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Quelques contributions à des problèmes variationnels géométriques impliquant des énergies non locales

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Some contributions to geometric variational problems involving nonlocal energies

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Titre : Quelques contributions à des problèmes variationnels géométriques impliquant des énergies non locales

Résumé : Cette thèse est dédiée à l'étude de deux problèmes variationnels géométriques impliquant des énergies non-locales : d'une part, la géométrie et les singularités des applications harmoniques fractionnaires, et d'autre part, un problème isopérimétrique avec un potentiel intégrable inspiré du modèle de goutte liquide pour le noyau atomique imaginé par Gamow. Concernant le premier sujet, nous améliorons des résultats de régularité partielle connus pour les applications 1/2-harmoniques minimisantes dans le cas où la variété d'arrivée est une sphère, en obtenant une estimation plus précise de la dimension de Hausdorff de l'ensemble singulier, c'est-à-dire l'ensemble des points de discontinuité. Nous caractérisons également les applications tangentes 1/2-harmoniques minimisantes de $\mathbb{R}^2$ dans le cercle unité $\mathbb{S}^1$, ce qui éclaire le comportement des applications 1/2-harmoniques minimisantes de $\mathbb{R}^2$ dans $\mathbb{S}^1$ près de leurs singularités. Pour $s \in]0,1[$, nous prouvons enfin des résultats de régularité partielle pour les applications s-harmoniques stationnaires ou minimisantes, et obtenons des estimées fines sur la dimension de Hausdorff de l'ensemble des singularités, en fonction de s. Concernant le deuxième sujet de la thèse, nous étudions un problème de minimisation sur les ensembles de périmètre fini sous contrainte de volume, dans lequel la fonctionnelle est constituée de la somme d'un terme de cohésion (le périmètre) et d'un terme répulsif donné par un noyau symétrique et intégrable sur $\mathbb{R}^n$. Nous montrons que sous des hypothèses raisonnables sur le comportement près de l'origine et sur certains des moments de ce noyau – qui incluent les potentiels de Bessel – le problème admet des minimiseurs de grande masse (ou volume). De plus, après renormalisation, ces minimiseurs convergent vers la boule unité lorsque la masse tend vers l'infini. En étudiant la stabilité de la boule, nous montrons que sans ces hypothèses, il peut y avoir rupture de symétrie, c'est-à-dire qu'il y a des cas pour lesquels le problème admet des minimiseurs qui ne sont pas la boule.

Mots-clés : problèmes isopérimétriques, applications harmoniques fractionnaires, énergies non locales, singularités, problèmes à bord libre, régularité partielle, surfaces minimales.

Title: Some contributions to geometric variational problems involving nonlocal energies

Abstract: This thesis is dedicated to the study of two separate geometric variational problems involving nonlocal energies: firstly, the geometry and singularities of fractional harmonic maps, and secondly, an isoperimetric problem with a repulsive integrable potential inspired by Gamow's liquid drop model for the atomic nucleus. On the first topic, we improve already-known results for minimizing 1/2-harmonic maps when the target manifold is a sphere by reducing the upper bound on the Haudorff dimension of the singular set, i.e., the set of points of discontinuity. We also characterize so-called minimizing 1/2-harmonic tangent maps from the plane into the unit circle $\mathbb{S}^1$, shedding light on the behavior of minimizing 1/2-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^1$ near singularities. Finally, when $s \in (0,1)$, we prove partial regularity results for s-harmonic maps into spheres in the stationary and minimizing case, obtaining sharp estimates on the Hausdorff dimension of the set of singularities, depending on the value of s. As for the second topic of the thesis, we study a minimization problem on sets of finite perimeter under a volume constraint, where the functional is the sum of a cohesive perimeter term and a repulsive term given by a general integrable symmetric kernel on $\mathbb{R}^n$. We show that under reasonable assumptions on the behavior near the origin and on some of the moments of this kernel – which include physically relevant Bessel potentials – the problem admits large mass (or volume) minimizers. In addition, after normalization, those minimizers converge to the unit ball as the mass goes to infinity. By studying the stability of the ball, we show that without these assumptions, symmetry breaking can occur, that is, there are cases when the problem admits minimizers which cannot be the ball.

Keywords: isoperimetric problems, fractional harmonic maps, nonlocal energies, singularities, free boundary problems, partial regularity, minimal surfaces.

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Introduction

In this thesis we present a few contributions to geometric variational problems involving nonlocal energies. Those contributions may be sorted into two main different topics: the first one is the geometry and singularities of fractional harmonic maps, and the second one is an isoperimetric problem with a nonlocal repulsive potential, in particular the study of its large mass minimizers. Each chapter is self-contained: Chapter 1 is a submitted paper written in collaboration with V. Millot, Chapter 2 is a paper to be submitted, in collaboration with V. Millot and A. Schikorra, and Chapter 3 is a work conducted by myself in parallel with the other two. Chapters 1 and 2 are devoted to the topic of fractional harmonic maps, while Chapter 3 is devoted to the aforementioned isoperimetric problem. In Appendix A, we collect some well-known facts about the fractional laplacian, which we define in a general, distributional setting which encompasses the functional setting adopted in Chapters 1 and 2, and give rigorous proofs of elliptic regularity for the distributional fractional laplacian.

We divide this introductory section into two parts: first, we introduce classical harmonic maps, their fractional counterparts, and present the main contributions Chapters 1 and 2 add to the field, and secondly we introduce the isoperimetric problem studied in Chapter 3 and the essential results therein.

I.1 Fractional harmonic maps

In what follows, we assume $\Omega \subseteq \mathbb{R}^n$ to be an open subset with Lipschitz boundary in $\mathbb{R}^n$, where $n \geqslant 1$, and $\mathcal{N}$ to be a smooth, compact, submanifold without boundary of $\mathbb{R}^d$, $d \geqslant 2$.

Before introducing fractional harmonic maps, we recall the definition, equivalent characterizations and some well-known regularity results for *classical* weakly harmonic maps. This will make the introduction of their fractional counterpart much more natural, and allow us to emphasize the strong analogies between them.

Classical harmonic maps

Definition and equivalent characterizations. Weakly harmonic maps can be seen both as natural generalizations of harmonic functions, replacing the target with a smooth compact manifold, or of geodesics, by considering a domain of dimension larger than 1. We recall that a function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be weakly harmonic in Ω if it satisfies

$$\Delta u = 0 \quad \text{in } \Omega$$

in a weak sense; to be precise, $u \in H^1(\Omega)$ is harmonic if and only if

$$\int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = 0, \quad \forall \varphi \in \mathscr{D}(\Omega),$$

where

$$\mathcal{E}(u, \Omega) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

is the usual Dirichlet energy in Ω, $\mathscr{D}(\Omega)$ denotes the space of smooth functions compactly supported in Ω, and $H^1(\Omega)$ the Sobolev space of square integrable functions in Ω whose weak partial derivatives are also square integrable. Observing that u is weakly harmonic in Ω if and only if it is a critical point of the Dirichlet energy in Ω, i.e.,

$$\Delta u = 0 \quad \text{weakly in } \Omega \iff \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u + t\varphi, \Omega)\right]_{|t=0} = 0, \quad \forall \varphi \in \mathscr{D}(\Omega),$$

it is natural to define harmonic maps into a smooth compact manifold without boundary $\mathcal{N}$ as critical points of the Dirichlet energy with respect to *variations on the target manifold*.

Definition I.1.1. Defining

$$H^1(\Omega; \mathcal{N}) := \left\{ u \in H^1(\Omega; \mathbb{R}^d) \ : \ u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

a map $u \in H^1(\Omega; \mathcal{N})$ is said to be a *weak harmonic map in* Ω if it satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\Pi_{\mathcal{N}}(u + t\varphi), \Omega)\right]_{|t=0} = 0, \quad \forall \varphi \in \mathscr{D}(\Omega; \mathbb{R}^d), \tag{I.1.1}$$

where $\Pi_{\mathcal{N}}$ denotes the nearest point projection on $\mathcal{N}$.

Here $\nabla u = (\nabla u^1, \ldots, \nabla u^d)$, where $u = (u^1, \ldots, u^d)$, $\nabla u \cdot \nabla v = \sum_{i=1}^d \nabla u^i \cdot \nabla v^i$, and $|\nabla u|^2 = \nabla u \cdot \nabla u$. Note that the nearest point projection on $\mathcal{N}$ is well defined and smooth in a tubular neighborhood of $\mathcal{N}$, since the manifold is smooth and compact, so that $t \mapsto \Pi_{\mathcal{N}}(u + t\varphi)$ is indeed a well-defined, smooth function in a neighborhood of 0. The harmonicity condition (I.1.1) can in fact be rewritten as

$$\Delta u \perp \mathrm{Tan}(u, \mathcal{N}) \tag{I.1.2}$$

in a weak sense, where $\mathrm{Tan}(p, \mathcal{N})$ denotes the tangent space of $\mathcal{N}$ at p. By weak sense, here we mean

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0, \quad \forall \varphi \in H^1(\Omega; u^* T\mathcal{N}),$$

where $H^1(\Omega; u^* T\mathcal{N})$ denotes the space of functions $\varphi \in H^1(\Omega; \mathbb{R}^d)$ compactly supported in Ω such that $\varphi(x) \in \mathrm{Tan}(u(x), \mathcal{N})$ for a.e. $x \in \Omega$. We can compute explicitly the Lagrange multiplier in (I.1.2) to get the equivalent formulation

$$\Delta u + A_u(\nabla u, \nabla u) = 0, \tag{I.1.3}$$

where A_p denotes the second fundamental form of $\mathcal{N}$ at the point p.

Example I.1.2. In the case $n = 1$, we see that $\gamma : (a, b) \to \mathcal{N}$ is weakly harmonic into Ω if $\gamma'' \cdot \gamma' = 0$, thus $|\gamma'|$ is constant, and γ is a constant-speed geodesic. The converse is true, so that in dimension 1, weak harmonic maps are given by constant-speed geodesics, which are known to be smooth.

This example shows one link between harmonic maps and minimal surfaces. In fact, when $n = 2$ and $\dim(\mathcal{N}) \geqslant 3$ we observe a similar relation between the two notions: any smooth harmonic map is a branched minimal surface. In fact, in that case, a smooth conformal map $u : \Omega \to \mathcal{N}$ is harmonic if and only if it is a branched minimal surface.

In arbitrary dimension, we are often particularly interested in the spherical case.

Example I.1.3. If $\mathcal{N} = \mathbb{S}^{d-1}$, where $d \geqslant 2$ and $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$, then (I.1.3) takes the simple form

$$-\Delta u = |\nabla u|^2 u \quad \text{in } \Omega \tag{I.1.4}$$

in the sense of distributions.

Looking at (I.1.4), we see that if $u \in H^1(\Omega)$, the term on the right-hand side belongs a priori only to $L^1(\Omega)$, so standard elliptic regularity theory does not apply. As the following example shows, weak harmonic maps may indeed present singularities (discontinuities), unlike harmonic functions which are known to be smooth.

Example I.1.4. The map $u : \mathbb{R}^3 \to \mathbb{S}^2$ defined by $u(x) = \frac{x}{|x|}$ is a weakly harmonic map from Ω into $\mathbb{S}^2$, for every open set $\Omega \subseteq \mathbb{R}^3$.

In fact, weak harmonic maps may be completely irregular: in [92], T. Rivière proved the existence of a weak harmonic map from B_1, the open unit ball of $\mathbb{R}^3$, into $\mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$, which is discontinuous everywhere. In dimension $n = 2$ however, F. Hélein proved in [61] that weakly harmonic maps are smooth whenever $\mathcal{N}$ has dimension at least 2.

Since there is no hope to get any kind of regularity for weak harmonic maps in dimension $n \geqslant 3$, we need to look at subclasses of weak harmonic maps. Let us first clarify what we mean by regularity. In fact, if a weak harmonic map $u \in H^1(\Omega)$ is continuous in Ω, then the regularity theory for quasilinear elliptic systems shows that u is actually locally Hölder continuous in Ω, and by a bootstrap procedure we can improve it to locally Lipschitz, and then C^∞; hence continuity implies smoothness, and the issue is always only to prove continuity of u. Hence we define the singular set of a weak harmonic map $u : \Omega \to \mathcal{N}$ by

$$\operatorname{sing}(u) := \{x \in \Omega \ : \ u \text{ is not continuous in any neighborhood of } x\}.$$

Minimizing & stationary harmonic maps. We first consider the subclass of so-called *minimizing* weak harmonic maps, made of those weak harmonic maps which minimize their Dirichlet energy with respect to their boundary data.

Definition I.1.5. We say that a map $u \in H^1(\Omega;\mathcal{N})$ is a *minimizing weak harmonic map in* Ω if

$$\mathcal{E}(u,\Omega) \leqslant \mathcal{E}(v,\Omega)$$

for any map $v \in H^1(\Omega;\mathcal{N})$ such that $u - v$ is compactly supported in Ω.

Such maps are easy to build. Indeed, assume that Ω has a Lipschitz boundary, and let $f \in H^1(\Omega;\mathcal{N})$. Then it is easy to see by the direct method of the calculus of variations that the problem

$$\min\left\{\mathcal{E}(u;\Omega) \ : \ u \in H^1(\Omega;\mathcal{N}) \text{ s.t. } u_{|\partial\Omega} = f\right\}$$

admits a minimizer, and this minimizer is in particular a minimizing weak harmonic map in Ω into $\mathcal{N}$.

Example I.1.6. Let B_1 be the open unit ball of $\mathbb{R}^3$, and let $g \in H^1(B_1;\mathbb{S}^2) \cap C^0(\partial B_1)$ such that the topological degree of $g_{|\partial B_1}$ is nonvanishing. Then given $u \in H^1(B_1,\mathbb{S}^2)$ a solution of

$$\min\left\{\mathcal{E}(u;B_1)\ :\ u \in H^1(B_1;\mathbb{S}^2) \text{ s.t. } u(x) = g\Big(\frac{x}{|x|}\Big) \text{ on } \partial B_1\right\},$$

u is obviously a minimizing weak harmonic map in B_1, and for topological reasons it must have at least one singular point.

Thus minimizing harmonic maps may have singularities, but the singular set of any minimizing map cannot be too large, in the sense that its Hausdorff dimension is at most $n-3$, as was proven by R. Schoen and K. Uhlenbeck in [103, 104] (when $n = 3$ we have the finer result that the singular set is locally finite). Let us remark that without further assumptions on $\mathcal{N}$, this upper bound is sharp in view of Example I.1.4. Let us say a few words on the case $\Omega \subseteq \mathbb{R}^3$ and $\mathcal{N} = \mathbb{S}^3$, which illustrates once again the tight link between harmonic maps and minimal surfaces. In that case, it can be shown that $\operatorname{sing}(u) = \emptyset$, and this is due to the famous result proven by F. J. Almgren[1] and E. Calabi[19], that any minimal 2-dimensional sphere in $\mathbb{S}^3$ is equatorial. Indeed, if a minimizing harmonic map $u : \Omega \to \mathbb{S}^3$ has a singular point x, then by a blowup argument around x, we can build a nontrivial 0-homogeneous (i.e., invariant under rotations) minimizing harmonic map φ from $\mathbb{R}^3$ into $\mathbb{S}^3$ which is singular only at the origin. This in turn gives a minimal 2-sphere in $\mathbb{S}^3$, which is necessarily equatorial. Making variations in the orthogonal direction of the equator, we contradict the minimality of φ, which proves that the singular set must be empty.

There is actually a larger class of weak harmonic maps for which we can obtain partial regularity results, and which includes the minimizing case: the class of *stationary harmonic maps*. It is made of those weak harmonic maps which are also critical points for the Dirichlet energy with respect to variations *on the domain*. To clarify what we mean by this, we consider a smooth vector field $X \in C_c^\infty(\Omega;\mathbb{R}^n)$, and its associated flow $\Phi_t(x) = \Phi(t,x)$ defined by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = X(\Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Definition I.1.7. We say that a weak harmonic map $u \in H^1(\Omega;\mathcal{N})$ is *stationary* if it satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u \circ \Phi_t, \Omega)\right]_{|t=0} = 0,$$

or equivalently

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n \Big(|\nabla u|^2\delta_{ij} - 2\partial_i u \cdot \partial_j u\Big)\, \partial_i X^j \,\mathrm{d}x = 0, \tag{I.1.5}$$

for every smooth vector field $X = (X^1,\dots,X^n) \in C_c^\infty(\Omega;\mathbb{R}^d)$, where Φ_t is the associated integral flow.

The essential property which makes stationary harmonic maps so particular is the so-called monotonicity formula, which can be obtained easily from the stationarity condition (I.1.5).

Proposition I.1.8 (Monotonicity formula)**.** *If $u \in H^1(\Omega;\mathcal{N})$ is a stationary harmonic map in Ω, then for every $x_0 \in \Omega$, and every $0 < \sigma < \rho$ such that $\rho < \operatorname{dist}(x_0, \partial\Omega)$, we have*

$$\rho^{2-n} \int_{B_\rho(x_0)} |\nabla u|^2 \,\mathrm{d}x - \sigma^{2-n} \int_{B_\sigma(x_0)} |\nabla u|^2 \,\mathrm{d}x = 2 \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} r^{2-n} |\partial_r u|^2 \,\mathrm{d}x,$$

where $r = |x - x_0|$, and $\partial_r u = \nabla u \cdot \left(\frac{x-x_0}{|x-x_0|}\right)$.

In particular, we see that the stationarity assumption implies that the function

$$r \in (0, \operatorname{dist}(x_0, \partial\Omega)) \mapsto r^{2-n} \int_{B_r(x_0)} |\nabla u|^2 \,\mathrm{d}x \quad \text{is nondecreasing}$$

for every $x_0 \in \Omega$. This monotonicity formula is important for proving the following partial regularity result, due to L. Evans[39] when $\mathcal{N} = \mathbb{S}^{d-1}$, and generalized by F. Bethuel[9] to a general target manifold.

Theorem I.1.9. *If $n \geqslant 2$ and $u \in H^1(\Omega;\mathcal{N})$ is a stationary harmonic maps, then $\mathscr{H}^{n-2}(\operatorname{sing}(u)) = 0$, where $\mathscr{H}^{n-2}$ denotes the $(n-2)$-dimensional Hausdorff measure, and $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$.*

How to prove partial regularity. Let us give the main ingredients of the proof by L. Evans of partial regularity for stationary harmonic maps into spheres.

To obtain partial regularity, the "usual" strategy is to prove an epsilon-regularity theorem, stating that if the (rescaled) energy of a map u in a ball $B_r(x)$ is small enough, then u is actually Hölder-continuous in a neighborhood of x.

Theorem I.1.10 (ε-regularity[39])**.** *Let $u \in H^1(\Omega;\mathbb{S}^{d-1})$ be a stationary harmonic map in Ω, where $n \geqslant 3$ and $d \geqslant 2$. Then there exists $\varepsilon_0 = \varepsilon_0(n)$ and $\alpha_0 = \alpha_0(n)$ such that, for any $B_r(x) \subseteq \Omega$, if*

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \,\mathrm{d}x \leqslant \varepsilon_0,$$

then $u \in C^{0,\alpha}(B_{\alpha_0 r}(x))$.

It is then well known that for any map $u \in H^1(\Omega;\mathbb{S}^{d-1})$, the set of points $x \in \Omega$ such that $\limsup_{r\to 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \,\mathrm{d}x > \varepsilon_0$ is of vanishing $(n-2)$-Hausdorff measure. The proof of this ε-regularity theorem in the stationary case relies mainly on the three following ingredients:

(i) the *monotonicity* formula;

(ii) a *div-curl structure* for the right-hand side of the harmonic maps equation (I.1.4);

(iii) Coifman-Lions-Meyer and Semmes' div-curl Lemma, stating that a div-curl product lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$;

(iv) the identification of $\mathcal{H}^1(\mathbb{R}^n)$ with the dual of BMO, the space of functions with bounded mean oscillation, due to C. Fefferman and E. M. Stein[44, 43].

Points (ii) and (iii) were already used by F. Hélein[60] in his proof of full regularity for harmonic maps when $n = 2$: in that case, the fact that the right-hand side of (I.1.4) belongs to $\mathcal{H}^1(\mathbb{R}^2)$ shows that $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ by Calderón-Zygmund theory, which gives continuity by Sobolev embedding. The observation that the right-hand side of (I.1.4) has a div-curl structure was made possible by J. Shatah's discovery[105], that the harmonicity condition is equivalent to the conservation law

$$\operatorname{div} \mathbf{\Omega}^{i,j} = 0, \quad \forall i,j \in \{1,\dots,d\}$$

where

$$\mathbf{\Omega}^{i,j} := u^i \nabla u^j - u^j \nabla u^i, \quad \forall i,j \in \{1,\dots,d\}.$$

Then F. Hélein's "trick" was to rewrite (I.1.4) as

$$\Delta u^i = \sum_{j=1}^{d} \mathbf{\Omega}^{i,j} \cdot \nabla u^j, \quad \forall i \in \{1.\dots,d\}, \tag{I.1.6}$$

which is specific to the spherical case.

In the minimizing case, the $(n-3)$ upper bound on the Hausdorff dimension of the singular set of u also relies on an ε-regularity theorem (which was originally proven using different arguments, relying on the minimality of u, by R. Schoen and K. Uhlenbeck[104]). One reduces the $(n-2)$ bound to $(n-3)$ by doing blowups around singular points (hinged on compactness of minimizing harmonic maps), which produces so-called tangent maps, and by stratification of the singular set (this is Federer's dimension argument). To be more precise, one shows that if the Hausdorff dimension of the singular set of some minimizing harmonic map u is larger than $(n-3)$, then one can build a minimizing 0-homogeneous harmonic map φ (a tangent map) from $\mathbb{R}^n$ into $\mathcal{N}$ whose singular set has nonvanishing $(n-2)$-Hausdorff measure, which is a contradiction.

To conclude this brief introduction on classical harmonic maps, we sum up in the following theorem some of the most well-known partial regularity results.

Theorem I.1.11. *Let $u \in H^1(\Omega;\mathcal{N})$ be a weak harmonic map in Ω, where $\mathcal{N}$ is of dimension $d-1$. Then $u \in C^\infty(u \setminus \operatorname{sing}(u))$ and we have*

(i) if $n=1$, then $\operatorname{sing}(u) = \emptyset$;

(ii) if $n=2$, then $\operatorname{sing}(u)$ is locally finite if $d=2$, and $\operatorname{sing}(u) = \emptyset$ if $d>2$;

(iii) if u is stationary and $n \geqslant 3$, then $\mathscr{H}^{n-2}(\operatorname{sing}(u)) = 0$;

(iv) if u is minimizing, $\operatorname{sing}(u)$ is locally finite when $n=3$, and $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n-3$ when $n>3$,

where $\dim_{\mathscr{H}} \operatorname{sing}(u)$ denotes the Hausdorff dimension of $\operatorname{sing}(u)$.

Fractional harmonic maps

In a series of articles [29, 28, 23, 24], F. Da Lio and T. Rivière have introduced and studied fractional 1/2-harmonic maps from the real line into a manifold. They naturally appear in several geometric problems such as minimal surfaces with free boundary[24, 26, 27, 47, 99, 110], and in some Ginzburg-Landau models for supraconductivity (see e.g. [8] and references therein). The notion of 1/2-harmonic maps has then been extended in [80, 86] to higher dimensions, and their natural generalization to s-harmonic maps for any $s \in (0,1)$ was studied e.g. in [82] in dimension 1 (in the minimizing case), in [81] in arbitrary dimension, as the asymptotic limit of solutions to a fractional Allen-Cahn equation, or in [93], although in a different setting. We introduce below the functional setting, the regularity issue for fractional harmonic maps and present the main contributions of Chapters 1 and 2.

The fractional Laplace operator. Similarly to the standard Laplace operator $(-\Delta)$, whose Fourier symbol is $(2\pi|\xi|)^2$, it is natural to define $(-\Delta)^s$ as the operator whose Fourier symbol is $(2\pi|\xi|)^{2s}$. That is, for any $u \in \mathscr{S}(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^n$, we define

$$(-\Delta)^s u := \mathcal{F}^{-1}((2\pi|\xi|)^{2s}\mathcal{F}(u)).$$

It is then well known (see e.g. [34]) that for any $u \in \mathscr{S}$, we have the integral representation

$$\begin{aligned}(-\Delta)^s u(x) &= \gamma_{n,s}\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\\ &= \frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(x)-u(x+h)+u(x-h)}{|h|^{n+2s}}\,\mathrm{d}h,\end{aligned}$$

where $\gamma_{n,s}$ is a constant depending only on n and $s \in (0,1)$. There are several ways to extend this definition to larger classes of functions (or even distributions, see Appendix A) than $\mathscr{S}(\mathbb{R}^n)$, and we present here the functional setting used in Chapters 1 and 2. Since we are interested in the action of $(-\Delta)^s u$ in Ω, we test $(-\Delta)^s u$ against maps $\varphi \in \mathscr{D}(\Omega)$, which gives

$$\langle (-\Delta)^s u, \varphi\rangle_{L^2} = \frac{\gamma_{n,s}}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y, \tag{I.1.7}$$

hence it is natural to define the fractional s-Dirichlet energy in Ω as

$$\mathcal{E}_s(u,\Omega) := \frac{\gamma_{n,s}}{4}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{I.1.8}$$

That way, we have the identity

$$\langle (-\Delta)^s u, \varphi\rangle_{L^2} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u+t\varphi,\Omega)\right]_{|t=0},$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\Omega)$. In view of (I.1.7) and (I.1.8), it is also natural to introduce the Hilbert space

$$\widehat{H}^s(\Omega;\mathbb{R}^d) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^d) \;:\; \mathcal{E}_s(u,\Omega) < +\infty\right\},$$

and for each $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$, to define $(-\Delta)^s u \in (\widehat{H}^s(\Omega;\mathbb{R}^d))'$ as the linear form

$$\langle (-\Delta)^s u, \varphi\rangle_{\Omega} := \frac{\gamma_{n,s}}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y,$$

for every $\varphi \in \widehat{H}^s(\Omega;\mathbb{R}^d)$. Let us emphasize the advantage of placing ourselves in a local setting, i.e., of considering the energy in Ω instead of the whole space. This is essentially because doing the latter is too restrictive: there are maps which belong to $\widehat{H}^s(\Omega;\mathbb{R}^d)$ for every bounded open set Ω, and however do not belong to $\dot{H}^s(\mathbb{R}^n;\mathbb{R}^d)$. It is easy for example to see that $C^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) \subseteq \widehat{H}^s(\Omega;\mathbb{R}^d)$ for every bounded open set $\Omega \subseteq \mathbb{R}^n$, thus this local setting allows us in particular to give a meaning to $(-\Delta)^s u$ for every $u \in C^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

Example I.1.12. Assume $n = 1$. Then the cosine and sine functions belong to $\widehat{H}^s(\Omega)$ for every bounded open set $\Omega \subseteq \mathbb{R}$, and we have $(-\Delta)^{\frac{1}{2}}\cos = \cos$ and $(-\Delta)^{\frac{1}{2}}\sin = \sin$.

Example I.1.13. Assume $n \geqslant 2$. Let $g \in C^1(\mathbb{S}^{n-1};\mathbb{R}^d)$ be nontrivial and $u : \mathbb{R}^n \to \mathbb{R}^d$ be the 0-homogeneous extension of g, i.e., $u(x) = g(\frac{x}{|x|})$. Then $u \in \widehat{H}^s(B_R;\mathbb{R}^d)$ for every $R > 0$, but $u \notin \dot{H}^s(\mathbb{R}^n;\mathbb{R}^d)$.

The previous example shows in particular that $x \mapsto \frac{x}{|x|} \in \widehat{H}^s(B_R;\mathbb{R}^n) \setminus \dot{H}^s(\mathbb{R}^n;\mathbb{R}^n)$ for every $n \geqslant 2$ and every $R > 0$.

We refer to Chapter 2 for more details on the space $\widehat{H}^s(\Omega;\mathbb{R}^d)$, and to Appendix A for a distributional, more general approach of the fractional Laplace operator.

Elliptic regularity. For the standard Laplacian, we know that a solution u of

$$-\Delta u = f, \quad \text{in } \Omega$$

morally gains 2 derivatives in Ω compared to f by the standard elliptic regularity theory. In analogy, one can show that solutions to

$$(-\Delta)^s u = f, \quad \text{in } \Omega$$

gain $2s$ "fractional" derivatives in Ω compared to f. We refer to [18], or to Appendix A for a rigorous statement and detailed proofs of this. In particular, solutions to $(-\Delta)^s u = 0$ in some open set Ω are smooth in Ω.

Fractional harmonic maps. Fractional harmonic maps are defined similarly to classical harmonic maps, as critical points of the fractional Dirichlet energy $\mathcal{E}_s$ with respect to variations on the target.

Definition I.1.14. A map $u \in \widehat{H}^s(\Omega;\mathcal{N})$ is said to be a weak s-harmonic map in Ω if it satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(\Pi_{\mathcal{N}}(u + t\varphi), \Omega)\right]_{|t=0} = 0, \quad \forall \varphi \in \mathscr{D}(\Omega;\mathbb{R}^d). \tag{I.1.9}$$

The condition of s-harmonicity (I.1.9) can be rewritten as

$$(-\Delta)^s u \perp \operatorname{Tan}(u, \mathcal{N})$$

in the weak sense:

$$\langle (-\Delta)^s u, \varphi \rangle_\Omega = 0, \quad \forall \varphi \in H^s_{00}(\Omega; u^*T\mathcal{N}),$$

where $H^s_{00}(\Omega; u^*T\mathcal{N})$ is the set of maps in $H^s(\Omega; u^*T\mathcal{N})$ vanishing a.e. in Ω^{c}.

Example I.1.15. When $\mathcal{N} = \mathbb{S}^{d-1}$, (I.1.9) takes the simple form

$$(-\Delta)^s u = |\mathrm{d_s}\, u|^2 u \quad \text{in } \Omega, \tag{I.1.10}$$

where

$$|\mathrm{d_s}\, u|^2(x) := \frac{\gamma_{n,s}}{2} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \,\mathrm{d}y.$$

Let us remark the strong analogy with Example I.1.3, where $|\mathrm{d_s}\, u|^2$ plays the role of the squared norm of the gradient in the equation of classical harmonic maps into spheres.

