bar{R} + \frac{6}{\ell^2}\right) + \int d^4x\,\partial_\mu(\hat{k}^\mu_{\mathrm{K}} + \hat{k}^\mu_{\mathrm{S}})\,,$$

1. Nous adoptons ici les conventions de l'article reproduit ci-dessous ; notons qu'elles diffèrent des notations utilisées dans le reste de ce manuscrit, que l'on retrouve via les substitutions $\phi \to 2\varphi$, $A^\mu \to 2A^\mu$ et $f \to f/2$.

où ℓ a la dimension d'une longueur, $A(\phi) = 1 + \mathcal{O}(\phi)$ pour $\phi \ll 1$, à l'instar des théories EMD, tandis que l'ajout du potentiel $U(\phi) = -6 - \phi^2 + \mathcal{O}(\phi^4)$ implique les comportements asymptotiquement AdS de la métrique, ainsi que logarithmique ("longue portée") du champ scalaire. Enfin, $\bar{g}_{\mu\nu}$ désigne la métrique d'un espace-temps de référence globalement AdS (non dynamique), de courbure $\bar{R} = -12\ell^{-2}$.

Le dernier terme de (6.1) est un terme de bord, ne contribuant pas aux équations du champ, mais dont le rôle sera primordial dans la suite. Il dépend du vecteur de Katz k^μ_{K} (chapeau désignant la multiplication par $\sqrt{-g}$), donné, e.g., en [188] et en introduction, cf. (32), que l'on propose de généraliser par l'ajout d'une contribution scalaire, k^μ_{S}, de la forme générique :

$$k^\mu_{\mathrm{S}} = f(\phi)\partial^\mu\phi \qquad \text{avec} \qquad f(\phi) = A + B\phi + \frac{1}{2}C\phi^2 + \mathcal{O}(\phi^3)\,, \tag{6.2}$$

où les termes d'ordre $\mathcal{O}(\phi^3)$ ne contribueront pas aux calculs ci-dessous, et où A, B et C sont des constantes à déterminer.

Dans l'article ci-dessous, on considère la classe des solutions générales, statiques et à symétrie sphérique des équations du champ associées à (6.1), i.e. d'Einstein, Maxwell et Klein-Gordon. Leur comportement à l'infini peut s'obtenir par itération des équations du champ, et s'écrit, en coordonnées de Droste :

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{\tilde{h}(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{6.3}$$

$$\text{où} \qquad h(r) = \ell^{-2}r^2 + 1 - \frac{2m_g}{r} + \mathcal{O}(1/r^2)\,, \quad \tilde{h}(r) = \ell^{-2}r^2 + 1 + \alpha^2 - \frac{2m_i}{r} + \mathcal{O}(1/r^2)\,,$$

$$\text{et} \qquad A_t = -\frac{4Q}{r} + \mathcal{O}(1/r^2)\,, \quad A_i = 0\,, \quad \phi(r) = \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + \mathcal{O}(1/r^3)\,,$$

$$\text{avec} \qquad \alpha = \frac{\phi_1}{2\ell}\,, \qquad m_i = m_g - \frac{\phi_1\phi_2}{3\ell^2}\,.$$

La solution (6.3) dépend de quatre constantes d'intégration : m_g, Q, ϕ_1, ϕ_2 (et même cinq en présence d'un charge magnétique P, cf. l'article ci-dessous), et son expression à cet ordre suffit pour déterminer tout ordre $1/r$ suivant. Notons que (6.3) inclut, e.g., le comportement asymtotique des trous noirs présentés en [189, 190].

Lorsque les équations du champ sont satisfaites, la variation de l'action (6.1) sous les déformations $\delta g^{\mu\nu}$, δA_μ et $\delta\phi$ s'écrit sous forme d'un terme de bord à l'infini, auquel les vecteurs de Katz contribuent, et que l'on peut évaluer connaissant (6.3). Si l'on impose alors que cette variation de l'action "on shell", i.e., consécutive à une variation des constantes d'intégration de (6.3), s'annule pour la classe de solutions la plus générale possible, on trouve que :

(i) les constantes A et B dans (6.2) doivent être fixées à $A = 0$ et $B = 1/2$, soit

$$f(\phi) = \frac{\phi}{2}(1 + C\phi) + \mathcal{O}(\phi^3)\,; \tag{6.4}$$

(ii) les constantes d'intégration ϕ_1 et ϕ_2 dans (6.3) doivent être reliées par la relation

$$\phi_2 = -3C\phi_1^2 + D\ell\phi_1\,, \tag{6.5}$$

où C est la constante apparaissant dans la définition de k^μ_{S}, et où D est une constante. Ainsi, notre principe variationnel impose la forme spécifique (6.4) du vecteur de Katz scalaire (où seule la constante C reste libre), et restreint les solutions à la sous-famille (6.5).

Ainsi équipés de l'action (6.1) avec (6.2, 6.4), le calcul des charges nœtheriennes d'un espace-temps est aisé. En effet, le superpotentiel de Katz-Bicak-Lynden-Bell [188] (donné en (35)), à présent muni d'une contribution scalaire,

$$8\pi\hat{J}^{[\mu\nu]} = \nabla^{[\mu}\hat{\xi}^{\nu]} - \overline{\nabla^{[\mu}\hat{\xi}^{\nu]}} + \xi^{[\mu}\hat{k}_K^{\nu]} + \xi^{[\mu}\hat{k}_S^{\nu]}\,, \tag{6.6}$$

permet de définir la masse de la famille de solutions (6.3, 6.5), de vecteur de Killing $\xi^\mu = \delta^\mu_t$, comme :

$$M = -\lim_{r\to\infty}\int d\theta d\varphi\, \hat{J}^{[0r]} = m_g + D\frac{\phi_1^2}{24\ell}\,. \tag{6.7}$$

Notons la finitude remarquable de ce résultat, malgré le comportement asymptotiquement AdS de la métrique, qui est assurée par les trois derniers termes du membre de droite de (6.6). Insistons, même, sur le rôle crucial du vecteur k_S^μ dans la définition de la masse (6.7) : non seulement il y ajoute une contribution du champ scalaire, comme attendu, mais on peut aussi facilement vérifier que sa forme spécifique, avec A et B donnés en (6.4), est la seule menant à une masse finie, confirmant ainsi la pertinence du principe variationnel adopté pour obtenir (6.4) et (6.5).

Mais la portée de l'article ci-dessous s'étend au-delà de la définition des charges nœtheriennes d'un espace-temps : une fois munis de notre masse (6.7), on montre que, remarquablement, les lois de la thermodynamique à l'équilibre des trous noirs sont automatiquement satisfaites.

En calculant par exemple l'action (6.1) "on shell", on trouve que, non seulement, elle est finie (la métrique $\bar{g}_{\mu\nu}$ de référence ainsi que les vecteurs de Katz jouant là encore leur rôle de "contre-termes"), mais elle satisfait aussi la relation de Gibbs [191, 192], pour la sous-famille de trous noirs vérifiant (6.5) uniquement :

$$I_{\text{onshell}} = S - \beta(M - Q\Phi_Q)\,, \tag{6.8}$$

où S désigne l'entropie du trou noir, β l'inverse de sa température, Φ_Q le potentiel électrique évalué sur son horizon, et où apparaît précisément notre masse M telle que définie en (6.7).

De même on montre sur un exemple spécifique de trou noir [190] que la sous-classe (6.5) satisfait la première loi de la thermodynamique, lorsque la masse M est définie, encore une fois, par (6.7) :

$$\delta M = T\delta S + \Phi_Q\delta Q + \Phi_P\delta P\,, \tag{6.9}$$

P étant la charge magnétique et Φ_P le potentiel associé, évalué sur l'horizon.

Récapitulons : afin d'étendre les lois de la thermodynamique à un trou noir muni d'un "cheveu" scalaire, il nous a suffi de généraliser la définition de sa masse par une modification du vecteur de Katz. Un principe variationnel simple détermine alors la forme spécifique de ce vecteur, et sélectionne éventuellement une sous-famille de solutions des équations du champ satisfaisant aux lois de la thermodynamique (cf. (i) et (ii) plus haut). Comme nous le verrons au chapitre suivant, il suffira de transposer ces étapes aux trous noirs EMD asymptotiquement plats, plus simples, pour en décrire la thermodynamique.

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Einstein-Katz action, variational principle, Noether charges and the thermodynamics of AdS-black holes

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Abstract: In this paper we describe 4-dimensional gravity coupled to scalar and Maxwell fields by the Einstein-Katz action, that is, the covariant version of the "Gamma-Gamma − Gamma-Gamma" part of the Hilbert action supplemented by the divergence of a generalized "Katz vector". We consider static solutions of Einstein's equations, parametrized by some integration constants, which describe an ensemble of asymptotically AdS black holes. Instead of the usual Dirichlet boundary conditions, which aim at singling out a specific solution within the ensemble, we impose that the variation of the action vanishes on shell for the broadest possible class of solutions. We will see that, when a long-range scalar "hair" is present, only sub-families of the solutions can obey that criterion. The Katz-Bicak-Lynden-Bell ("KBL") superpotential built on this (generalized) vector will then give straightforwardly the Noether charges associated with the spacetime symmetries (that is, in the static case, the mass). Computing the action on shell, we will see next that the solutions which obey the imposed variational principle, and with Noether charges given by the KBL superpotential, satisfy the Gibbs relation, the Katz vectors playing the role of "counterterms". Finally, we show on the specific example of dyonic black holes that the sub-class selected by our variational principle satisfies the first law of thermodynamics when their mass is defined by the KBL superpotential.

Keywords: Black Holes, Classical Theories of Gravity, AdS-CFT Correspondence

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Contents

1 The Einstein-Katz action coupled to scalar and Maxwell fields

Let us consider the following ("Einstein frame") action I, functional of the fields $g^{\mu\nu}(x^\rho)$, $A_\mu(x^\rho)$ and $\phi(x^\rho)$, with greek indices running from 0 to 3 and coordinates $x^\rho \equiv \{t, r, \theta, \varphi\}$:

$$2\kappa\, I = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{U(\phi)}{\ell^2} - \frac{1}{4}A(\phi)F^2\right) - \int d^4x\sqrt{-\bar{g}}\left(\bar{R} + \frac{6}{\ell^2}\right) + \int d^4x\, \partial_\mu(\hat{k}^\mu_{\rm K} + \hat{k}^\mu_{\rm S})\,. \tag{1.1}$$

Here $\kappa \equiv 8\pi$ (we have set $G = c = 1$); ℓ has the dimension of a length; $d^4x \equiv dt\, dr\, d\theta\, d\varphi$; R is the scalar curvature of the metric $g_{\mu\nu}$, with inverse $g^{\mu\nu}$, determinant g and signature $(-,+,+,+)$; $(\partial\phi)^2 \equiv \partial^\mu\phi\,\partial_\mu\phi \equiv g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$; $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. To keep our presentation simple, we will assume (for reasons explained later) that the expansions for small ϕ of the potential $U(\phi)$ and the coupling function $A(\phi)$ are

$$U(\phi) = -6 - \phi^2 + \mathcal{O}(\phi^4) \quad , \quad A(\phi) = 1 + \mathcal{O}(\phi)\,. \tag{1.2}$$

In the second term of (1.1) $\bar{g}_{\mu\nu}$ and $\bar{R} = -12\ell^{-2}$ are the metric and scalar curvature of a regularizing anti-de Sitter (AdS) background and all overlined quantities are built with $\bar{g}_{\mu\nu}$. That background is chosen in order that the total action vanishes for a global AdS spacetime, that is when $g_{\mu\nu} = \bar{g}_{\mu\nu}$, $\phi = 0$, $A_\mu = 0$.