As for classical harmonic maps, fractional harmonic maps may always be regular in low dimension, but we expect that they can be totally irregular starting from dimension $n = 2$. For $s = \frac{1}{2}$, it was shown in [29, 28] that weak 1/2-harmonic maps from the real line are smooth, and another proof in the spirit of F. Hélein's proof for classical harmonic maps in dimension 2 was obtained later in [77] by K. Mazowiecka and A. Schikorra. We suspect that a construction similar to the one by T. Rivière in [92] could be done to produce a 1/2-harmonic map from the 2-dimensional disk into $\mathbb{S}^1$ which is discontinuous everywhere, but this has yet to be made. If $s \in (\frac{1}{2}, 1)$, by Sobolev embedding we see that for $n = 1$, s-harmonic maps are necessarily Hölder continuous, and by bootstrapping we can show that they are smooth (the steps from Hölder to Lipschitz continuity and then from Lipschitz continuity to full regularity is done e.g. in Chapter 2). Still in dimension 1, when $s \in (0, \frac{1}{2})$, s-harmonic maps may however have singularities.

Since we expect that there is no hope to get any kind of regularity for general fractional harmonic maps starting from dimension 2, we introduce the *minimizing* and *stationary* subclasses.

Definition I.1.16. We say that a map $u \in \widehat{H}^s(\Omega; \mathcal{N})$ is a *minimizing s-harmonic map in* Ω if

$$\mathcal{E}_s(u, \Omega) \leqslant \mathcal{E}_s(v, \Omega),$$

for any map $v \in \widehat{H}^s(\Omega; \mathcal{N})$ such that $u - v$ is compactly supported in Ω.

Definition I.1.17. We say that a weak s-harmonic map $u \in \widehat{H}^s(\Omega; \mathcal{N})$ is *stationary* if it satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u \circ \Phi_t, \Omega)\right]_{|t=0} = 0,$$

for every smooth vector field $X = (X^1, \ldots, X^n) \in C_c^\infty(\Omega; \mathbb{R}^d)$, where Φ_t is the associated integral flow.

In [82], the authors prove smoothness of minimizing s-harmonic maps in dimension 1, but the issue of regularity was still open in the stationary case for $s \in (0, \frac{1}{2})$. In Chapter 2 we answer partially this question, proving that when $s \in (0, \frac{1}{2})$ and $n = 1$, the singular set of stationary s-harmonic maps is locally finite.

Compared with classical harmonic maps, stationary s-harmonic maps do not necessarily have the property that $r \mapsto r^{2s-n}\mathcal{E}_s(u, B_r(x))$ is monotone, but it was shown in [80, 81] that a monotonicity formula holds for the fractional harmonic extension of u to the upper half-space, introduced in [18] by L. Caffarelli and L. Silvestre, whose definition we recall below.

Fractional harmonic extension. From now on we denote by $\mathbb{R}^{n+1}_+$ the upper half-space $\mathbb{R}^n \times (0, +\infty)$, and for any $\mathbf{x} = (x, 0) \in \partial\mathbb{R}^{n+1}_+$, we let $B^+_R(\mathbf{x})$ be the open upper half-ball in $\mathbb{R}^{n+1}_+$ of radius R centered at $\mathbf{x}$.

Given $u \in \widehat{H}^s(\Omega; \mathbb{R}^d)$, the *fractional harmonic extension* of u, denoted by $u^{\mathrm{e}} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^d$, is defined by

$$u^{\mathrm{e}}(x, z) := \sigma_{n,s} \int_{\mathbb{R}^n} \frac{z^{2s} u(y)}{(|x-y|^2 + z^2)^{\frac{n+2s}{2}}} \,\mathrm{d}y.$$

Note that u^{e} is simply the convolution of u (in y) with the “fractional Poisson kernel”

$$\mathbf{P}_{n,s}(y, z) := \sigma_{n,s} \frac{z^{2s}}{(|y|^2 + z^2)^{\frac{n+2s}{2}}},$$

where $\sigma_{n,s}$ is chosen so that

$$\int_{\mathbb{R}^n} \mathbf{P}_{n,s}(y, z)\,\mathrm{d}y = 1,$$

and u^{e} is the unique solution of

$$\begin{cases} \operatorname{div}(z^a \nabla u^{\mathrm{e}}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u^{\mathrm{e}} = u & \text{on } \partial\mathbb{R}^{n+1}_+ \simeq \mathbb{R}^n, \end{cases}$$

where $a := 1-2s$. When $s = \frac{1}{2}$, u^{e} is simply the harmonic extension of u, i.e., the solution of

$$\begin{cases} \Delta u^{\mathrm{e}} = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u^{\mathrm{e}} = u & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

It is well known that u^{e} is well-defined on $\mathbb{R}^{n+1}_+$ whenever $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$ and belongs to a weighted Sobolev space in $G \subseteq \mathbb{R}^{n+1}_+$, provided G is sufficiently regular and its "trace" on $\partial\mathbb{R}^{n+1}_+$ is compactly included in Ω. Let us point out that, defining the weighted Dirichlet energy in $G \subseteq \mathbb{R}^{n+1}_+$ by

$$\mathbf{E}_s(u^{\mathrm{e}}, G) := \frac{\delta_s}{2} \int_G |\nabla u^{\mathrm{e}}|^2 \, |z|^a \mathrm{d}\mathbf{x},$$

with δ_s well-chosen we have

$$[u]^2_{H^s(\mathbb{R}^n)} = \mathbf{E}_s(u^{\mathrm{e}}, \mathbb{R}^{n+1}_+), \quad \forall u \in H^s(\mathbb{R}^n;\mathbb{R}^d).$$

On a bounded open set Ω, there is no simple equivalent to the above equality, however we can control the H^s seminorm of u in $B_R(x)$ by the energy of u^{e} in $B^+_{2R}(\mathbf{x})$, where $\mathbf{x} = (x, 0)$, and conversely we can control the energy of u^{e} in $B^+_R(\mathbf{x})$ by $\mathcal{E}_s(u, B_{2R}(x))$, so that we almost have "equivalence" of the energy of u and of its fractional harmonic extension. The harmonic extension has many interesting properties. First, the fractional Laplacian is realized as the Dirichlet-to-Neumann operator induced by the extension procedure, that is, the distributional normal trace of $z^a\nabla u^{\mathrm{e}}$ on $\partial\mathbb{R}^{n+1}_+ \simeq \mathbb{R}^n$ is $(-\Delta)^s u$ (up to a multiplicative constant), i.e. $-\delta_s z^a \partial_z \, u^{\mathrm{e}} = (-\Delta)^s u$ on $\partial\mathbb{R}^{n+1}_+$. When $s = \frac{1}{2}$, this implies

$$\begin{cases} \Delta u^{\mathrm{e}} = 0, & \text{in } G \\ u^{\mathrm{e}} \in \mathcal{N}, & \text{on } \Omega \\ \partial_z \, u^{\mathrm{e}} \perp \operatorname{Tan}(u^{\mathrm{e}}, \mathcal{N}) & \text{on } \Omega, \end{cases}$$

for any 1/2-harmonic map u in Ω, where G is a suitable smooth extension of Ω in $\mathbb{R}^{n+1}_+$, so that u^{e} is a so-called *harmonic map with partially free boundary in* $\mathcal{N}$, for which there is a regularity theory up to the boundary. Taking advantage of this, V. Millot and Y. Sire obtained in [80] the following partial regularity result.

Theorem I.1.18 ([80, Theorem 4.18])**.** *If $u \in \widehat{H}^{1/2}(\Omega;\mathbb{S}^{d-1})$ is a 1/2-harmonic map in Ω, then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$, and*

(i) if $n = 1$, then $\operatorname{sing}(u) = \emptyset$;

(ii) if $n \geqslant 2$ and u is stationary, then $\mathscr{H}^{n-1}(\operatorname{sing}(u)) = 0$;

(iii) if u is minimizing, then $\operatorname{sing}(u)$ is locally finite when $n = 2$, and $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n-2$ when $n \geqslant 3$.

One of our main results in Chapter 1 refines the upper bound of the previous theorem on the Hausdorff dimension of the singular set in the minimizing case when the target in a sphere, and states that minimizing 1/2-harmonic maps into spheres (of dimension larger than 1) are smooth in dimension $n = 2$.

A second interesting property of the extension is what we often refer to as the "criticality transfer" property. It states that the fractional harmonic extension of a fractional harmonic map u is also a critical point of the weighted Dirichlet energy with respect to a specific type of variations on the target, which we do not detail here. Modulo some technicalities, the converse holds as well: if the map u^{e} is a critical point of the weighted Dirichlet energy in sufficiently regular open sets with respect to those variations, then u is a fractional harmonic map in some open set $\Omega \subseteq \mathbb{R}^n$. Even more interesting, the stationarity and minimality properties of u are transferred to its extension, and conversely. In view of these transfer properties and the "equivalence" between the fractional Dirichlet energy $\mathcal{E}_s$ and the weighted Dirichlet energy $\mathbf{E}_s$, in many cases we may chose indifferently to work either with u or with its extension u^{e}, whichever is more handy.

However, working with the extension has one huge advantage: there is a monotonicity formula for the extension of stationary harmonic maps (originally proven in [80]). Precisely, if u is a stationary s-harmonic map in Ω, then for every $\mathbf{x} = (x, 0)$ such that $x \in \Omega$, the function $r \mapsto r^{2s-n}\mathbf{E}_s(u^{\mathrm{e}}, B_R^+(\mathbf{x}))$ is nondecreasing. In analogy with classical harmonic maps, this allows for the construction of tangent maps, stratification of the singular set, and the use of Federer's dimension reduction argument, but also a control of the BMO seminorm of stationary s-harmonic maps.

Minimizing 1/2-harmonic maps. In Chapter 1, we focus on minimizing 1/2-harmonic maps into spheres. First we improve Theorem I.1.18 in the minimizing case when $d \geqslant 3$, proving the following theorem.

Theorem I.1.19. *Assume that $d \geqslant 3$. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. If $u \in \widehat{H}^{1/2}(\Omega; \mathbb{S}^{d-1})$ is a minimizing 1/2-harmonic map in Ω, then $\operatorname{sing}(u) = \emptyset$ for $n \leqslant 2$, $\operatorname{sing}(u)$ is locally finite in Ω for $n = 3$, and $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n - 3$ for $n \geqslant 4$.*

Using Federer's dimension reduction argument, proving the theorem amounts to showing that there exists no nontrivial 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^{d-1}$. As in the classical case, where it was possible to show smoothness for weak harmonic maps from $\mathbb{R}^3$ into $\mathbb{S}^3$ using a geometric argument involving minimal surfaces, our proof in the setting of 1/2-harmonic maps relies on a geometric result on minimal surfaces with partially free boundary. To be precise we use the recent result by A. Fraser and R. Schoen[48], that a minimal disk whose boundary lies in the $(d-1)$-dimensional unit sphere, for $d \geqslant 3$, must be a flat disk through the origin, extending a famous result of J. C. C. Nitsche[87] for $d = 3$ to arbitrary spheres. This implies that any 1/2-harmonic from the real line into $\mathbb{S}^{d-1}$ lies in an equator, and as a consequence that any nonconstant 0-homogeneous 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^{d-1}$ is equatorial. When can then destabilize this tangent map by doing variations in the orthogonal direction of the equator, and show that it contradicts its minimality. The contradiction arises from the knowledge of the sharp constant in a Hardy-type inequality, and the fact that 1/2-harmonic maps into $\mathbb{S}^1$ are completely described by so-called Blaschke products (see [80]), so that their energy is necessarily an integer multiple of π.

Theorem I.1.20. *The map $u_\star : \mathbb{R}^2 \to \mathbb{S}^1$ given by $u_\star(x) := \frac{x}{|x|}$ is a minimizing 1/2-harmonic map in $\mathbb{R}^2$. Moreover, it is the unique nonconstant 0-homogeneous minimizing 1/2-harmonic map up to an orthogonal transformation.*

The second main result of Chapter 1 states that a minimizing 1/2-harmonic map from a two dimensional domain into $\mathbb{S}^1$ must have a degree ± 1 at each singularity, where the topological degree at a singular point is defined as the degree of the restriction to any small circle surrounding the point.

Theorem I.1.21. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded open set. If $u \in \widehat{H}^{1/2}(\Omega; \mathbb{S}^1)$ is a minimizing $1/2$-harmonic map in Ω and $a \in \Omega \cap \operatorname{sing}(u)$, then $\deg(u, a) \in \{+1, -1\}$.*

To prove the minimality of $u_\star$, we follow a strategy similar to the one employed by H. Brezis, J.-M. Coron, and E. H. Lieb[13]. Introducing the distributional Jacobian of $H^{1/2}$ maps via the harmonic extension, we obtain a lower bound on the energy of any minimizing harmonic map with partially free boundary in $\mathbb{S}^{d-1}$ which agrees with $u_\star^{\mathrm{e}}$ outside a compact subset of $B_1^+ \cup (D_1 \times \{0\})$, where D_1 denotes the n-dimensional open unit ball in $\mathbb{R}^n$, and show that this bound agrees with the energy of $u_\star^{\mathrm{e}}$. Then to prove uniqueness, we take advantage of the fact that $1/2$-harmonic maps from the real line into $\mathbb{S}^1$ are given by Blaschke products, so that we explicitly know the form of 0-homogeneous $1/2$-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^1$. Direct computations show that $u_\star$ is the unique minimizer among 0-homogeneous maps with degree ± 1, up to orthogonal transformations. Then, as in [13], we exclude maps of higher degree by using suitable competitors, but the construction is much more technical and in the end relies on the numerical computation of some integrals.

Stationary and minimizing s-harmonic maps. Chapter 2 is essentially devoted to the study of regularity for stationary and minimizing s-harmonic maps into spheres $\mathbb{S}^{d-1}$ of arbitrary dimension, where $s \in (0, 1)$. We prove partial regularity results for stationary s-harmonic maps into spheres, which are, to our knowledge, new results, and improve already-known partial regularity results in the minimizing case when $s \neq \frac{1}{2}$. Our main results are summarized in the following theorems.

Theorem I.1.22. *Assume that $n \leqslant 2s$. If $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is a weakly s-harmonic map in Ω, then $u \in C^\infty(\Omega)$.*

Theorem I.1.23. *Assume that $n > 2s$. If $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is a stationary weakly s-harmonic map in Ω, then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ and*

(1) for $s > 1/2$ and $n \geqslant 3$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n - 2$;

(2) for $s > 1/2$ and $n = 2$, $\operatorname{sing}(u)$ is locally finite in Ω;

(3) for $s = 1/2$ and $n \geqslant 2$, $\mathscr{H}^{n-1}(\operatorname{sing}(u)) = 0$;

(4) for $s < 1/2$ and $n \geqslant 2$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n - 1$;

(5) for $s < 1/2$ and $n = 1$, $\operatorname{sing}(u)$ is locally finite in Ω.

Theorem I.1.24. *Assume that $n > 2s$. If $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is a minimizing s-harmonic map in Ω, then $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ and*

(1) for $n \geqslant 3$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n - 2$;

(2) for $n = 2$, $\operatorname{sing}(u)$ is locally finite in Ω;

(3) for $n = 1$, $\operatorname{sing}(u) = \emptyset$ (i.e., $u \in C^\infty(\Omega)$).

Our approach is inspired by the one carried out by L. Evans in [39], where the author proves that the singular set of any stationary classical harmonic map from $\Omega \subseteq \mathbb{R}^n$ into $\mathbb{S}^{d-1}$ has null $(n-2)$-Hausdorff measure. In fact, we almost have access to the same ingredients described above in the case of classical harmonic maps:

(i) we have a monotonicity formula, through the fractional harmonic extension;

(ii) using the fractional divergence and gradient quantities recently introduced in [77], the right-hand side of (I.1.10) "almost" has a fractional div-curl structure;

(iii) in [77], K. Mazowiecka and A. Schikorra prove a fractional div-curl lemma similar to the one by Coifman, Lions, Meyer and Semmes, so that the right-hand side of (I.1.10) is the sum of an element in $\mathcal{H}^1(\mathbb{R}^n)$ and of an extra term, which needs to be carefully examined.

We say that the right-hand side of (I.1.10) "almost" has a fractional div-curl structure, because, contrary to the classical case, the s-harmonic maps equation cannot be rewritten as simply as (I.1.6), but an extra term appears. In fact, formally, (I.1.10) may be rewritten as

$$(-\Delta)^s v^i = \sum_{j=1}^{d} (\Omega^{i,j} \odot \mathrm{d_s}\, v^j) + T, \quad \forall i \in \{1, \dots, d\},$$

where $\mathrm{d_s}\, u$ plays the role of a fractional s-gradient, and $\Omega^{i,j}$ and T are defined by

$$\begin{aligned} \Omega^{i,j}(x,y) &:= v^i(x)\, \mathrm{d_s}\, v^j(x,y) - v^j(x)\, \mathrm{d_s}\, u^i(x,y) \\ T(x) &:= \Big(\mathrm{d_s}\, v^i \odot (v^j(x)\, \mathrm{d_s}\, v^j)\Big)(x). \end{aligned}$$

Here we do not elaborate on the meaning of the $\odot$ product, the idea being that $\Omega^{i,j}$ has vanishing fractional s-divergence, and that $\Omega^{i,j} \odot \mathrm{d_s}\, v^j$ has a fractional div-curl structure. The extra T term can be handled using a nontrivial embedding between $\mathcal{Q}_p^{\alpha,q}$-spaces (see [114]) which is a direct consequence of continuous embeddings between Morrey-Lorentz flavors of Triebel-Lizorkin spaces proven in [63], as well as identification of those spaces with $\mathcal{Q}_p^{\alpha,q}$-spaces in some cases (see [114, 115]). Combining all these tools, we can prove an ε-regularity theorem for stationary s-harmonic maps into spheres. An immediate consequence of the ε-regularity theorem is that the singular set of any stationary s-harmonic map into spheres has null $(n-2s)$-Hausdorff measure.

We then improve this result further when $s \neq \frac{1}{2}$ by proving compactness of stationary s-harmonic maps into $\mathbb{S}^{d-1}$ with bounded fractional Dirichlet energy and then applying Federer's dimension reduction principle. In order to prove compactness of those maps, we need to improve Hölder regularity to Lipschitz regularity away from singular points, which we do by adapting some proofs by J. Roberts[93], who produced a thorough regularity theory for minimizing fractional harmonic maps in a different setting. In fact, to be completely exhaustive, we prove that fractional harmonic maps are smooth outside their singular set, by a rather technical bootstrap procedure based on the fractional harmonic maps equation. The compactness of stationary s-harmonic maps with bounded fractional Dirichlet energy when $s \neq \frac{1}{2}$ then follows by Marstrand's theorem, which states that a Radon measure μ cannot have a positive and finite noninteger density on a set of nonvanishing μ measure. When $s = \frac{1}{2}$, non-compactness is well-known, and we give an example of a noncompact bounded sequence.

Open questions and perspectives. Let us conclude this summary of our results by stating a few open questions and perspectives on fractional harmonics maps.

- It would be interesting to see if a construction similar to the one by Tristian Rivière is possible for 1/2-harmonic maps, i.e., if it is possible to build a 1/2-harmonic map from $\mathbb{D}$ into $\mathbb{S}^1$ which is everywhere discontinuous.
- In the case $s \neq 1/2$, very few properties are known about minimizing s-harmonic maps for $n > 1$, except for partial regularity results. We do not even know if $x/|x|$ is minimizing in dimension 2, that is, if the approach using the distributional Jacobian is reproductible in that case. Concerning the characterization of tangent

maps from $\mathbb{R}^2$ into $\mathbb{S}^1$, most of the work done for the case $s = 1/2$ relies on the full knowledge of 1/2-harmonic maps from the real line, that is, the fact that they are given by Blaschke products (see [80]). For $s \neq 1/2$, we do not know if a similar characterization can be obtained in dimension 1.

- In the minimizing case, we know that both 1/2-harmonic maps and "classical" harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^{d-1}$ are smooth (for $d \geqslant 3$). We have reasons to believe that, by a compactness argument, we might be able to show that minimizing s-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^{d-1}$ are smooth (for $d \geqslant 3$) when s is close to 1/2 or close to 1. Then we could investigate whether this is actually true for any $s \in (1/2, 1)$. It would in addition immediately allows us to further reduce the dimension of the singular set for minimizing s-harmonic maps into spheres to $n-3$ for any $s \in (1/2, 1)$.

I.2 An isoperimetric problem with a nonlocal repulsive potential

Chapter 3 is dedicated to the study of large mass minimizers of the problem

$$\min \left\{ P(E) + \iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \ : \ |E| = m \right\}, \tag{P1}$$

where m is a positive number and $P(E)$ denotes the perimeter of E, defined by

$$P(E) := \sup \left\{ \int_E \operatorname{div} \varphi \ : \ \varphi \in \mathscr{D}(\mathbb{R}^n;\mathbb{R}^n) \text{ such that } \|\varphi\|_{L^\infty(\mathbb{R}^n)} \leqslant 1 \right\}.$$

Sets whose perimeter is finite are naturally called *sets of finite perimeter*. Observe that in Problem (P1), we have a competition between two terms: the (local) perimeter term constrains the set E to concentrate as much as possible, while the nonlocal term acts as a repulsive term, forcing E to spread. Indeed, it is known that the perimeter is *minimized* by balls under volume constraint, while the nonlocal term is *maximized* by balls. Before stating the assumptions we make on the kernel K, let us say a few words about this problem for some specific kernels.

The Riesz case. Problems such as (P1) are motivated by a model for the atomic nucleus introduced by George Gamow in the late 1920s, which is now referred to as Gamow's liquid drop model for the atomic nucleus. This denomination is due to the fact that in this simple model (then refined by Heisenberg, von Wiezsäcker and Bohr in the 1930s), the protons and neutrons inside the atomic nucleus are treated as an incompressible and uniformly charged fluid. In this model, the atomic nucleus is represented by a set $\Omega \subseteq \mathbb{R}^3$ of volume m (which we also call the mass), and its energy is given by

$$P(\Omega) + \frac{1}{8\pi} \iint_{\Omega\times\Omega} \frac{1}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

The perimeter term represents the energy associated with the attractive short-range nuclear force, while the Coulombic repulsive term is due to the positively charged protons pushing themselves away from each other. This model successfully explained the phenomenon of nuclear fission: indeed, there are two critical masses $0 < m_1 \leqslant m_2 < \infty$ such that, below m_1, the problem admits a minimizer (no fission), and above m_2, there is no minimizer (fission). In fact, there exists another threshold $0 < m_0 \leqslant m_1$ such that below it, the ball is the unique minimizer (up to translation). These results were first rigorously proven in [66] (see also [67] for the planar case). Many variants and generalizations of this model have been proposed and extensively studied since then (see e.g. [67, 66, 68, 65, 10, 45]), one of the most natural being to replace the Newton potential $\frac{1}{|x|^{n-2}}$ in dimension 3 by Riesz potentials in arbitrary dimension $n \geqslant 2$, that is,

$$K(x) = \frac{1}{|x|^{n-\alpha}}, \quad \alpha \in (0,n).$$

The Newton case $\alpha = 2$ in dimension $n \geqslant 3$ was treated e.g. in [65], the Riesz cases with $\alpha \in (0, n-1)$ in [10], and the complete Riesz case $\alpha \in (0,n)$ in any dimension in [45], where the perimeter $P(E)$ can also be replaced by fractional perimeters $P_s(E)$, $s \in (0,1)$.

In the two following theorems we give a nonexhaustive summary of some results known in the Riesz case.

Theorem I.2.1 ([67, 66, 65, 10, 45])**.** *Given $n \geqslant 2$ and $\alpha \in (0,n)$, there exists $m_0 = m_0(n,\alpha)$ such that for any $m < m_0$ the ball of volume m, denoted by $[B]_m$, is the unique minimizer, up to translations, of Problem (P1) for $K(x) = |x|^{-(n-\alpha)}$. Here $[B]_m$ denotes the unit open ball of volume m in $\mathbb{R}^n$, centered at the origin.*

There are also some nonexistence results.

Theorem I.2.2 ([67, 66, 72])**.** *Given $n \geqslant 2$ and $\alpha \in (n-2,n)$, then there exists $m_1 = m_1(n,\alpha)$ such that for any $m > m_1$, Problem (P1) admits no minimizer for $K(x) = |x|^{-(n-\alpha)}$.*

We can give clues as to why Problem (P1) admits no large mass minimizers in the Riesz case. First, observe that without the perimeter term, the problem

$$\min\left\{\iint_{E\times E} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \ : \ |E| = m\right\}$$

admits no minimizer, since it is always better (Riesz kernels being stricly radially decreasing) to split a set E into infinitely many pieces and send them farther from each other at infinity. Secondly, the relatively slow decay at infinity of the Riesz kernels make them nonintegrable, which would indicate that the repulsive potential takes over the perimeter term in Problem (P1) for large masses, and would explain the nonexistence of large mass minimizers.

As for the thresholds m_0, m_1, and m_2, physical evidence indicate that in dimension $n = 3$ at least, they should be equal, but this has yet to be proven.

The Bessel case. Other physically relevant potentials have been suggested (e.g. in [68]), such as Bessel potentials, which are rapidly decreasing kernels compared with Riesz kernels, and are in particular integrable at infinity. Bessel kernels are given by the operators $(I-\Delta)^{-\frac{\alpha}{2}}$ for $\alpha \in (0,n)$, i.e., the Bessel kernel of order α is the fundamental solution of

$$(I-\Delta)^{\frac{\alpha}{2}} f = \delta_0,$$

where δ_0 is the Dirac distribution at the origin. In fact, we can consider the "generalized" Bessel-type potentials given by $(I-\kappa\Delta)^{-\frac{\alpha}{2}}$, where $\alpha,\kappa \in (0,+\infty)$. As far as we know there is little literature on Problem (P1) when K is a Bessel kernel, and especially on the asymptotic behavior for large masses. Unlike Riesz kernels (which are fundamental solutions of $(-\Delta)^{\frac{\alpha}{2}} f = \delta_0$), Bessel kernels are generally not explicit, in the sense that they only have an integral representation, and they do not behave as nicely as Riesz kernels under scaling. Near the origin, Riesz and Bessel kernels of the same order α behave similarly, however at infinity Bessel kernels decay much faster: their decay at infinity is *exponential*.

For small masses, the similarity between Riesz and Bessel kernels near the origin suggests that Problem (P1) presents the same kind of behavior whether K is a Riesz or a Bessel kernel of order α, that is, there exists a critical mass below which, up to translations, the ball of volume m is the unique minimizer. In this "small volume" case we believe the approach for the Riesz case in [45] can be adapted without major difficulties, but this is not the subject of Chapter 3. We are more interested in the case of large volumes. For Riesz kernels of order $\alpha \in (n-2, n)$, by Theorem I.2.2 we know that above a critical mass, Problem (P1) admits no minimizer. Here, as we prove in Chapter 3, the better integrability of the Bessel kernels changes the asymptotic behavior when the mass goes to infinity: if κ is small enough, Problem (P1) admits large mass minimizers, and up to translations, any sequence of normalized (to unit mass) minimizers converges to the unit ball as the mass goes to infinity.

Compactly supported potentials. Since we see that the Riesz and the Bessel cases present a very different behavior, which we attribute to the faster decay at infinity of the Bessel kernels, an interesting question is: what happens when the decay is even faster than exponential, e.g. when K is compactly supported?

Recalling our informal discussion on nonexistence for Riesz potentials, we see that in the compact support case, sending disjoint pieces of a set E at infinity does not decrease the energy of the nonlocal term: when the pieces are far enough, they simply have no interaction between each other. Thus we may imagine that we should be able to build a minimizing sequence lying in a fixed ball, hence get some compactness and prove the existence of minimizers by the direct method in the calculus of variations. In dimension $n = 2$ this strategy can be implemented quite easily (the advantage being that sets of finite perimeter are essentially bounded, i.e. included in a ball), but in higher dimension it is not that simple.

Using the link between minimizers of (P1) and so-called "quasi-minimizers" of the perimeter (in Chapter 3 we prefer the terminology of "almost-minimizers"), that case was nonetheless successfully treated by S. Rigot in [88], yielding the following result.

Theorem I.2.3 ([88]). *If K is compactly supported, then Problem (P1) always admits minimizers. In addition, for any minimizer E, $\partial^* E$ is a $C^{1,\frac{1}{2}}$-hypersurface, and, up to a renormalization, E has a finite number of connected components N, where N and the regularity constant of $\partial^* E$ can be bounded depending only on K and n.*

General assumptions. In Chapter 3, we consider general kernels K satisfying the following assumptions:

(H1) $K \in L^1(\mathbb{R}^n) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, and

$$\int_{\mathbb{R}^n} |x| K(x)\,\mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n} |\nabla K(x)||x|\,\mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n} |\nabla K(x)||x|^2\,\mathrm{d}x < +\infty;$$

(H2) there exists a nonnegative and nonincreasing function $k : (0, +\infty) \to \mathbb{R}$ such that $K(x) = k(|x|)$ for $\mathscr{L}^n - a.e.\ x \in \mathbb{R}^n$;

(H3) k' is continuous on $(0, +\infty)$.

Sometimes, we add the extra assumption

(H4) $K(x) = o(|x|^{n+1})$ at infinity, and there exists some $\alpha > 0$ such that $K(x) = o(|x|^{\alpha-n})$ near the origin.

It is easy to see that all four assumptions are satisfied by the generalized Bessel kernels for all $\alpha, \kappa \in (0, +\infty)$. To state our main results, we define

$$I_K^{l,p} := \int_{\mathbb{R}^n} |x|^p |\partial_r^l K(x)| \, \mathrm{d}x = n|B_1| \int_0^\infty |k'(r)| r^{p+n-1} \, \mathrm{d}r$$

for $l \in \{0, 1\}$ and $p \in \{0, 1, 2\}$, where $\partial_r K$ is the radial derivative of K, and

$$\mathbf{K}_{p,n} := \fint_{\mathbb{S}^{n-1}} |e \cdot x|^p \, d\mathscr{H}_x^{n-1},$$

which does not depend on $e \in \mathbb{S}^{n-1}$ by symmetry.

One of the keys to study Problem (P1) is to notice that

$$\iint_{E\times E} K(x-y) \, \mathrm{d}x \, \mathrm{d}y = mI_K^{0,0} - \iint_{E\times E^c} K(x-y) \, \mathrm{d}x \, \mathrm{d}y,$$

since K is integrable, so that Problem (P1) is equivalent to

$$\min \{ Per_K(E) \ : \ |E| = m \},$$

where

$$\mathrm{Per}_K(E) := \iint_{E\times E^c} K(x-y) \, \mathrm{d}x \, \mathrm{d}y,$$

which should be seen as a "nonlocal perimeter" term. Another key is to write $\mathrm{Per}_K(E)$ as

$$\mathrm{Per}_K(E) = \frac{1}{2} \iint_{\mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|} \eta(x-y) \, \mathrm{d}x \, \mathrm{d}y,$$

where χ_E is the indicator function of E and $\eta(x) := |x|K(x)$. We can then use the results of J. Bourgain, H. Brezis, and P. Mironescu[12] (more precisely its extension to BV functions obtained in [32]) to show that the nonlocal perimeter Γ-converges to a positive constant C times the classical perimeter when m goes to infinity. Hence we can hope that there exists a critical mass above which Problem (P1) admits a minimizer whenever the constant C is small enough. This is our first main result.