The last term is the integral of a divergence and will not contribute to the field equations (a hat means multiplication by $\sqrt{-g}$ as in $\hat{k}^\mu \equiv \sqrt{-g}\, k^\mu$); $k^\mu_{\rm K}$ is the "Katz vector" (see [1] and, e.g., [2], for short reviews of its virtues):

$$k^\mu_{\rm K} \equiv -(g^{\nu\rho}\Delta^\mu_{\nu\rho} - g^{\mu\nu}\Delta^\rho_{\nu\rho}) \quad \text{where} \quad \Delta^\mu_{\nu\rho} \equiv \Gamma^\mu_{\nu\rho} - \overline{\Gamma}^\mu_{\nu\rho}\,, \tag{1.3}$$

$\Gamma^\mu_{\nu\rho}$ and $\bar\Gamma^\mu_{\nu\rho}$ being respectively the Christoffel symbols associated to the dynamical metric $g_{\mu\nu}$ and the background AdS metric $\bar g_{\mu\nu}$. Again, this vector vanishes when $g_{\mu\nu} = \bar g_{\mu\nu}$.[1] The divergence of the vector $k^\mu_{\rm S}$ is a new, "scalar", contribution to the action:

$$k^\mu_{\rm S} \equiv f(\phi)\partial^\mu\phi \tag{1.4}$$

where the appropriate function $f(\phi)$ will turn out to be

$$f(\phi) = \frac{\phi}{2}(1 + C\phi) \tag{1.5}$$

with C a dimensionless constant (which may depend on the parameters entering the potential $U(\phi)$).[2]

2 The conditions for an extremal action

The variation of the action (1.1) under a deformation of the metric, $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$ is:

$$2\kappa\,\delta_{(g)}I = \int d^4x\sqrt{-g}\left(G_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu} + \int d^4x\,\partial_\mu \hat V^\mu_{(g)} \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, where $T_{\mu\nu}$ is the stress-energy tensor of the scalar and Maxwell fields,

$$T_{\mu\nu} \equiv \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 + \ell^{-2}U(\phi)\right) + A(\phi)\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^2\right), \tag{2.2}$$

and where

$$\hat V^\mu_{(g)} \equiv \Delta^\rho_{\nu\rho}\delta\hat g^{\mu\nu} - \Delta^\mu_{\nu\rho}\delta\hat g^{\nu\rho} + f\partial_\nu\phi\,\delta\hat g^{\mu\nu}\,. \tag{2.3}$$

(The role of the Katz-vector $k^\mu_{\rm K}$ is to eliminate all terms in $\delta(\partial_\rho g_{\mu\nu})$ generated by the variation of R, see [1].)

Similarly the variation of I under a deformation of the scalar field, $\phi \to \phi + \delta\phi$, is

$$\begin{aligned} 2\kappa\,\delta_{(\phi)}I &= \int d^4x\sqrt{-g}\left(\Box\,\phi - \frac{1}{\ell^2}\frac{dU}{d\phi} - \frac{1}{4}\frac{dA}{d\phi}F^2\right)\delta\phi + \int d^4x\,\partial_\mu\hat V^\mu_{(\phi)} \\ \text{with } \hat V^\mu_{(\phi)} &= \hat\partial^\mu\phi\left(\frac{df}{d\phi} - 1\right)\delta\phi + f\hat g^{\mu\nu}\delta(\partial_\nu\phi)\,. \end{aligned} \tag{2.4}$$

[1]The gravity part of the action can be rewritten as (see, e.g. [2])

$$\int d^4x\,(\hat R - \overline{\hat R} + \partial_\mu\hat k^\mu_{\rm K}) = \int d^4x\,[\hat g^{\mu\rho}(\Delta^\lambda_{\mu\sigma}\Delta^\sigma_{\rho\lambda} - \Delta^\sigma_{\mu\rho}\Delta^\lambda_{\sigma\lambda}) + (\hat g^{\mu\nu} - \overline{\hat g^{\mu\nu}})\bar R_{\mu\nu}]\,,$$

hence its name "Einstein-Katz" action. The Katz vector is closely related to the Gibbons-Hawking-York boundary term: in Gaussian normal coordinates where the metric reads $ds^2 = \epsilon dw^2 + \gamma_{ij}dx^i dx^j$ with γ the determinant of the induced metric γ_{ij}, the GHY boundary term is $2\epsilon\int K\sqrt{|\gamma|}d^3x$ where $K \equiv \gamma^{ij}K_{ij}$ is the trace of the extrinsic curvature $K_{ij} = \frac{1}{2}\partial_w\gamma_{ij}$, whereas the Katz boundary term reads $\int k^w\sqrt{|\gamma|}d^3x$ with $k^w = 2\epsilon\big[K - \frac{1}{2}\bar K_{ij}(h^{ij} + \bar h^{ij})\big]$.

[2]Other divergences of vectors involving A_μ can be added to the Lagrangian, such as $\partial_\mu(B(\phi)\hat F^{\mu\nu}A_\nu)$; they will not be required in the examples we shall consider below.

Finally the variation of I with respect to the gauge field A_μ is

$$2\kappa\,\delta_{(A)}I = \int d^4x\sqrt{-g}\,D_\mu\left(A(\phi)F^{\mu\nu}\right)\delta A_\nu + \int d^4x\,\partial_\mu \hat{V}^\mu_{(A)} \quad \text{with} \quad \hat{V}^\mu_{(A)} = -A(\phi)\hat{F}^{\mu\nu}\delta A_\nu\,. \tag{2.5}$$

As a first condition to extremize the action, we impose that the fields obey the Einstein, Klein-Gordon and Maxwell equations:

$$G_{\mu\nu} = \frac{1}{2}T_{\mu\nu} \quad , \quad \Box\phi - \frac{1}{\ell^2}\frac{dU}{d\phi} - \frac{1}{4}\frac{dA}{d\phi}F^2 = 0 \quad , \quad D_\mu\left(A(\phi)F^{\mu\nu}\right) = 0\,, \tag{2.6}$$

(the Klein-Gordon equation, say, following from the others thanks to the Bianchi identity).

A second condition to guarantee that the variation of the action vanishes is that the vector densities $\hat{V}^\mu$ be zero on the boundary, which is composed of 2 spacelike hypersurfaces Σ_1 and Σ_2 (whose equations, in adapted coordinates, are $x^0 = t_1$ and $x^0 = t_2$, with $t_2 - t_1 = \beta$), together with a timelike cylinder formed by the piling of the 2-spheres at infinity with equations $x^0 = const$, $x^1 = r \to \infty$.

A way to implement that is to impose Dirichlet conditions, that is that the fields are fixed on the boundary a priori: $\delta g^{\mu\nu} = 0$, $\delta\phi = 0$ and $\delta A_\mu = 0$ on the boundary. As in point-mechanics, such conditions are aimed at selecting a particular solution of the field equations, that is at fixing to arbitrary, but specific, values its integration constants (to wit the mass and charge parameters which play the role of initial and final "positions"). If such Dirichlet conditions are imposed, then the function $f(\phi)$ which appears in $\hat{V}^\mu_{(\phi)}$ in (2.4) must be absent and the boundary term in the action (1.1) then reduces to the Katz-vector.

Those are not the conditions we will choose here. Since, rather than specific, we want to consider *families* of (black hole) solutions in order to study their thermodynamics we shall impose that the vectorial densities $\hat{V}^\mu_{(g)}$, $\hat{V}^\mu_{(\phi)}$, $\hat{V}^\mu_{(A)}$ vanish on the boundary when evaluated on shell, that is upon variation of the integration constants that characterize the solutions (their mass, possibly angular momentum, gauge and scalar parameters).

As we shall see below in the case of static, asymptotically AdS, "hairy" black holes, the introduction of these Katz vectors together with the proposed variational principle will simplify some calculations and may shed a new light on the thermodynamics of AdS black holes.

3 Constraints on static, spherically symmetric and asymptotically AdS solutions imposed by the variational principle

Consider a static and spherically symmetric configuration of the fields. In Schwarzschild-Droste coordinates, the metric and fields read:

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{\tilde{h}(r)} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2), \tag{3.1}$$

$$\phi = \phi(r) \quad \text{and} \quad A_\mu = (-V(r), 0, 0, 4P\cos\theta)\,, \tag{3.2}$$

(where the A_ϕ component of A_μ, with P a constant with dimension of a length, allows for a magnetic charge P).

The Klein-Gordon and Maxwell equations, see (2.6), then read (a prime denoting derivation w.r.t. r)

$$\phi'\tilde{h}\left[\ln\left(\phi' r^2\sqrt{h\tilde{h}}\right)\right]' = \ell^{-2}\frac{dU}{d\phi} + \frac{1}{4}\frac{dA}{d\phi}F^2 \quad , \quad \left(r^2\sqrt{\tilde{h}/h}\,AV'\right)' = 0 \tag{3.3}$$

and the t-t, together with the difference between the t-t and r-r components of Einstein's equations close the system:

$$r\tilde{h}' + \tilde{h} - 1 + \frac{r^2}{4}\left[\tilde{h}\,\phi'^2 + \frac{2U}{\ell^2} + A(\phi)\left(\frac{16P^2}{r^4} + \frac{\tilde{h}}{h}(V')^2\right)\right] = 0 \quad , \quad \phi'^2 = \frac{2}{r}\left(\ln\frac{h}{\tilde{h}}\right)' . \tag{3.4}$$

Suppose now that $A(\phi) = 1 + \mathcal{O}(\phi)$ and $U(\phi) = -6 - \phi^2 + \mathcal{O}(\phi^4)$, as assumed in (1.2).[3] Solving iteratively the system (3.3)–(3.4), one finds that the solution behaves asymptotically as, see, e.g., [7] (the Coulomb potential $V(r)$ is defined up to a constant which is chosen so that it vanishes at infinity):

$$\phi(r) = \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + \mathcal{O}(1/r^3) \quad , \qquad V(r) = \frac{4Q}{r} + \mathcal{O}(1/r^2) \tag{3.5}$$

$$h(r) = \ell^{-2}r^2 + 1 - \frac{2m_g}{r} + \mathcal{O}(1/r^2) \quad , \quad \tilde{h}(r) = \ell^{-2}r^2 + 1 + \alpha^2 - \frac{2m_i}{r} + \mathcal{O}(1/r^2) . \tag{3.6}$$

with:

$$\alpha = \frac{\phi_1}{2\ell} \quad \text{and} \quad m_i = m_g - \frac{\phi_1\phi_2}{3\ell^2} . \tag{3.7}$$

Once the coupling function $A(\phi)$ and the potential $U(\phi)$ are explicitly given, all subsequent coefficients in the $1/r$ expansion of the metric, as well as of the scalar and Maxwell fields, can be expressed in function of the 5 integration constants, ϕ_1, ϕ_2, m_g, together with the magnetic and electric charges P and Q. (Of course, specific black hole solutions may depend on fewer parameters, as we shall see in section VI.)

The solution (3.5)–(3.6)–(3.7) of the field equations will extremize the action if the radial components of $\hat{V}^\mu_{(g)}$, $\hat{V}^\mu_{(\phi)}$ and $\hat{V}^\mu_{(A)}$, defined in (2.1)–(2.4)–(2.5), vanish on the boundary (their t, θ and φ components are zero because the solution is static and spherically symmetric.)

To evaluate $\hat{V}^r_{(g)}$ we write the AdS background metric in the same, Schwarzschild-Droste, coordinates, that is:

$$d\bar{s}^2 = -\bar{h}(r)dt^2 + \frac{dr^2}{\bar{\tilde{h}}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{with} \quad \bar{h} = \bar{\tilde{h}} = \ell^{-2}r^2 + 1 . \tag{3.8}$$

To evaluate $\hat{V}^r_{(\phi)}$, we take the following ansatz for $f(\phi)$:

$$f(\phi) = A + B\phi + C\frac{\phi^2}{2}\cdots \tag{3.9}$$

[3] A generic behaviour of the scalar potential, $U(\phi) = -6 + \mu^2\phi^2 + \lambda\phi^3 + \cdots$, changes the fall-off of the scalar field and renders the analysis more involved, see, e.g., [3, 5, 6], and references therein.