Theorem I.2.4. *Assume $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. Then there exists $m_e = m_e(n, K)$ such that, for any $m > m_e$, Problem (P1) admits a minimizer, and any minimizer E is, up to a translation, included in $4[B]_m$ up to a set of vanishing Lebesgue measure, where $[B]_m$ is the ball of volume m centered at the origin.*

The proof of existence is not trivial, the main obstacle for using the direct method in the calculus of variations being the possibility for a minimizing sequence to have some mass escape at infinity. We solve this problem of lack of compactness by showing that for large masses, a minimizing sequence may be constrained inside a ball via a truncation lemma.

We then study the Γ-limit of the rescaled functional of Problem (P1), and show that if $I_K^{0,1}$ is small enough, the Γ-limit is a positive multiple of the perimeter. This implies our second main result.

Theorem I.2.5. *Assume $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers going to infinity, and for all $k \in \mathbb{N}$, let E_k be a minimizer of Problem (P1) of mass m_k such that $\int_{E_k} x \,\mathrm{d}x = 0$. Then letting $F_k := \left(\frac{|B_1|}{m_k}\right)^{\frac{1}{n}} E_k$, the sequence $(F_k)_{k\in\mathbb{N}}$ of sets of finite perimeter of volume $|B|$ converges to the unit ball B centered at the origin w.r.t. to the L^1 norm, i.e.,*

$$|F_k \triangle B| \xrightarrow{k\to\infty} 0.$$

Applying results from [88], when $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, we obtain that minimizers have a $C^{1,\frac{1}{2}}$ reduced boundary, and we show that they are necessarily connected whenever K is nontrivial. Then we recall a well-suited notion of stability and show that if $I_K^{0,1}$ is above the threshold given in the statements of the previous theorems, then for large masses, balls are not stable, and thus cannot be minimizers of Problem (P1).

Theorem I.2.6. *If $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ and K satisfies (H4), there exists m_u such that for any $m > m_u$ the ball $[B]_m$ is not stable for the functional of Problem (P1).*

In particular, if $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, then balls cannot be minimizers (if one exists) for large masses. This is particularly interesting in the case where K is compactly supported. Indeed for such kernels we know that Problem (P1) admits minimizers for all masses, and for large masses none of them can be a ball, i.e., symmetry breaking occurs.

The proof for the instability of large balls relies essentially on the study of the Jacobi operator associated with the minimized functional, and on a result similar to the one by J. Bourgain, H. Brezis, and P. Mironescu in [12] for $W^{1,2}(\mathbb{S}^{n-1})$, i.e., computation of the limit,

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1},$$

where $(\eta_\varepsilon)_{\varepsilon>0}$ is a family of "$(n-1)$-dimensional mollifiers".

Open questions and perspectives. Here again, we conclude this introductory part by stating a few open questions and perspectives on this isoperimetric problem.

- As stated in the exploratory Section 3.4.5, it should be possible to improve the convergence and show that the reduced boundary of rescaled minimizers converge to the ball w.r.t. the Hausdorff distance, but for this one would need to obtain density estimates for rescaled minimizers of Problem (P1) uniform in $m^{1/n}$, which has yet to be done.

- Then we would like to address the question of whether there exists a critical mass such that for larger masses, the ball is a minimizer, when $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. For this, a first step is to study the stability of the ball in the different regimes (small masses, large masses, and the intermediate case). By Theorem I.2.6 shown in Chapter 3, we know that large balls are not stable if $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, but we do not know if large balls can be stable if $I_K^{0,1}$ is small enough. We believe stability could be proven by obtaining uniform upper bounds for the quantity

$$\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1}$$

 in terms of $\int_{\mathbb{S}^{n-1}}|\nabla f|^2\, d\mathscr{H}^{n-1}$ for $f \in W^{1,2}(\mathbb{S}^{n-1})$. If large balls are stable for any $I_K^{0,1}$ small enough, it could be possible to show that they are local minimizers w.r.t.

L^1 perturbations, and then perhaps global minimizers, but these are only conjectures for now.

- It could also be interesting to make numerical simulations in dimension 2 for different Bessel potentials, to test whether minimizers exist in the intermediate regime, and if they are close to balls or not.

Chapter 1

Minimizing 1/2-harmonic maps into spheres

1.1 Introduction

In a series of articles [29, 28, 23, 24], F. Da Lio & T. Rivière have introduced and studied fractional 1/2-*harmonic maps* from the real line into a manifold. Given a compact smooth submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ without boundary, 1/2-harmonic maps into $\mathcal{N}$ are defined as critical points of the so-called 1/2-Dirichlet energy under the constraint to be $\mathcal{N}$-valued. They naturally appear in several geometric problems such as minimal surfaces with free boundary, see [24, 26, 27, 47, 99, 110] and Section 1.4.2. They also come into play in some Ginzburg-Landau models for supraconductivity, see e.g. [8] and references therein. The Euler-Lagrange equation satisfied by 1/2-harmonic maps is in strong analogy with the standard harmonic map system. Instead of the usual Laplace operator, the equation involves the square root Laplacian as defined in Fourier space (i.e., the multiplier operator of symbol $2\pi|\xi|$), and it suffers the same pathologies regarding regularity. A main issue was then to prove the smoothness *a priori* of weak solutions. It has been achieved in [29, 28], thus extending the famous regularity result of F. Hélein for harmonic maps from surfaces [61]. The notion of 1/2-harmonic maps has been extended in [80, 86] to higher dimensions, and partial regularity for minimizing or stationary 1/2-harmonic maps established (again in analogy with minimizing/stationary harmonic maps [9, 39, 103]). Before going further, let us now describe in detail the mathematical framework.

Given a bounded open set $\Omega \subseteq \mathbb{R}^n$, the 1/2-Dirichlet energy in Ω of a measurable map $u : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$\mathcal{E}_{\frac{1}{2}}(u,\Omega) := \frac{\gamma_n}{4} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y\,,$$

where $\Omega^c := \mathbb{R}^n \setminus \Omega$. The normalization constant $\gamma_n := \pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2})$ is chosen in such a way that

$$\mathcal{E}_{\frac{1}{2}}(u,\Omega) = \frac{1}{2}\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{1}{4}}u\right|^2 \mathrm{d}x \quad \text{for every } u \in \mathscr{D}(\Omega)\,.$$

Following [80, Section 2], we denote by $\widehat{H}^{1/2}(\Omega;\mathbb{R}^m)$ the Hilbert space made of all $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\mathcal{E}_{\frac{1}{2}}(u,\Omega) < \infty$, and we set

$$\widehat{H}^{1/2}(\Omega;\mathcal{N}) := \left\{u \in \widehat{H}^{1/2}(\Omega;\mathbb{R}^m) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^n\right\}.$$

Definition 1.1.1. A map $u \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is said to be a weakly 1/2-harmonic map in Ω with values in $\mathcal{N}$ if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\frac{1}{2}}\left(\pi_{\mathcal{N}}(u+t\varphi),\Omega\right)\right]_{|t=0} = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega;\mathbb{R}^m)\,,$$

where $\pi_{\mathcal{N}}$ denotes the nearest point projection on $\mathcal{N}$.

According to [80, Section 4], a weakly 1/2-harmonic map in Ω satisfies the variational Euler-Lagrange equation

$$\frac{\gamma_n}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y = 0 \tag{1.1.1}$$

for every $\varphi \in H^{1/2}_{00}(\Omega;u^*T\mathcal{N})$. In other words, (1.1.1) holds for every $\varphi \in H^{1/2}_{00}(\Omega;\mathbb{R}^m)$ satisfying $\varphi(x) \in \mathrm{Tan}(u(x),\mathcal{N})$ for a.e. $x \in \Omega$ (recall that $H^{1/2}_{00}(\Omega)$ is the completion of $\mathscr{D}(\Omega)$ in $H^{1/2}(\mathbb{R}^n)$ for the norm topology). This equation is the weak formulation of the nonlinear system

$$(-\Delta)^{\frac{1}{2}}u \perp \mathrm{Tan}(u,\mathcal{N}) \quad \text{in } \Omega\,, \tag{1.1.2}$$

where $(-\Delta)^{\frac{1}{2}}$ is the integro-differential operator given by

$$(-\Delta)^{\frac{1}{2}}u(x) := \gamma_n\,\mathrm{p.v.}\left(\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+1}}\,\mathrm{d}y\right)\,.$$

(The notation p.v. means that the integral is taken in the Cauchy principal value sense.) In the case $\mathcal{N} = \mathbb{S}^{m-1}$ (the unit sphere of $\mathbb{R}^m$), the Lagrange multiplier relative to the constraint to be $\mathbb{S}^{m-1}$-valued takes a very simple form, and equation (1.1.2) rewrites (see [80, Remark 4.3])

$$(-\Delta)^{\frac{1}{2}}u(x) = \left(\frac{\gamma_n}{2}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y\right)u(x) \quad \text{in } \Omega\,. \tag{1.1.3}$$

In this case, it is clear that the right-hand side in (1.1.3) has a priori no better integrability than $L^1(\Omega)$, and thus linear elliptic theory does not apply to determine the smoothness of solutions. In [29, 28] and subsequently in [77], the authors have shown that the source term can actually be rewritten in some “fractional div-curl form". As a consequence, nonlinear compensations appear and the right-hand side of (1.1.3) belongs in fact to the Hardy space. In dimension 1, it leads to continuity and then full regularity as it happens for harmonic maps in dimension 2 [61]. In higher dimensions, we do not expect any kind of regularity for weakly 1/2-harmonic maps into a general manifold, again by analogy with weakly harmonic maps in dimensions greater than three [92]. However, some partial regularity does hold for minimizing (or at least stationary) 1/2-harmonic maps.

Definition 1.1.2. A map $u \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is said to be a minimizing 1/2-harmonic map in Ω with values in $\mathcal{N}$ if

$$\mathcal{E}_{\frac{1}{2}}(u,\Omega) \leqslant \mathcal{E}_{\frac{1}{2}}(v,\Omega)$$

for every competitor $v \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ such that $\mathrm{spt}(v-u) \subseteq \Omega$.

The result of [80, 86] asserts that a minimizing 1/2-harmonic map u in Ω belongs to $C^\infty(\Omega \setminus \mathrm{sing}\,(u))$ where $\mathrm{sing}\,(u)$ is the *singular set* of u in Ω defined as

$$\mathrm{sing}\,(u) := \Omega \setminus \{x \in \Omega : u \text{ is continuous in a neighborhood of } x\}\,, \tag{1.1.4}$$

which is a relatively closed subset of Ω. Moreover, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n-2$ for $n \geqslant 3$, and $\operatorname{sing}(u)$ is locally finite in Ω for $n=2$ (the notation $\dim_{\mathscr{H}}$ stands for the Hausdorff dimension), see Corollary 1.3.7.

The main purpose of this article is to improve this general regularity result in the case of minimizing 1/2-harmonic maps into the sphere $\mathbb{S}^{m-1}$. In a first direction, we prove that the size of the singular set can be reduced in case of two or higher dimensional spheres.

Theorem 1.1.3. *Assume that $m \geqslant 3$. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. If $u \in \widehat{H}^{1/2}(\Omega;\mathbb{S}^{m-1})$ is a minimizing 1/2-harmonic map in Ω, then $\operatorname{sing}(u) = \emptyset$ for $n \leqslant 2$, $\operatorname{sing}(u)$ is locally finite in Ω for $n=3$, and $\dim_{\mathcal{H}} \operatorname{sing}(u) \leqslant n-3$ for $n \geqslant 4$.*

For $m=2$, i.e., in the case of minimizing 1/2-harmonic maps into $\mathbb{S}^1$, such improved regularity cannot hold for topological reasons, even in dimension 2. To illustrate this fact, let us consider the following variational problem

$$\min\left\{\mathcal{E}_{\frac{1}{2}}(u,\mathbb{D}) : u \in \widehat{H}^{1/2}(\mathbb{D};\mathbb{S}^1)\,, u(x) = g(x/|x|) \text{ for a.e. } x \in \mathbb{D}^c\right\},$$

where $\mathbb{D}$ denotes the open unit disk in $\mathbb{R}^2$, and $g:\mathbb{S}^1 \to \mathbb{S}^1$ is a smooth given map of nonvanishing topological degree. Existence of minimizers easily follows from the direct method of calculus of variations, and any minimizer is obviously a minimizing 1/2-harmonic map in $\mathbb{D}$. On the other hand, the degree condition on g implies that g does not admit a continuous extension to the whole disk $\overline{\mathbb{D}}$, and thus any minimizer must have at least one singular point. In dimension 2, we already know that the set of singularities is locally finite, and our purpose is to give a description of "their shape". This description relies on a blow-up analysis near a singular point (see Section 1.5.4), and the study of all possible blow-up limits, usually called *tangent maps*. They turn out to be 0-homogeneous and minimizing 1/2-harmonic maps over the whole space (i.e., minimizing in every ball). Our next theorem provides the classification of all 0-homogeneous minimizing 1/2-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^1$.

Theorem 1.1.4. *The map $u_\star : \mathbb{R}^2 \to \mathbb{S}^1$ given by $u_\star(x) := \frac{x}{|x|}$ is a minimizing 1/2-harmonic map in $\mathbb{R}^2$. Moreover, it is the unique nonconstant 0-homogeneous minimizing 1/2-harmonic map up to an orthogonal transformation. In other words, if $u \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{S}^1)$ is a nonconstant 0-homogeneous minimizing 1/2-harmonic map in $\mathbb{R}^2$, then there exists $A \in O(2,\mathbb{R})$ such that $u(x) = u_\star(Ax)$ for every $x \in \mathbb{R}^2 \setminus \{0\}$.*

As a corollary of Theorem 1.1.4, we obtain that a minimizing 1/2-harmonic map from a two dimensional domain into $\mathbb{S}^1$ must have a degree ± 1 at each singularity. The topological degree at a singular point is here defined as the degree of the restriction to any small circle surrounding the point.

Theorem 1.1.5. *Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded open set. If $u \in \widehat{H}^{1/2}(\Omega;\mathbb{S}^1)$ is a minimizing 1/2-harmonic map in Ω and $a \in \Omega \cap \operatorname{sing}(u)$, then $\deg(u,a) \in \{+1,-1\}$.*

The results and proofs presented in this note represent fractional $H^{1/2}$-counterparts of classical results on minimizing harmonic maps into spheres. First, to prove Theorem 1.1.3, we show that a 0-homogeneous minimizing 1/2-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^{m-1}$ must be constant if $m \geqslant 3$. This can be seen as the analogue of R. Schoen & K. Uhlenbeck result [104, Proposition 1.2] about the constancy of 0-homogeneous minimizing harmonic maps from $\mathbb{R}^3$ into $\mathbb{S}^3$. Their result relies on the fact that a harmonic 2-sphere into $\mathbb{S}^3$ must be equatorial, a consequence of a theorem of F.J. Almgren [1] and E. Calabi [19].

Constancy then follows through a second variation argument, destabilizing nonconstant maps in the orthogonal direction to the image. In our context, any 1/2-harmonic circle (see Section 1.4.1) turns out to be the boundary of a minimal disk with free boundary in $\mathbb{S}^{m-1}$. Recently, A.M. Fraser & R. Schoen [48] proved that such a minimal disk must be a flat disk through the origin, extending a famous result of J.C.C. Nitsche [87] for $m = 3$ to arbitrary spheres. As a consequence, any 1/2-harmonic circle is equatorial (see Corollary 1.4.6), and we use this fact to destabilize nonconstant 0-homogeneous 1/2-harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^{m-1}$ using again variations in the orthogonal direction to their image (see Proposition 1.4.7). Let us mention that, surprisingly, the same strategy applies to prove smoothness of minimizing "fractional s-harmonic maps" from the line into a sphere for $s \in (0, 1/2)$, see [82].

Concerning Theorem 1.1.4 and Theorem 1.1.5, we have obtained the $H^{1/2}$-analogue of a classical result of H. Brezis, J.M. Coron, and E.H. Lieb [13] (see also [2]). In the spirit of [13], the minimality of $x/|x|$ is obtained by means of sharp energy lower bounds, which in turn rely on the distributional Jacobian for $H^{1/2}$-maps into $\mathbb{S}^1$, see [11, 79, 91]. To prove the uniqueness part, we use the fact that all 0-homogeneous 1/2-harmonic maps in $\mathbb{R}^2$ can be written in terms of finite Blaschke products, which are rational functions of the complex variable. This fact has been established in [80] (see also [8, 23]). Using this representation, we prove rigidity among degree ± 1 maps by domain deformations. Then we exclude maps with higher degree by suitable constructions of competitors in the spirit of [13, Proof of Theorem 7.4]. Compared to [13], the construction turns out to be more involved as it requires additional steps and the numerical evaluation of certain integrals. Finally, Theorem 1.1.5 is obtained through the aforementioned blow-up analysis near a singularity. More precisely, we prove that homothetic expansions of a minimizing 1/2-harmonic map near a singular point converge up to subsequences to a nontrivial 0-homogeneous minimizing 1/2-harmonic map, so that the conclusion follows from Theorem 1.1.4. Compared to [13] again, we do not know if a minimizing 1/2-harmonic map u satisfies $u(x) \sim A(x-a)/|x-a|$ near a singular point $a \in \Omega$ for some $A \in O(2, \mathbb{R})$, or equivalently if uniqueness of the blow-up limits holds. For classical minimizing harmonic maps (into analytic manifolds), uniqueness of blow-ups (i.e., of tangent maps) at isolated singularities has been proved in [107, 108]. It rests on the so-called Łojasiewicz-Simon inequality, which is not known in our context.

In most of the proofs, we follow the approach of [80] using of the harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$ given by the convolution with the Poisson kernel. This allows us to realize the 1/2-Laplacian as the associated Dirichlet-to-Neumann map (see Section 1.2), and then rephrase the 1/2-harmonic map equation as a harmonic map system with (partially) free boundary condition, see Section 1.3. In particular, we make use of the existing regularity and compactness results of R. Hardt & F.H. Lin [58], F. Duzaar & K. Steffen [37, 38], and F. Duzaar & J.F. Grotowski [36, 35], see Section 1.3.1.

Notation

Throughout the paper, $\mathbb{R}^{n+1}_+$ is the open upper half-space $\mathbb{R}^n \times (0, \infty)$, and $\mathbb{R}^n$ can be identified with $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$. More generally, a set $A \subseteq \mathbb{R}^n$ can be identified with $A \times \{0\} \subseteq \partial\mathbb{R}^{n+1}_+$. Points in $\mathbb{R}^{n+1}$ are written $\mathbf{x} = (x, x_{n+1})$ with $x \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$. We shall denote by $B_r(\mathbf{x})$ the open ball in $\mathbb{R}^{n+1}$ of radius r centered at $\mathbf{x} = (x, x_{n+1})$, while $D_r(x)$ is the open ball (or disk) in $\mathbb{R}^n$ centered at x (and thus $D_r(x) \times \{0\} = B_r((x,0)) \cap (\mathbb{R}^n \times \{0\})$). If the center is at the origin, we simply write B_r and D_r the corresponding balls. In case $n = 2$, we write $\mathbb{D} := D_1$.

• For an arbitrary set $G \subseteq \mathbb{R}^{n+1}$, we define

$$G^+ := G \cap \mathbb{R}^{n+1}_+ \quad \text{and} \quad \partial^+ G := (\partial G)^+ = \partial G \cap \mathbb{R}^{n+1}_+ .$$

• If $G \subseteq \mathbb{R}^{n+1}_+$ is a bounded open set, we shall say that G is **admissible** whenever

(i) ∂G is Lipschitz regular;

(ii) the (relative) open set $\partial^0 G \subseteq \mathbb{R}^n \times \{0\}$ defined by

$$\partial^0 G := \Big\{ \mathbf{x} \in \partial G \cap \partial \mathbb{R}^{n+1}_+ : B_r^+(\mathbf{x}) \subseteq G \text{ for some } r > 0 \Big\}$$

is non-empty and has a Lipschitz boundary in $\mathbb{R}^n$;

(iii) $\partial G = \partial^+ G \cup \overline{\partial^0 G}$.

According to this definition, an half ball B_r^+ is admissible, and $\partial^0 B_r^+ = D_r \times \{0\}$.
• The tangent space to a manifold $\mathcal{N}$ at a point $p \in \mathcal{N}$ is denoted by $\mathrm{Tan}(p, \mathcal{N})$ (while the tangent bundle of $\mathcal{N}$ is simply denoted by $T\mathcal{N}$).
• We often identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and if $x = (x_1, x_2) \in \mathbb{R}^2$, the complex variable is written $z := x_1 + ix_2$. Functions taking values into $\mathbb{R}^2$ are also understood as complex valued functions. The product of two such functions are thus understood in the sense of complex multiplication.

Finally, we always denote by C a generic positive constant which may only depend on the dimension n, and possibly changing from line to line. If a constant depends on additional given parameters, we shall write those parameters using the subscript notation.

1.2 Harmonic extension & the 1/2-Laplacian

1.2.1 Harmonic extension

For a measurable function $u : \mathbb{R}^n \to \mathbb{R}^m$, we denote by u^{e} its extension to the upper half-space $\mathbb{R}^{n+1}_+$ given by the convolution of u with the Poisson kernel, i.e.,

$$u^{\mathrm{e}}(\mathbf{x}) := \gamma_n \int_{\mathbb{R}^n} \frac{x_{n+1} u(y)}{(|x-y|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} \, \mathrm{d}y \quad \text{for } \mathbf{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}_+ .$$

This extension is well defined whenever u belongs to the Lebesgue L^p over $\mathbb{R}^n$ with respect to the finite measure $(1+|x|^2)^{-\frac{n+1}{2}} \, \mathrm{d}x$ for some $1 \leqslant p \leqslant \infty$. In this case, it is well known that u^{e} provides an harmonic extension of u to $\mathbb{R}^{n+1}_+$. In other words, u^{e} solves

$$\begin{cases} \Delta u^{\mathrm{e}} = 0 & \text{in } \mathbb{R}^{n+1}_+ , \\ u^{\mathrm{e}} = u & \text{on } \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\} . \end{cases}$$

Moreover, $u^{\mathrm{e}} \in L^\infty(\mathbb{R}^{n+1}_+)$ whenever $u \in L^\infty(\mathbb{R}^n)$, and

$$\|u^{\mathrm{e}}\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leqslant \|u\|_{L^\infty(\mathbb{R}^n)} . \tag{1.2.1}$$

We shall make use of the following lemma about the harmonic extension. Using the Fourier transform[1], its proof is elementary and it is left to the reader.

[1] Recall that the Fourier transform of the Poisson kernel is given by $\exp(-2\pi x_{n+1}|\xi|)$.

Lemma 1.2.1. *If* $u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$*, then*

$$\int_{\mathbb{R}^n} |u^{\mathrm{e}}(x, x_{n+1})|^2 \,\mathrm{d}x \leqslant \|u\|^2_{L^2(\mathbb{R}^n)} \quad \forall x_{n+1} > 0\,,$$

and

$$\int_{\mathbb{R}^n} |u^{\mathrm{e}}(x, x_{n+1})|^2 \,\mathrm{d}x \leqslant \frac{C\|u\|^2_{L^1(\mathbb{R}^n)}}{x^n_{n+1}} \quad \forall x_{n+1} > 0\,,$$

for a constant C *depending only on* n*.*

We complete this subsection recalling the classical identity relating the $H^{1/2}$-seminorm over $\mathbb{R}^n$ with the Dirichlet energy of the harmonic extension:

$$\begin{aligned}\frac{\gamma_n}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}} \,\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} |\nabla u^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \\ &= \min\left\{ \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} |\nabla v|^2 \,\mathrm{d}\mathbf{x} : v \in \dot{H}^1(\mathbb{R}^{n+1}_+;\mathbb{R}^d)\,,\ v = u \text{ on } \partial\mathbb{R}^{n+1}_+ \right\} \quad (1.2.2)\end{aligned}$$

for every u in the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^n;\mathbb{R}^m)$.

1.2.2 The 1/2-Laplacian and the Dirichlet-to-Neumann map

Given a smooth bounded open set $\Omega \subseteq \mathbb{R}^n$, the 1/2-Laplacian $(-\Delta)^{\frac{1}{2}} : \widehat{H}^{1/2}(\Omega;\mathbb{R}^m) \to \big(\widehat{H}^{1/2}(\Omega;\mathbb{R}^m)\big)'$ is defined as the continuous linear operator induced by the quadratic form $\mathcal{E}_{\frac{1}{2}}(\cdot,\Omega)$. For $u \in \widehat{H}^{1/2}(\Omega;\mathbb{R}^m)$, the action of $(-\Delta)^{\frac{1}{2}}u$ on an element $\varphi \in \widehat{H}^{1/2}(\Omega;\mathbb{R}^m)$ is denoted by $\langle(-\Delta)^{\frac{1}{2}}u,\varphi\rangle_\Omega$, and it is given by

$$\langle(-\Delta)^{\frac{1}{2}}u,\varphi\rangle_\Omega = \frac{\gamma_n}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+1}} \,\mathrm{d}x\mathrm{d}y\,. \quad (1.2.3)$$

Note that, when restricted to $H^{1/2}_{00}(\Omega;\mathbb{R}^m)$, the distribution $(-\Delta)^{\frac{1}{2}}u$ actually belongs to $H^{-1/2}(\Omega;\mathbb{R}^m)$.

It is well known that the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ coincides with the *Dirichlet-to-Neumann operator* associated with the harmonic extension to $\mathbb{R}^{n+1}_+$. To be more specific, if $u \in \widehat{H}^{1/2}(\Omega;\mathbb{R}^m)$, then u^{e} is well defined, and $u^{\mathrm{e}} \in H^1(G;\mathbb{R}^m)$ for every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\partial^0 G} \subseteq \Omega\times\{0\}$. Hence, u^{e} admits a distributional exterior normal derivative $\partial_\nu u^{\mathrm{e}}$ on $\Omega\times\{0\}$. By harmonicity of u^{e}, its action on $\varphi \in \mathscr{D}(\Omega;\mathbb{R}^m)$ can be defined as

$$\langle\partial_\nu u^{\mathrm{e}},\varphi\rangle_\Omega := \int_{\mathbb{R}^{n+1}_+} \nabla u^{\mathrm{e}}\cdot\nabla\Phi \,\mathrm{d}\mathbf{x}\,, \quad (1.2.4)$$

where Φ is any smooth extension of φ compactly supported in $\mathbb{R}^{n+1}_+ \cup (\Omega\times\{0\})$. By approximation, the same identity holds for any $\Phi \in H^1(\mathbb{R}^{n+1}_+;\mathbb{R}^m)$ compactly supported in $\mathbb{R}^{n+1}_+ \cup (\Omega\times\{0\})$. In this way, the distribution $\partial_\nu u^{\mathrm{e}}$ appears to belong to $H^{-1/2}_{00}(\Omega;\mathbb{R}^m)$, and the following identity holds (see [80, Lemma 2.9])

$$\langle\partial_\nu u^{\mathrm{e}},\varphi\rangle_\Omega = \langle(-\Delta)^{\frac{1}{2}}u,\varphi\rangle_\Omega \qquad \forall\varphi \in H^{1/2}_{00}(\Omega;\mathbb{R}^m)\,. \quad (1.2.5)$$

All details can be found in [80, Section 2].

1.3 1/2-harmonic maps vs harmonic maps with free boundary

1.3.1 Minimizing harmonic maps with free boundary

For an admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$, we consider the Dirichlet energy $\mathbf{E}_{\frac{1}{2}}(\cdot, G)$ defined on $H^1(G;\mathbb{R}^m)$ by

$$\mathbf{E}_{\frac{1}{2}}(v, G) := \frac{1}{2}\int_G |\nabla v|^2 \,\mathrm{d}\mathbf{x}\,. \tag{1.3.1}$$

We also consider a given smooth submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ that we assume to be compact and without boundary.

Definition 1.3.1. Let $G \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set, and consider a map $v \in H^1(G;\mathbb{R}^m)$ satisfying $v(\mathbf{x}) \in \mathcal{N}$ $\mathscr{H}^n$-a.e. on $\partial^0 G$. We say that v is a minimizing harmonic map in G with respect to the partially free boundary condition $v(\partial^0 G) \subseteq \mathcal{N}$ if

$$\mathbf{E}_{\frac{1}{2}}(v, G) \leqslant \mathbf{E}_{\frac{1}{2}}(w, G)$$

for every competitor $w \in H^1(G;\mathbb{R}^m)$ satisfying $w(\mathbf{x}) \in \mathcal{N}$ for $\mathscr{H}^n$-a.e. $\mathbf{x} \in \partial^0 G$, and such that $\mathrm{spt}(w - v) \subseteq G \cup \partial^0 G$. *In short, we may say that v is a minimizing harmonic map with free boundary in G.*

Using variations supported in the open set G, one obtains that a minimizing harmonic map v with free boundary is harmonic in G, i.e.,

$$\Delta v = 0 \quad \text{in } G\,.$$

In particular, $v \in C^\infty(G)$ by standard elliptic theory. Hence the regularity issue is at the (partially) free boundary $\partial^0 G$. As in [37, 58], one obtains from minimality the boundary condition

$$\frac{\partial v}{\partial \nu} \perp \mathrm{Tan}(v, \mathcal{N}) \quad \text{on } \partial^0 G\,,$$

which has to be understood in the weak sense, that is

$$\int_G \nabla v \cdot \nabla \zeta \,\mathrm{d}\mathbf{x} = 0$$

for every $\zeta \in H^1(G;\mathbb{R}^m)$ satisfying $\zeta(\mathbf{x}) \in \mathrm{Tan}(v(\mathbf{x}), \mathcal{N})$ for $\mathscr{H}^n$-a.e. $\mathbf{x} \in \partial^0 G$ and such that $\mathrm{spt}(\zeta) \subseteq G \cup \partial^0 G$.

Assuming that $v \in L^\infty(G)$, one may apply the (partial) regularity results of [37, 58] to derive the following theorem (see [80, Section 4] or [86]). In its statement, $\mathrm{sing}\,(v)$ denotes the so-called *singular set* of v (in $\partial^0 G$), i.e.,

$$\mathrm{sing}\,(v) := \partial^0 G \setminus \{\mathbf{x} \in \partial^0 G : v \text{ is continuous in a neighborhood of } \mathbf{x}\}\,,$$

which turns out to be a relatively closed subset of $\partial^0 G$.

Theorem 1.3.2. *Let $v \in H^1(G;\mathbb{R}^m) \cap L^\infty(G)$ satisfying $v(\partial^0 G) \subseteq \mathcal{N}$ be a minimizing harmonic map with free boundary in G. Then $v \in C^\infty((G \cup \partial^0 G) \setminus \mathrm{sing}\,(v))$, $\mathrm{sing}\,(v)$ is locally finite in $\partial^0 G$ for $n = 2$, and $\dim_{\mathscr{H}} \mathrm{sing}\,(v) \leqslant n - 2$ for $n \geqslant 3$.*

By means of Federer's dimension reduction principle, the size of the singular set can be further reduced according to the existence or nonexistence of nontrivial *tangent maps*. Those maps are defined as all possible blow-up limits of minimizing harmonic maps with free boundary at a point of the free boundary $\partial^0 G$, see [58, Section 3.5]. In our setting, they appear to be 0-homogeneous maps $v_0 \in H^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+};\mathbb{R}^m) \cap L^\infty(\mathbb{R}^{n+1}_+)$ satisfying $v_0(\partial\mathbb{R}^{n+1}_+) \subseteq \mathcal{N}$ which are minimizing harmonic maps with free boundary in B^+_R for every $R > 0$. Applying [58, Theorem 3.6] (see also [38, Remark 4.3]), we readily obtain the following result.

Theorem 1.3.3. *Let $\ell = \ell(\mathcal{N})$ be the largest integer such that any bounded and 0-homogeneous minimizing harmonic map with free boundary v_0 from $\mathbb{R}^{j+1}_+$ to $\mathbb{R}^m$ with $v_0(\partial\mathbb{R}^{j+1}_+) \subseteq \mathcal{N}$ is a constant for each $j = 1, \ldots, \ell$. For any minimizing harmonic map v with free boundary as in Theorem 1.3.2, we have $\operatorname{sing}(v) = \emptyset$ if $n \leqslant \ell$, $\operatorname{sing}(v)$ is locally finite in $\partial^0 G$ if $n = \ell + 1$, and $\dim_{\mathscr{H}} \operatorname{sing}(v) \leqslant n - \ell - 1$ if $n \geqslant \ell + 2$.*

Remark 1.3.4. Note that, in applying [58], we use the fact that any bounded and 0-homogeneous minimizing harmonic map with free boundary v_0 satisfies the uniform bound

$$\|v_0\|_{L^\infty(\mathbb{R}^{j+1}_+)} = \|v_{0|\mathbb{R}^j\times\{0\}}\|_{L^\infty(\mathbb{R}^j)} \leqslant C_{\mathcal{N}}\,,$$

where $C_{\mathcal{N}}$ is (essentially) the width of $\mathcal{N}$ (assuming that $0 \in \mathcal{N}$). This estimate follows from the fact v_0 is precisely given by the harmonic extension to $\mathbb{R}^{j+1}_+$ of its restriction to $\mathbb{R}^j \times \{0\}$. In other words, if we set $u_0 := v_{0|\mathbb{R}^j\times\{0\}}$, then $v_0 = (u_0)^{\rm e}$ (the convolution product of u_0 with the j-dimensional Poisson kernel). Indeed, the difference $v_0 - (u_0)^{\rm e}$ is a bounded harmonic function in $\mathbb{R}^{j+1}_+$. Since it vanishes on $\partial\mathbb{R}^{j+1}_+$, it has to vanish identically by the classical Liouville theorem.

We conclude this subsection with an important compactness result for minimizing harmonic maps with free boundary (on which Theorem 1.3.2 and Theorem 1.3.3 are based). It corresponds to a weaker version of a more general compactness theorem obtained in [36, Theorem 2.2] (see also [35, Theorem 2.2]).

Theorem 1.3.5 (compactness). *Let $(v_k) \subseteq H^1(G;\mathbb{R}^m)$ be a bounded sequence of minimizing harmonic maps in G with respect to the partially free boundary condition $v_k(\partial^0 G) \subseteq \mathcal{N}$. There exist a (not relabeled) subsequence and $v \in H^1(G;\mathbb{R}^m)$ a minimizing harmonic map with free boundary in G such that $v_k \to v$ strongly in $H^1_{\rm loc}(G \cup \partial^0 G)$.*

1.3.2 Harmonic extension of minimizing 1/2-harmonic maps

In this subsection, our aim is to prove that minimizing 1/2-harmonic maps and minimizing harmonic maps with free boundary can be made in one-to-one correspondance by means of the harmonic extension. It has been proven in [80, Proposition 4.9] that the harmonic extension of a minimizing 1/2-harmonic map returns a minimizing harmonic map with free boundary in the upper half-space. We shall improve this result showing that a converse statement holds true. Here again, $\mathcal{N} \subseteq \mathbb{R}^m$ denotes a given smooth and compact submanifold without boundary.

Theorem 1.3.6 (minimality transfer). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth open set. A map $u \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is a minimizing 1/2-harmonic map in Ω if and only if its harmonic extension $u^{\rm e}$ is a minimizing harmonic map with free boundary in every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0 G} \subseteq \Omega \times \{0\}$.*

Proof. According to [80, Corollary 2.10 & Proposition 4.9], if $u \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is a minimizing 1/2-harmonic map in Ω, then $u^{\rm e}$ is a minimizing harmonic map with free boundary in every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0 G} \subseteq \Omega \times \{0\}$. It hence remains to prove the converse statement. We thus assume that $u^{\rm e}$ is minimizing harmonic map with free boundary in every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0 G} \subseteq \Omega \times \{0\}$.

Step 1. We consider an arbitrary competitor $w \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$, and we assume that $h := w - u$ is compactly supported in an open set $\Omega' \subseteq \mathbb{R}^n$ with $\overline{\Omega'} \subseteq \Omega$. The map h being compactly supported in Ω', it belongs to $H^{1/2}_{00}(\Omega;\mathbb{R}^m)\cap L^1(\mathbb{R}^n)$. In view of identity (1.2.2), its harmonic extension $h^{\rm e}$ belongs to the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^{n+1}_+;\mathbb{R}^m)$.

We would like to use directly $w^{\rm e}$ as a competitor for $u^{\rm e}$ and use the minimality of $u^{\rm e}$, however we cannot do so, since $h^{\rm e} = (w-u)^{\rm e}$ is not compactly supported in some bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\partial^0 G \subseteq \Omega \times \{0\}$ (in fact, it is not even necessarily compactly supported in $\overline{\mathbb{R}^{n+1}_+}$). Instead, we claim that there exists a sequence $(h_k) \subseteq H^1(\mathbb{R}^{n+1}_+;\mathbb{R}^m)$ such that each h_k is supported in $G_k \cup \partial^0 G_k$ for some admissible bounded open set $G_k \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\partial^0 G_k} \subseteq \Omega \times \{0\}$, $h_{k|\mathbb{R}^n\times\{0\}} = h$,

$$\int_{\mathbb{R}^{n+1}_+} |\nabla h_k|^2 \,{\rm d}\mathbf{x} \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^{n+1}_+} |\nabla h^{\rm e}|^2 \,{\rm d}\mathbf{x}\,, \tag{1.3.2}$$

and we will use $u^{\rm e} + h_k$ as competitors for $u^{\rm e}$. Before proving this claim, we complete the proof of the theorem.

By assumption $u^{\rm e}$ is a minimizing harmonic map with free boundary in G_k. Since $u^{\rm e} + h_k$ is an admissible competitor for the minimality of $u^{\rm e}$ in G_k, we infer that

$$\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} |\nabla h_k|^2 \,{\rm d}\mathbf{x} + \int_{\mathbb{R}^{n+1}_+} \nabla h_k \cdot \nabla u^{\rm e} \,{\rm d}\mathbf{x} = \mathbf{E}_{\frac{1}{2}}(u^{\rm e}+h_k, G_k) - \mathbf{E}_{\frac{1}{2}}(u^{\rm e}, G_k) \geqslant 0\,. \tag{1.3.3}$$

On the other hand, (1.2.4) and (1.2.5) yield

$$\int_{\mathbb{R}^{n+1}_+} \nabla h_k \cdot \nabla u^{\rm e} \,{\rm d}\mathbf{x} = \langle (-\Delta)^{\frac{1}{2}} u, h\rangle_\Omega\,, \tag{1.3.4}$$

since $h_k = h$ on $\mathbb{R}^n \times \{0\}$. Letting $k \to \infty$ in (1.3.3), we deduce from (1.3.2) and (1.3.4) that

$$\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} |\nabla h^{\rm e}|^2 \,{\rm d}\mathbf{x} + \langle (-\Delta)^{\frac{1}{2}} u, h\rangle_\Omega \geqslant 0\,. \tag{1.3.5}$$

In view of (1.2.2) and (1.2.3), we have

$$\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} |\nabla h^{\rm e}|^2 \,{\rm d}\mathbf{x} = \mathcal{E}_{\frac{1}{2}}(h,\Omega) = \mathcal{E}_{\frac{1}{2}}(w,\Omega) + \mathcal{E}_{\frac{1}{2}}(u,\Omega) - \langle (-\Delta)^{\frac{1}{2}} u, w\rangle_\Omega\,,$$

and since

$$\langle (-\Delta)^{\frac{1}{2}} u, h\rangle_\Omega = \langle (-\Delta)^{\frac{1}{2}} u, w\rangle_\Omega - \langle (-\Delta)^{\frac{1}{2}} u, u\rangle_\Omega = \langle (-\Delta)^{\frac{1}{2}} u, w\rangle_\Omega - 2\mathcal{E}_{\frac{1}{2}}(u,\Omega)\,,$$

inequality (1.3.5) yields

$$\mathcal{E}_{\frac{1}{2}}(w,\Omega) - \mathcal{E}_{\frac{1}{2}}(u,\Omega) \geqslant 0\,.$$

Thus u is indeed a minimizing 1/2-harmonic map in Ω.

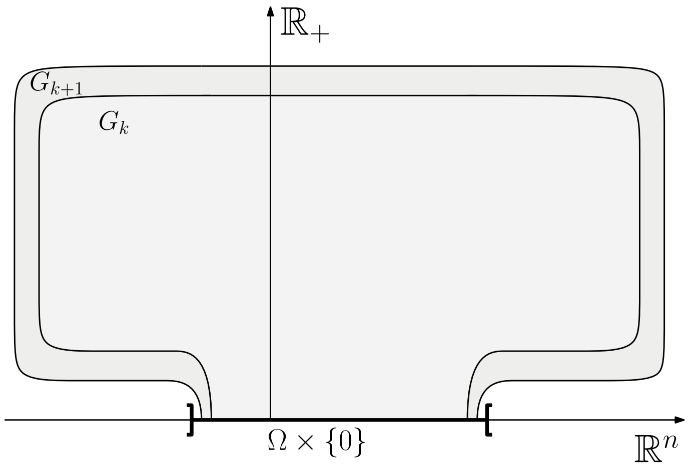

Figure 1.1: Construction of (h_k) s.t. $\operatorname{spt} h_k \subseteq G_k \cup \partial^0 G_k$

Step 2. We now proceed to the construction of the sequence (h_k) satisfying (1.3.2), cutting-off h appropriately (see Figure 1.1). For an integer $i \geqslant 1$, we denote by $\chi_i \in C^\infty(\mathbb{R}; [0,1])$ a smooth cutoff function satisfying $\chi_i(t) = 1$ for $|t| \leqslant i$, and $\chi_i(t) = 0$ for $|t| \geqslant i+1$, with $|\chi_i'| \leqslant C$ for some constant C independent of i. We first define

$$h_i^{(1)}(\mathbf{x}) := \chi_i(x_{n+1}) h^{\mathrm{e}}(\mathbf{x}) \,.$$

By Lemma 1.2.1, $h_i^{(1)} \in L^2(\mathbb{R}^{n+1}_+)$, so that $h_i^{(1)} \in H^1(\mathbb{R}^{n+1}_+; \mathbb{R}^m)$. Moreover, $h_i^{(1)} = h$ on $\mathbb{R}^n \times \{0\}$, and

$$\int_{\mathbb{R}^{n+1}_+} |\nabla h_i^{(1)}|^2 \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{n+1}_+} \chi_i^2 |\nabla h^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} + 2 \int_{\{i < x_{n+1} < i+1\}} \chi_i \chi_i' \, h^{\mathrm{e}} \cdot \partial_{n+1} h^{\mathrm{e}} \,\mathrm{d}\mathbf{x} + \int_{\{i < x_{n+1} < i+1\}} |\chi_i'|^2 |h^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \,.$$

From Lemma 1.2.1 and Fubini's theorem, we infer that

$$\int_{\{i < x_{n+1} < i+1\}} |\chi_i'|^2 |h^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{C}{i^n} \|h\|^2_{L^1(\mathbb{R}^n)} \,. \tag{1.3.6}$$

Since $h^{\mathrm{e}} \in \dot{H}^1(\mathbb{R}^{n+1}_+; \mathbb{R}^m)$, it follows by dominated convergence, (1.3.6), and Hölder inequality, that

$$\int_{\mathbb{R}^{n+1}_+} |\nabla h_i^{(1)}|^2 \,\mathrm{d}\mathbf{x} \underset{i \to \infty}{\longrightarrow} \int_{\mathbb{R}^{n+1}_+} |\nabla h^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \,.$$

We can thus find an integer $i_k \geqslant 1$ such that

$$\left| \int_{\mathbb{R}^{n+1}_+} |\nabla h_{i_k}^{(1)}|^2 \,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{n+1}_+} |\nabla h^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \right| \leqslant 2^{-k-2} \,. \tag{1.3.7}$$

Next we define for an integer $j \geqslant 1$,

$$h_j^{(2)}(\mathbf{x}) := \chi_j(|\mathbf{x}|) h_{i_k}^{(1)}(\mathbf{x}) \,.$$

Then $h_j^{(2)} \in H^1(\mathbb{R}^{n+1}_+; \mathbb{R}^m)$, and one classically shows (using $h_{i_k}^{(1)} \in H^1(\mathbb{R}^{n+1}_+; \mathbb{R}^m)$) that

$$\int_{\mathbb{R}^{n+1}_+} |\nabla h_j^{(2)}|^2 \,\mathrm{d}\mathbf{x} \underset{j \to \infty}{\longrightarrow} \int_{\mathbb{R}^{n+1}_+} |\nabla h_{i_k}^{(1)}|^2 \,\mathrm{d}\mathbf{x} \,.$$

In view of (1.3.7), we can find an integer $j_k \geqslant 1$ in such a way that

$$\left| \int_{\mathbb{R}^{n+1}_+} |\nabla h^{(2)}_{j_k}|^2 \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{n+1}_+} |\nabla h^{\mathrm{e}}|^2 \, \mathrm{d}\mathbf{x} \right| \leqslant 2^{-k-1} , \tag{1.3.8}$$

and $\Omega \times \{0\} \subseteq \partial^0 B_{j_k}$ to ensure that $h^{(2)}_{j_k} = h^{(1)}_{i_k} = h$ on $\mathbb{R}^n \times \{0\}$.

Let us now fix a small parameter $\delta > 0$ such that $\operatorname{dist}(\partial\Omega, \Omega') > 3\delta$, and consider a smooth cutoff function $\psi \in C^\infty(\mathbb{R}; [0,1])$ satisfying $\psi(t) = 0$ for $|t| < \delta$, and $\psi(t) = 1$ for $|t| \geqslant 2\delta$. For an integer $\ell \geqslant 1$, we consider a further cutoff function $\eta_\ell \in C^\infty(\mathbb{R}; [0,1])$ such that $\eta_\ell(t) = 1$ for $|t| \leqslant 2^{-\ell}$, $\eta_\ell(t) = 0$ for $|t| \geqslant 2^{-\ell+1}$, and $|\eta'_\ell| \leqslant C 2^\ell$ for some constant C independent of ℓ. Setting

$$\zeta_\ell(\mathbf{x}) := 1 - \eta_\ell(x_{n+1}) \psi\big(\operatorname{dist}(x, \Omega')\big) ,$$

we define

$$h^{(3)}_\ell(\mathbf{x}) := \zeta_\ell(\mathbf{x}) h^{(2)}_{j_k}(\mathbf{x}) .$$

Setting G_ℓ to be the interior of the set

$$\Big(\big\{ \operatorname{dist}(x, \Omega') \leqslant 2\delta \, , \; 0 \leqslant x_{n+1} \leqslant 2^{-\ell} \big\} \cup \big\{ x_{n+1} \geqslant 2^{-\ell} \big\} \Big) \cap B_{j_k} ,$$

then G_ℓ is an admissible bounded open set satisfying $\overline{\partial^0 G_\ell} \subseteq \Omega \times \{0\}$. The map $h^{(3)}_\ell$ belongs to $H^1(\mathbb{R}^{n+1}_+; \mathbb{R}^m)$, it is supported in $G_\ell \cup \partial^0 G_\ell$, and $h^{(3)}_\ell = h^{(2)}_{j_k} = h$ on the boundary $\mathbb{R}^n \times \{0\}$. Then, we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} |\nabla h^{(3)}_\ell|^2 \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{n+1}_+} & \zeta^2_\ell |\nabla h^{(2)}_{j_k}|^2 \, \mathrm{d}\mathbf{x} \\ & + 2 \int_{\mathbb{R}^{n+1}_+} \zeta_\ell (\nabla \zeta_\ell \cdot \nabla h^{(2)}_{j_k}) \cdot h^{(2)}_{j_k} \, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^{n+1}_+} |h^{(2)}_{j_k}|^2 |\nabla \zeta_\ell|^2 \, \mathrm{d}\mathbf{x} . \end{aligned} \tag{1.3.9}$$

Writing $A_\ell := \big\{ \operatorname{dist}(x, \Omega') > \delta \, , 2^{-\ell} < x_{n+1} < 2^{-\ell+1} \big\}$, we estimate

$$\int_{\mathbb{R}^{n+1}_+} |h^{(2)}_{j_k}|^2 |\nabla \zeta_\ell|^2 \, \mathrm{d}\mathbf{x} \leqslant C_\delta \int_{A_\ell} \frac{|h^{(2)}_{j_k}|^2}{x^2_{n+1}} \, \mathrm{d}\mathbf{x} . \tag{1.3.10}$$

Since $h^{(2)}_{j_k} = h = 0$ on $\big\{ \operatorname{dist}(x, \Omega') > \delta \big\} \times \{0\}$, we infer from Hardy's inequality that

$$\begin{aligned} \int_{\big\{ \operatorname{dist}(x,\Omega') > \delta \big\} \times \mathbb{R}_+} \frac{|h^{(2)}_{j_k}|^2}{x^2_{n+1}} \, \mathrm{d}\mathbf{x} &\leqslant C \int_{\big\{ \operatorname{dist}(x,\Omega') > \delta \big\} \times \mathbb{R}_+} |\nabla h^{(2)}_{j_k}|^2 \, \mathrm{d}\mathbf{x} \\ &\leqslant C \int_{\mathbb{R}^{n+1}_+} |\nabla h^{(2)}_{j_k}|^2 \, \mathrm{d}\mathbf{x} . \end{aligned}$$

As a consequence,

$$\int_{A_\ell} \frac{|h^{(2)}_{j_k}|^2}{x^2_{n+1}} \, \mathrm{d}\mathbf{x} \underset{\ell \to \infty}{\longrightarrow} 0 ,$$

by dominated convergence. In turn, (1.3.10) implies

$$\int_{\mathbb{R}^{n+1}_+} |h^{(2)}_{j_k}|^2 |\nabla \zeta_\ell|^2 \, \mathrm{d}\mathbf{x} \underset{\ell \to \infty}{\longrightarrow} 0 .$$

Back to (1.3.9), we deduce (still by dominated convergence and Hölder inequality) that

$$\int_{\mathbb{R}^{n+1}_+} |\nabla h^{(3)}_\ell|^2\,\mathrm{d}\mathbf{x} \underset{\ell\to\infty}{\longrightarrow} \int_{\mathbb{R}^{n+1}_+} |\nabla h^{(2)}_{j_k}|^2\,\mathrm{d}\mathbf{x}\,.$$

In view of (1.3.8), we may now select a subsequence $\{\ell_k\}$ such that

$$\left|\int_{\mathbb{R}^{n+1}_+} |\nabla h^{(3)}_{\ell_k}|^2\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{n+1}_+} |\nabla h^{\mathrm{e}}|^2\,\mathrm{d}\mathbf{x}\right| \leqslant 2^{-k}\,,$$

and the conclusion follows for $h_k := h^{(3)}_{\ell_k}$ and $G_k := G_{\ell_k}$. □

As a consequence of Theorem 1.3.6, we can derive a partial regularity theory for minimizing 1/2-harmonic from the regularity of minimizing harmonic maps with free boundary (see [80, 86]). Notice that, in applying Theorem 1.3.2 and Theorem 1.3.3, we use that $u^{\mathrm{e}} \in L^\infty(\mathbb{R}^{n+1}_+)$ by (1.2.1) and the fact that u is taking values in the compact manifold $\mathcal{N}$. Recall that $\mathrm{sing}\,(u)$ denotes the singular set of u in Ω (see (1.1.4)), which is a relatively closed subset of Ω.

Corollary 1.3.7 ([86] and [80, Theorem 1.2 & Remark 4.24])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth open set. If $u \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is a minimizing 1/2-harmonic map in Ω, then $u \in C^\infty(\Omega\setminus \mathrm{sing}\,(u))$, $\mathrm{sing}\,(u)$ is locally finite in Ω for $n=2$, and $\dim_{\mathscr{H}} \mathrm{sing}\,(u) \leqslant n-2$ for $n \geqslant 3$.*

Exactly as in Theorem 1.3.3, the estimate on the Hausdorff dimension of $\mathrm{sing}\,(u)$ can be improved according to the existence or nonexistence of 0-homogeneous minimizing 1/2-harmonic maps, i.e., maps in $H^{1/2}_{\mathrm{loc}}(\mathbb{R}^n;\mathcal{N})$ which are minimizing in every ball.

Definition 1.3.8. A map $u_0 \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^n;\mathcal{N})$ is said to be a 0-homogenous 1/2-harmonic map if u_0 is 0-homogeneous and a weakly 1/2-harmonic map in every ball of $\mathbb{R}^n$. Similarly, u_0 is said to be a 0-homogenous minimizing 1/2-harmonic map if it is 0-homogeneous and a minimizing 1/2-harmonic map in every ball of $\mathbb{R}^n$.

Corollary 1.3.9. *Let $\bar\ell = \bar\ell(\mathcal{N})$ be the largest integer such that any 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^j$ into $\mathcal{N}$ is a constant for each $j = 1,\dots,\bar\ell$. For any minimizing 1/2-harmonic map u as in Corollary 1.3.7, we have $\mathrm{sing}\,(u) = \emptyset$ if $n \leqslant \bar\ell$, $\mathrm{sing}\,(u)$ is locally finite in Ω if $n = \bar\ell+1$, and $\dim_{\mathscr{H}} \mathrm{sing}\,(u) \leqslant n-\bar\ell-1$ if $n \geqslant \bar\ell+2$. Moreover, $\bar\ell = \ell$ where ℓ is given by Theorem 1.3.3.*

Proof. By Theorem 1.3.6, if u_0 is a 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^j$ into $\mathcal{N}$, then $(u_0)^{\mathrm{e}}$ is a bounded minimizing harmonic map with free boundary in every half ball B^+_R. Since the harmonic extension preserves homogeneity, $(u_0)^{\mathrm{e}}$ is also 0-homogeneous. Hence $(u_0)^{\mathrm{e}}$ is constant whenever $j \leqslant \ell$ with ℓ given by Theorem 1.3.3, and so is u_0. This shows that $\ell \leqslant \bar\ell$. The other way around, if $v_0 : \mathbb{R}^{j+1}_+ \to \mathbb{R}^d$ with $v_0(\mathbb{R}^j\times\{0\}) \subseteq \mathcal{N}$ is a bounded and 0-homogeneous minimizing harmonic map with free boundary, then $v_0 = (v_{0|\mathbb{R}^j\times\{0\}})^{\mathrm{e}}$ according to Remark 1.3.4. By Theorem 1.3.6, it follows that $v_{0|\mathbb{R}^j\times\{0\}}$ is a 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^j$ into $\mathcal{N}$. By definition of $\bar\ell$, $v_{0|\mathbb{R}^j\times\{0\}}$ is constant whenever $j \leqslant \bar\ell$. Hence v_0 is constant for $j \leqslant \bar\ell$, which shows that $\bar\ell \leqslant \ell$. We have thus proved that $\bar\ell = \ell$.

Now, if u is as in Corollary 1.3.7, then Theorem 1.3.6 tells us that u^{e} is a bounded minimizing harmonic map with free boundary in every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0 G} \subseteq \Omega\times\{0\}$. Hence the conclusion follows from Theorem 1.3.2 knowing that $\bar\ell = \ell$. □

1.4 Minimizing 1/2-harmonic maps into a sphere

1.4.1 1/2-harmonic circles

The purpose of this first subsection is to recall the notion 1/2-harmonic circle into a manifold $\mathcal{N}$, and its relation established in [80] with 0-homogeneous 1/2-harmonic maps from $\mathbb{R}^2$ into $\mathcal{N}$. Once again, $\mathcal{N}$ is assumed to be a smooth and compact submanifold of $\mathbb{R}^m$ without boundary. Let us start with the definition of a 1/2-harmonic circle into $\mathcal{N}$. First, the 1/2-Dirichlet energy of a map $g \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^m)$ is defined as

$$\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) := \frac{\gamma_1}{4}\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|g(x)-g(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \quad \text{with} \quad \gamma_1 = \frac{1}{\pi}\,. \tag{1.4.1}$$

The choice of the constant γ_1 in (1.4.1) is made in such a way that

$$\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) = \frac{1}{2}\int_{\mathbb{D}} |\nabla w_g|^2\,\mathrm{d}x \qquad \forall g \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^m)\,, \tag{1.4.2}$$

where $w_g \in H^1(\mathbb{D};\mathbb{R}^m)$ denotes the (unique) harmonic extension of g to the unit disk $\mathbb{D}$ of the plane $\mathbb{R}^2$, i.e., the unique solution of

$$\begin{cases} \Delta w_g = 0 & \text{in } \mathbb{D} \\ w_g = g & \text{on } \partial\mathbb{D} = \mathbb{S}^1\,, \end{cases} \tag{1.4.3}$$

see e.g. [80, Section 4.2].

Definition 1.4.1. A map $g \in H^{1/2}(\mathbb{S}^1;\mathcal{N})$ is said to be a (weakly) 1/2-harmonic circle into $\mathcal{N}$ if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\frac{1}{2}}\left(\frac{g+t\varphi}{|g+t\varphi|},\mathbb{S}^1\right)\right]_{|t=0} = 0 \qquad \forall \varphi \in C^\infty(\mathbb{S}^1;\mathbb{R}^m)\,.$$

Remark 1.4.2. Any 1/2-harmonic circle g is smooth, i.e., $g \in C^\infty(\mathbb{S}^1)$. This follows directly from the regularity theory for weakly 1/2-harmonic maps in one space dimension of [29, 28] (see also [80, Theorem 4.18 & Remark 4.24]). Indeed, as in [80, Remark 4.29], $g \in H^{1/2}(\mathbb{S}^1;\mathcal{N})$ is weakly 1/2-harmonic if and only if $g\circ\mathfrak{C}_{|\mathbb{R}} \in \dot{H}^{1/2}(\mathbb{R};\mathcal{N})$ is a weakly 1/2-harmonic map on $\mathbb{R}$, where $\mathfrak{C} : \overline{\mathbb{R}^2_+} \to \overline{\mathbb{D}}\setminus\{(1,0)\}$ is the (conformal) Cayley transform (see (1.5.36)) and $\mathfrak{C}_{|\mathbb{R}} : \mathbb{R} \to \mathbb{S}^1\setminus\{(1,0)\}$ its restriction to $\mathbb{R} \simeq \partial\mathbb{R}^2_+$. Hence the regularity result of [29, 28] applies, and it yields $g \in C^\infty(\mathbb{S}^1\setminus\{(1,0)\})$. On the other hand, the map $\tilde{g}(x) := g(-x)$ is clearly 1/2-harmonic (by invariance of the energy under the symmetry $x \mapsto -x$), so that $\tilde{g} \in C^\infty(\mathbb{S}^1\setminus\{(1,0)\})$. Thus g is in fact also smooth near $(1,0)$, and the conclusion follows.

We are interested in 1/2-harmonic circles since they appear as angular profiles of 0-homogeneous 1/2-harmonic maps on $\mathbb{R}^2$. More precisely, we have the following proposition proved in [80, Proposition 4.30]. (Note that this proposition is stated for $\mathcal{N} = \mathbb{S}^1$, but the proof actually applies to any target manifold $\mathcal{N}$.)

Proposition 1.4.3 ([80])**.** *A map $u_0 \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2;\mathcal{N})$ is a 0-homogeneous 1/2-harmonic map if and only if $u_0(x) = g(\frac{x}{|x|})$ for some 1/2-harmonic circle $g : \mathbb{S}^1 \to \mathcal{N}$.*

Remark 1.4.4. Note that, by Proposition 1.4.3 and Remark 1.4.2, a 0-homogeneous minimizing 1/2-harmonic map on $\mathbb{R}^2$ is smooth away from the origin.

1.4.2 1/2-harmonic circles into spheres

The goal of this subsection is to establish a crucial classification result for 1/2-harmonic circles into spheres, a cornerstone in the proofs of both Theorem 1.1.3 and Theorem 1.1.4. From now on, we restrict ourselves to the case $\mathcal{N} = \mathbb{S}^{m-1}$ with $m \geqslant 3$.