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where the constants A, B and C will be chosen in order that $\hat{V}^r_{(g)}$ and $\hat{V}^r_{(\phi)}$ vanish on the boundary. (The Maxwellian boundary term, $\hat{V}^r_{(A)}$, turns out to vanish at spatial infinity because $F^{tr} = -F^r_t/h$ contains the extra factor $h^{-1} = \mathcal{O}(1/r^2)$ as compared to $\partial^r\phi$.)

The calculation of $\hat{V}^r_{(g)}$ and $\hat{V}^r_{(\phi)}$ uses (omitting the irrelevant factor $\sin\theta$ in $\sqrt{-g} = r^2\sin\theta\sqrt{h/\tilde{h}}$):

$$\hat{g}^{tt} = -\frac{r^2}{\sqrt{h\tilde{h}}}, \qquad \hat{g}^{\theta\theta} = \sqrt{\frac{h}{\tilde{h}}}, \qquad \hat{g}^{rr} = r^2\sqrt{h\tilde{h}} \tag{3.10}$$

$$\Delta^r_{tt} = \frac{\tilde{h}h' - \overline{\tilde{h}h'}}{2}, \qquad \Delta^r_{\theta\theta} = -r(\tilde{h} - \bar{\tilde{h}}), \qquad \Delta^t_{rt} = \frac{h'}{2h} - \frac{\overline{h'}}{2\bar{h}} \tag{3.11}$$

$$\begin{aligned} \delta h &= -2\frac{\delta m_g}{r} + \mathcal{O}(1/r^2), \qquad \delta\tilde{h} = 2\alpha\delta\alpha - 2\frac{\delta m_i}{r} + \mathcal{O}(1/r^2), \\ \delta\phi &= \frac{\delta\phi_1}{r} + \frac{\delta\phi_2}{r^2} + \mathcal{O}(1/r^3)\,. \end{aligned} \tag{3.12}$$

(Of course κ and ℓ are not varied as they are "universal" constants entering the action.) The result is — using (3.5) and (3.6) but without the need to impose (3.7):

$$\hat{V}^r_{(g)} = -A\phi_1\alpha\delta\alpha + \mathcal{O}(1/r)\,, \tag{3.13}$$

$$\hat{V}^r_{(\phi)} = r^2\,\ell^{-2}A\,\delta\phi_1 + r\,\ell^{-2}(2A\,\delta\phi_2 + \phi_1\delta\phi_1(2B-1)) + \mathcal{O}(1)\,. \tag{3.14}$$

We hence first recover from (3.13) the well-known result that, in the absence of scalar field (and thus in the absence of f), the variation of the action on shell is zero $\forall\ \delta m$ (and $\forall\ \delta Q$ and δP). To ensure now that $\hat{V}^r_{(g)}$ and $\hat{V}^r_{(\phi)}$ vanish at spatial infinity *for the largest possible family of solutions*, i.e. when neither ϕ_1 nor ϕ_2 are given a priori (or, else, when neither $\delta\phi_1$ nor $\delta\phi_2$ are imposed to be a priori zero "à la" Dirichlet), we must have:

(i) first, see (3.14), that $A = 0$ (which implies $\hat{V}^r_{(g)} = \mathcal{O}(1/r)$), $B = 1/2$ and, thus, as anticipated in (1.5):

$$f(\phi) = \frac{\phi}{2}(1 + C\phi) + \mathcal{O}(\phi^3)\,, \tag{3.15}$$

where C is an arbitrary constant,

(ii) and, these conditions being fulfilled, we must have, second, that the $\mathcal{O}(1)$-term of $\hat{V}^r_{(\phi)}$, which reduces then to

$$\hat{V}^r_{(\phi)} = -\frac{1}{2\ell^2}\left(\phi_1\delta\phi_2 - (\phi_2 - 3C\phi_1^2)\delta\phi_1\right) + \mathcal{O}(1/r)\,, \tag{3.16}$$

vanishes as well at spatial infinity, $\hat{V}^r_{(\phi)} = 0$, which implies that ϕ_1 and ϕ_2 cannot in fact be entirely free but must be related as

$$\phi_2 = -3C\phi_1^2 + D\ell\phi_1 \tag{3.17}$$

where D is another arbitrary number.

Therefore ϕ_1 and ϕ_2 can never be *independent* constants of integration of the Einstein equations.

The numbers C and D are to be seen as "universal", that is determined by the *theory*, and not, contrarily to ϕ_1, m_g, P or Q, by its *solutions*. They do not have the same status however: C enters the action through the boundary term $k_S^\mu = f(\phi)\partial^\mu\phi$, and should be put on the same footing as κ and ℓ, whereas D does not. One may therefore argue that D must be taken to be zero, in which case the scalar field asymptotic behaviour preserves the AdS symmetry, see e.g. [3] or [8].

As is well known, scalar fields saturating the Breitenlohner-Freedman bound, $\mu^2 \geq -9/4$ (see footnote 4), ensure the linear stability of AdS when $\phi_1 = 0$ [9, 10]. When $\phi_1 \neq 0$ a detailed analysis shows that the condition for linear stability, in the case when the scalar field mass is $\mu^2 = -2$, is $\frac{1}{\ell}\frac{d\phi_2}{d\phi_1}\,|_{\phi_1=0} \geq -2/\pi$. Hence D is constrained by stability as $D \geq -2/\pi$ [11].

To summarize, by introducing an adequate function f, we have built an action whose variation is zero on shell for the broadest possible family of solutions. The fact that this family has to be restricted by the constraint (3.17) is the first result of this paper.

4 Mass as a Noether charge

The Katz-Bicak-Lynden-Bell superpotential is the "covariantization" of Freud's superpotential and hence is based on a covariantization of Einstein's pseudo-tensor, see [1] and [2] or [12]. That Lagrangian definition of Noether charges has been successfully applied to various spacetimes over the years, including D-dimensional, rotating, asymptotically flat or AdS black hole solutions of pure Einstein or Einstein-Gauss-Bonnet equations, see [13–16]; for a comparison with other approaches and, in particular, the Ashtekar-Magon-Das definition of the mass, see [17].

If the gravity theory is Einstein's, the following identity, see e.g. [13], is obtained by exploiting the invariance under diffeomorphisms $x^\rho \to x^\rho + \xi^\rho$ of the action $I_g = \int d^4x(\hat{R} - \hat{\bar{R}}) + \int d^4x\,\partial_\mu(\sqrt{-g}\,k^\mu)$ (matter fields do not play any role):

$$\int d^4x\,\partial_{\mu\rho}\hat{J}^{[\mu\rho]} \equiv 0 \quad \text{where} \quad \kappa\hat{J}^{[\mu\rho]} = D^{[\mu}\hat{\xi}^{\rho]} - \overline{D^{[\mu}\hat{\xi}^{\rho]}} + \xi^{[\mu}\hat{k}^{\rho]} \quad \text{with} \quad k^\rho = k_K^\rho + k_S^\rho\,. \tag{4.1}$$

The "superpotential" $J^{[\mu\nu]}$ was first introduced by Katz in the case of a flat background, see [1], and then extended by Katz, Bicak and Lynden-Bell to an arbitrary background in [2] and thus is usually called the "KBL superpotential", see also [12]; brackets denote antisymmetrization (e.g. $\xi^{[\mu}k^{\rho]} \equiv (\xi^\mu k^\rho - \xi^\rho k^\mu)/2$); the overline over $D^{[\mu}\hat{\xi}^{\rho]}$ means that it is evaluated using the background metric $\bar{g}_{\mu\nu}$; finally k_K^ρ and k_S^ρ are given in (1.3) and (1.4). One recognizes in the first two terms half of the (regularized) Komar superpotential. The role of the Katz vector k_K^ρ is, in particular, to correct the "anomalous factor 2" of Komar's formulae, see [1]. The contribution of the vector k_S^ρ is new, and due to the presence of a long-range scalar field.

Equation (4.1) yields conservation laws. If spacetime is stationary they read (in adapted coordinates):

$$\lim_{r\to\infty}\int_S d\theta\, d\phi\, \hat{J}^{[0r]} = \text{Const.} \tag{4.2}$$

where S is the 2-sphere at spatial infinity with equation $r \to \infty$.

If one is interested in defining the mass M, ξ^μ is taken to be the Killing vector for time translations, $\xi^\mu = (1,0,0,0)$, and the constant is identified with $-M$.

Consider now the asymptotically AdS configurations (3.5) and (3.6) for the fields and the metric. Using (3.10)–(3.11) and choosing the function $f(\phi)$ given in (3.15) a short calculation gives

$$M = \frac{1}{8}(\ell^{-2}\phi_1^2 - 4\alpha^2)r + \left[m_i + \frac{1}{8}\ell^{-2}\phi_1(3\phi_2 + C\phi_1^2)\right] + \mathcal{O}(1/r)\,. \tag{4.3}$$

Imposing now that (i) the configuration (3.5)–(3.6) indeed solves Einstein's equation, that is, imposing conditions (3.7), and (ii) that the solution extremizes the action, that is, imposing condition (3.17), then yields

$$M = m_g + D\frac{\phi_1^2}{24\ell}\,. \tag{4.4}$$

Hence, the mass of the solutions of the field equations which extremize the action is given by (4.4). When $D \neq 0$ that extends the results obtained in ref [8]. The expression (4.4) for the mass of asymptotically AdS hairy black holes is the second result of this paper.

5 Euclidean action and Gibbs' relation

The value of the action on shell plays an important role in the study of black hole thermodynamics. We compute it here and show that it can be identified to a Gibbs potential, the Katz vectors playing the role of the counterterms introduced in [4] and used in, e.g., [6, 7], and references therein.

Let us first treat the bulk part of the action (1.1), that is

$$2\kappa I_{\text{bulk}} \equiv \int d^4x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{U(\phi)}{\ell^2} - \frac{1}{4}A(\phi)F^2\right)\,. \tag{5.1}$$

I_{bulk} reduces to a boundary term on shell. Indeed the Gauss-Codazzi equations (or a direct manipulation of the Einstein equations (3.3) and (3.4)), reduce, in the case of a static, spherically symmetric metric, written in Schwarzschild coordinates as in (3.2), to:

$$\hat{R} + 2\hat{G}^t_t = -\left(r^2h'\sqrt{\tilde{h}/h}\right)'\,. \tag{5.2}$$

Now the t-t equation of motion, after adding $\hat{R}$ to both sides reads $\hat{R}+2\hat{G}^t_t = \hat{R}+\hat{T}^t_t$, that is, see (2.2)

$$\hat{R} + 2\hat{G}^t_t = \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{U(\phi)}{\ell^2} - \frac{1}{4}A(\phi)F^2\right) + A_0\,\partial_\mu(A\hat{F}^{0\mu}) - (A\hat{F}^{0r}A_0)' \tag{5.2}$$

where one recognizes in the first term the integrand of I_{bulk}. As for Maxwell's equations they are $\partial_\mu(A\hat{F}^{\mu\nu}) = 0$, so that the second term on the r.h.s. of (5.2) is zero on shell; as for their first integral it gives $A\hat{F}^{0r} = 4Q$, which simplifies the last term (recalling that $A_0 = -V$ and that $V = 4Q/r + \mathcal{O}(1/r^2)$, see (3.5)). Therefore

$$2\kappa I_{\text{bulk}}|_{\text{onshell}} = -4\pi\beta\left(r^2h'\sqrt{\tilde{h}/h} + 4QV\right)^{r\to\infty}_{r=r_+} \tag{5.3}$$

where 4π is the integral over the angles, where $\beta \equiv t_2 - t_1$ is the time integral, and where the spatial boundaries of the manifold are taken to be spatial infinity and, as usual, the black hole horizon.