If $g : \mathbb{S}^1 \to \mathbb{S}^{m-1}$ is a 1/2-harmonic circle into $\mathbb{S}^{m-1}$ (and thus smooth), then w_g defines a smooth map from the closed unit disk $\overline{\mathbb{D}}$ into the closed unit ball $\overline{B^m}$ of $\mathbb{R}^m$. By the maximum principle w_g maps the open disk $\mathbb{D}$ into the unit open ball B^m, and of course $w_g(\partial\mathbb{D}) \subseteq \mathbb{S}^{m-1} = \partial B^m$ by the boundary condition. In terms of w_g, the Euler-Lagrange equation for g being 1/2-harmonic writes (see e.g. [80, Remark 4.29])

$$\frac{\partial w_g}{\partial \nu} \wedge w_g = 0 \quad \text{on } \partial\mathbb{D}\,. \tag{1.4.4}$$

It has been (independently) proved in [8, 23, 24, 27], and [80, Lemma 4.27 & Remark 4.29] that g being 1/2-harmonic implies that w_g is (weakly) conformal or anti-conformal, i.e., it satisfies

$$\begin{cases} \left|\dfrac{\partial w_g}{\partial x_1}\right| = \left|\dfrac{\partial w_g}{\partial x_2}\right| \\ \dfrac{\partial w_g}{\partial x_1} \cdot \dfrac{\partial w_g}{\partial x_2} = 0 \end{cases} \quad \text{in } \mathbb{D}\,.$$

In addition, $|\nabla w_g|$ does not vanish near $\partial\mathbb{D}$ whenever g is not constant (by the Hopf boundary lemma applied to $|w_g|^2$, see e.g. [24, Proof of Theorem 2.7])[2]. As a consequence, if g is not constant, then w_g is a *(branched) minimal immersion* of the unit disk up to the boundary (with branched points only in the interior), and the boundary condition (1.4.4) tells us that $w_g(\overline{\mathbb{D}})$ meets ∂B^m orthogonally. For $m = 3$, a celebrated result of J.C.C. Nitsche [87] says that $w_g(\overline{\mathbb{D}})$ has to be the intersection of $\overline{B^3}$ with a plane through the origin. This result has been extended recently to arbitrary dimensions $m \geqslant 3$ in [48, Theorem 2.1]. In conclusion, if $g : \mathbb{S}^1 \to \mathbb{S}^{m-1}$ is a nonconstant 1/2-harmonic circle, then $g(\mathbb{S}^1)$ is an equatorial circle of $\mathbb{S}^{m-1}$. By invariance of the energy under rotations on the image, we can assume that such 1/2-harmonic map g takes values into $\mathbb{R}^2 \times \{0\}^{m-2} \subseteq \mathbb{R}^m$, so that it takes the form $g = (\widehat{g}, 0)$ where $\widehat{g} : \mathbb{S}^1 \to \mathbb{S}^1$ is a nonconstant 1/2-harmonic circle. On the other hand, the classification of all 1/2-harmonic circles into $\mathbb{S}^1$ has been obtained in [8, 23, 24, 80]: they are given by finite Blaschke products (see also [84] for a preliminary result where Blaschke products were first identified). The result can be stated as follows.

Theorem 1.4.5. *A map $\widehat{g} : \mathbb{S}^1 \to \mathbb{S}^1$ is a nonconstant 1/2-harmonic circle if and only if there exist an integer $d \geqslant 1$, $\theta \in [0, 2\pi]$, and $\alpha_1, \dots, \alpha_d \in \mathbb{D}$ such that $w_{\widehat{g}}$ or its complex conjugate equals*

$$z \mapsto e^{i\theta} \prod_{k=1}^{d} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}\,.$$

In particular, $\mathcal{E}_{\frac{1}{2}}(\widehat{g}, \mathbb{S}^1) = \pi d$.

[2] One can also prove that $\partial_\nu w_g$ does not vanish on $\partial\mathbb{D}$ as follows. Using (1.4.2) and (constrained) outer variations of $\mathcal{E}_{\frac{1}{2}}(\cdot, \mathbb{S}^1)$ at g, we can argue as in [80, Remark 4.3] to derive the equation

$$\frac{\partial w_g}{\partial \nu}(x) = \left(\frac{\gamma_1}{2} \int_{\mathbb{S}^1} \frac{|g(x) - g(y)|^2}{|x - y|^2} \mathrm{d}y\right) g(x) \quad \text{for } x \in \mathbb{S}^1\,.$$

Then, assuming by contradiction that $\partial_\nu w_g$ vanishes at some point $x_0 \in \mathbb{S}^1$, this equation implies that g is equal to the constant $g(x_0)$ (since $|g| = 1$).

Gathering the above results, we may now state the following corollary.

Corollary 1.4.6. *Assume that $m \geqslant 3$. If $g : \mathbb{S}^1 \to \mathbb{S}^{m-1}$ is a nonconstant 1/2-harmonic circle, then $g(\mathbb{S}^1)$ is an equatorial circle of $\mathbb{S}^{m-1}$, and $\mathcal{E}_{\frac{1}{2}}(g, \mathbb{S}^1) = \pi d$ with $d = |\deg(g)| \in \mathbb{N} \setminus \{0\}$.*

1.4.3 Proof of Theorem 1.1.3

We are now ready to prove Theorem 1.1.3. According to Corollary 1.3.9, it is enough to prove Proposition 1.4.7 below.

Proposition 1.4.7. *Assume that $m \geqslant 3$. If $u_0 \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{S}^{m-1})$ is a 0-homogeneous minimizing 1/2-harmonic map, then u_0 is constant.*

Proof. Assume by contradiction that u_0 is not constant. From Proposition 1.4.3, we know that

$$u_0(x) = g\left(\frac{x}{|x|}\right),$$

for some nonconstant 1/2-harmonic circle $g : \mathbb{S}^1 \to \mathbb{S}^{m-1}$. According to Corollary 1.4.6, $g(\mathbb{S}^1)$ is an equatorial circle of $\mathbb{S}^{m-1}$, and

$$\mathcal{E}_{\frac{1}{2}}(g, \mathbb{S}^1) = \pi d \quad \text{for some integer } d \geqslant 1.$$

Rotating coordinates in the image if necessary, we may assume without loss of generality that $g(\mathbb{S}^1) = \mathbb{S}^1 \times \{0\}^{m-2}$.

Let us now fix an arbitrary *radial* function $\zeta \in C^\infty_c(\mathbb{R}^2)$, and define $\varphi(x) := \zeta(x) e_m$, where $(e_1, \ldots, e_m)$ denotes the canonical basis of $\mathbb{R}^m$. Then $\varphi \in C^\infty_c(\mathbb{R}^2; \mathbb{R}^m)$, and consider a radius $R = R(\zeta) > 0$ such that $\mathrm{spt}(\zeta) \subseteq D_R$. For $\varepsilon \in (-1, 1)$, we define

$$u_\varepsilon := \frac{u_0 + \varepsilon\varphi}{\sqrt{1 + \varepsilon^2 |\varphi|^2}}.$$

Note that $u_\varepsilon \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^m)$, and since $\varphi(x) \cdot u_0(x) = 0$ for every $x \neq 0$, we actually have $u_\varepsilon \in H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{S}^{m-1})$. By construction we have $\mathrm{spt}(u_\varepsilon - u) \subseteq D_R$, so that

$$\mathcal{E}_{\frac{1}{2}}(u_\varepsilon, D_R) \geqslant \mathcal{E}_{\frac{1}{2}}(u, D_R)$$

for every $\varepsilon \in (-1, 1)$ by minimality of u_0. Equality obviously holds at $\varepsilon = 0$, and thus

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \mathcal{E}_{\frac{1}{2}}(u_\varepsilon, D_R)\right]_{|\varepsilon=0} \geqslant 0. \tag{1.4.5}$$

Straightforward computations yield

$$\dot{u} := \left(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}\varepsilon}\right)_{|\varepsilon=0} = \varphi \quad \text{and} \quad \ddot{u} := \left(\frac{\mathrm{d}^2 u_\varepsilon}{\mathrm{d}\varepsilon^2}\right)_{|\varepsilon=0} = -|\varphi|^2 u_0 \in H^{1/2}_{00}(D_R; \mathbb{R}^m),$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \mathcal{E}_{\frac{1}{2}}(u_\varepsilon, D_R)\right]_{|\varepsilon=0} = \frac{\gamma_2}{2} \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus (D^c_R \times D^c_R)} \frac{|\dot{u}(x) - \dot{u}(y)|^2}{|x - y|^3} \,\mathrm{d}x\mathrm{d}y$$
$$+ \frac{\gamma_2}{2} \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus (D^c_R \times D^c_R)} \frac{(u_0(x) - u_0(y)) \cdot (\ddot{u}(x) - \ddot{u}(y))}{|x - y|^3} \,\mathrm{d}x\mathrm{d}y.$$

Since $|\varphi|^2 = \zeta^2$ and ζ is compactly supported in D_R, we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{E}_{\frac{1}{2}}(u_\varepsilon, D_R)\right]_{|\varepsilon=0} = \frac{\gamma_2}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y - \left\langle(-\Delta)^{\frac{1}{2}}u_0, \zeta^2 u_0\right\rangle_{D_R}. \quad (1.4.6)$$

Recalling the weak formulation of (1.1.3) (or [80, Remark 4.3]), we have

$$(-\Delta)^{\frac{1}{2}}u_0(x) = \left(\frac{\gamma_2}{2}\int_{\mathbb{R}^2}\frac{|u_0(x)-u_0(y)|^2}{|x-y|^3}\,\mathrm{d}y\right)u_0(x) \quad \text{in } H^{-1/2}(D_R)\,.$$

Using the above equation in (1.4.6) and the fact that $|u_0| = 1$, we deduce from (1.4.5) that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \geqslant \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|u_0(x)-u_0(y)|^2}{|x-y|^3}\,\mathrm{d}y\right)\zeta^2(x)\,\mathrm{d}x\,. \quad (1.4.7)$$

Computing the right-hand side of this inequality in polar coordinates leads to (recall that ζ is assumed to be radial, i.e., $\zeta(x) = \zeta(|x|)$)

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|u_0(x)-u_0(y)|^2}{|x-y|^3}\,\mathrm{d}y\right)\zeta^2(x)\,\mathrm{d}x$$
$$= \int_0^\infty \zeta^2(r)\left(\iint_{\mathbb{S}^1\times\mathbb{S}^1}|g(\sigma_1)-g(\sigma_2)|^2\left[\int_0^\infty \frac{\rho}{|\sigma_1-\rho\sigma_2|^3}\,\mathrm{d}\rho\right]\mathrm{d}\sigma_1\mathrm{d}\sigma_2\right)\mathrm{d}r\,.$$

By formula [54, GW (213)(5b) p. 326], one has

$$\int_0^\infty \frac{\rho}{|\sigma_1-\rho\sigma_2|^3}\,\mathrm{d}\rho = \int_0^\infty \frac{\rho}{(1-2\rho\sigma_1\cdot\sigma_2+\rho^2)^{3/2}}\,\mathrm{d}\rho$$
$$= \frac{1}{1-\sigma_1\cdot\sigma_2} = \frac{2}{|\sigma_1-\sigma_2|^2} \qquad \forall \sigma_1 \neq \sigma_2\,.$$

Therefore,

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|u_0(x)-u_0(y)|^2}{|x-y|^3}\,\mathrm{d}y\right)\zeta^2(x)\,\mathrm{d}x = \frac{8}{\gamma_1}\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1)\int_0^\infty \zeta^2(r)\,\mathrm{d}r = 4\pi d\int_{\mathbb{R}^2}\frac{\zeta^2}{|x|}\,\mathrm{d}x\,,$$

and we conclude from (1.4.7) that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \geqslant 4\pi d\int_{\mathbb{R}^2}\frac{\zeta^2}{|x|}\,\mathrm{d}x\,. \quad (1.4.8)$$

In view of the arbitrariness of ζ, we conclude that (1.4.8) holds for every radial function $\zeta \in C_c^\infty(\mathbb{R}^2)$. On the other hand, Hardy's inequality in $H^{1/2}(\mathbb{R}^2)$ (see e.g. [46, 62]) says that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \geqslant C_\sharp\int_{\mathbb{R}^2}\frac{\zeta^2}{|x|}\,\mathrm{d}x \quad \forall \zeta \in C_c^\infty(\mathbb{R}^2)\,, \quad (1.4.9)$$

with optimal constant

$$C_\sharp := 8\pi\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2.$$

Moreover, the constant $C_\sharp$ is still sharp when restricting (1.4.9) to radial functions (by symmetric decreasing rearrangement, see e.g. [48]). In view of (1.4.8), we finally deduce that

$$4\pi d \leqslant C_\sharp\,,$$

that is $d \leqslant 2\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2 < 1$, a contradiction. □

1.5 Minimizing 1/2-harmonic maps into the circle

The aim of this section is now to prove Theorem 1.1.4 and Theorem 1.1.5. We thus assume that $n = m = 2$. In the first subsection, we recall the construction and properties of the distributional Jacobian in $H^{1/2}$-spaces (see [11, 91] or [79]). In the spirit of [13], the distributional Jacobian appears to be the main tool to derive energy lower bounds, and in particular to prove the minimality of $\frac{x}{|x|}$, see Section 1.5.2. The uniqueness part of Theorem 1.1.4 is proved in Section 1.5.3. It relies on Theorem 1.4.5 and subtle constructions of competitors, again in the spirit of [13]. Compared to [13], the argument is more intricate as it requires a preliminary construction (see Lemma 1.5.8) and the numerical evaluation of certain integrals. The last subsection is devoted to the proof of Theorem 1.1.5. The proof here is more classical and it is essentially based on Theorem 1.1.4.

1.5.1 The distributional Jacobian

For a map $g \in H^{1/2}(\partial B_1^+;\mathbb{R}^2)$, we define a distribution $T(g) \in (\mathrm{Lip}(\partial B_1^+))'$ in the following way. Consider $u \in H^1(B_1^+;\mathbb{R}^2)$ such that $u = g$ on ∂B_1^+, and set

$$H(u) := 2(\partial_2 u \wedge \partial_3 u, \partial_3 u \wedge \partial_1 u, \partial_1 u \wedge \partial_2 u) \in L^1(B_1^+;\mathbb{R}^3)\,,$$

where $\wedge$ denotes the wedge product on $\mathbb{R}^2$ (i.e., $a \wedge b := \det(a,b)$ for $a, b \in \mathbb{R}^2$).

For a scalar function $\varphi \in \mathrm{Lip}(\partial B_1^+)$ and an arbitrary extension $\Phi \in \mathrm{Lip}(B_1^+)$ of φ to the closed half ball $\overline{B_1^+}$, we define the action of $T(g)$ on φ by setting

$$\langle T(g), \varphi\rangle := \int_{B_1^+} H(u)\cdot \nabla\Phi \,\mathrm{d}\mathbf{x}\,.$$

Noticing that

$$\operatorname{div} H(u) = 0 \quad \text{in } \mathscr{D}'(B_1^+)\,,$$

it is routine to check that $T(g)$ is well defined, i.e., it does not depend on the extensions u and Φ, see e.g. [11, Lemma 3]. In addition, the mapping $T : g \mapsto T(g)$ is continuous, see [11, Lemma 9].

Lemma 1.5.1. *The mapping $T : H^{1/2}(\partial B_1^+;\mathbb{R}^2) \to (\mathrm{Lip}(\partial B_1^+))'$ is strongly continuous. More precisely, there exists a constant C such that*

$$|\langle T(g_1) - T(g_2), \varphi\rangle| \leqslant C\Big([g_1]_{H^{1/2}(\partial B_1^+)} + [g_2]_{H^{1/2}(\partial B_1^+)}\Big)[g_1 - g_2]_{H^{1/2}(\partial B_1^+)}[\varphi]_{\mathrm{lip}}$$

for every $g_1, g_2 \in H^{1/2}(\partial B_1^+;\mathbb{R}^2)$ and $\varphi \in \mathrm{Lip}(\partial B_1^+)$, where

$$[g]^2_{H^{1/2}(\partial B_1^+)} := \iint_{\partial B_1^+\times\partial B_1^+} \frac{|g(\mathbf{x}) - g(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}\mathscr{H}^2_{\mathbf{x}}\mathrm{d}\mathscr{H}^2_{\mathbf{y}}$$

and

$$[\varphi]_{\mathrm{lip}} := \sup_{\substack{\mathbf{x},\mathbf{y}\in\partial B_1^+\\ \mathbf{x}\neq\mathbf{y}}} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}\,.$$

We shall make use of the following explicit representation of $T(g)$ for maps g belonging to the following class of partially regular maps

$$\begin{aligned}\mathscr{R} := \Big\{ g \in H^{1/2}(\partial B_1^+;\mathbb{R}^2) : g_{|\mathbb{D}\times\{0\}} \in W^{1,1}(\mathbb{D};\mathbb{S}^1)\,,\ & g \text{ is smooth on } \overline{\partial^+ B_1},\\ &\text{smooth in a neighborhood of } \partial\mathbb{D}\times\{0\},\\ &\text{and smooth away from finitely many points in } \mathbb{D}\times\{0\}\Big\}\,.\end{aligned}$$

For a map $g \in \mathscr{R}$ and $a \in \mathbb{D}$ a singular point of $g_{|\mathbb{D}\times\{0\}} : \mathbb{D} \to \mathbb{S}^1$, we shall denote by $\deg(g, a)$ the topological degree of g restricted to any small circle around a (oriented in the counterclockwise sense). We have the following representation of $T(g)$ for g in the class $\mathscr{R}$.

Proposition 1.5.2. *Let $g \in \mathscr{R}$ be such that $g \in C^\infty\big((\overline{\mathbb{D}} \times \{0\}) \setminus \{a_1, \dots, a_K\}\big)$ for some distinct points $a_1, \dots, a_K \in \mathbb{D} \times \{0\}$. If $d_i := \deg(g, a_i)$, then*

$$\langle T(g), \varphi\rangle = 2\int_{\partial^+ B_1} \det(\nabla_\tau g)\varphi \,\mathrm{d}\mathscr{H}^2 - 2\pi \sum_{i=1}^K d_i \varphi(a_i) \quad \forall \varphi \in \mathrm{Lip}(\partial B_1^+)\,, \tag{1.5.1}$$

where $\nabla_\tau g$ denotes the tangential gradient[3] *of g on $\partial^+ B_1$.*

Proof. By the smoothness assumption on g, we may find an extension u of g which is smooth in $\overline{B_1^+} \setminus \{a_1, \dots, a_K\}$. We first claim that

$$\langle T(g), \varphi\rangle = 2\int_{\partial^+ B_1} \det(\nabla_\tau g)\varphi \,\mathrm{d}\mathscr{H}^2 + \int_{\mathbb{D}} (g\wedge\nabla g)\cdot\nabla^\perp\varphi\,\mathrm{d}x - \int_{\partial\mathbb{D}} (g\wedge\partial_\tau g)\varphi\,\mathrm{d}\mathscr{H}^1\,, \tag{1.5.2}$$

where $\nabla := (\partial_{x_1}, \partial_{x_2})$ and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ on $\mathbb{D}$, and ∂_τ denotes the tangential derivation on $\partial\mathbb{D}$ (oriented in the counterclockwise sense). Smoothing u near the $a_i's$, we can find a sequence (u_k) of smooth maps over $\overline{B}_1^+$ such that $u_k = u$ in a neighborhood of $\partial^+ B_1$, $u_k \to u$ strongly in $H^1(B^+)$, and $u_{k|\mathbb{D}\times\{0\}} \to g_{|\mathbb{D}\times\{0\}}$ strongly in $W^{1,1}(\mathbb{D})$ with $\|u_{k|\mathbb{D}\times\{0\}}\|_{L^\infty(\mathbb{D})} \leqslant 1$. In particular, given an extension $\Phi \in \mathrm{Lip}(B_1^+)$ of φ, we have

$$\int_{B_1^+} H(u_k)\cdot\nabla\Phi\,\mathrm{d}\mathbf{x} \underset{k\to\infty}{\longrightarrow} \langle T(g), \varphi\rangle\,. \tag{1.5.3}$$

Since div $H(u_k) = 0$, by the divergence theorem we have

$$\begin{aligned}\int_{B_1^+} H(u_k)\cdot\nabla\Phi\,\mathrm{d}\mathbf{x} &= \int_{\partial^+ B_1} \mathbf{x}\cdot H(u_k)\varphi\,\mathrm{d}\mathscr{H}^2 - 2\int_{\mathbb{D}} (\partial_1 u_k\wedge\partial_2 u_k)\varphi\,\mathrm{d}x \\ &= 2\int_{\partial^+ B_1} \det(\nabla_\tau g)\varphi\,\mathrm{d}\mathscr{H}^2 - 2\int_{\mathbb{D}} (\partial_1 u_k\wedge\partial_2 u_k)\varphi\,\mathrm{d}x\,. \end{aligned}\tag{1.5.4}$$

Noticing that $2\partial_1 u_k \wedge \partial_2 u_k = \mathrm{curl}(u_k\wedge\nabla u_k)$, a further integration by parts yields

$$\begin{aligned}2\int_{\mathbb{D}} (\partial_1 u_k\wedge\partial_2 u_k)\varphi\,dx &= -\int_{\mathbb{D}} (u_k\wedge\nabla u_k)\cdot\nabla^\perp\varphi\,\mathrm{d}x + \int_{\partial\mathbb{D}} (u_k\wedge\partial_\tau u_k)\varphi\,\mathrm{d}\mathscr{H}^1 \\ &= -\int_{\mathbb{D}} (u_k\wedge\nabla u_k)\cdot\nabla^\perp\varphi\,\mathrm{d}x + \int_{\partial\mathbb{D}} (g\wedge\partial_\tau g)\varphi\,\mathrm{d}\mathscr{H}^1\,. \end{aligned}\tag{1.5.5}$$

Gathering (1.5.3)-(1.5.4)-(1.5.5) and letting $k \to \infty$ now leads to (1.5.2) by dominated convergence.

To prove (1.5.1), it is now enough to show that

$$\int_{\mathbb{D}} (g\wedge\nabla g)\cdot\nabla^\perp\varphi\,\mathrm{d}x = \int_{\partial\mathbb{D}} (g\wedge\partial_\tau g)\varphi\,\mathrm{d}\mathscr{H}^1 - 2\pi\sum_{i=1}^K \varphi(a_i)\,. \tag{1.5.6}$$

[3] For $x \in \partial^+ B_1$ and $\tau_1, \tau_2 \in \mathrm{Tan}(x, \partial^+ B)$ such that (τ_1, τ_2, x) is a direct orthonormal basis of $\mathbb{R}^3$, we have $\nabla_\tau g(x) := (\partial_{\tau_1} g(x), \partial_{\tau_2} g(x))$, and $\det(\nabla_\tau g(x))$ does not depend on the choice of τ_1 and τ_2.

To this purpose we consider a sequence (φ_k) of Lipschitz functions over $\overline{\mathbb{D}}$ such that φ_k is constant in a neighborhood of each a_i, $\varphi_k \to \varphi$ uniformly on $\overline{\mathbb{D}}$, and $\nabla\varphi_k \rightharpoonup \nabla\varphi$ weakly* in $L^\infty(\mathbb{D})$. In this way,

$$\int_{\mathbb{D}} (g \wedge \nabla g) \cdot \nabla^\perp \varphi_k \,\mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{D}} (g \wedge \nabla g) \cdot \nabla^\perp \varphi \,\mathrm{d}x \,.$$

Given k, we consider $\varepsilon_k > 0$ small enough in such a way that $D_{2\varepsilon_k}(a_i) \cap D_{2\varepsilon_k}(a_j) = \emptyset$ for $i \neq j$, $D_{2\varepsilon_k}(a_i) \cap \partial\mathbb{D} = \emptyset$ for each i, and $\varphi_k = \varphi_k(a_i)$ in $D_{\varepsilon_k}(a_i)$. Then,

$$\begin{aligned}\int_{\mathbb{D}} (g \wedge \nabla g) \cdot \nabla^\perp \varphi_k \,\mathrm{d}x &= \int_{\mathbb{D}\setminus\bigcup_{i=1}^K D_{\varepsilon_k}(a_i)} (g \wedge \nabla g) \cdot \nabla^\perp \varphi_k \,\mathrm{d}x \\ &= -2\int_{\mathbb{D}\setminus\bigcup_{i=1}^K D_{\varepsilon_k}(a_i)} (\partial_1 g \wedge \partial_2 g)\varphi_k \,\mathrm{d}x + \int_{\partial\mathbb{D}} (g\wedge\partial_\tau g)\varphi_k \,\mathrm{d}\mathscr{H}^1 \\ &\quad - \sum_{i=1}^K \varphi_k(a_i) \int_{\partial D_{\varepsilon_k}(a_i)} (g\wedge\partial_\tau g)\,\mathrm{d}\mathscr{H}^1 \\ &= \int_{\partial\mathbb{D}} (g\wedge\partial_\tau g)\varphi_k \,\mathrm{d}\mathscr{H}^1 - 2\pi\sum_{i=1}^K d_i\varphi_k(a_i)\,. \end{aligned} \tag{1.5.7}$$

In the last identity, we have used the fact that $\partial_1 g \wedge \partial_2 g = \det(\nabla g) = 0$ in the region $\mathbb{D}\setminus\bigcup_{i=1}^K D_{\varepsilon_h}(a_i)$, since g is $\mathbb{S}^1$-valued and smooth in that region. Letting $k \to \infty$ in (1.5.7) finally leads to (1.5.6). □

1.5.2 Proof of Theorem 1.1.4, part 1.

By Theorem 1.3.6, to prove the minimality of $u_\star(x) := \frac{x}{|x|}$, it is enough to prove that its harmonic extension is minimizing, and this is the way we proceed. First, we need to compute explicitly its harmonic extension. To this purpose, it is useful to consider the inverse stereographic projection $\mathfrak{S} : \overline{\mathbb{D}} \to \overline{\mathbb{S}^2_+}$ given by

$$\mathfrak{S}(x) := \left(\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)\,, \tag{1.5.8}$$

and its inverse $\mathfrak{S}^{-1} : \overline{\mathbb{S}^2_+} \to \overline{\mathbb{D}}$ (which is the stereographic projection from the south pole):

$$\mathfrak{S}^{-1}(\mathbf{x}) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\right) = \frac{x_1 + ix_2}{1+x_3}\,. \tag{1.5.9}$$

Let us recall that $\mathfrak{S}$ is a conformal transformation.

Lemma 1.5.3. *The harmonic extension of the map $u_\star(x) := x/|x|$ is given by*

$$u^{\mathrm{e}}_\star(\mathbf{x}) = \frac{x}{|\mathbf{x}| + x_3}\,.$$

Proof. Since $u_\star$ is 0-homogeneous, its harmonic extension $u^{\mathrm{e}}_\star$ is also 0-homogeneous. Being harmonic in $\mathbb{R}^3_+$, it satisfies

$$\begin{cases} \Delta_{\mathbb{S}^2} u^{\mathrm{e}}_\star = 0 & \text{on } \mathbb{S}^2_+\,, \\ u^{\mathrm{e}}_\star = u_\star & \text{on } \partial\mathbb{S}^2_+ = \mathbb{S}^1 \times \{0\}\,, \end{cases} \tag{1.5.10}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^2$. Next we define $w : \overline{\mathbb{D}} \to \mathbb{R}^2$ by setting

$$w(x) := u_\star^{\rm e}(\mathfrak{S}(x)) \,,$$

where $\mathfrak{S}$ is the inverse stereographic projection from the closed unit disk into $\overline{\mathbb{S}^2_+}$ defined in (1.5.8). Since $\mathfrak{S}$ is conformal, and $\mathfrak{S}(x) = (x, 0)$ for $x \in \partial\mathbb{D}$, we infer from (1.5.10) that

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{D}\,, \\ w(x) = x & \text{on } \partial\mathbb{D}\,. \end{cases}$$

By uniqueness of the harmonic extension, we deduce that $w(x) = x$ for every $x \in \mathbb{D}$, and consequently

$$u_\star^{\rm e}(\mathbf{x}) = \mathfrak{S}^{-1}(\mathbf{x}) = \frac{x}{1+x_3} \quad \text{for every } \mathbf{x} = (x, x_3) \in \mathbb{S}^2_+\,.$$

The conclusion follows by 0-homogeneity of $u_\star^{\rm e}$. ☐

In what follows, we keep the notation $u_\star(x) := x/|x|$. In the following lemma, we provide an approximation result to reduce the class of competitors (to test the minimality of $u_\star$) to the ones belonging to the class $\mathscr{R}$.

Lemma 1.5.4. *Let $u \in H^{1/2}(\mathbb{D};\mathbb{S}^1)$ be such that $u = u_\star$ in a neighborhood of $\partial\mathbb{D}$. There exists a sequence $(u_k) \subseteq H^{1/2}(\mathbb{D};\mathbb{S}^1) \cap W^{1,1}(\mathbb{D})$ such that $u_k = u_\star$ in a neighborhood of $\partial\mathbb{D}$, u_k is smooth away from finitely many points, and $u_k \to u$ strongly in $H^{1/2}(\mathbb{D})$.*

Proof. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we recall that both $H^{1/2}(\mathbb{D};\mathbb{C}) \cap L^\infty(\mathbb{D})$ and $W^{1,1}(\mathbb{D};\mathbb{C}) \cap L^\infty(\mathbb{D})$ are Banach algebras. If $\bar{u}_\star$ denotes the complex conjugate of $u_\star$, the map $w := \bar{u}_\star u$ belongs to $H^{1/2}(\mathbb{D};\mathbb{S}^1) \cap W^{1,1}(\mathbb{D})$, and it is identically equal to one in a neighborhood of $\partial\mathbb{D}$. Extending w by the value one outside $\mathbb{D}$, we can apply the method in [79, Proof of Theorem 2.16] to produce a sequence $(w_k) \subseteq H^{1/2}_{\rm loc}(\mathbb{R}^2;\mathbb{S}^1) \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ such that w_k is smooth outside a finite subset of $\mathbb{R}^2$, and $w_k \to w$ strongly in $H^{1/2}_{\rm loc}(\mathbb{R}^2)$. Using that w equals one near $\partial\mathbb{D}$, a quick inspection of the construction (which is based on a convolution argument with a sequence of mollifiers) shows that w_k is also equal to one near $\partial\mathbb{D}$ (at least for k large enough). Therefore, setting $u_k := u_\star w_k$, we have $u_k \in H^{1/2}(\mathbb{D};\mathbb{S}^1) \cap W^{1,1}(\mathbb{D})$, u_k is equal to $u_\star$ near $\partial\mathbb{D}$, u_k is smooth away from a finite set, and $u_k \to u_\star$ strongly in $H^{1/2}(\mathbb{D})$. ☐

We shall need the following theorem which is a slight generalization of [13, Theorem 7.5]. Since the proof follows closely [13] with only minor modifications, we shall omit it.

Theorem 1.5.5 ([13])**.** *Let $(\mathscr{M}, \boldsymbol{\delta})$ be a compact metric space, and μ a nonnegative Radon measure on $\mathscr{M}$ satisfying $\mu(\mathscr{M}) = 1$. Given a closed subset $A \subseteq \mathscr{M}$, $N \geqslant 1$ distinct points $a_1, \ldots, a_N \subseteq A$, and $d_1, \ldots, d_N \in \mathbb{Z}$ satisfying $\sum_i d_i = 1$, define for $\nu := \sum_i d_i \delta_{a_i}$,*

$$\mathbf{I}(\nu) := \sup\Big\{ \int_{\mathscr{M}} \varphi \,{\rm d}\mu - \int_{\mathscr{M}} \varphi \,{\rm d}\nu : \varphi \in {\rm Lip}(\mathscr{M})\,,\ [\varphi]_{\rm lip} \leqslant 1 \Big\}\,,$$

with $[\varphi]_{\rm lip} := \sup_{x \neq y} \frac{|\varphi(x)-\varphi(y)|}{\boldsymbol{\delta}(x,y)}$. Then,

$$\mathbf{I}(\nu) \geqslant \min_{c \in A} \int_{\mathscr{M}} \boldsymbol{\delta}(x, c) \,{\rm d}\mu_x\,.$$

Proof of Theorem 1.1.4: minimality of $u_\star$. By Theorem 1.3.6, to prove that $u_\star$ is a 0-homogeneous minimizing 1/2-harmonic map, it is enough to show that $u_\star^{\rm e}$ is a minimizing harmonic map with free boundary in every bounded admissible open set $G \subseteq \mathbb{R}^3_+$. In turn, it reduces to prove that $u_\star^{\rm e}$ is a minimizing harmonic map with free boundary in B_R^+ for every radius $R > 0$. By 0-homogeneity of $u_\star^{\rm e}$, it is enough to show that $u_\star^{\rm e}$ is a minimizing harmonic map with free boundary in B_1^+.

First, we compute using Lemma 1.5.3,

$$\mathbf{E}_{\frac{1}{2}}(u_\star^{\rm e}, B_1^+) = \int_{B_1^+} \frac{{\rm d}\mathbf{x}}{(|\mathbf{x}| + x_3)^2} = \int_{\partial^+ B_1} \frac{{\rm d}\mathscr{H}^2}{(1 + x_3)^2} = \pi \,. \tag{1.5.11}$$

In view of (1.5.11), it is thus enough to show that

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \pi \tag{1.5.12}$$

for every map $v \in H^1(B_1^+; \mathbb{R}^2)$ such that $v = u_\star^{\rm e}$ in a neighborhood of $\partial^+ B_1$ and $|v| = 1$ on $\mathbb{D} \times \{0\}$.

Let us consider such a map v. From the pointwise inequality $|\nabla v|^2 \geqslant |H(v)|$, we first infer that

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \frac{1}{2} \int_{B_1^+} |H(v)| \, {\rm d}\mathbf{x} \,. \tag{1.5.13}$$

Then, consider an arbitrary function $\varphi \in {\rm Lip}(\partial B_1^+)$ satisfying $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leqslant |\mathbf{x} - \mathbf{y}|$ for every $\mathbf{x}, \mathbf{y} \in \partial B_1^+$. By the McShane-Whitney extension theorem, we can find a 1-Lipschitz function $\Phi \in {\rm Lip}(B_1^+)$ such that $\Phi_{|\partial B_1^+} = \varphi$. Since $|\nabla \Phi| \leqslant 1$ a.e. in B_1^+, we deduce from (1.5.13) that

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \frac{1}{2} \int_{B_1^+} H(v) \cdot \nabla \Phi \, {\rm d}\mathbf{x} = \frac{1}{2} \langle T(g), \varphi \rangle \,, \tag{1.5.14}$$

where $g := v_{|\partial B_1^+} \in H^{1/2}(\partial B_1^+; \mathbb{R}^2)$ is equal to $u_\star^{\rm e}$ in a neighborhood of $\partial^+ B_1$.

By Lemma 1.5.4, we can find a sequence $(u_k) \subseteq H^{1/2}(\mathbb{D}; \mathbb{S}^1) \cap W^{1,1}(\mathbb{D})$ such that $u_k = u_\star$ in a neighborhood of $\partial \mathbb{D}$, u_k is smooth away from finitely many points in $\mathbb{D}$, and $u_k \to g_{|\mathbb{D} \times \{0\}}$ strongly in $H^{1/2}(\mathbb{D})$. Setting

$$g_k(\mathbf{x}) := \begin{cases} g(\mathbf{x}) & \text{if } \mathbf{x} \in \partial^+ B_1 \,, \\ u_k(x) & \text{if } \mathbf{x} = (x, 0) \in \mathbb{D} \times \{0\} \,, \end{cases}$$

we have $g_k \in \mathscr{R}$, $g_k = u_\star^{\rm e}$ in a neighborhood of $\partial^+ B_1$, and $g_k \to g$ strongly in $H^{1/2}(\partial B_1^+)$.

Let us now fix the index k. Since $g_k \in \mathscr{R}$, we can find distinct points $a_1, \ldots, a_{N_k}$ in $\mathbb{D}$ such that g_k is smooth away from the a_i's. In addition, if $d_i := \deg(g_k, a_i)$, then

$$\sum_{i=1}^{N_k} d_i = \deg(g_k, \partial \mathbb{D}) = \deg(u_\star, \partial \mathbb{D}) = 1 \,.[4]$$

[4] Indeed, defining $f_k : \overline{\mathbb{D}} \setminus \{a_1, \ldots, a_{N_k}\} \to \mathbb{S}^1$ by $f_k(z) = \Pi_{i=1}^{N_k} \left(\frac{\overline{z - a_i}}{|z - a_i|} \right)^{d_i} g_k(z)$, then $\deg(f_k, \partial \mathbb{D}) = \deg(g_k, \partial \mathbb{D}) - \sum_{i=1}^{N_k} d_i$, using the well-known properties of the topological degree that $\deg(fg, \partial \mathbb{D}) = \deg(f, \partial \mathbb{D}) + \deg(g, \partial \mathbb{D})$ and $\deg(\overline{f}, \partial \mathbb{D}) = -\deg(f, \partial \mathbb{D})$ for every $f, g \in C(\partial \mathbb{D}, \mathbb{S}^1)$. In addition, $\deg(f_k, \partial(\mathbb{D}_{r_i}(a_i))) = 0$ for some small $r_1, \ldots, r_{N_k} \in (0,1)$ by construction, thus there exists $\widetilde{f}_k \in C(\mathbb{D}, \mathbb{S}^1)$ agreeing with f_k on $\overline{\mathbb{D}} \setminus \big(\cup_{i=1}^{N_k} \mathbb{D}_{r_i}(a_i) \big)$. Since $\widetilde{f}_k$ is a continuous map from $\overline{\mathbb{D}}$ into $\mathbb{S}^1$, it is homotopic to a point and $\deg(\widetilde{f}_k, \partial \mathbb{D}) = 0$. Hence $0 = \deg(\widetilde{f}_k, \partial \mathbb{D}) = \deg(f_k, \partial \mathbb{D}) = \deg(g_k, \partial \mathbb{D}) - \sum_{i=1}^{N_k} d_i$ since $\widetilde{f}_{k\,|\partial \mathbb{D}} = f_{k\,|\partial \mathbb{D}}$.

Applying Proposition 1.5.2 to g_k and using that $\det(\nabla_\tau g_k) = \mathbf{x}\cdot H(u_\star^{\rm e})$ on $\partial^+ B_1$ yields

$$\frac{1}{2}\langle T(g_k),\varphi\rangle = \pi\left(\frac{1}{\pi}\int_{\partial^+ B_1}\frac{\varphi}{(1+x_3)^2}\,{\rm d}\mathscr{H}^2 - \sum_{i=1}^{N_k} d_i\varphi(a_i)\right).$$

In turn, applying Theorem 1.5.5 with $\mathscr{M} = \partial B_1^+$ endowed with the Euclidean metric, $A = \overline{\mathbb{D}}\times\{0\}$, $\mu = \frac{1}{\pi}(1+x_3)^{-2}\mathscr{H}^2 \llcorner \partial^+ B_1$, and $\nu = \sum_i d_i\delta_{a_i}$, yields

$$\sup_{[\varphi]_{\rm lip}\leqslant 1}\frac{1}{2}\langle T(g_k),\varphi\rangle \geqslant \min_{c\in\overline{\mathbb{D}}\times\{0\}}\int_{\partial^+ B_1}\frac{|\mathbf{x}-c|}{(1+x_3)^2}\,{\rm d}\mathscr{H}^2\,. \tag{1.5.15}$$

Next, observe that the minimum value above is achieved at $c = 0$. Indeed, the function

$$V : z\in\overline{\mathbb{D}}\mapsto \int_{\partial^+ B_1}\frac{|\mathbf{x}-(z,0)|}{(1+x_3)^2}\,{\rm d}\mathscr{H}^2$$

is clearly convex, and

$$\nabla V(0) = -\int_{\partial^+ B_1}\frac{x}{(1+x_3)}\,{\rm d}\mathscr{H}^2 = 0\,.$$

Going back to (1.5.15), we have thus proved that

$$\sup_{[\varphi]_{\rm lip}\leqslant 1}\frac{1}{2}\langle T(g_k),\varphi\rangle \geqslant \int_{\partial^+ B_1}\frac{1}{(1+x_3)^2}\,{\rm d}\mathscr{H}^2 = \pi\,. \tag{1.5.16}$$

Now we deduce from Lemma 1.5.1 that

$$\sup_{[\varphi]_{\rm lip}\leqslant 1}\frac{1}{2}\langle T(g),\varphi\rangle \geqslant \sup_{[\varphi]_{\rm lip}\leqslant 1}\frac{1}{2}\langle T(g_k),\varphi\rangle - C[g-g_k]_{H^{1/2}(\partial B_1^+)}\,, \tag{1.5.17}$$

for a constant C independent of k. Gathering (1.5.14), (1.5.17), and (1.5.16), we obtain

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \pi - C[g-g_k]_{H^{1/2}(\partial B_1^+)}\,.$$

Letting $k\to\infty$ leads to (1.5.12), which completes the proof. □

1.5.3 Proof of Theorem 1.1.4, part 2.

The goal of this subsection is to prove that $u_\star(x) = \frac{x}{|x|}$ is the unique 0-homogeneous 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$, up to an orthogonal transformation. This is achieved in two steps. The first one consists in proving that $u_\star$ is the unique 0-homogeneous 1/2-harmonic map of degree ± 1 (at the origin), up to an orthogonal transformation (see Proposition 1.5.7). In the second step, we prove that a 0-homogeneous 1/2-harmonic map with a degree (at the origin) different from ± 1 is not minimizing (see Proposition 1.5.9).

Lemma 1.5.6. *If u_0 is a nontrivial 0-homogenous 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$, then*

$$u_0^{\rm e}(\mathbf{x}) = w\circ\mathfrak{S}^{-1}\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big)\,,$$

where $\mathfrak{S}^{-1}$ is the stereographic projection (1.5.9), and w is a finite Blaschke product or the complex conjugate of a finite Blaschke product. In other words,

$$w(z) \;\textit{or}\; \overline{w}(z) \;= e^{i\theta}\prod_{j=1}^{d}\frac{z-\alpha_j}{1-\overline{\alpha_j}z} \tag{1.5.18}$$

for some $\theta\in[0,2\pi[$, $d\in\mathbb{N}\setminus\{0\}$, and $\alpha_1,\dots,\alpha_d\in\mathbb{D}$. As a consequence,

$$\mathbf{E}_{\frac{1}{2}}(u_0^{\rm e}, B_1^+) = \pi d\,. \tag{1.5.19}$$

Proof. By Proposition 1.4.3, $u_0(x) = g(\frac{x}{|x|})$ for every $x \neq 0$, for some nonconstant 1/2-harmonic circle $g : \mathbb{S}^1 \to \mathbb{S}^1$. By Theorem 1.4.5, the harmonic extension w_g of g to the unit disk $\mathbb{D}$ (i.e., the solution of (1.4.3)) is of the form (1.5.18). Hence, we only have to prove that $u_0^{\rm e}(\mathbf{x}) = w_g \circ \mathfrak{S}^{-1}(\frac{\mathbf{x}}{|\mathbf{x}|})$. The argument is exactly as in the proof of Lemma 1.5.3. By 0-homogeneity, $u_0^{\rm e}$ solves

$$\begin{cases} \Delta_{\mathbb{S}^2} u_0^{\rm e} = 0 & \text{on } \mathbb{S}^2_+\,, \\ u_0^{\rm e}(\mathbf{x}) = g & \text{on } \partial\mathbb{S}^2_+ = \mathbb{S}^1 \times \{0\}\,. \end{cases}$$

As a consequence, $u_0^{\rm e} \circ \mathfrak{S}$ is harmonic in $\mathbb{D}$, and it equals g on $\partial\mathbb{D}$. In other words, $u_0^{\rm e} \circ \mathfrak{S} = w_g$, and (1.5.18) follows.

Next, by 0-homogeneity of $u_0^{\rm e}$, conformal invariance, (1.4.2), and Theorem 1.4.5,

$$\mathbf{E}_{\frac{1}{2}}(u_0^{\rm e}, B_1^+) = \frac{1}{2}\int_{\partial^+ B_1} |\nabla_\tau u_0^{\rm e}|^2 \,{\rm d}\mathscr{H}^2 = \frac{1}{2}\int_{\mathbb{D}} |\nabla w_g|^2 \,{\rm d}z = \mathcal{E}_{\frac{1}{2}}(g, \mathbb{S}^1) = \pi d\,,$$

which completes the proof. □

Proposition 1.5.7. *Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be a 1/2-harmonic circle such that $\deg(g) \in \{\pm 1\}$. Assume that $u_0 := g(\frac{x}{|x|})$ is a 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$. Then g is an orthogonal transformation, i.e., $g(x) = Ax$ for some $A \in O(2, \mathbb{R})$.*

Proof. Step 1. By Theorem 1.3.6, $u_0^{\rm e}$ is a minimizing harmonic map with free boundary in B_1^+. Therefore, $u_0^{\rm e}$ is stationary in B_1^+ in the sense of [80, Definition 4.10], see [80, Remark 4.13]. In turn, by [80, Remark 4.11] it implies that

$$\int_{B_1^+} \left(|\nabla u_0^{\rm e}|^2 \,{\rm div}\, X - 2\sum_{i,j=1}^{3} (\partial_i u_0^{\rm e} \cdot \partial_j u_0^{\rm e})\partial_j X_i \right) {\rm d}\mathbf{x} = 0 \tag{1.5.20}$$

for every $X := (X_1, X_2, X_3) \in C^1(\overline{B}_1^+; \mathbb{R}^3)$ compactly supported in $B_1^+ \cup \partial^0 B_1^+$ and such that $X_3 = 0$ on $\partial^0 B_1^+$.

We now consider a unit vector $e \in \mathbb{S}^1 \times \{0\}$ and an even function $\eta \in C^1(\mathbb{R})$ compactly supported in $(-1, 1)$. Using the vector field $X(\mathbf{x}) := \eta(|\mathbf{x}|)e$ in (1.5.20), we obtain

$$\int_{B_1^+} \Big(|\nabla u_0^{\rm e}|^2 \,\mathbf{x}\cdot e - 2(e \cdot \nabla u_0^{\rm e}) \cdot (\mathbf{x} \cdot \nabla u_0^{\rm e}) \Big) \eta'(|\mathbf{x}|) \frac{{\rm d}\mathbf{x}}{|\mathbf{x}|} = 0\,. \tag{1.5.21}$$

On the other hand, since u_0 is 0-homogeneous, $u_0^{\rm e}$ is also 0-homogeneous. Hence $\mathbf{x}\cdot\nabla u_0^{\rm e} = 0$, and by Fubini's theorem, (1.5.21) yields

$$\left(\int_{\partial^+ B_1} |\nabla u_0^{\rm e}|^2 \mathbf{x} \cdot e \,{\rm d}\mathscr{H}^2 \right) \left(\int_0^1 \eta'(r)\,{\rm d}r \right) = 0\,,$$

since $\nabla u_0^{\rm e}$ is homogeneous of degree -1. By arbitrariness of η and e, we conclude that

$$\int_{\partial^+ B_1} |\nabla u_0^{\rm e}|^2 x \,{\rm d}\mathscr{H}^2 = 0 \tag{1.5.22}$$

(recall that $\mathbf{x} = (x, x_3)$).

Step 2. Since minimality is preserved under complex conjugation (i.e., $\overline{u}_0$ is also a 0-homogeneous minimizing 1/2-harmonic map), we may assume that $\deg(g) = 1$ (otherwise we consider $\overline{g}$ instead of g). Then we infer from Lemma 1.5.6 that

$$u_0^{\rm e}(\mathbf{x}) = w \circ \mathfrak{S}^{-1}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \quad \text{with} \quad w(z) = e^{i\theta}\frac{z - \alpha}{1 - \bar{\alpha} z}\,,$$

for some $\theta \in [0, 2\pi[$ and $\alpha \in \mathbb{D}$ (where $\mathfrak{S}^{-1}$ is the stereographic projection (1.5.9)).

By conformal invariance, we have

$$\int_{\partial^+ B_1} |\nabla u_0^{\mathrm{e}}|^2 x \,\mathrm{d}\mathscr{H}^2 = 2 \int_{\mathbb{D}} |\nabla w(z)|^2 \frac{z}{1+|z|^2} \,\mathrm{d}z\,. \tag{1.5.23}$$

In addition, since w is holomorphic in $\mathbb{D}$, we have

$$|\nabla w(z)|^2 = 2|w'(z)|^2 = \frac{(1-|\alpha|^2)}{|1-\overline{\alpha} z|^4}\,. \tag{1.5.24}$$

Hence, combining (1.5.22), (1.5.23), and (1.5.24) yields

$$\int_{\mathbb{D}} \frac{z}{(1+|z|^2)|1-\overline{\alpha} z|^4} \,\mathrm{d}z = 0\,,$$

which in turn implies that $\alpha = 0$. In other words, $g(z) = e^{i\theta} z$, i.e., g is a rotation. □

Lemma 1.5.8. *Let u_0 be a 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$. If $u_0^{\mathrm{e}}(\mathbf{x}) = w \circ \mathfrak{S}^{-1}(\frac{\mathbf{x}}{|\mathbf{x}|})$ with $\mathfrak{S}^{-1}$ the stereographic projection (1.5.9), and*

$$w(z) = e^{i\theta} \prod_{j=1}^{d} \frac{z-\alpha_j}{1-\overline{\alpha_j} z}\,,$$

with $d \in \mathbb{N} \setminus \{0\}$, and $\alpha_1, \dots, \alpha_d \in \mathbb{D}$, then

$$|w(z)| \leqslant \left(\frac{3|z|+1}{|z|+3}\right)^d \quad \textit{for every } z \in \mathbb{D}\,.$$

Proof. The case $d = 1$ is a direct consequence of Proposition 1.5.7, so it remains to consider the case $d \geqslant 2$. Set $\boldsymbol{\delta} := \max_j |\alpha_j| \in [0,1)$. We may assume without loss of generality that $\boldsymbol{\delta} = |\alpha_d|$. Since minimality is preserved under rotations on the image (i.e., Au_0 is a 0-homogeneous minimizing 1/2-harmonic map for every $A \in SO(2,\mathbb{R})$), we can also assume that $\alpha_d \in [0,1)$, so that $\boldsymbol{\delta} = \alpha_d$. Then we write

$$w(z) = \frac{z-\boldsymbol{\delta}}{1-\boldsymbol{\delta} z}\,\widetilde{w}(z) \quad \text{with} \quad \widetilde{w}(z) = e^{i\theta} \prod_{j=1}^{d-1} \frac{z-\alpha_j}{1-\overline{\alpha_j} z}\,.$$

We aim to prove that

$$\boldsymbol{\delta} \leqslant 1/3\,, \tag{1.5.25}$$

which immediately leads to the conclusion since

$$\frac{|z-\alpha_j|}{|1-\overline{\alpha_j} z|} \leqslant \frac{|z|+|\alpha_j|}{|\alpha_j||z|+1} \leqslant \frac{|z|+\boldsymbol{\delta}}{\boldsymbol{\delta}|z|+1} \leqslant \frac{3|z|+1}{|z|+3}$$

for each j and every $z \in \mathbb{D}$ [5].

[5] We may see that $\frac{|z-\alpha_j|}{|1-\overline{\alpha_j} z|} \leqslant \frac{|z|+|\alpha_j|}{|\alpha_j||z|+1}$ for all $z \in \mathbb{D}$, by writing $\frac{|z-\alpha_j|}{|1-\overline{\alpha_j} z|} = \frac{|ze^{-i\theta_j} - |\alpha_j||}{|1-|\alpha_j| ze^{-i\theta_j}|}$, where $\theta_j = \arg(\alpha_j)$, so that it is enough to show $\frac{|z-\alpha|}{|1-\alpha z|} \leqslant \frac{|z|+\alpha}{\alpha|z|+1}$ for all $z \in \mathbb{D}$ and $\alpha \in (0,1)$. Taking this inequality to the square and writing $z = re^{i\theta}$ this is equivalent to $\frac{r^2+\alpha^2-2r\alpha\cos\theta}{1+\alpha^2 r^2 - 2\alpha r\cos\theta} \leqslant \frac{r^2+\alpha^2+2\alpha r}{1+\alpha^2 r^2+2\alpha r}$ whose left-hand side is maximal when $\cos\theta = -1$, by studying its variations, which gives the result. In addition the right-hand side of this last inequality is an increasing function of α.

To prove (1.5.25), we shall construct suitable competitors to test the minimality of $u_0^{\rm e}$ in B_1^+ (recall that $u_0^{\rm e}$ is a minimizing harmonic map with free boundary in B_1^+ by Theorem 1.3.6). Given a parameter $\varepsilon \in (0,1)$, we consider a smooth function $\boldsymbol{\beta} : [0,1] \to [0,1]$ such that $\boldsymbol{\beta}(r) = \boldsymbol{\delta}$ in a neighborhood of $r = 1$, $\boldsymbol{\beta}(r) < 1$ for $r > \varepsilon$, and $\boldsymbol{\beta}(r) = 1$ for $r \leqslant \varepsilon$. Next we consider the smooth map on $\mathbb{D} \times [0,1]$ given by

$$\widehat{w}(z,r) := \frac{z - \boldsymbol{\beta}(r)}{1 - \boldsymbol{\beta}(r) z}\, \widetilde{w}(z)\,.$$

By construction, $\widehat{w}(\cdot, r)$ is a Blaschke product with d factors for $r > \varepsilon$, and $(d-1)$ factors for $r \leqslant \varepsilon$ (more precisely, $\widehat{w}(\cdot, r) = \widetilde{w}$ for $r \leqslant \varepsilon$). Setting $g_r := \widehat{w}(\cdot, r)_{|\partial\mathbb{D}}$, we then have $\deg(g_r) = d$ for $r > \varepsilon$, and $\deg(g_r) = d-1$ for $r \leqslant \varepsilon$. From (1.4.2) and Theorem 1.4.5, we infer that

$$\frac{1}{2}\int_{\mathbb{D}} \left|\nabla_z \widehat{w}(z,r)\right|^2 {\rm d}z = \mathcal{E}_{\frac{1}{2}}(g_r, \mathbb{S}^1) = \begin{cases} \pi d & \text{for } r > \varepsilon\,, \\ \pi(d-1) & \text{for } r \leqslant \varepsilon\,. \end{cases} \tag{1.5.26}$$

In addition, since $|\widetilde{w}| \leqslant 1$, we have the pointwise estimate

$$\left|\frac{\partial \widehat{w}}{\partial r}(z,r)\right|^2 \leqslant \frac{|z^2-1|^2}{|1-\boldsymbol{\beta}(r)z|^4}|\boldsymbol{\beta}'(r)|^2 = \frac{(1+|z|^2)^2 - 4z_1^2}{(1-2\boldsymbol{\beta}(r)z + \boldsymbol{\beta}^2(r)|z|^2)^2}|\boldsymbol{\beta}'(r)|^2\,. \tag{1.5.27}$$

We define a map $v \in H^1(B_1^+;\mathbb{R}^2)$ by setting

$$v(\mathbf{x}) := \widehat{w}\left(\mathfrak{S}^{-1}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), |\mathbf{x}|\right) \quad \text{for } \mathbf{x} \in B_1^+\,.$$

Note that $|v| = 1$ on $\partial^0 B_1^+$, and that $v = u_0^{\rm e}$ in a neighborhood of $\partial^+ B_1$. Hence v is an admissible competitor to test the minimality of $u_0^{\rm e}$ in B_1^+, i.e.,

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \mathbf{E}_{\frac{1}{2}}(u_0^{\rm e}, B_1^+) = \pi d\,, \tag{1.5.28}$$

where we have used (1.5.19) in the last equality.

Computing the energy of v in polar coordinates, we obtain

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) = \int_0^1 \left(\frac{1}{2}\int_{\partial^+ B_1} \left|\nabla_\tau v(r\mathbf{x})\right|^2 {\rm d}\mathscr{H}^2\right) {\rm d}r + \int_\varepsilon^1 \left(\frac{r^2}{2}\int_{\partial^+ B_1} \left|\partial_r v(r\mathbf{x})\right|^2 {\rm d}\mathscr{H}^2\right) {\rm d}r\,. \tag{1.5.29}$$

By conformal invariance, we have

$$\frac{1}{2}\int_{\partial^+ B_1} \left|\nabla_\tau v(r\mathbf{x})\right|^2 {\rm d}\mathscr{H}^2 = \frac{1}{2}\int_{\mathbb{D}} \left|\nabla_z \widehat{w}(z,r)\right|^2 {\rm d}z\,. \tag{1.5.30}$$

Combining (1.5.26), (1.5.29), and (1.5.30) yields

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) = \pi(d-\varepsilon) + \int_\varepsilon^1 \left(\frac{r^2}{2}\int_{\partial^+ B_1} \left|\partial_r v(r\mathbf{x})\right|^2 {\rm d}\mathscr{H}^2\right) {\rm d}r\,. \tag{1.5.31}$$

Then, recalling that

$$\mathfrak{S}^{-1}{}_\sharp \mathscr{H}^2 \llcorner \mathbb{S}^2 = \frac{4}{(1+|z|^2)^2}\,{\rm d}z\,, \tag{1.5.32}$$

we obtain
$$\frac{r^2}{2}\int_{\partial^+ B_1} \left|\partial_r v(r\mathbf{x})\right|^2 \mathrm{d}\mathscr{H}^2 = 2r^2 \int_{\mathbb{D}} \frac{\left|\partial_r \widehat{w}(z,r)\right|^2}{(1+|z|^2)^2}\,\mathrm{d}z\,.$$
In turn, this last identity together with (1.5.27) and Lemma 1.A.1 yields
$$\frac{r^2}{2}\int_{\partial^+ B_1} \left|\partial_r v(r\mathbf{x})\right|^2 \mathrm{d}\mathscr{H}^2 \leqslant 2\pi r^2 F(\boldsymbol{\beta}^2(r))\left|\boldsymbol{\beta}'(r)\right|^2 \tag{1.5.33}$$
with
$$F(t) := \left(\frac{t^2-10t+1}{(1+t)^4}\right)\log\left(\frac{(1-t)^2}{4}\right) - \frac{t^2+11t-2}{(1+t)^3}\,.$$
Notice that $F:[0,1)\to\mathbb{R}$ is an increasing function, and that $F(0)=2-2\log(2)>0$.

Gathering (1.5.28), (1.5.31), and (1.5.33) leads to
$$\pi\varepsilon \leqslant 2\pi\int_{\varepsilon}^{1} r^2 F(\boldsymbol{\beta}^2(r))\left|\boldsymbol{\beta}'(r)\right|^2 \mathrm{d}r\,. \tag{1.5.34}$$
Next we set $\boldsymbol{\beta}(r) =: \boldsymbol{\gamma}(\varepsilon/r)$, so that $\boldsymbol{\gamma}:[\varepsilon,1]\to[0,1]$ satisfies $\boldsymbol{\gamma}(1)=1$, $\boldsymbol{\gamma}(t)<1$ for $t<1$, and $\boldsymbol{\gamma}(t)=\boldsymbol{\delta}$ in a neighborhood of $t=\varepsilon$. Changing variables in (1.5.34), we infer that
$$1 \leqslant 2\int_{\varepsilon}^{1} F(\boldsymbol{\gamma}^2(t))\left|\boldsymbol{\gamma}'(t)\right|^2 \mathrm{d}t\,.$$
In view of our arbitrary choice of ε and $\boldsymbol{\gamma}$, we conclude that
$$1 \leqslant 2\int_{0}^{1} F(\boldsymbol{\gamma}^2(t))\left|\boldsymbol{\gamma}'(t)\right|^2 \mathrm{d}t \tag{1.5.35}$$
for every C^1-function $\boldsymbol{\gamma}:[0,1]\to[0,1]$ satisfying $\boldsymbol{\gamma}(0)=\boldsymbol{\delta}$ and $\boldsymbol{\gamma}(1)=1$. Setting
$$G(s) := \int_0^s \sqrt{F(t^2)}\,\mathrm{d}t\,,$$
inequality (1.5.35) must hold for $\boldsymbol{\gamma}(t) := G^{-1}\big(G(1)t + G(\boldsymbol{\delta})(1-t)\big)$, which returns the inequality $1\leqslant 2\big(G(1)-G(\boldsymbol{\delta})\big)^2$. Therefore,
$$1 \leqslant \sqrt{2}\int_{\boldsymbol{\delta}}^{1}\sqrt{F(t^2)}\,\mathrm{d}t =: J(\boldsymbol{\delta})\,.$$
Since $J(1/3)\approx 0.971<1$, we finally reach the conclusion that $\boldsymbol{\delta}\leqslant 1/3$. □

Proposition 1.5.9. *Let $g:\mathbb{S}^1\to\mathbb{S}^1$ be a 1/2-harmonic circle. If $d := |\deg(g)| \geqslant 2$, then the map $u_0 := g(\frac{x}{|x|})$ is not a 0-homogeneous minimizing 1/2-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$.*

Proof. We argue by contradiction assuming that u_0 is a 0-homogeneous minimizing 1/2-harmonic map in $\mathbb{R}^2$. Once again, it implies that u_0^{e} is a minimizing harmonic map with free boundary in B_1^+ by Theorem 1.3.6. By Lemma 1.5.6, u_0^{e} is of the form (1.5.18), and without loss of generality we can assume that the map w in (1.5.18) is equal to the right-hand side of (1.5.18) (otherwise we consider the complex conjugate of u_0 instead of u_0, which is also minimizing).