The lower boundary of (5.3) is evaluated on the (supposedly non-degenerate) horizon where $h = h'_+(r - r_+) + \cdots$, $\tilde{h} = \tilde{h}'_+(r - r_+) + \cdots$, so that $4\pi\beta\Big(r^2 h'\sqrt{\tilde{h}/h}\Big)|_{r_+} = 4\pi\beta r_+^2\sqrt{\tilde{h}'_+ h'_+}$. The temperature T of the black hole, defined as, e.g., $1/2\pi$ the horizon surface gravity, is given by $T = \sqrt{\tilde{h}'_+ h'_+}/(4\pi)$. Its entropy S is defined as one-fourth of its area, that is: $S = \pi r_+^2$. Finally the time interval β is taken to be the inverse of the temperature. Hence, all in all, we get the standard result, see, e.g., [7]:

$$4\pi\beta\left(r^2 h'\sqrt{\tilde{h}/h} + 4QV\right)_{r_+} = 16\pi(S + \beta Q\Phi_Q)\,, \tag{5.4}$$

where Φ_Q is the value of the electric potential on the horizon.

Inserting now the asymptotic solution (3.5), (3.6) and (3.7) in (5.3), the upper boundary reads

$$\lim_{r\to\infty} -4\pi\beta\left(r^2 h'\sqrt{\tilde{h}/h} + 4QV\right) = -\frac{8\pi\beta}{\ell^2}r^3 - 4\pi\beta\,\frac{\phi_1^2}{4\ell^2}\,r + 4\pi\beta\left(-2m_g - \frac{2}{3}\frac{\phi_1\phi_2}{\ell^2}\right)\,. \tag{5.5}$$

As one can see, and is well-known, that upper boundary term diverges, first because the metric is asymptotically AdS, and, second, because of the presence of a long range scalar field (as for the electromagnetic field, it does not contribute). Usually those divergences are eliminated by means of counterterms added to the GHY surface term, that is boundary terms which depend on its curvature, see e.g. [6] or [7]. The addition of such terms does not prove necessary in the present framework, as the divergences are cancelled by the background bulk action and the Katz boundary terms, as we now show in detail.

To the bulk action on shell, that is the sum of (5.5) and (5.4), we must indeed now add the contributions on shell of the background action, $2\kappa\bar{I} = -\int d^4x\sqrt{-\bar{g}}\,(\bar{R} + \frac{6}{\ell^2})$, and the generalized Katz term, $2\kappa I_{\rm b} = \int d^4x\,\partial_\mu(\hat{k}^\mu_{\rm K} + \hat{k}^\mu_{\rm S})$. The background being AdS in Schwarzschild coordinates, see (3.8), its action on shell is trivially given by

$$2\kappa\bar{I}|_{\rm onshell} = \frac{6}{\ell^2}4\pi\beta\int_0^r dr\,r^2 = \frac{8\pi\beta}{\ell^2}r^3 \tag{5.6}$$

where the notation makes it clear that, AdS spacetime being regular, the lower bound is taken to be $r = 0$. That term cancels the first (divergent) term in (5.5).

As for the computation of $2\kappa I_{\rm b} = \int d^4x\,\partial_\mu(\hat{k}^\mu_{\rm K} + \hat{k}^\mu_{\rm S})$, it proceeds as follows (after integration on time and on the angles and omitting the factor $\sin\theta$ in $\sqrt{-g}$):

$$2\kappa I_{\rm b} = 4\pi\beta(\hat{k}^r_{\rm K} + \hat{k}^r_{\rm S})|^{r\to\infty} \tag{5.7}$$

where here, as above, the inner boundary is taken to be the center of the manifold where the vectors, by symmetry, are taken to vanish. Using (3.5) and (3.6) one has

$$\hat{k}^r_{\rm S} \equiv f(\phi)\hat{\partial}^r\phi = \frac{\phi}{2}(1 + C\phi)r^2\sqrt{h\tilde{h}}\,\phi' = -\frac{\phi_1^2}{2\ell^2}r - \frac{\phi_1}{2\ell^2}(3\phi_2 + C\phi_1^2) + \mathcal{O}(1/r)\,, \tag{5.8}$$

and, using (3.10) and (3.11) as well as the metrics (3.6) and (3.8):

$$\hat{k}^r_{\rm K} \equiv -(\hat{g}^{\nu\rho}\Delta^r_{\nu\rho} - \hat{g}^{r\nu}\Delta^\rho_{\nu\rho}) = 3\alpha^2 r + (4m_g - 6m_i) + \mathcal{O}(1/r)\,. \tag{5.9}$$

Since the e.o.m. impose $\alpha = \phi_1/(2\ell)$ and $m_i = m_g - 2\phi_1\phi_2/(3\ell^2)$, see (2.6), the boundary term finally reads

$$2\kappa I_{\rm b}|_{\rm onshell} = 4\pi\beta\,\frac{\phi_1^2}{4\ell^2}\,r + 4\pi\beta\left(-2m_g + \frac{\phi_1\phi_2}{2\ell^2} - C\frac{\phi_1^3}{2\ell^2}\right)\,. \tag{5.10}$$

As the detailed calculation presented here makes it clear, both vectors $k^\mu_{\rm K}$ and $k^\mu_{\rm S}$, with the function $f(\phi)$ imposed by our variational principle, play a role in cancelling the second, divergent, term in (5.5).

Adding (5.5), (5.4), (5.6) and (5.7), the action on shell is finite and reads

$$I_{\rm onshell} = S + \beta Q\Phi_Q - \beta\left(m_g + \frac{\phi_1\phi_2}{24\ell^2} + C\frac{\phi_1^3}{8\ell^2}\right)\,. \tag{5.11}$$

The last piece of information we have not used yet is that our variational principle imposes that $\phi_2 = -3C\phi_1^2 + D\ell\phi_1$, see (3.17), and that the KBL superpotential defines the mass as $M = m_g + D\frac{\phi_1^2}{24\ell}$, see (4.4), so that the last term in (5.11) is nothing but the mass. Hence:

$$I_{\rm onshell} = S - \beta(M - Q\Phi_Q)\,, \tag{5.12}$$

which is the Gibbs relation when, as usual, one interprets $-I_{\rm onshell}/\beta$ as the black hole Gibbs potential[4] (note the absence of the magnetic charge, as in [7], a result which, as noted in [7], can be modified by a proper adjunction to the action of boundary terms involving Maxwell's field, see footnote 2). Consider as an example the asymptotically AdS black hole solution (with no electric nor magnetic charges) discovered by one of us in [21]: it obeys the Gibbs relation simply because the asymptotic fall-off of the scalar field is such that $\phi_2 = -3C\phi_1^2 + D\ell\phi_1$ with $D = 0$ and $2C = -\sqrt{\nu^2-1}$, ν being a parameter entering the definition of the scalar potential $U(\phi)$.

Arriving at (5.12) by means of the Katz vectors is the third, and main, result of this paper.[5]

6 Dyonic black-holes and their thermodynamics

In [7] Lü et al. (henceforth LPP) considered the following bulk action:

$$I[g^{\mu\nu},\phi,A_\mu] \equiv \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + 6\ell^{-2}\cosh\left(\frac{\phi}{\sqrt{3}}\right) - \frac{1}{4}e^{-\sqrt{3}\phi}F_{\mu\nu}F^{\mu\nu}\right) \tag{6.1}$$

[4]The Gibbs potential is related to the Euclidean action I^E by $G = I^E/\beta$, where $I^E = -iI$ and I is the Lorentzian action integrated in imaginary time $t \in [0, -i\beta]$, i.e. $I = -iI_{onshell}$ and thus $I^E = -I_{onshell}$. We note that, since we imposed that the variation of the action on shell be zero, the on-shell action itself does not depend on the integration constants (m_g, ϕ_1, ϕ_2, Q and P) and hence depends only on Σ_1 and Σ_2, that is β.

[5]Hence the Katz vectors replace, in the present framework, the sum of the GHY surface term, see footnote 1, and the following counterterms: $-\frac{1}{\kappa}\int d^3x\sqrt{|\gamma|}\left(\frac{2}{\ell} + \frac{\mathcal{R}\ell}{2}\right)$, where γ_{ij} is the metric on the boundary and $\mathcal{R}$ its curvature scalar, augmented by a scalar contribution, taken to be $\frac{1}{6\kappa}\int d^3x\sqrt{|\gamma|}\left(\phi\, n^\nu\partial_\nu\phi - \frac{\phi^2}{2\ell}\right)$, see e.g. [7] and references therein, and see [6] for an alternative proposal.

which falls in the class studied above since $U(\phi) \equiv -6\cosh\left(\phi/\sqrt{3}\right) = -6 - \phi^2 + \cdots$ and $A(\phi) \equiv \exp(-\sqrt{3}\,\phi) = 1 - \sqrt{3}\,\phi + \cdots$.

Restricting one's attention, as we did above, to static, spherically symmetric, asymptotically AdS solutions of the derived equations of motion, the leading orders in $1/r$ of the metric and the scalar and electromagnetic fields are given by (3.5), (3.6) and (3.7). As previoulsy discussed the solutions obtained by solving iteratively the field equations depend on 5 integration constants, ϕ_1, ϕ_2, m_g, Q and P.

Now, Lü et al. found a remarkable sub-family of black-hole solutions with that asymptotic behaviour, characterized by 3 parameters only, which can be taken to be ϕ_1, ϕ_2 and m_g, Q and P becoming specific functions of ϕ_1, ϕ_2 and m_g, see eq. (2.1) of [7] for their explicit expression.[6]

The horizon r_+ of the black hole is the common zero of g_{tt} and g^{rr}, that is is such that

$$h[r_+] = 0 \quad , \quad \tilde{h}[r_+] = 0\,. \tag{6.2}$$

Hence $r_+ = r_+(\phi_1, \phi_2, m_g)$; see LPP eq. (2.10) for its (implicit) expression.

The electric and magnetic potentials are also easily defined and are also functions of ϕ_1, ϕ_2 and m_g, see LPP eq. (2.12) for their values (Φ_Q and Φ_P) on the horizon.

The temperature T and entropy S of the black-hole are defined as is usual in Einstein's theory and are, as well, known functions of ϕ_1, ϕ_2 and m_g, see LPP eq. (2.11).

Finally, Lü et al. calculate the mass of the black hole using the Astekhar-Magnon-Das method which yields

$$M_{\rm LPP} = m_g\,. \tag{6.3}$$

(The Hamiltonian-Wald approach, see [18–20, 22], yields the same result.)

With all that information in hand it is an exercise to study the thermodynamics of the black hole. The first law is satisfied if

$$L_{\rm LPP} \equiv TdS - (dM_{\rm LPP} - \Phi_P dP - \Phi_Q dQ) \tag{6.4}$$

vanishes. Expanding dS etc in $dS = (\partial S/\partial\beta_1)d\beta^1 + (\partial S/\partial\beta_2)d\beta^2 + (\partial S/\partial\mu)d\mu$ etc, the explicit calculation shows that $L_{\rm LPP}$ does *not* vanish. As shown in [7], see also [5] and [19], one rather has that

$$L_{\rm LPP} = \frac{1}{12\ell^2}(2\phi_2\, d\phi_1 - \phi_1\, d\phi_2)\,. \tag{6.5}$$

Various solutions for that puzzle have been proposed.

Lü et al. in [7] rewrite the right-hand-side of (6.5) as XdY with X and Y being some functions of β_1 and β_2 given in LPP eq. (2.14). Despite the fact that X and Y are not uniquely defined, LPP interpret Y as some "scalar charge" and X as the corresponding scalar potential evaluated on the horizon. That solution to the problem has been criticized in, e.g., [23]. It is indeed unclear how a scalar charge can be defined as there is no global symmetry associated to scalar fields.