We shall build competitors to test the minimality of $u_0^{\rm e}$, and to this purpose we consider the extended complex plane $\mathbb{C}\cup\{\infty\}$. We also identify $\mathbb{R}^2_+$ with the complex upper half-plane $\mathbb{C}_+ := \{z\in\mathbb{C} : \mathfrak{Im}(z)>0\}$. We consider the Cayley transform $\mathfrak{C} : \overline{\mathbb{C}_+}\to\overline{\mathbb{D}}\setminus\{1\}$ given by

$$\mathfrak{C}(z) := \frac{z-i}{z+i}\,, \tag{1.5.36}$$

and its inverse

$$\mathfrak{C}^{-1}(z) = \frac{i(1+z)}{1-z}\,. \tag{1.5.37}$$

Note that $\mathfrak{C}$ maps the real line $\mathbb{R}\times\{0\} = \partial\mathbb{C}_+$ into $\mathbb{S}^1\setminus\{1\} = \partial\mathbb{D}\setminus\{1\}$. In the sequel, we use the (standard) convention

$$\mathfrak{C}^{-1}(1) = \infty \quad\text{and}\quad \mathfrak{C}(\infty) = 1\,.$$

We define a map $f : \overline{\mathbb{D}}\to\overline{\mathbb{C}_+}\cup\{\infty\}$ by setting

$$f(z) := (\mathfrak{C}^{-1}\circ w)(z)\,. \tag{1.5.38}$$

As a complex-valued function, f is a rational function of z with poles (exactly) at the finite set $Z_w^+ := w^{-1}(\{1\})\subseteq\mathbb{S}^1$. In particular, f is smooth in $\overline{\mathbb{D}}\setminus Z_w^+$. In addition, $f(\mathbb{D}) = \mathbb{C}_+$, and $f(\mathbb{S}^1\setminus Z_w^+) = \mathbb{R}\times\{0\}$.

Given a parameter $\varepsilon\in(0,1)$, we consider a smooth function $\boldsymbol{\theta} : [0,1]\to[0,1]$ such that $\boldsymbol{\theta}(r) = 1$ in a neighborhood of $r=1$, $\boldsymbol{\theta}(r)>0$ for $r>\varepsilon$, and $\boldsymbol{\theta}(r) = 0$ for $r\leqslant\varepsilon$. Next we define the smooth map on B_1^+ given by

$$v(\mathbf{x}) := \mathfrak{C}\left(\frac{1}{\boldsymbol{\theta}(|\mathbf{x}|)} f\circ\mathfrak{S}^{-1}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\right)\,,$$

where $\mathfrak{S}^{-1}$ is the stereographic projection (1.5.9). With the convention $0/0=\infty$, we observe that v extends smoothly up to ∂B_1^+ except for finitely many points in $\partial^0 B_1^+$. More precisely, setting $Z_w^- := w^{-1}(\{-1\})\subseteq\mathbb{S}^1$, the set Z_w^- is finite, and v is smooth in $\overline{B_1^+}\setminus(\varepsilon Z_w^-\times\{0\})$. By construction, $v=1$ in B_ε^+, $|v|=1$ on $\partial^0 B_1^+$, and $v=u^{\rm e}$ in a neighborhood of $\partial^+ B_1$. As our computations will show, $v\in H^1(B_1^+;\mathbb{R}^2)$ so that v is an admissible competitor to test the minimality of $u_0^{\rm e}$ in B_1^+, i.e.,

$$\mathbf{E}_{\frac{1}{2}}(v, B_1^+) \geqslant \mathbf{E}_{\frac{1}{2}}(u_0^{\rm e}, B_1^+) = \pi d\,, \tag{1.5.39}$$

where we have used (1.5.19) in the last equality.

To compute the energy of v, it is useful to rewrite v as

$$v(\mathbf{x}) = \widehat{w}\left(\mathfrak{S}^{-1}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), |\mathbf{x}|\right)\,,$$

where $\widehat{w}$ is the smooth map defined on $\overline{\mathbb{D}}\times(\varepsilon,1)$ by

$$\widehat{w}(z,r) := \mathfrak{C}\left(\frac{1}{\boldsymbol{\theta}(r)}\mathfrak{C}^{-1}(w(z))\right) = \mathfrak{C}\left(\frac{1}{\boldsymbol{\theta}(r)} f(z)\right)\,.$$

Notice that for each $r\in(\varepsilon,1)$, $\widehat{w}(\cdot,r)$ is a Blaschke product with d factors. Indeed, for each $r\in(\varepsilon,1)$, $\widehat{w}(\cdot,r)$ is clearly holomorphic on $\mathbb{D}$, it is smooth up to $\partial\mathbb{D}$, and $|w(\cdot,r)|=1$ on $\partial\mathbb{D}$. By a classical result of Fatou [41], it implies that $w(\cdot,r)$ is a finite Blaschke product. Since the restriction g_r of $w(\cdot,r)$ to $\partial\mathbb{D}$ is an $\mathbb{S}^1$-valued function of degree d, it must be a product of precisely d factors. Therefore, we can infer from (1.4.2) and Theorem 1.4.5 that

$$\frac{1}{2}\int_{\mathbb{D}}\left|\nabla_z\widehat{w}(z,r)\right|^2\mathrm{d}z=\mathcal{E}_{\frac{1}{2}}(g_r,\mathbb{S}^1)=\pi d\qquad\forall r\in(\varepsilon,1)\,.\tag{1.5.40}$$

On the other hand, a straightforward computation yields for $r\in(\varepsilon,1)$,

$$\begin{aligned}\left|\frac{\partial\widehat{w}}{\partial r}(z,r)\right|^2&=\left|\mathfrak{C}'\left(\frac{f(z)}{\boldsymbol{\theta}(r)}\right)\right|^2\frac{|f(z)|^2}{\boldsymbol{\theta}^2(r)}|\boldsymbol{\theta}'(r)|^2\\&=\frac{4|f(z)|^2|\boldsymbol{\theta}'(r)|^2}{\left(\boldsymbol{\theta}^2(r)+2\boldsymbol{\theta}(r)f_2(z)+|f(z)|^2\right)^2}\,,\end{aligned}\tag{1.5.41}$$

where f_2 denotes the imaginary part of f.

Computing the energy of v in polar coordinates, we obtain

$$\begin{aligned}\mathbf{E}_{\frac{1}{2}}(v,B_1^+)=\int_\varepsilon^1&\left(\frac{1}{2}\int_{\partial^+B_1}\left|\nabla_\tau v(r\mathbf{x})\right|^2\mathrm{d}\mathscr{H}^2\right)\mathrm{d}r\\&+\int_\varepsilon^1\left(\frac{r^2}{2}\int_{\partial^+B_1}\left|\partial_r v(r\mathbf{x})\right|^2\mathrm{d}\mathscr{H}^2\right)\mathrm{d}r\,.\end{aligned}\tag{1.5.42}$$

Using the conformal invariance of $\mathfrak{S}^{-1}$ and (1.5.40), we derive

$$\frac{1}{2}\int_{\partial^+B_1}\left|\nabla_\tau v(r\mathbf{x})\right|^2\mathrm{d}\mathscr{H}^2=\frac{1}{2}\int_{\mathbb{D}}\left|\nabla_z\widehat{w}(z,r)\right|^2dz=\pi d\quad\forall r\in(\varepsilon,1)\,.\tag{1.5.43}$$

Next, (1.5.41) together with (1.5.32) leads to

$$\begin{aligned}\frac{r^2}{2}\int_{\partial^+B_1}\left|\partial_r v(r\mathbf{x})\right|^2\mathrm{d}\mathscr{H}^2&=2\int_{\mathbb{D}}\left|\frac{\partial\widehat{w}}{\partial r}(z,r)\right|^2\frac{r^2}{(1+|z|^2)^2}\,\mathrm{d}z\\&=8\int_{\mathbb{D}}\frac{|f(z)|^2|\boldsymbol{\theta}'(r)|^2r^2}{\left(\boldsymbol{\theta}^2(r)+2\boldsymbol{\theta}(r)f_2(z)+|f(z)|^2\right)^2(1+|z|^2)^2}\,\mathrm{d}z\,.\end{aligned}\tag{1.5.44}$$

for every $r\in(\varepsilon,1)$.

Combining (1.5.39), (1.5.42), (1.5.43), and (1.5.44), we deduce that

$$\frac{\pi d\varepsilon}{8}\leqslant\int_\varepsilon^1\left(\int_{\mathbb{D}}\frac{|f(z)|^2|\boldsymbol{\theta}'(r)|^2r^2}{\left(\boldsymbol{\theta}^2(r)+2\boldsymbol{\theta}(r)f_2(z)+|f(z)|^2\right)^2(1+|z|^2)^2}\,\mathrm{d}z\right)\mathrm{d}r\,.\tag{1.5.45}$$

Next we set $\boldsymbol{\theta}(r)=:\boldsymbol{\alpha}(\varepsilon/r)$, so that $\boldsymbol{\alpha}:[\varepsilon,1]\to[0,1]$ satisfies $\boldsymbol{\alpha}(1)=0$, $\boldsymbol{\alpha}(t)>0$ for $t<1$, and $\boldsymbol{\alpha}(t)=1$ in a neighborhood of $t=\varepsilon$. Changing variables in (1.5.45) gives

$$\frac{\pi d}{8}\leqslant\int_\varepsilon^1 H_f(\boldsymbol{\alpha}(t))|\boldsymbol{\alpha}'(t)|^2\,\mathrm{d}t\tag{1.5.46}$$

with

$$H_f(a):=\int_{\mathbb{D}}\frac{|f(z)|^2}{\left(a^2+2af_2(z)+|f(z)|^2\right)^2(1+|z|^2)^2}\,\mathrm{d}z\,,\quad a\in(0,1]\,.$$

In view of (1.5.38), we can rewrite $H_f(a)$ as

$$H_f(a)=\int_{\mathbb{D}}\frac{K_a(w(z))}{(1+|z|^2)^2}\,\mathrm{d}z\,,$$

where $K_a : \mathbb{D} \to [0, \infty)$ is given by

$$K_a(z) := \frac{|\mathfrak{C}^{-1}(z)|^2}{(a^2 + 2a\mathfrak{C}_2^{-1}(z) + |\mathfrak{C}^{-1}(z)|^2)^2} ,$$

and $\mathfrak{C}_2^{-1}$ denotes the imaginary part of $\mathfrak{C}^{-1}$.

Since minimality is preserved under rotations on the image, σu_0 is a minimizing 0-homogeneous 1/2-harmonic map for each $\sigma \in \mathbb{S}^1$. As a consequence, (1.5.46) must hold with f replaced by $f_\sigma := \mathfrak{C}^{-1}(\sigma w)$ for every $\sigma \in \mathbb{S}^1$. Averaging the resulting inequalities over all $\sigma \in \mathbb{S}^1$ yields

$$\frac{\pi d}{8} \leqslant \frac{1}{2\pi} \int_{\mathbb{S}^1} \left(\int_\varepsilon^1 H_{f_\sigma}(\boldsymbol{\alpha}(t)) |\boldsymbol{\alpha}'(t)|^2 \, \mathrm{d}t \right) \mathrm{d}\sigma = \int_\varepsilon^1 \widetilde{H}_w(\boldsymbol{\alpha}(t)) |\boldsymbol{\alpha}'(t)|^2 \, \mathrm{d}t \tag{1.5.47}$$

with

$$\widetilde{H}_w(a) = \int_{\mathbb{D}} \frac{\widetilde{K}_a(w(z))}{(1+|z|^2)^2} \, \mathrm{d}z \quad \text{and} \quad \widetilde{K}_a(z) := \frac{1}{2\pi} \int_{\mathbb{S}^1} K_a(\sigma z) \, \mathrm{d}\sigma .$$

Then observe that $\widetilde{K}_a(z)$ only depends on $|z|$, i.e., $\widetilde{K}_a(z) = \widetilde{K}_a(|z|)$. Hence Lemma 1.A.2 tells us that

$$\widetilde{K}_a(w(z)) = \frac{1}{2\pi} \int_{\mathbb{S}^1} K_a(|w(z)|\sigma) \, \mathrm{d}\sigma = J(a, |w(z)|) ,$$

where the function $\lambda \mapsto J(a, \lambda)$, given by formula (1.A.5), is an increasing function. Using that $d \geqslant 2$, we infer from Lemma 1.5.8 that

$$|w(z)| \leqslant \left(\frac{3|z|+1}{|z|+3} \right)^2 \quad \forall z \in \mathbb{D} ,$$

and as a consequence,

$$\widetilde{H}_w(a) \leqslant 2\pi \int_0^1 J\left(a, \frac{(3r+1)^2}{(r+3)^2}\right) \frac{r}{(1+r^2)^2} \, \mathrm{d}r =: 2\pi F_1(a) \quad \forall a \in (0,1] .$$

Inserting this last inequality in (1.5.47) leads to

$$\frac{d}{16} \leqslant \int_\varepsilon^1 F_1(\boldsymbol{\alpha}(t)) |\boldsymbol{\alpha}'(t)|^2 \, \mathrm{d}t .$$

In view of the arbitrariness of ε and $\boldsymbol{\alpha}$, we conclude that

$$\frac{d}{16} \leqslant \int_0^1 F_1(\boldsymbol{\alpha}(t)) |\boldsymbol{\alpha}'(t)|^2 \, \mathrm{d}t \tag{1.5.48}$$

for every C^1-function $\boldsymbol{\alpha} : [0,1] \to [0,1]$ satisfying $\boldsymbol{\alpha}(0) = 1$ and $\boldsymbol{\alpha}(1) = 0$.

Setting

$$G(\alpha) := \int_0^\alpha \sqrt{F_1(a)} \, \mathrm{d}a$$

inequality (1.5.48) must hold for $\boldsymbol{\alpha}(t) = G^{-1}(G(1)(1-t))$, which returns the inequality $d/16 \leqslant (G(1))^2$. In other words,

$$\sqrt{d} \leqslant 4 \int_0^1 \sqrt{F_1(a)} \, \mathrm{d}a . \tag{1.5.49}$$

Now we change variable in this integral setting $t = \frac{1-a}{1+a}$. Using formula (1.A.5), we obtain

$$4\int_0^1 \sqrt{F_1(a)}\,\mathrm{d}a = 2\int_0^1 \sqrt{F_2(t)}\,\mathrm{d}t \leqslant 2\left(\int_0^1 F_2(t)\,\mathrm{d}t\right)^{1/2} \tag{1.5.50}$$

with

$$F_2(t) := \int_0^1 \left(\frac{(2t^2+1)t^2(3r+1)^{12} - (6t^2-1)(3r+1)^8(r+3)^4}{((r+3)^4 - (3r+1)^4t^2)^3} + \frac{t^2(3r+1)^4(r+3)^8 + (r+3)^{12}}{((r+3)^4 - (3r+1)^4t^2)^3}\right)\frac{r\,\mathrm{d}r}{(1+r^2)^2}\,.$$

From (1.5.49) and (1.5.50), we conclude that $d \leqslant 4\int_0^1 F_2(t)\,\mathrm{d}t$. However, a direct (numerical) computation provides the estimate $4\int_0^1 F_2(t)\,\mathrm{d}t \simeq 1.93 < 2$, which contradicts $d \geqslant 2$, and the proof is complete. □

1.5.4 Proof of Theorem 1.1.5

We complete this section with the proof of Theorem 1.1.5, and to this purpose we consider $u \in \widehat{H}^{1/2}(\Omega;\mathbb{S}^1)$ a minimizing 1/2-harmonic map in a smooth bounded open set $\Omega \subseteq \mathbb{R}^2$. By Corollary 1.3.7, u is smooth in Ω away from a locally finite subset of Ω. Assume that $a \in \Omega$ is a singular point of u, and assume without loss of generality that $a = 0$. Fix $R > 0$ such that $D_{2R} \subseteq \Omega$ and $u \in C^\infty(D_{2R} \setminus \{0\})$. Then,

$$d := \deg(u,0) = \deg(u_{|\partial D_\rho}) \qquad \forall \rho \in (0,2R)\,. \tag{1.5.51}$$

By Theorem 1.3.6, u^{e} is a minimizing harmonic map with free boundary in B_R^+. Therefore, u^{e} is stationary in B_R^+ in the sense of [80, Definition 4.10], see [80, Remark 4.13]. In turn, by [80, Remark 4.11] it implies that

$$\int_{B_R^+}\left(|\nabla u^{\mathrm{e}}|^2 \operatorname{div} X - 2\sum_{i,j=1}^3 (\partial_i u^{\mathrm{e}}\cdot\partial_j u^{\mathrm{e}})\partial_j X_i\right)\mathrm{d}\mathbf{x} = 0 \tag{1.5.52}$$

for every $X := (X_1,X_2,X_3) \in C^1(\overline{B}_R^+;\mathbb{R}^3)$ compactly supported in $B_R^+ \cup \partial^0 B_R^+$ and such that $X_3 = 0$ on $\partial^0 B_R^+$. Arguing as in [80, Proof of Lemma 5.2, Step 2], we infer from (1.5.52) that

$$\frac{1}{r}\mathbf{E}_{\frac12}(u^{\mathrm{e}},B_r^+) - \frac{1}{s}\mathbf{E}_{\frac12}(u^{\mathrm{e}},B_s^+) = \int_s^r \frac{1}{t}\left(\int_{\partial^+ B_t}\left|\frac{\partial u^{\mathrm{e}}}{\partial\nu}\right|^2 \mathrm{d}\mathscr{H}^2\right)\mathrm{d}t \quad \forall\, 0<s<r<R\,. \tag{1.5.53}$$

As a consequence, $r \mapsto \frac{1}{r}\mathbf{E}_{\frac12}(u^{\mathrm{e}},B_r^+)$ is nondecreasing, and the limit

$$\boldsymbol{\Theta} := \lim_{r\downarrow 0}\frac{1}{r}\mathbf{E}_{\frac12}(u^{\mathrm{e}},B_r^+)$$

exists. Since 0 is a singular point of u (and thus of u^{e}), it follows that $\boldsymbol{\Theta} > 0$ by e.g. [58, Theorem 3.4] (recall our discussion before Theorem 1.3.2).

We now consider a sequence $\rho_k \downarrow 0$ with $\rho_k \leqslant R$, and we set for $x \in D_{2R/\rho_k}$,

$$u_k(x) := u(\rho_k x)\,.$$

Then, $u_k \in \widehat{H}^{1/2}(D_{2R/\rho_k};\mathbb{S}^1)$, $u_k^{\rm e}(\mathbf{x}) = u^{\rm e}(\rho_k\mathbf{x})$, and $u_k^{\rm e} \in H^1(B^+_{2R/\rho_k})$ is a minimizing harmonic map with free boundary in B^+_{2R/ρ_k}. Since

$$\frac{1}{r^{n-1}}\mathbf{E}_{\frac{1}{2}}(u_k^{\rm e}, B_r^+) = \frac{1}{(\rho_k r)^{n-1}}\mathbf{E}_{\frac{1}{2}}(u^{\rm e}, B^+_{\rho_k r}) \qquad \forall\, 0 < r < \frac{R}{\rho_k}\,, \tag{1.5.54}$$

we infer from (1.5.53) that $\mathbf{E}_{\frac{1}{2}}(u_k^{\rm e}, B_r^+)$ is bounded with respect to k for every $r < R/\rho_k$. Recalling that $|u_k^{\rm e}| \leqslant 1$ (since u_k is $\mathbb{S}^1$-valued), we can apply Theorem 1.3.5 to find a (not relabeled) subsequence such that $u_k^{\rm e} \to v$ strongly in $H^1(B_r^+)$ for every $r > 0$, where v is minimizing harmonic map with free boundary in B_r^+ for every $r > 0$. Setting $u_0 := v_{|\partial\mathbb{R}^3_+}$, we have $u_k \to u_0$ strongly in $H^{1/2}(D_r)$ for every $r > 0$. Hence $u_k^{\rm e} \to u_0^{\rm e}$ in $L^2(B_r^+)$ for every $r > 0$ by [80, Lemma 2.4], which shows that $v = u_0^{\rm e}$. In view of (1.5.54) and the strong convergence of $u_k^{\rm e}$, we have

$$\frac{1}{r^{n-1}}\mathbf{E}_{\frac{1}{2}}(u_0^{\rm e}, B_r^+) = \lim_{k\to\infty}\frac{1}{r^{n-1}}\mathbf{E}_{\frac{1}{2}}(u_k^{\rm e}, B_r^+) = \boldsymbol{\Theta} \quad \forall r > 0\,. \tag{1.5.55}$$

In turn, rescaling (1.5.53) yields

$$\begin{aligned}\int_s^r \frac{1}{t}\left(\int_{\partial^+ B_t}\left|\frac{\partial u_0^{\rm e}}{\partial\nu}\right|^2 {\rm d}\mathscr{H}^2\right){\rm d}t &= \lim_{k\to\infty}\int_s^r \frac{1}{t}\left(\int_{\partial^+ B_t}\left|\frac{\partial u_k^{\rm e}}{\partial\nu}\right|^2 {\rm d}\mathscr{H}^2\right){\rm d}t \\ &= \lim_{k\to\infty}\left(\frac{1}{r^{n-1}}\mathbf{E}_{\frac{1}{2}}(u_k^{\rm e}, B_r^+) - \frac{1}{r^{n-1}}\mathbf{E}_{\frac{1}{2}}(u_k^{\rm e}, B_s^+)\right) \\ &= 0\end{aligned}$$

for every $r > s > 0$. Therefore, $u_0^{\rm e}$ is 0-homogeneous, and thus $u_0^{\rm e}$ is a 0-homogeneous minimizing harmonic map with free boundary. Since $\boldsymbol{\Theta} > 0$, we deduce from (1.5.55) that $u_0^{\rm e}$ is not constant. Then u_0 is a nontrivial 0-homogeneous minimizing 1/2-harmonic map on $\mathbb{R}^2$ by Theorem 1.3.6. Then Theorem 1.1.4 tells us that $u_0(x) = \frac{Ax}{|x|}$ for some orthogonal matrix $A \in O(2,\mathbb{R})$. In particular,

$$\deg(u_{0|\partial D_r}) \in \{\pm 1\} \qquad \forall r > 0\,. \tag{1.5.56}$$

Now, by the strong H^1-convergence of $(u_k^{\rm e})$ and Fubini's theorem, (up to a further subsequence if necessary) we can find $r_* \in (0,1)$ such that $u_k^{\rm e} \to u_0^{\rm e}$ strongly in $H^1(\partial^+ B_{r_*})$. By continuity of the trace operator, we have $u_k \to u_0$ strongly in $H^{1/2}(\partial D_{r_*})$. The degree being continuous with respect to the strong $H^{1/2}$-convergence (see [14]), we deduce from (1.5.56) that $\deg(u_{k|\partial D_{r_*}}) \in \{\pm 1\}$ for k large enough, that is $\deg(u_{|\partial D_{\rho_k r_*}}) \in \{\pm 1\}$. In view of (1.5.51), we have thus proved that $d \in \{\pm 1\}$, which completes the proof.

Appendix

1.A Detailed computations

We provide in this appendix some details about the computations performed in Section 1.5.3.

Lemma 1.A.1. *For every* $\gamma \in [0,1)$,

$$I(\gamma) := \int_{\mathbb{D}} \frac{(1+|z|^2)^2 - 4z_1^2}{(1-2\gamma z_1 + \gamma^2|z|^2)(1+|z|^2)^2}\,{\rm d}z = \pi F(\gamma^2)$$

with

$$F(t) := \left(\frac{t^2-10t+1}{(1+t)^4}\right)\log\left(\frac{(1-t)^2}{4}\right) - \frac{t^2+11t-2}{(1+t)^3}\,.$$

Proof. Write $I(\boldsymbol{\gamma}) = A(\boldsymbol{\gamma}) - 4B(\boldsymbol{\gamma})$ with

$$A(\boldsymbol{\gamma}) := \int_{\mathbb{D}} \frac{1}{(1 - 2\boldsymbol{\gamma} z_1 + \boldsymbol{\gamma}^2 |z|^2)} \, \mathrm{d}z$$

and

$$B(\boldsymbol{\gamma}) := \int_{\mathbb{D}} \frac{z_1^2}{(1 - 2\boldsymbol{\gamma} z_1 + \boldsymbol{\gamma}^2 |z|^2)(1 + |z|^2)^2} \, \mathrm{d}z \, .$$

Using polar coordinates, we further rewrite

$$A(\boldsymbol{\gamma}) = \int_0^1 M(\boldsymbol{\gamma} r) r \, \mathrm{d}r \quad \text{and} \quad B(\boldsymbol{\gamma}) = \int_0^1 \frac{N(\boldsymbol{\gamma} r) r^3}{(1 + r^2)^2} \, \mathrm{d}r \, ,$$

where

$$M(a) := \int_0^{2\pi} \frac{\mathrm{d}\theta}{(1 - 2a\cos(\theta) + a^2)^2} \quad \text{and} \quad N(a) := \int_0^{2\pi} \frac{\cos^2(\theta)}{(1 - 2a\cos(\theta) + a^2)^2} \, \mathrm{d}\theta$$

are defined for $a \in [0, 1)$.

Lengthy but elementary computations yield

$$M(a) = 2\pi \frac{1 + a^2}{(1 - a^2)^3} \quad \text{and} \quad N(a) = 2\pi \left(\frac{1 + a^2}{(1 - a^2)^3} - \frac{1}{2(1 - a^2)} \right) .$$

Then we first obtain

$$A(\boldsymbol{\gamma}) = 2\pi \int_0^1 \frac{\boldsymbol{\gamma}^2 r^3 + r}{(1 - \boldsymbol{\gamma}^2 r^2)^3} \, \mathrm{d}r = \pi \left[\frac{r^2}{(1 - \boldsymbol{\gamma}^2 r^2)^2} \right]_0^1 = \frac{\pi}{(1 - \boldsymbol{\gamma}^2)^2} \, . \tag{1.A.1}$$

Concerning $B(\boldsymbol{\gamma})$, we can rewrite it as

$$B(\boldsymbol{\gamma}) = \pi \big(2U(\boldsymbol{\gamma}^2) - V(\boldsymbol{\gamma}^2) \big) \tag{1.A.2}$$

with

$$U(t) := \int_0^1 \frac{(1 + tr^2) r^3}{(1 - tr^2)^3 (1 + r^2)^2} \, \mathrm{d}r \quad \text{and} \quad V(t) := \int_0^1 \frac{r^3}{(1 - tr^2)(1 + r^2)^2} \, \mathrm{d}r \, .$$

Once again, elementary computations lead to

$$V(t) = \frac{1}{2(1 + t)^2} \log\left(\frac{2}{1 - t} \right) - \frac{1}{4(1 + t)}$$

and

$$U(t) = \left(\frac{t^2 - 4t + 1}{2(1 + t)^4} \right) \log\left(\frac{2}{1 - t} \right) + \frac{1}{8(1 - t)^2} + \frac{1}{2} P(t) \, ,$$

with

$$\begin{aligned} P(t) := \frac{1}{4(1 - t)} + \frac{1}{4(1 + t)} - \frac{3}{4(1 + t)^2} + \frac{t^2 + 2t}{(1 + t)^2 (1 - t)} \\ - \frac{t}{(1 + t)(1 - t)} - \frac{4t^2}{(1 + t)^3 (1 - t)} - \frac{1 - t}{2(1 + t)^3} \, . \end{aligned}$$

Therefore,

$$2U(t) - V(t) = \left(\frac{t^2 - 10t + 1}{2(1+t)^4}\right)\log\left(\frac{2}{1-t}\right) + \frac{1}{4(1-t)^2} + P(t) + \frac{1}{4(1+t)}\,. \tag{1.A.3}$$

A direct computation shows that

$$P(t) + \frac{1}{4(1+t)} = \frac{t^2 + 11t - 2}{4(1+t)^3}\,. \tag{1.A.4}$$

Gathering (1.A.1)-(1.A.2)-(1.A.3)-(1.A.4) now leads to $I(\gamma) = \pi F(\gamma^2)$ as announced. □

Lemma 1.A.2. *Let $\mathfrak{C}$ be the Cayley transform (defined in (1.5.36)). For $a \in (0,1]$ and $z \in \overline{\mathbb{D}}$, let*

$$K_a(z) := \frac{|\mathfrak{C}^{-1}(z)|^2}{(a^2 + 2a\mathfrak{C}_2^{-1}(z) + |\mathfrak{C}^{-1}(z)|^2)^2}\,,$$

where $\mathfrak{C}_2^{-1}$ denotes the imaginary part of $\mathfrak{C}^{-1}$. Define for $\lambda \in (0,1)$,

$$J(a,\lambda) := \frac{1}{2\pi}\int_{\mathbb{S}^1} K_a(\lambda\sigma)\,\mathrm{d}\sigma\,.$$

Then,

$$J(a,\lambda) = \frac{(1+t)^4}{16}\left(\frac{(2t^2+1)t^2\lambda^6 - (6t^2-1)\lambda^4 + t^2\lambda^2 + 1}{(1-\lambda^2 t^2)^3}\right) \text{ with } t := \frac{1-a}{1+a} \tag{1.A.5}$$

for every $\lambda \in [0,1]$. In addition, $\lambda \in [0,1] \mapsto J(a,\lambda)$ is increasing for every $a \in (0,1]$.

Proof. Recalling that

$$\mathfrak{C}^{-1}\sharp\mathscr{H}^1 \llcorner \mathbb{S}^1 = \frac{2}{1+x^2}\,\mathrm{d}x\,,$$

we change variables to obtain

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} K_a(\lambda z)\,\mathrm{d}\mathscr{H}^1 = \frac{1}{\pi}\int_{\mathbb{R}} \frac{K_a(\lambda\mathfrak{C}(x))}{1+x^2}\,\mathrm{d}x\,.$$

Next we set

$$c := \frac{1-\lambda}{1+\lambda} \in (0,1)\,, \quad A := \frac{a+c}{1+ac}\,, \quad B := c^2 + \frac{1}{c^2}\,,$$

to compute

$$K_a(\lambda\mathfrak{C}(x)) = \left(\frac{c^2}{(1+ac)^4}\right)\frac{x^4 + Bx^2 + 1}{(x^2+A^2)^2}\,.$$

By Lemma 1.A.4 below, we have

$$J(a,\lambda) = \frac{c^2}{(1+ac)^4}\left(\frac{1+A^2}{2A^3} + \frac{B-2}{2A(A+1)^2}\right)\,.$$

In terms of the variables t and $\mu := \lambda^2 \in (0,1)$, we obtain

$$J(a,\lambda) = \frac{(1+t)^4}{16}\left(\frac{(2t^2+1)t^2\mu^3 - (6t^2-1)\mu^2 + t^2\mu + 1}{(1-\mu t^2)^3}\right)\,,$$

which is the announced formula. Next, if

$$f : \mu \in (0,1) \mapsto \frac{(2t^2+1)t^2\mu^3 - (6t^2-1)\mu^2 + t^2\mu + 1}{(1-\mu t^2)^3},$$

we have

$$f'(\mu) = \frac{4t^2(1-\mu)^2 + 2\mu(1-t^2)^2}{(1-\mu t^2)^4} > 0\,,$$

which shows that $\lambda \mapsto J(a,\lambda)$ is indeed increasing for every $a \in (0,1)$. □

Remark 1.A.3. Note that the function $J(a,\lambda)$ defined in (1.A.5) can be rewritten as

$$J(a,\lambda) = \frac{(1+t)^4}{32}\left(\frac{(1-\lambda^2)^2}{(1+\lambda t)(1-\lambda t)^3} + \frac{(1-\lambda^2)^2}{(1+\lambda t)^3(1-\lambda t)} + \frac{4\lambda^2}{1-\lambda^2 t^2}\right).$$

From this formula, one easily determines the behavior of J as $a \sim 0$ and $\lambda \sim 1$.