[6]Note that in [7] ρ, and not r, is the Schwarzschild coordinate. Note too that, instead of ϕ_1, ϕ_2 and m_g, LPP use 3 other parameters called β_1, β_2 and μ which are implicitely related to ϕ_1, ϕ_2 and m_g through their eq. (2.7-9).

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Cardenas et al. impose in [22] that the asymptotic scalar field (which behaves as $\phi = \phi_1/r + \phi_2/r^2 + \cdots$) preserve the asymptotic AdS symmetries. That imposes, see [3], that $\phi_2 = -3C\phi_1^2$ with C an arbitrary number, see also [19]. The same result is obtained along a different route in [20].

We propose here still another solution to the puzzle: (1) impose that the LPP back hole solutions obey the variational principle advocated in this paper, hence proper boundary terms have to be added to the bulk action (6.1); (2) define the mass "à la" Katz rather than "à la" AMD.

Let us analyze these 2 points:

(1) After adding to the action (6.1) the boundary terms built from the vectors $k^\mu_{\rm K}$ and $k^\mu_{\rm S}$ — see (1.3) and (1.4) with $f(\phi)$ given in (3.15) —, and subtracting a regularizing AdS background, the LPP black-hole solutions extremize the action if ϕ_1 and ϕ_2 are not independent but related as, see (3.17):

$$\phi_2 = -3C\phi_1^2 + D\ell\phi_1\,. \tag{6.6}$$

(2) The mass of the LPP black hole solutions satisfying that constraint, as obtained by means of the KBL superpotential, is given by (4.4):

$$M = m_g + D\frac{\phi_1^2}{24\ell} \quad \text{that is} \quad M = M_{\rm LPP} + D\frac{\phi_1^2}{24\ell}\,. \tag{6.7}$$

Therefore, we have on one hand: $dM_{\rm LPP} = dM - D\phi_1\,d\phi_1/12\ell$, and, on the other hand: $(2\phi_2\,d\phi_1 - \phi_1\,d\phi_2)/(12\ell^2) = D\phi_1\,d\phi_1/12\ell$, so that (6.4) and (6.5) yield

$$dM = TdS + \Phi_P dP + \Phi_Q dQ \tag{6.8}$$

and the first law is then satisfied, whatever the value of the two numbers C and D.

That result generalizes to $D \neq 0$ those of [7] where the mass is defined using the Ashtekar-Magon-Das formula, as well as those of [22] which are based on the Hamiltonian approach of [3], and shows, as emphasized in [20], that enforcing that the action be extremum on shell may constrain the integration constants which appear in the general solution of the field equations.

7 Concluding remarks

Solving Einstein's equations to find new black hole solutions has been for decades a huge challenge. Hence, the discovery that only sub-classes of some honest-to-god solutions obeyed the firmly established laws of black hole thermodynamics came as a bad surprise. What was made hopefully translucid in the present paper is that the whole family of solutions is in fact thermodynamically acceptable, under the condition however that a distinction be made between thermodynamical variables and parameters. Indeed the constraint $\phi_2 = -3C\phi_1^2 + D\ell\phi_1$, see (3.17), does not numerically constrain ϕ_2 since C and/or D are arbitrary; it only tells us that ϕ_2 cannot be varied independently of ϕ_1. It is possible that these boundary conditions on the scalar field be generalized to other families of solutions by modifying the Katz vector accordingly. For instance, an arbitrary function of the scalar field times the Katz vector can be considered.

Another conclusion of this paper is that the Einstein-Katz action offers a straightforward way to compute the (automatically finite) black hole action on shell. Moreover, when implementing the constraint on the parameters imposed by the variational principle (that is, $\phi_2 = -3C\phi_1^2 + D\ell\phi_1$) and using the definitions of the Noether charges as deduced from the KBL superpotential (that is, $M = m_g + D\phi_1^2/4\ell$), that action on shell, $I|_{\text{onshell}}$, can be related to a Gibbs potential which (automatically) obeys the Gibbs relation (in the case studied here: $I|_{\text{onshell}} = S - \beta(M - Q\Phi_Q)$). That approach has to be contrasted to the usual one where the action is taken to be the Einstein-Hilbert action complemented by the Gibbons-Hawking-York boundary term. Since that action diverges on shell, various counterterms, involving the curvature of the boundary, have to be added, leading to well-controlled and well-understood, albeit fairly heavy calculations, see [24] or [25] and [26] in the case of pure gravity, or [5, 7] and [6] where extra counterterms are added when scalar fields are present.

A question that we leave to future work is how does the Katz boundary action (together with the background bulk action) compare, in general, to the Gibbons-Hawking-York surface term as regularized by the counterterms (which involve curvature tensors). Insight may be gained from [17] where the KBL vector and superpotential, which, like the GHY term, involve only first derivatives of the metrics, were related to the Ashtekar-Magnon-Das mass formula which, like the counterterms, involves the curvature tensor, that is second derivatives of the metric.

Another question, also left to further studies, is how to generalize the analysis presented here to more general scalar potentials, to higher dimensions, to Gauss-Bonnet or higher derivative theories of gravity.

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Chapitre 7

Thermodynamique d'un trou noir membre d'un système binaire

Au fil de cette thèse, nous avons illustré le rôle clé de la "skeletonisation" dans le traitement des systèmes binaires, qui décrit la trajectoire de chaque corps par la ligne d'univers d'une particule ponctuelle. En théories EMD par exemple, nous avons introduit au chapitre 4 l'action "skeleton", valable lorsque tout effet de "taille finie" (e.g., de marée ou hors équilibre) et de spin sont négligés :

$$I_{\rm m}^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x_A^\mu\}] = -\sum_A \int m_A(\varphi) ds_A + \sum_A q_A \int A_\mu \, dx_A^\mu \,, \tag{7.1}$$

avec $ds_A = \sqrt{-g_{\mu\nu} dx_A^\mu dx_A^\nu}$, et où, une fois calculées, les fonctions $m_A(\varphi)$ et les constantes q_A encodent les caractéristiques de chaque corps A suffisantes pour en décrire le mouvement autour de son compagnon.

L'action (7.1) est un point de départ vers l'obtention des équations du mouvement d'un système binaire (cf. chapitre 4) ainsi que de son rayonnement gravitationnel (cf. chapitre 5). Cependant, bien qu'elle soit de la forme la plus générale possible préservant la covariance ainsi que la symétrie $U(1)$ de l'action fondamentale EMD, elle repose néanmoins sur l'hypothèse, qu'il faut justifier, que tant qu'il ne s'agit que de décrire leur trajectoire, les corps compacts peuvent être remplacés par des particules ponctuelles.

Considérons à présent le trou noir statique et à symétrie sphérique des théories EMD, étudié en partie II, dont les champs sont donnés, rappelons-le, par

$$\begin{aligned}
&ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} d\Omega^2 \,, \\
&A_t = -\frac{Q\, e^{2a\varphi_\infty}}{r} \,, \quad A_i = 0 \,, \qquad \text{avec} \qquad Q^2 = \frac{r_+ r_-}{1+a^2}\, e^{-2a\varphi_\infty} \,, \\
&\varphi = \varphi_\infty + \frac{a}{1+a^2} \ln\left(1-\frac{r_-}{r}\right) \,.
\end{aligned} \tag{7.2}$$

Le calcul de sa sensibilité $m_A(\varphi)$ et de son couplage q_A a été effectué au chapitre 4, par identification de la solution (7.2) aux champs générés par une particule décrite par (7.1), et a donné que q_A s'identifiait à Q et que la fonction $m_A(\varphi)$ dépendait d'une constante d'intégration, notée μ_A.

Ainsi, nous avions conclu que la dynamique d'un trou noir (7.2) ne dépend que de deux paramètres, q_A et μ_A, devant rester constants lors des réajustements du trou noir, i.e. de r_+ et de r_-, lorsque son environnement φ_∞ varie lentement, suite au mouvement de son compagnon lointain. Par conséquent, sa "masse ADM" [71], c.-à-d. la moitié du coefficient $\mathcal{O}(1/r)$ de g_{rr} à l'infini, n'est pas conservée.

Or, il existe un domaine d'étude des trous noirs, a priori distinct, mais précisément conçu pour décrire la façon dont ces derniers se réajustent lors d'une interaction avec leur environnement : celui de leur thermodynamique.

Dans ce chapitre final, on montre que la première loi de la thermodynamique des trous noirs EMD permet non seulement de justifier leur "skeletonisation", mais est aussi le cadre de pensée adéquat pour interpréter l'apparition des constantes q_A et μ_A caractérisant leur mouvement.

Nous reproduisons ci-dessous un article réalisé en collaboration avec M. Cárdenas et N. Deruelle [193] qui établit, dans un premier temps, la première loi de la thermodynamique du trou noir (7.2). Pour cela, il suffit de transposer les étapes présentées en détail dans le chapitre précédent, pour en définir la masse M à l'aide de la "technologie" katzienne,

$$M = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}\, r_-\right) - \frac{a}{1+a^2}\int r_-\, d\varphi_\infty\,, \tag{7.3}$$

somme de la "masse ADM" et d'une contribution scalaire. Son calcul est donné en détail dans l'appendice B de l'article ci-dessous [1]. On note d'ailleurs que son résultat (7.3) coïncide avec la définition de la masse dans le cadre du formalisme Hamiltonien, voir, e.g., [194, 195], présentée en appendice A.

En outre, on montre que la variation de M, relative à r_+, r_- et φ_∞, ainsi que celles de la charge Q et de l'entropie S du trou noir satisfont automatiquement l'identité suivante :

$$T\delta S = \delta M - \Phi \delta Q\,, \tag{7.4}$$

où T est la température du trou noir, et Φ son potentiel électrique évalué sur son horizon r_+.

Dans un second temps, on montre que cette première loi de la thermodynamique permet de justifier, ainsi que d'interpréter, la "skeletonisation" du trou noir (7.2). En effet, on trouve que :

(i) puisque la charge du trou noir Q s'identifie au paramètre constant q_A apparaissant dans (7.1), elle doit être fixe lorsqu'il orbite autour d'un compagnon, $\delta Q = 0$;

(ii) de plus, en vertu des conditions d'identification des champs (7.2) à ceux générés par la particule (7.1) obtenues au chapitre 4 (cf. 4.3.2), sa masse M, telle que définie à la Katz (7.3), est elle aussi constante, $\delta M = 0$.

La "skeletonisation" décrit donc un trou noir "isolé", i.e. n'échangeant ni charge ($\delta Q = 0$), ni masse ($\delta M = 0$) (7.3), lorsque, par exemple, il interagit avec son compagnon. La thermodynamique, et plus précisément sa première loi (7.4), nous enseigne alors que dans ces conditions (appelées parfois "adiabatiques", au sens thermodynamique du terme), son entropie S doit elle aussi être conservée : $\delta S = 0$.

Ainsi, lors d'une variation de son environnement scalaire φ_∞, due au mouvement de son compagnon lointain, le trou noir (7.2) réajuste sa configuration à l'équilibre (i.e., r_+ et

1. Notons au passage que, lorsque M est définie selon (7.3), il est facile de vérifier que l'action Einstein-Maxwell-dilaton-Katz "on shell" satisfait encore la relation de Gibbs, présentée au chapitre précédent, bien que ce calcul ne fasse pas l'objet de l'article ci-dessous.

Dans le cas des étoiles, on trouvera une justification de leur "skeletonisation" dans [23] ainsi que, pour les théories scalaire-tenseur, dans l'appendice A de [88]. Notons que la méthode qui y est employée, aisément adaptable aux théories EMD, nécessite d'exprimer la "masse" de l'étoile sous la forme du Lagrangien d'Einstein ("$\Gamma\Gamma - \Gamma\Gamma$") "on shell", intégré de l'infini ($r \to \infty$) jusqu'au centre de l'étoile ($r = 0$), supposé régulier. Dans le cas d'un trou noir, qui est singulier en $r = 0$, l'intégration du Lagrangien d'Einstein (où de celui d'Einstein-Katz, qui en est la généralisation covariante), ne peut s'étendre que de l'infini ($r \to \infty$) à son horizon ($r = r_H$). Ainsi resurgit l'"énergie libre" de Gibbs, suggérant là encore la thermodynamique comme outil privilégié pour interpréter la "skeletonisation" d'un trou noir.

r_-) à charge Q et masse (nœtherienne) M constantes, en respectant la première loi de la thermodynamique, de sorte que son entropie S reste elle aussi constante. Le fait que son mouvement soit entièrement décrit par les deux constantes q_A et μ_A trouve donc une explication naturelle dans le cadre de la thermodynamique : si q_A s'identifie à sa charge $U(1)$ conservée, la constante d'intégration notée μ_A dans le chapitre 4, s'avérait coïncider avec la masse irréductible du trou noir, $\mu_A = \sqrt{A_+/16\pi}$ (avec A_+ l'aire de son l'horizon) pour la bonne raison que son entropie, $S = A_+/4$, est un invariant adiabatique : $\mu_A = \sqrt{S/4\pi}$.

L'article qui suit ouvre donc une interface entre la thermodynamique des trous noirs d'une part, et leur "skeletonisation" d'autre part. Elle permet en effet d'interpréter, et peut-être même de guider, la façon dont la trajectoire d'un trou noir peut être réduite à une ligne d'univers.

Elle montre en particulier que les échanges de masse sous forme d'ondes gravitationnelles et de charges sont ignorés. De plus, il est clair que, dans les derniers stades précédant la fusion d'un système binaire, où la période orbitale devient comparable au temps de relaxation d'un trou noir, il n'est plus possible de le considérer comme étant à l'équilibre, aussi bien en théories EMD qu'en relativité générale d'ailleurs : son comportement ne satisfait plus la première loi de la thermodynamique, se réajustant lentement d'une configuration (7.2) à une autre. Et, de ce fait, en vertu de la seconde loi, son entropie doit augmenter. L'action "skeleton" (7.1) doit donc, probablement, être étendue pour pouvoir décrire une particule sensible aussi aux gradients $\{\partial_t, \partial_i\}$ des champs, ce qui permettra d'encoder, cette fois-ci, la réponse du trou noir à un environnement non plus quasi-statique, mais dynamique, cf. [196–199].

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Thermodynamics sheds light on black hole dynamics

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We propose to unify two *a priori* distinct aspects of black hole physics: their thermodynamics, and their description as point particles, which is an essential starting point in the post-Newtonian approach to their dynamics. We will find that, when reducing a black hole to a point particle endowed with its specific effective mass, one in fact describes a black hole satisfying the first law of thermodynamics, such that its global charges, and hence its entropy, remain constant. This gives a thermodynamical interpretation of its effective mass, thus opening a promising synergy between black hole thermodynamics and the analytical approaches to the two-body problems in gravity theories. To illustrate this relationship, the Einstein-Maxwell-dilaton theory, which contains simple examples of asympotically flat, hairy black hole solutions, will serve as a laboratory.

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I. INTRODUCTION

Thermodynamics has proven to be a powerful tool to give a physical interpretation of the integration constants of Einstein's equations characterizing a black hole spacetime, in introducing its extensive parameters (global charges and entropy) and its intensive ones defined on the horizon (temperature, electric potential, etc.). The first law then tells us how a black hole readjusts its equilibrium configuration when interacting with its environment.

On the other hand, the dynamics of interacting (nonrotating) compact objects relies on reducing them to point particles endowed with an effective mass parameter. In the case of a general relativistic Schwarzschild black hole, the interpretation of this parameter seems straightforward since it cannot be anything but the Schwarzschild mass, which is the only integration constant at hand.

Consider now as an example Einstein-Maxwell-dilaton theories, which consist in supplementing general relativity with a scalar field and a (non-minimally coupled) vector field. Such theories allow for the existence of hairy black hole solutions depending no longer on one but on three integration constants. Their reduction to point particles was recently performed in Ref. [1] and involves a scalar-field-sensitive mass $m(\varphi)$ *à la* Eardley [2]. The explicit calculation of this black hole "sensitivity" includes a constant parameter μ which is equated with the Schwarzschild mass when the hairs are cut off.

We will show that this constant μ must be defined in terms of the entropy of the black hole alone. This can be understood thus: Eardley's $m(\varphi)$ modeling means that the black hole is moving adiabatically in the fields of its companion; this will imply that it satisfies the first law of thermodynamics; moreover, the specific form of $m(\varphi)$ previously obtained in Ref. [1] will impose that it exchanges no mass nor charge with its environment. Therefore, its entropy will remain constant and hence can be related to μ.

Returning then to the general relativistic Schwarzschild black hole, this result shows that the constant parameter describing it as a point particle must not be interpreted as its mass but rather be related to its entropy. Since the second law tells us that the entropy must increase, this means that, at a better, nonadiabatic approximation, its effective mass can no longer be taken as a constant.

II. EXAMPLE OF EMD BLACK HOLES AND THEIR REDUCTION TO POINT PARTICLES

The vacuum Einstein-Maxwell-dilaton (EMD) action of gravity is taken to be, see Refs. [3–5],

$$16\pi I[g_{\mu\nu},A_\mu,\varphi]=\int d^4x\sqrt{-g}(R-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-e^{-2a\varphi}F^2), \tag{2.1}$$

where g is the determinant of the metric $g_{\mu\nu}$; R is the Ricci scalar, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ with $F^2=F^{\mu\nu}F_{\mu\nu}$; and a parametrizes the theory.

The field equations derived from the action (2.1) are

$$R_{\mu\nu}=2\partial_\mu\varphi\partial_\nu\varphi+2e^{-2a\varphi}\left(F_\mu{}^\lambda F_{\nu\lambda}-\frac{1}{4}g_{\mu\nu}F^2\right), \tag{2.2a}$$

$$D_\mu(e^{-2a\varphi}F^{\mu\nu})=0, \tag{2.2b}$$

$$\Box\varphi=-\frac{a}{2}e^{-2a\varphi}F^2, \tag{2.2c}$$

where D_μ is the covariant derivative associated to $g_{\mu\nu}$ and $\Box = D_\mu D^\mu$.

The "electrically" charged, static, spherically symmetric black hole solutions of the equations above which will best illustrate the correspondence between their thermodynamics and dynamics were found in Refs. [3–5] and read, with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$,

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}}dt^2 \\ &\quad + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}}dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}d\Omega^2, \\ A_t &= -\sqrt{\frac{r_+r_-}{1+a^2}}\frac{e^{a\varphi_\infty}}{r}, \qquad A_i = 0, \\ \varphi &= \varphi_\infty + \frac{a}{1+a^2}\ln\left(1-\frac{r_-}{r}\right), \end{aligned} \tag{2.3}$$

where an irrelevant sign and additive constant were chosen in the definition of A_t. This family of solutions depends on three integration constants: the radius r_+ of the horizon, the location r_- of the curvature singularity, and the asymptotic value φ_∞ of the scalar field which must be equated to the local, adiabatically varying, value of the scalar field created by the environment, e.g., a faraway companion.

In order to be able to address the post-Newtonian dynamics of two interacting EMD black holes, we have to explicitly include them as sources in the action. They were phenomenologically replaced in Ref. [1] by point particles described by the following "skeleton" action,

$$\begin{aligned} I^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x_A^\mu\}] &= I - \sum_A \int m_A(\varphi) ds_A \\ &\quad + \sum_A q_A \int A_\mu dx_A^\mu, \end{aligned} \tag{2.4}$$

where I is given in (2.1), $ds_A = \sqrt{-g_{\mu\nu}dx_A^\mu dx_A^\nu}$, and $x_A^\mu[s_A]$ is the worldline of the skeletonized black hole A. The parameter q_A is taken to be a constant in order to preserve the $U(1)$ symmetry of the full action. As for the scalar-field-sensitive, effective, "mass" function $m_A(\varphi)$, it is taken to be a function of the value of the scalar field at the location of the particle A. The calculation of such mass functions is standard when the compact object is a neutron star; see, e.g., Refs. [6,7]. Let us recall here briefly how it was, for the first time, computed in Ref. [1] when the compact object A is the EMD black hole described above.

The field equations derived from (2.4) are the same as (2.2) but supplemented by point source terms. They were solved in Ref. [1] in the near-worldline region of the particle A, in its rest frame, and at linear order around a background solution consisting of an asymptotically flat spacetime, a vector field which can be "gauged away" to zero, and an asymptotic scalar field environment φ_∞ that is imposed by the faraway companion B. The solutions were then equated to the EMD black hole solution (2.3) at leading, $\mathcal{O}(1/r)$, order to yield (dropping, from now on, the index A)

$$q = \sqrt{\frac{r_+r_-}{1+a^2}}e^{-a\varphi_\infty}, \tag{2.5a}$$

$$m(\varphi_\infty) = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}r_-\right), \tag{2.5b}$$

$$\frac{dm}{d\varphi}(\varphi_\infty) = \frac{ar_-}{1+a^2}. \tag{2.5c}$$

The system (2.5) is integrable; indeed, expressing r_+ and r_- in terms of m and $dm/d\varphi$ using (2.5b) and (2.5c) and injecting the result into (2.5a) gives the first order differential equation

$$\left[\left(\frac{dm}{d\varphi}\right)\left(m(\varphi) - \frac{1-a^2}{2a}\frac{dm}{d\varphi}\right)\right]_{\varphi=\varphi_\infty} = \frac{a}{2}q^2 e^{2a\varphi_\infty}. \tag{2.6}$$

This differential equation can be solved for all a; see Ref. [1]. The solution reads as $F[m(\varphi_\infty), q, \varphi_\infty, a] = \mu^2$, where the explicit form of F can be found using, e.g., *Mathematica* and where μ is an integration constant. In the case $a = 1$, which is enough to illustrate our purposes, the solution simply is

$$m(\varphi) = \sqrt{\mu^2 + q^2\frac{e^{2\varphi}}{2}}, \tag{2.7}$$

where the index ∞ has been dropped since the scalar background φ_∞, imposed by B, can have any value. One can then address the post-Newtonian dynamics of the two black holes and, for example, compute the two-body PN Lagrangian; see Ref. [1].

A question left pending at this stage is the relationship between the constants q and μ characterizing the skeletonized black hole and its extensive parameters, that is, its electric charge Q, mass M, and entropy S.

III. THERMODYNAMICS VERSUS DYNAMICS OF EMD BLACK HOLES

The first law of thermodynamics obeyed by EMD black holes is found in the standard way.

Their temperature T is defined as

$$T \equiv \frac{\kappa}{2\pi} = \frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{\frac{1-a^2}{1+a^2}}, \tag{3.1}$$

where κ is their surface gravity, with $\kappa^2 = -\frac{1}{2}(\nabla_\mu\xi_\nu\nabla^\mu\xi^\nu)_{r_+}$ and $\xi^\mu = (1,0,0,0)$ being the timelike Killing vector.

Their electric potential is

$$\Phi \equiv A_t(r \to \infty) - A_t(r_+) = \sqrt{\frac{r_-}{(1+a^2)r_+}} e^{a\varphi_\infty}. \quad (3.2)$$

The action for the metric being Einstein-Hilbert's, the entropy S of the black holes is the fourth of their horizon area A_+:

$$S \equiv \frac{A_+}{4} = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}}. \quad (3.3)$$

As for the global charges associated to these solutions, that is, their electric charge Q and mass M, they can be obtained within various approaches, e.g., the Hamiltonian one as developed by Regge-Teitelboim [8] or the Lagrangian one as developed by Katz [9,10]. As usual in scalar-tensor theories of gravity and as shown in Appendixes A and B, the scalar field contributes to the on-shell Hamiltonian so that

$$Q = \sqrt{\frac{r_+ r_-}{1+a^2}} e^{-a\varphi_\infty}, \quad (3.4a)$$

$$M = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2} r_-\right) - \frac{a}{1+a^2} \int r_- d\varphi_\infty. \quad (3.4b)$$

As one can see, M is the sum of the Arnowitt-Deser-Misner (ADM) mass [which is one-half the $\mathcal{O}(1/r)$ coefficient of g_{rr} at spatial infinity] and of a scalar contribution [11–13] (which is called the scalar charge in Ref. [14]); see, e.g., Refs. [15,16].

With all these definitions in hand, it is easily checked that the variations of S, Q, and M with respect to r_+, r_-, and φ_∞ are such that the following identity holds (which is equivalent to that obtained in Ref. [14]):

$$T\delta S = \delta M - \Phi \delta Q. \quad (3.5)$$

This first law of black hole thermodynamics implies in particular that when the black hole does not exchange any mass ($\delta M = 0$) nor charge ($\delta Q = 0$) with its environment, its entropy remains constant.

We show now how the first law of thermodynamics (3.5) *justifies* the skeletonization of EMD black holes introduced in Sec. II and provides an interpretation of the constants q and μ that characterize it.

Comparing (3.4a) and (2.5a), we first see that we must identify the constant q, called q_A in the skeleton action (2.4), to the global electric charge Q of the black hole. The significance of this identification is that the dynamical evolution of the skeletonized black hole is such that its charge remains constant, $\delta Q = 0$.

Second, the variation of the global black hole mass M, when interacting with its environment, follows from (3.4b) and reads

$$\delta M = \frac{1}{2}\delta\left(r_+ + \frac{1-a^2}{1+a^2} r_-\right) - \frac{a r_-}{1+a^2}\delta\varphi_\infty, \quad (3.6)$$

which is zero when taking into account (2.5b) and (2.5c). This means that the black hole global mass remains constant as well during its dynamical evolution, $\delta M = 0$ (while the ADM mass and "scalar charge" are not separately conserved).

Therefore, the skeletonization of black holes proposed in (2.4) amounts to describing them as remaining isolated when, for example, they orbit around a companion.

Finally, the first law (3.5) tells us that, since $\delta Q = 0$ and $\delta M = 0$, the entropy of the black hole remains constant as well: $S = \text{const}$. This is the main result of this paper, which shows that when one reduces a black hole to a point particle *à la* Eardley (2.4), one in fact describes a black hole of which the equilibrium configuration readjusts adiabatically when interacting with its companion, such that its mass M, charge Q, and hence entropy S remain constant.

Therefore, it must be possible to define the parameter μ appearing in the mass function $m(\varphi)$ when integrating (2.6) as a function of the entropy S only (or, equivalently, of the "irreducible mass" M_{irr} only [17,18]). That this is so can be shown for all a: indeed, inserting the expressions (2.5a) and (2.5b) for q and $m(\varphi)$ in the solution $F[m(\varphi), q, \varphi, a] = \mu^2$, one finds that μ^2 must be equated with $S/4\pi \equiv M_{\text{irr}}^2$ as given in (3.3). Since, moreover, the parameter q must be equated to the graviphoton charge Q, the solution $F[m(\varphi), Q, \varphi, a] = S/4\pi$ implicitly gives $m(\varphi)$ in terms of Q, S, and φ. In the illustrative example $a = 1$ where $m(\varphi)$ is given in (2.7), this yields

$$\mu^2 = \frac{S}{4\pi} \quad \text{so that } m(\varphi) = \sqrt{\frac{S}{4\pi} + Q^2 \frac{e^{2\varphi}}{2}}. \quad (3.7)$$

Note that when $r_- = 0$ ($\forall a$) the black hole solution is Schwarzschild's so that $m(\varphi)$ is reduced, as it should be, to its (constant) mass $m = \sqrt{S/4\pi} = r_+/2$. The same is true in the Reissner-Nordström limit, $a = 0$, for which $m = (r_+ + r_-)/2$; see (2.5). On the other hand, when a nontrivial scalar field is present, the phenomenological, "Eardley-inspired," scalar-field-sensitive mass function $m(\varphi)$ for black holes which was shown in Ref. [1] to satisfy (2.6) is in fact explained and justified by their thermodynamics, and the parameters q and μ become related to their global electric charge and *entropy*.

IV. CONCLUSION

The results above indicate that the conservative dynamics of a (hairy) black hole when skeletonized *à la* Eardley, as in (2.4), is generically such that it does not exchange energy (nor electric charge) with its environment.

Therefore, because of the first law of thermodynamics, the black hole adiabatically readjusts its equilibrium configuration in such a way that its entropy (or area in the case at hand) remains constant. We conjecture that this fact holds in any scalar-vector-tensor theory of gravity and that the parameters entering the scalar-field-sensitive mass functions attributed to skeletonized black holes can always be related to their global gauge charges and their Wald entropy [19], which remain constant in their motion around their companion.

Of course, our results no longer apply in the late stages of a binary system coalescence. In particular, when the period of the orbit becomes comparable to the readjustment time of a black hole, the adiabatic approximation breaks down, and the entropy must increase. Perhaps a way to capture this phenomenon would be to generalize our point particle ansatz by introducing a more elaborate one depending, for example, on the Weyl tensor or on the four-gradient of the scalar field as well [20–22]. We leave this to further work. Other extensions of our work would be to see how spins and magnetic "charges" can be included.

Finally, the scalar environment of the black hole could be imposed by a time dependent cosmological environment, rather than a faraway companion. Our results show that in that case as well the readjustment of the black hole is always such that, at the adiabatic approximation, its entropy remains constant.

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APPENDIX A: GLOBAL CHARGES IN A HAMILTONIAN FRAMEWORK

The Hamiltonian generator associated to the Lagrangian (2.1) reads

$$H[\eta^\perp,\eta^i,\eta^{(A)}] = \int d^3x(\eta^\perp\mathcal{H}_\perp + \eta^i\mathcal{H}_i - \eta^{(A)}\mathcal{G}) + Q[\eta^\perp,\eta^i,\eta^{(A)}]. \tag{A1}$$

It is obtained from the standard ADM Hamiltonian (from which the equations of motion can be deduced) by replacing the Lagrange multipliers (that is, A_t together with the lapse N and shift N^i of the standard $1+3$ decomposition of the metric) by the asymptotic surface spacetime deformations $\eta^\perp$, η^i, and the gauge parameter of the Abelian symmetry $\eta^{(A)}$.

The constraints $\mathcal{H}_\perp$, $\mathcal{H}_i$, and $\mathcal{G}$ are given by

$$\begin{aligned}\mathcal{H}_\perp = {}&\frac{16\pi}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i{}_i)^2\right) - \frac{\sqrt{\gamma}}{16\pi}R^{(3)}\\ &+2\pi\frac{\pi_\varphi^2}{\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{8\pi}\partial^i\varphi\partial_i\varphi - 2\pi e^{2a\varphi}\frac{\pi^i\pi_i}{\sqrt{\gamma}}\\ &+\frac{\sqrt{\gamma}}{16\pi}e^{-2a\varphi}F^{ij}F_{ij},\end{aligned} \tag{A2}$$

$$\mathcal{H}_i = 2\nabla_j\pi^j{}_i + \pi_\varphi\partial_i\varphi + \pi^jF_{ij}, \qquad \mathcal{G} = \partial_i\pi^i. \tag{A3}$$

Here, γ_{ij} is the spatial metric with determinant γ, scalar curvature $R^{(3)}$, and covariant derivative ∇_i, and the conjugate momenta of γ_{ij}, φ, and A_i (with $F_{ij} = \partial_iA_j - \partial_jA_i$ and a dot representing time derivatives) are

$$\begin{aligned}\pi^{ij} &= -\frac{\sqrt{\gamma}}{16\pi}(K^{ij} - \gamma^{ij}K)\\ \text{where } K_{ij} &= \frac{1}{2N}(\nabla_iN_j + \nabla_jN_i - \dot{\gamma}_{ij}),\end{aligned} \tag{A4}$$

$$\begin{aligned}\pi_\varphi &= \frac{\sqrt{\gamma}}{4\pi N}(\dot{\varphi} - N^i\partial_i\varphi),\\ \pi^i &= -\frac{\sqrt{\gamma}e^{-2a\varphi}}{4\pi N}(-\gamma^{ij}F_{0j} + N^j\gamma^{ik}F_{jk}).\end{aligned} \tag{A5}$$

As for the variation δQ of the surface term Q, it is obtained by demanding that the on-shell variation of the Hamiltonian, with respect to the dynamical variables and $\eta^\perp$, η^i, $\eta^{(A)}$, vanishes: $\delta H = 0$. We get $\delta Q = \delta Q^g + \delta Q^\varphi + \delta Q^{(A)}$ with

$$\begin{aligned}\delta Q^g &= \lim_{r\to\infty}\frac{1}{16\pi}\int dS_lG^{ijkl}(\eta^\perp\nabla_k\delta\gamma_{ij} - \partial_k\eta^\perp\delta\gamma_{ij})\\ &\quad+\int dS_l[2\eta_k\delta\pi^{kl} + (2\eta^k\pi^{jl} - \eta^l\pi^{kj})\delta\gamma_{jk}],\\ &\quad\text{with}\quad G^{ijkl} = \frac{1}{2}\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}),\\ \delta Q^\phi &= -\lim_{r\to\infty}\int dS_i\left(\frac{\sqrt{\gamma}}{4\pi}\eta^\perp\partial^i\varphi\delta\varphi + \eta^i\pi_\varphi\delta\varphi\right),\\ \delta Q^{(A)} &= -\lim_{r\to\infty}\int dS_i\left[\frac{\sqrt{\gamma}}{4\pi}\eta^\perp e^{-2a\varphi}F^{ij}\delta A_j\right.\\ &\quad\left.+(\eta^i\pi^j - \pi^j\eta^i)\delta A_j - \eta^{(A)}\delta\pi^i\right].\end{aligned} \tag{A6}$$

For the black hole spacetimes (2.3), the timelike Killing vector is $\xi^\mu = (1,0,0,0)$, so that the only deformation parameters to consider are $\eta^{(A)}$ and

$$\eta^\perp = N\xi^t \tag{A7}$$

(where N is 1 at infinity), and one finally obtains

$$\delta Q = \xi^t \left[\frac{1}{2}\left(\delta r_+ + \frac{1-a^2}{1+a^2}\delta r_-\right) - \frac{a r_-}{1+a^2}\delta\varphi_\infty\right] - \eta^{(A)}\delta\left(\sqrt{\frac{r_+ r_-}{1+a^2}}e^{-a\varphi_\infty}\right). \tag{A8}$$

In this approach, the variation δM of the mass is the coefficient of ξ^t, and the variation δQ of the electric charge is the coefficient of $-\eta^{(A)}$; see Ref. [15]. Hence, integration yields

$$M = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}r_-\right) - \frac{a}{1+a^2}\int r_-\delta\varphi_\infty, \tag{A9}$$

$$Q = \sqrt{\frac{r_+ r_-}{1+a^2}}e^{-a\varphi_\infty}. \tag{A10}$$

APPENDIX B: MASS AS A NOETHER CHARGE

The Einstein-Maxwell-dilaton-Katz action is defined by

$$16\pi I_{\mathrm{K}} = \int d^4x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi}F^2) - \int d^4x\sqrt{-\bar{g}}\bar{R} + \int d^4x\partial_\mu(\hat{k}^\mu_K + \hat{k}^\mu_S), \tag{B1}$$

where barred quantities refer to a reference spacetime that will be taken to be flat; see Ref. [16]. The last term is (a hat meaning multiplication by $\sqrt{-g}$)

$$\hat{k}^\mu_K = -(\hat{g}^{\nu\rho}\Delta^\mu_{\nu\rho} - \hat{g}^{\mu\nu}\Delta^\rho_{\nu\rho}) \quad \text{with} \quad \Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \bar{\Gamma}^\mu_{\nu\rho} \quad \text{and} \quad \hat{k}^\mu_S = A\hat{\partial}^\mu\varphi, \tag{B2}$$

where $\Gamma^\mu_{\nu\rho}$ and $\bar{\Gamma}^\mu_{\nu\rho}$ are the Christoffel symbols associated to $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ and A is an arbitrary constant.

The variation of (B1) with respect to the scalar field reads, on shell,

$$16\pi\delta_\varphi I_{\mathrm{K}} = \int d^4x\partial_\mu\hat{V}^\mu_\varphi \quad \text{with} \quad \hat{V}^\mu_\varphi = -4(\hat{\partial}^\mu\varphi)\delta\varphi + A\hat{g}^{\mu\nu}\delta(\partial_\nu\varphi). \tag{B3}$$

For the black hole family (2.3), we have, upon variation of the integration constants r_+, r_-, and φ_∞,

$$16\pi\delta_\varphi I_K^{\mathrm{onshell}} = \lim_{r\to\infty}\int_{t_1}^{t_2} dt\int_{S^2} d\theta d\phi\hat{V}^r_\varphi \quad \text{with} \quad \hat{V}^r_\varphi = \frac{a\sin\theta}{1+a^2}(A\delta r_- - 4r_-\delta\varphi_\infty) + \mathcal{O}(1/r). \tag{B4}$$

Imposing now that the variation of the action vanishes for the broadest possible family of black holes yields a relationship between r_- and φ_∞:

$$A\delta r_- = 4r_-\delta\varphi_\infty. \tag{B5}$$

Note that no further conditions emerge from varying (B1) with respect to $g_{\mu\nu}$ and A_μ; indeed, one finds $\hat{V}^r_g = \mathcal{O}(1/r)$ and $\hat{V}^r_A = \mathcal{O}(1/r)$ on shell and at spatial infinity.

Finally, exploiting the invariance of (B1) under diffeomorphisms, $x^\mu \to x^\mu + \xi^\mu$, one obtains "*à la* Noether" a conserved current $\hat{j}^\mu$ deriving from the generalized Katz-Bicak-Lynden-Bell superpotential, see, e.g., Ref. [23]:

$$\partial_\mu\hat{j}^\mu = 0 \quad \text{with} \quad \hat{j}^\mu = \partial_\nu\hat{J}^{[\mu\nu]} \quad \text{and} \quad 8\pi\hat{J}^{[\mu\nu]} = \nabla^{[\mu}\hat{\xi}^{\nu]} - \overline{\nabla^{[\mu}\hat{\xi}^{\nu]}} + \xi^{[\mu}\hat{k}^{\nu]}_K + \xi^{[\mu}\hat{k}^{\nu]}_S. \tag{B6}$$

When spacetime is stationary, $\xi^\mu \equiv (1,0,0,0)$ is the time-like Killing vector, and the mass of the black hole (2.3) is defined as

$$M = -\lim_{r\to\infty}\int d\theta d\phi\hat{J}^{[0r]} = M_K + M_S$$

$$\text{where } M_K = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}r_-\right) \text{ and } M_S = -\frac{A}{4}\frac{ar_-}{1+a^2}. \tag{B7}$$

Note that the Katz mass M_K coincides with the ADM mass. Finally, M is easily written independently of the constant A, using (B5), as

$$M = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}r_-\right) - \frac{a}{1+a^2}\int r_- d\varphi_\infty, \tag{B8}$$

which coincides with (A9).

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Conclusion

Le formalisme EOB, le calcul d'un lagrangien à deux corps et d'un flux d'ondes gravitationnelles à des ordres post-newtoniens élevés, la "skeletonisation"... sont autant d'outils théoriques jouant un rôle central dans l'obtention de formes d'ondes émises par les systèmes binaires compacts en relativité générale, à la fois précises et complètes, à confronter aux données des interféromètres de détection des ondes gravitationnelles.

Mais, comme le suggèrent les exemples des théories scalaire-tenseur et Einstein-Maxwell-dilaton (EMD) développés dans cette thèse, il n'est pas si difficile de transposer lesdits outils aux gravités modifiées. Par ailleurs, afin de contraindre ces dernières il n'est peut-être pas nécessaire d'atteindre des degrés de sophistication aussi élevés qu'en relativité générale : la question primordiale étant plutôt de déterminer à quels ordres post-kepleriens (dans le lagrangien à deux corps comme dans les flux) apparaissent des déviations à la relativité générale dans des quantités observables. Concluons donc cette thèse en mentionnant, dans cette optique, quelques développements futurs possibles.

Comme nous l'avons montré au chapitre 5, en théories EMD (incluant les théories scalaire-tenseur comme cas particulier), la présence d'un rayonnement de nature dipolaire modifie l'expression de $\dot{f}$ en fonction de la fréquence f du signal dans la phase spiralante initiale, par une contribution d'ordre -1PK, absente en relativité générale, cf. (5.6). Le fait qu'aucune contribution dipolaire au "chirp" n'ait été détectée constitue donc une première contrainte, dans le régime de champ faible, des théories EMD (aux redshifts observés tout du moins).

La question de quantifier ces contraintes n'a été qu'effleurée dans cette thèse. C'est là l'objet, par exemple, du formalisme "post-einsteinien paramétrisé" (PPE) introduit par N. Yunes et F. Pretorius [200]. Il consiste à paramétrer de façon "agnostique" les modifications des formes d'ondes de la relativité générale (incluant une contribution -1PK au "chirp"), à l'aide de quatre paramètres, aujourd'hui contraints [201]. Il sera donc utile de calculer ces paramètres "PPE" en théories EMD, afin de remonter aux contraintes sur les paramètres $\alpha^0_{A/B}$ et $e_{A/B}$ caractérisant les corps.

En revanche, lorsque les dipôles sont négligeables, i.e. $\alpha^0_A \simeq \alpha^0_B$ et $e_A \simeq e_B$, le "chirp" d'ordre 0PK est dégénéré avec celui de la relativité générale. Il faut donc aller au-delà de l'approche PPE. Le formalisme EOB s'avérera alors particulièrement utile, pour extraire cette fois des contraintes dans le régime de champ fort.

Précisons comment procéder. En relativité générale, si l'on connait les masses des deux corps (les spins étant négligés), le signal gravitationnel est entièrement déterminé jusque dans le régime de champ fort. Quant aux valeurs numériques des deux masses, on les extrait en comparant le $\dot{f}(f)$ observé à l'expression théorique 1PN [202] :

$$\dot{f} = \frac{96}{5}\pi^{8/3}\mathcal{M}^{5/3}f^{11/3}\left[1 - \left(\frac{743}{336} + \frac{11\nu}{4}\right)\nu^{-2/5}(\pi\mathcal{M}f)^{2/3} + \cdots\right], \tag{7.5}$$

c.-à-d., par la mesure de la "chirp mass" de la relativité générale $\mathcal{M} = \nu^{3/5}M$, ainsi que du rapport de masses symétrique $\nu = \frac{m_A m_B}{M^2}$, avec $M = m_A + m_B$. En ce qui concerne les théories EMD, nous connaissons déjà l'expression de la "chirp mass" (habillée par les quadrupôles scalaire et "électrique") si le dipôle est négligé, cf. (5.9). Pour accéder au rapport de masses symétrique habillé, il faudra donc généraliser notre formule $\dot{f}(f)$, cf. (5.6), à l'ordre suivant en calculant le flux d'ondes gravitationnelles à l'ordre 1PK (l'approximation du dipôle nul suffisant). La dégénérescence avec la relativité générale étant ainsi levée, le formalisme EOB permettra de contraster les prédictions des théories EMD à celles de la relativité générale, même lorsque le dipôle est nul, dans le régime de champ fort.

Ce flux 1PK serait d'ailleurs opportun dans la limite scalaire-tenseur pure, par cohérence avec le lagrangien à deux corps 2PK que nous avons étudié ici. Notons aussi que les équations du mouvement scalaire-tenseur ont été obtenues à l'ordre 3PK par L. Bernard [97]. Cette information s'avérera utile dans le régime de champ fort, et pourra faire l'objet de travaux futurs afin de l'incorporer au sein du formalisme EOB. De la même façon, il serait utile de calculer le lagrangien EMD à l'ordre 2PK.

Une autre piste importante à explorer est l'inclusion des effets de taille finie, comme nous l'avons suggéré dans le chapitre 7 par l'exemple des trous noirs EMD. Afin de rendre compte de l'augmentation de l'entropie des trous noirs à laquelle on s'attend près de leur coalescence, une façon de procéder est de faire dépendre leurs sensibilités des gradients des champs, par exemple, du champ scalaire : comme l'ont montré T. Damour et G. Esposito-Farèse dans [197], cela consiste à généraliser l'action les caractérisant en

$$S^{\mathrm{pp}}_{\mathrm{m,A}} = -c^2 \int ds_A \left[m_A(\varphi) + n_A(\varphi)(\partial\varphi)^2\right] \ . \tag{7.6}$$

La fonction $n_A(\varphi)$ a la dimension d'une masse fois une longueur au carré, ce qui permet de l'estimer, pour un corps compact, à $n_A/m_A = \mathcal{O}(G_* m_A/c^2)^2$. Comme de plus $\varphi = \mathcal{O}(v^2/c^2)$, on trouve que ce terme en gradients est accompagné d'un préfacteur $1/c^6$. C'est donc un terme d'ordre 3PK, contrairement à 5PN pour celui décrivant les effets de taille finie en relativité générale, faisant intervenir par covariance le tenseur de Weyl au carré [199]. Notons que par un raisonnement similaire, le premier terme de taille finie lié au "graviphoton", à la fois covariant et préservant la symétrie $U(1)$, doit dépendre de $F_{\mu\nu}F^{\mu\nu}$ ($F_{\mu\nu}$ étant le tenseur de Faraday), et intervient donc lui aussi dès l'ordre 3PK. Proche de l'ISCO d'un trou noir binaire, on peut donc s'attendre à des effets hors équilibre bien plus importants qu'en relativité générale, voir, e.g., [203].

Une autre extension possible de cette thèse concerne l'étude de la relaxation du trou noir final issu de la fusion d'un système binaire en théories EMD. En effet, dans le chapitre 5, nous avons obtenu les formes d'ondes émises par un système binaire de trous noirs "chevelus". Mais si nous voulons pousser son évolution au-delà de l'ISCO, il faut, tout comme en relativité générale (cf. chapitre 1), raccorder l'onde gravitationnelle aux modes quasi-normaux du trou noir final. Un projet futur (en collaboration avec E. Berti) est donc d'étudier la théories des perturbations des trous noirs EMD, afin d'en déterminer les modes quasi-normaux (notons d'ailleurs que des travaux préliminaires ont été effectués dans cette direction par R. Brito et C. Pacilio [204]).

Enfin, les méthodes développées ici pourront être étendues à d'autres gravités modifiées, telles que les théories scalaire-tenseur couplées à l'invariant de Gauss-Bonnet, dans lesquelles les trous noirs manifestent des phénomènes de scalarisation spontanée similaires à ceux des théories scalaire-tenseur [118]. Une autre possibilité à explorer, dans la continuité des théories EMD, serait celle où le couplage du champ scalaire φ au "graviphoton" A_μ devient une fonction générique ; voir, par exemple, les travaux récents de C. Herdeiro [205].

Ainsi, cette thèse, qui s'est attaquée à la dynamique "EOBisée" de corps compacts (y compris de trous noirs) en théories de gravité modifiée, et qui, vu la nouveauté du sujet, s'est arrétée aux premières étapes, mènera à des développements que l'on peut espérer fructueux.

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