Lemma 1.A.4. *For $A, B > 0$, we have*

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{x^4 + Bx^2 + 1}{(1+x^2)(x^2+A^2)^2}\,dx = \frac{1+A^2}{2A^3} + \frac{B-2}{2A(A+1)^2}\,. \tag{1.A.6}$$

Proof. Write $X := x^2$, and observe that

$$\begin{aligned}\frac{X^2+BX+1}{(X+1)(X+A^2)^2} =& \frac{2-B}{(1-A^2)^2}\frac{1}{X+1} \\ &+ \left(1 + \frac{B-2}{(1-A^2)^2}\right)\frac{1}{X+A^2} \\ &+ \left((1-A^2) + (2-B)\frac{A^2}{1-A^2}\right)\frac{1}{(X+A^2)^2}\,.\end{aligned}$$

On the other hand,

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{dx}{1+x^2} = 1\,,\quad \frac{1}{\pi}\int_{\mathbb{R}}\frac{dx}{x^2+A^2} = \frac{1}{A}\,,\quad \frac{1}{\pi}\int_{\mathbb{R}}\frac{dx}{(x^2+A^2)^2} = \frac{1}{2A^3}\,,$$

and (1.A.6) follows. □

Chapter 2

Partial regularity for fractional harmonic maps into spheres

2.1 Introduction

The theory of fractional harmonic maps into a manifold is quite recent. It is has been initiated some years ago by F. Da Lio and T. Rivière in [29, 28]. In those first articles, they have introduced and studied 1/2-harmonic maps from the real line into a smooth and compact closed submanifold $\mathcal{N} \subseteq \mathbb{R}^d$. A map $u : \mathbb{R} \to \mathcal{N}$ is said to be a 1/2-harmonic map into $\mathcal{N}$ if it is a critical point of the 1/2-Dirichlet energy

$$\mathcal{E}_{\frac{1}{2}}(u, \mathbb{R}) := \frac{1}{2} \int_{\mathbb{R}} \left|(-\Delta)^{\frac{1}{4}} u\right|^2 \mathrm{d}x = \frac{1}{4\pi} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, \mathrm{d}x\mathrm{d}y\,,$$

among all maps with values into $\mathcal{N}$, or equivalently, if it satisfies the Euler-Lagrange equation

$$(-\Delta)^{\frac{1}{2}} u \perp \mathrm{Tan}(u, \mathcal{N}) \tag{2.1.1}$$

in the distributional sense. Here $(-\Delta)^s$ denotes the integro-differential (multiplier) operator associated to the Fourier symbol $(2\pi|\xi|)^{2s}$, $s \in (0, 1)$. The notion of 1/2-harmonic map into $\mathcal{N}$ appears in several geometrical problems, such as free boundary minimal surfaces or Steklov eigenvalue problems, see [24] and references therein. The special case $\mathcal{N} = \mathbb{S}^{d-1}$ is important for both geometrical and analytical issues. From the analytical point of view, it enlightens the internal structure of equation (2.1.1). Indeed, the Lagrange multiplier associated to the constraint to be $\mathbb{S}^{d-1}$-valued takes a very simple form, and (2.1.1) reduces to the equation

$$(-\Delta)^{\frac{1}{2}} u(x) = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, \mathrm{d}y\right) u(x)\,, \tag{2.1.2}$$

which is in clear analogy with the equation for usual harmonic maps from a $2d$-domain into the sphere. In particular, there is a similar analytical issue concerning regularity of solutions since the right-hand side of (2.1.2) has *a priori* no better integrability than L^1, and elliptic linear theory does not apply. In their pioneering work [29], F. Da Lio and T. Rivière proved complete smoothness of 1/2-harmonic maps through a reformulation of equation (2.1.2) in terms of algebraic quantities, the "3-terms commutators", exhibiting some compensation phenomena. In [28] (dealing with arbitrary targets), smoothness of 1/2-harmonic maps follows from a more general compensation result for nonlocal systems with antisymmetric potential, in the spirit of [90]. In the same stream of ideas, K. Mazowiecka and A. Schikorra obtained in [77] a new proof of the regularity of 1/2-harmonic maps, very close to the original argument of F. Hélein [60] to prove smoothness of harmonic maps from surfaces into spheres (see also [61]). Once again, the key point in [77] is to rewrite the right-hand side of (2.1.2) to discover a suitable "fractional div-curl structure". From the new form of the equation, they deduce that $(-\Delta)^{\frac{1}{2}}u$ belongs (essentially) to the Hardy space $\mathscr{H}^1$ by applying their main result [77, Theorem 2.1], a generalization to the fractional setting of the div-curl estimate of R. Coifman, P.L. Lions, Y. Meyer, and S. Semmes [21]. Continuity of solutions is then a consequence of Calderón-Zygmund theory, from which it is possible to deduce C^∞-regularity.

Several generalizations of the regularity result of [29, 28] have been obtained, e.g. for critical points of higher order or/and p-power type energies (still in the corresponding critical dimension), see [25, 30, 31, 77, 101, 100, 98]. The regularity theory for 1/2-harmonic maps into a manifold in higher dimensions has been addressed in [86] and [80] (see also Chapter 1 and [78]). In higher dimensions, the theory provides partial regularity (i.e. regularity away from a "small" singular set) for stationary 1/2-harmonic maps (i.e. critical points for both *inner and outer variations*), and energy minimizing 1/2-harmonic maps. It can be seen as the analogue of the partial regularity theory for harmonic maps by R. Schoen and K. Uhlenbeck [103, 104] in the minimizing case, and by L.C. Evans [39] and F. Bethuel [9] in the stationary case. In [80], the argument consists in considering the harmonic extension to the upper half-space in one more dimension provided by the convolution with the Poisson kernel. The extended map is then harmonic and satisfies a nonlinear Neumann boundary condition which fits within the (previously known) theory of harmonic maps with partially free boundary, see [37, 38, 64, 58, 97].

The purpose of this article is to extend the regularity theory for fractional harmonic maps in arbitrary dimensions to the context of *s-harmonic maps*, i.e., when the operator $(-\Delta)^{\frac{1}{2}}$ is replaced by $(-\Delta)^s$ with arbitrary power $s \in (0,1)$. As a first attempt in this direction, we only consider the case where the target manifold $\mathcal{N}$ is the standard unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$, $d \geqslant 2$. We now describe the functional setting.

Given $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ a bounded open set, the fractional s-Dirichlet energy in Ω of a measurable map $u : \mathbb{R}^n \to \mathbb{R}^d$ is defined by

$$\mathcal{E}_s(u,\Omega) := \frac{\gamma_{n,s}}{4} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,, \tag{2.1.3}$$

where Ω^c denotes the complement of Ω, i.e. $\Omega^c := \mathbb{R}^n \setminus \Omega$. The normalization constant $\gamma_{n,s} > 0$, whose precise value is given by (2.2.1), is chosen in such a way that

$$\mathcal{E}_s(u,\Omega) = \frac{1}{2}\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x \qquad \forall u \in \mathscr{D}(\Omega;\mathbb{R}^d)\,.$$

Following [80, 81], we denote by $\widehat{H}^s(\Omega;\mathbb{R}^d)$ the Hilbert space made of $L^2_{\mathrm{loc}}(\mathbb{R}^n)$-maps u such that $\mathcal{E}_s(u,\Omega) < \infty$, and we set

$$\widehat{H}^s(\Omega;\mathbb{S}^{d-1}) := \Big\{u \in \widehat{H}^s(\Omega;\mathbb{R}^d) : u(x) \in \mathbb{S}^{d-1} \text{ for a.e. } x \in \mathbb{R}^n\Big\}\,.$$

We then define weakly s-harmonic maps in Ω as critical points of $\mathcal{E}_s(u,\Omega)$ in the (nonlinear) space $\widehat{H}^s(\Omega;\mathbb{S}^{d-1})$. More precisely, we say that a map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is a *weakly s-harmonic map* in Ω into $\mathbb{S}^{d-1}$ if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s\Big(\frac{u+t\varphi}{|u+t\varphi|},\Omega\Big)\right]_{|t=0} = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega,\mathbb{R}^d)\,.$$

Exactly as (2.1.2), the Euler-Lagrange equation reads

$$(-\Delta)^s u(x) = \left(\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\right)u(x) \quad \text{in } \mathscr{D}'(\Omega)\,, \tag{2.1.4}$$

where $(-\Delta)^s$ is the integro-differential operator given by

$$(-\Delta)^s u(x) := \text{p.v.}\left(\gamma_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right)\,,$$

and the notation p.v. means that the integral is taken in the Cauchy principal value sense. We refer to Sections 2.2 and 2.3 for the precise weak (variational) formulation of equation (2.1.4).

Once again, the right-hand side in (2.1.4) has a priori no better integrability than L^1, and linear elliptic theory does not apply to determine the regularity of solutions. However, in the case $n \leqslant 2s$, that is $n = 1$ and $s \in [1/2, 1)$, the equation is *subcritical*. For $n = 1$ and $s = 1/2$, this is the result of [29, 28]. For $n = 1$ and $s \in (1/2, 1)$, solutions are at least Hölder continuous by the embedding $H^s \hookrightarrow C^{0,s-1/2}$, and this is to enough to reach C^∞-smoothness by applying Schauder type estimates for the fractional Laplacian.

Theorem 2.1.1. *Assume that $n = 1$ and $s \in [1/2, 1)$. If $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is a weakly s-harmonic map in Ω, then $u \in C^\infty(\Omega)$.*

On the other hand, the case $n > 2s$ is *supercritical*, and by analogy with (usual) weakly harmonic maps in dimension $\geqslant 3$, we do not expect any regularity without further assumptions. Indeed, in his groundbreaking article [92], T. Rivière has constructed a weakly harmonic map from the 3-dimensional ball into $\mathbb{S}^2$ which is everywhere discontinuous. A natural extra assumption one can add to weakly s-harmonic maps is *stationarity*, that is

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u\circ\phi_t,\Omega)\right]_{|t=0} = 0 \qquad \forall X \in C^1_c(\Omega;\mathbb{R}^n)\,,$$

where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the integral flow of the vector field X. According to the standard terminology in calculus of variations, a weakly s-harmonic map in Ω is a critical point of $\mathcal{E}_s(\cdot,\Omega)$ with respect to outer variations (i.e. in the target), a stationary map is a critical point of $\mathcal{E}_s(\cdot,\Omega)$ with respect to inner variations (i.e. in the domain), and thus a *stationary weakly s-harmonic map* in Ω is a critical point of $\mathcal{E}_s(\cdot,\Omega)$ with respect to both inner and outer variations.

Our second main result provides partial regularity for such maps. In its statement, the *singular set* of u in Ω is defined as

$$\text{sing}(u) := \Omega \setminus \{x\in\Omega : u \text{ is continuous in a neighborhood of } x\}\,,$$

$\dim_{\mathscr{H}}$ denotes the Hausdorff dimension, and $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Theorem 2.1.2. *Assume that $s \in (0,1)$ and $n > 2s$. If $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is a stationary weakly s-harmonic map in Ω, then $u \in C^\infty(\Omega\setminus\text{sing}(u))$ and*

1. *for $s > 1/2$ and $n \geqslant 3$, $\dim_{\mathscr{H}}\text{sing}(u) \leqslant n-2$;*
2. *for $s > 1/2$ and $n = 2$, $\text{sing}(u)$ is locally finite in Ω;*
3. *for $s = 1/2$ and $n \geqslant 2$, $\mathscr{H}^{n-1}(\text{sing}(u)) = 0$;*

4. *for* $s < 1/2$ *and* $n \geqslant 2$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n-1$;

5. *for* $s < 1/2$ *and* $n = 1$, $\operatorname{sing}(u)$ *is locally finite in* Ω.

The other common assumption to consider is energy minimality. We say that a map $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is a minimizing s-harmonic map in Ω if

$$\mathcal{E}_s(u, \Omega) \leqslant \mathcal{E}_s(v, \Omega)$$

for every competitor $v \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ such that $v - u$ is compactly supported in Ω. Notice that minimality implies criticality with respect to inner and outer variations, so that a minimizing s-harmonic map in Ω is in particular a stationary weakly s-harmonic map in Ω. However, minimality implies a stronger partial regularity, at least for $s \in (0, 1/2)$.

Theorem 2.1.3. *Assume that* $s \in (0,1)$ *and* $n > 2s$. *If* $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ *is a minimizing* s*-harmonic map in* Ω, *then* $u \in C^\infty(\Omega \setminus \operatorname{sing}(u))$ *and*

1. *for* $n \geqslant 3$, $\dim_{\mathscr{H}} \operatorname{sing}(u) \leqslant n-2$;

2. *for* $n = 2$, $\operatorname{sing}(u)$ *is locally finite in* Ω;

3. *for* $n = 1$, $\operatorname{sing}(u) = \emptyset$ *(i.e.,* $u \in C^\infty(\Omega)$*)*.

Before describing the way we prove Theorem 2.1.2 and Theorem 2.1.3, let us comment on the sharpness of the results above.

Remark 2.1.1. In the case $s \in (0, 1/2)$, essentially no better regularity than the one coming from the energy space can be expected from a weakly s-harmonic map in Ω. Indeed, for an arbitrary set $E \subseteq \mathbb{R}^n$ such that the characteristic function χ_E belongs to $\widehat{H}^s(\Omega)$, consider the function $u := \chi_E - \chi_{E^c}$. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we can see u as a map from $\mathbb{R}^n$ into $\mathbb{S}^1$, and it belongs to $\widehat{H}^s(\Omega; \mathbb{S}^1)$. It has been observed in [81, Remark 1.7] that u is a weakly s-harmonic map in Ω into $\mathbb{S}^1$, i.e., it satisfies (2.1.4). For $s = 1/2$, we believe that, in the spirit of [92], it should be possible to construct an example of 1/2-harmonic map from the 2-dimensional disk into $\mathbb{S}^1$ which is discontinuous everywhere using the material in [79]. However, for $s \in (1/2, 1)$ and $n = 2$, it remains open whether or not such a pathological example exists.

Remark 2.1.2. For $s \in (0, 1/2)$, the partial regularity for stationary weakly s-harmonic maps is sharp in the sense that the size of the singular set can not be improved. Following Remark 2.1.1 above and [81, Remark 1.7], for a set $E \subseteq \mathbb{R}^n$ such that $\chi_E \in \widehat{H}^s(\Omega)$, the map $u := \chi_E - \chi_{E^c}$ is a weakly s-harmonic map in Ω into $\mathbb{S}^1$, and

$$\mathcal{E}_s(u, \Omega) = \gamma_{n,s} P_{2s}(E, \Omega)\,,$$

where $P_{2s}(E, \Omega)$ is the fractional $2s$-perimeter of E in Ω introduced by L. Caffarelli, J.M. Roquejoffre, and O. Savin [17], and given by

$$P_{2s}(E, \Omega) = \left(\iint_{(E\cap\Omega)\times(E^c\cap\Omega)} + \iint_{(E\cap\Omega^c)\times(E^c\cap\Omega)} + \iint_{(E\cap\Omega)\times(E^c\cap\Omega^c)} \right) \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+2s}}\,.$$

Therefore, u is a stationary weakly s-harmonic map in Ω if and only if E is stationary in Ω for $P_{2s}(\cdot, \Omega)$ (see [81]). This includes the case where ∂E is a nonlocal minimal surface in the sense of [17]. In particular, if E is a half-space, then u is a stationary weakly s-harmonic map in Ω and $\operatorname{sing}(u) = \partial E \cap \Omega$ is an hyperplane.

Remark 2.1.3. For arbitrary spheres, Theorem 2.1.3 is sharp for $s = 1/2$. Indeed, we know from Theorem 1.1.4 ([78, Theorem 1.4]) that the map $x/|x|$ is a minimizing 1/2-harmonic map into $\mathbb{S}^1$ in the unit disk $D_1 \subseteq \mathbb{R}^2$. The minimality of $x/|x|$ for $s \neq 1/2$ is open, but one can check that it is at least a stationary s-harmonic map into $\mathbb{S}^1$ in D_1, showing that Theorem 2.1.2 is sharp also for $s \in [1/2, 1)$.

For arbitrary $s \in (0, 1)$, the following classical example suggests that Theorem 2.1.3 might be sharp anyway. Consider the minimization problem (still in dimension $n = 2$),

$$\min \Big\{ \mathcal{E}_s(u, D_1) : u \in \widehat{H}^s(D_1, \mathbb{S}^1)\,,\ u(x) = x/|x| \text{ in } \mathbb{R}^2 \setminus D_1 \Big\}\,.$$

Existence of solutions follows easily from the direct method of calculus of variations, and any solution is obviously a minimizing s-harmonic map in D_1. Since $x/|x|$ does not admit any $\mathbb{S}^1$-valued continuous extension to D_1, any solution must have at least one singular point in $\overline{D}_1$.

Remark 2.1.4. For $s = 1/2$ and $d \geqslant 3$ (i.e., for $\mathbb{S}^2$ or higher dimensional target spheres), the size of the singular set of a minimizing 1/2-harmonic map can be reduced. It has been proved in Theorem 1.1.3 ([78, Theorem 1.3]) that in this case, $\mathrm{sing}(u) = \emptyset$ for $n = 2$, it is locally finite for $n = 1$, and $\dim_{\mathscr{H}} \mathrm{sing}(u) \leqslant n - 3$ for $n \geqslant 4$. It would be interesting to know if this improvement persists for $s \neq 1/2$.

The proof of Theorems 2.1.1, 2.1.2 and 2.1.3 relies on several ingredients that we now briefly describe. The first one consists in applying the so-called *Caffarelli-Silvestre extension* procedure [18] to the open upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$. This extension (which may have originated in the probability literature [85]) allows us to represent $(-\Delta)^s$ as the Dirichlet-to-Neumann operator associated with the degenerate elliptic operator $L_s := -\mathrm{div}(z^{1-2s}\nabla\cdot)$, where $z \in (0, +\infty)$ denotes the extension variable. In this way (after extension), we can reformulate the s-harmonic map equation as a degenerate harmonic map equation with *partially* free boundary, very much like in [80, 81]. Under the stationarity assumption, the extended map satisfies a fundamental monotonicity formula, which in turn implies local controls in BMO (bounded mean oscillation) of the s-harmonic map under consideration by its energy.

Probably the main step in the proof is an epsilon-regularity result where we show that under a (standard) smallness assumption on the energy $\mathcal{E}_s$ in a ball, then the (stationary) s-harmonic map is Hölder continuous in a smaller ball. The strategy we follow here is quite inspired from the argument of L.C. Evans [39] making use of the conservation laws discovered by F. Hélein [60] and the duality $\mathscr{H}^1$/BMO. In our fractional setting, we make use of the fractional conservation laws together with the "fractional div-curl lemma" of K. Mazowiecka and A. Schikorra [77]. A main difference with [39] lies in the fact that an additional "error term" appears when rewriting the s-harmonic map equation in the suitable form where compensation can be seen. To control this error term in arbitrary dimensions, we make use of a recent embedding result between Triebel-Lizorkin-Morrey type spaces [63] and various characterizations of these spaces [96, 114].

Once Hölder continuity is obtained, we prove Lipschitz continuity in an even smaller ball using an adjustment of the classical "harmonic replacement" technique, see [102]. More precisely, using the extension, we adapt an argument due to J. Roberts [93] in the case of degenerate harmonic maps with free boundary (i.e., with homogeneous - degenerate - Neumann boundary condition). With Lipschitz continuity in hands, we are then able to derive C^∞-regularity from Schauder estimates for the fractional Laplacian.

To obtain the bounds on the size of the singular set, we follow somehow the usual dimension reduction argument of Almgren & Federer for harmonic maps (see [107]), which is based on the strong compactness of blowups around points. Here compactness (for $s \neq 1/2$) is obtained as in [81], and it is a consequence of the monotonicity formula together with Marstrand's Theorem (see e.g. [76]). Finally, in the minimizing case and $s \in (0, 1/2)$, we obtain an improvement on the size of the singular set (compared with the stationary case) from the triviality of the so-called "tangent maps" (i.e. blowup limits), a consequence of the regularity of minimizing s-harmonic maps in one dimension proved in [82].

Notation

Throughout the paper, $\mathbb{R}^n$ is often identified with $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$. More generally, sets $A \subseteq \mathbb{R}^n$ can be identified with $A \times \{0\} \subseteq \partial\mathbb{R}^{n+1}_+$. Unlike Chapter 1, points in $\mathbb{R}^{n+1}$ are here written $\mathbf{x} = (x, z)$[1] with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. We shall denote by $B_r(\mathbf{x})$ the open ball in $\mathbb{R}^{n+1}$ of radius r centered at $\mathbf{x} = (x, z)$, while $D_r(x) := B_r(\mathbf{x}) \cap \mathbb{R}^n$ is the open ball (or disk) in $\mathbb{R}^n$ centered at x. For an arbitrary set $G \subseteq \mathbb{R}^{n+1}$, we write

$$G^+ := G \cap \mathbb{R}^{n+1}_+ \quad \text{and} \quad \partial^+ G := \partial G \cap \mathbb{R}^{n+1}_+ \,.$$

If $G \subseteq \mathbb{R}^{n+1}_+$ is a bounded open set, we shall say that G is **admissible** whenever

- ∂G is Lipschitz regular;
- the (relative) open set $\partial^0 G \subseteq \mathbb{R}^n$ defined by

  $$\partial^0 G := \Big\{ \mathbf{x} \in \partial G \cap \partial\mathbb{R}^{n+1}_+ : B^+_r(\mathbf{x}) \subseteq G \text{ for some } r > 0 \Big\} \,,$$

  is non-empty and has Lipschitz boundary;
- $\partial G = \partial^+ G \cup \overline{\partial^0 G}\,.$

Finally, we usually denote by C a generic positive constant which only depends on the dimension n and $s \in (0, 1)$, and possibly changing from line to line. If a constant depends on additional given parameters, we shall write those parameters using the subscript notation.

2.2 Functional spaces, fractional operators, and compensated compactness

2.2.1 Fractional H^s-spaces

For an open set $\Omega \subseteq \mathbb{R}^n$, the Sobolev-Slobodeckij space $H^s(\Omega)$ is made of all functions $u \in L^2(\Omega)$ such that[2]

$$[u]^2_{H^s(\Omega)} := \frac{\gamma_{n,s}}{2} \iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx dy < \infty \,, \quad \gamma_{n,s} := s\, 2^{2s} \pi^{-\frac{n}{2}} \frac{\Gamma\big(\frac{n+2s}{2}\big)}{\Gamma(1 - s)} \,. \tag{2.2.1}$$

[1] An extra weight is associated with the $(n + 1)$-th dimension whenever $s \neq 1/2$. To emphasize the particular role played by the $(n + 1)$-th variable and to simplify notations, we prefer to write z instead of x_{n+1} in this chapter.

[2] The normalization constant $\gamma_{n,s}$ is chosen in such a way that $[u]^2_{H^s(\mathbb{R}^n)} = \displaystyle\int_{\mathbb{R}^n} (2\pi|\xi|)^{2s} |\widehat{u}|^2 \, d\xi$, where $\widehat{u}$ denotes the (ordinary frequency) Fourier transform of u.

It is a separable Hilbert space normed by $\|\cdot\|^2_{H^s(\Omega)} := \|\cdot\|^2_{L^2(\Omega)} + [\cdot]^2_{H^s(\Omega)}$. The space $H^s_{\mathrm{loc}}(\Omega)$ denotes the class of functions whose restriction to any relatively compact open subset Ω' of Ω belongs to $H^s(\Omega')$. The linear subspace $H^s_{00}(\Omega) \subseteq H^s(\mathbb{R}^n)$ is in turn defined by

$$H^s_{00}(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\,.$$

Endowed with the induced norm, $H^s_{00}(\Omega)$ is also a Hilbert space, and

$$[u]^2_{H^s(\mathbb{R}^n)} = \frac{\gamma_{n,s}}{2}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = 2\mathcal{E}_s(u,\Omega) \quad \forall u \in H^s_{00}(\Omega)\,,$$

where $\mathcal{E}_s(\cdot,\Omega)$ is the s-Dirichlet energy defined in (2.1.3).

If Ω is bounded and its boundary is smooth enough (e.g. if $\partial\Omega$ is Lipschitz regular), then

$$H^s_{00}(\Omega) = \overline{\mathscr{D}(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}} \tag{2.2.2}$$

(see [57, Theorem 1.4.2.2]) . The topological dual space of $H^s_{00}(\Omega)$ is denoted by $H^{-s}(\Omega)$.

We are mostly interested in the class of functions

$$\widehat{H}^s(\Omega) := \Big\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{E}_s(u,\Omega) < \infty\Big\}\,.$$

The following properties hold for any open subsets Ω and Ω' of $\mathbb{R}^n$:

- $\widehat{H}^s(\Omega)$ is a linear space;
- $\widehat{H}^s(\Omega) \subseteq \widehat{H}^s(\Omega')$ whenever $\Omega' \subseteq \Omega$, and $\mathcal{E}_s(\cdot,\Omega') \leqslant \mathcal{E}_s(\cdot,\Omega)$;
- if Ω' is bounded, then $\widehat{H}^s(\Omega) \cap H^s_{\mathrm{loc}}(\mathbb{R}^n) \subseteq \widehat{H}^s(\Omega')$;
- if Ω is bounded, then $H^s_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq \widehat{H}^s(\Omega)$.

From Lemma 2.2.1 below, it follows that $\widehat{H}^s(\Omega)$ is a Hilbert space for the scalar product induced by the Hilbertian norm $u \mapsto \|u\|_{\widehat{H}^s(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + \mathcal{E}_s(u,\Omega)\big)^{1/2}$ (see e.g. [81] and [80, proof of Lemma 2.1]).

Lemma 2.2.1. *Let $x_0 \in \Omega$ and $\rho > 0$ be such that $D_\rho(x_0) \subseteq \Omega$. There exists a constant $C_\rho = C_\rho(n,s) > 0$, independent of x_0, such that*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(|x-x_0|+1)^{n+2s}}\,\mathrm{d}x \leqslant C_\rho\left(\mathcal{E}_s(u,D_\rho(x_0)) + \|u\|^2_{L^2(D_\rho(x_0))}\right)$$

for every $u \in \widehat{H}^s(\Omega)$.

Remark 2.2.2. From the Hilbertian structure of $\widehat{H}^s(\Omega)$, it follows that any bounded sequence $\{u_k\}$ in $\widehat{H}^s(\Omega)$ admits a subsequence converging weakly in $\widehat{H}^s(\Omega)$. In addition, if $u_k \rightharpoonup u$ weakly in $\widehat{H}^s(\Omega)$, then $u_k \to u$ strongly in $L^2(\Omega)$ by the compact embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$. In particular, $\|u_k\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}$. Since $\liminf_k \|u_k\|_{\widehat{H}^s(\Omega)} \geqslant \|u\|_{\widehat{H}^s(\Omega)}$, it follows that $\liminf_k \mathcal{E}_s(u_k,\Omega) \geqslant \mathcal{E}_s(u,\Omega)$.

2.2.2 Fractional operators and compensated compactness

Given an open set $\Omega \subseteq \mathbb{R}^n$, the fractional Laplacian $(-\Delta)^s$ in Ω is defined as the continuous linear operator $(-\Delta)^s : \widehat{H}^s(\Omega) \to (\widehat{H}^s(\Omega))'$ induced by the quadratic form $\mathcal{E}_s(\cdot, \Omega)$. In other words, the (distributional) fractional Laplacian $(-\Delta)^s u$ of a given function $u \in \widehat{H}^s(\Omega)$ is defined through its action on $\widehat{H}^s(\Omega)$ by

$$\left\langle (-\Delta)^s u, \varphi \right\rangle_\Omega := \frac{\gamma_{n,s}}{2} \iint_{(\mathbb{R}^n \times \mathbb{R}^n)\setminus(\Omega^c \times \Omega^c)} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \,. \tag{2.2.3}$$

Notice that the restriction of the linear form $(-\Delta)^s u$ to the subspace $H^s_{00}(\Omega)$ belongs to $H^{-s}(\Omega)$ with the estimate $\|(-\Delta)^s u\|^2_{H^{-s}(\Omega)} \leqslant 2\mathcal{E}_s(u, \Omega)$.

Remark 2.2.3. Notice that the operator $(-\Delta)^s$ has the following local property: if $u \in \widehat{H}^s(\Omega)$ and $\Omega' \subseteq \Omega$ is an open subset, then

$$\left\langle (-\Delta)^s u, \varphi \right\rangle_\Omega = \left\langle (-\Delta)^s u, \varphi \right\rangle_{\Omega'} \quad \forall \varphi \in H^s_{00}(\Omega') \,.$$

Following [77], we now relate the fractional Laplacian $(-\Delta)^s$ to suitable notions of fractional gradient and fractional divergence. To this purpose, we first need to recall from [77] the notion of (fractional) *"s-vector field"* over a domain. The space of s-vector fields in Ω, that we shall denote by $L^2_{\mathrm{od}}(\Omega)$ (in agreement with [77]), is defined as the Lebesgue space of *L^2-scalar functions* over the open set $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c) \subseteq \mathbb{R}^{2n}$ with respect to the measure $|x-y|^{-n}\mathrm{d}x\mathrm{d}y$. In other words,

$$L^2_{\mathrm{od}}(\Omega) := \left\{ F : (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c) \to \mathbb{R} : \|F\|_{L^2_{\mathrm{od}}(\Omega)} < \infty \right\},$$

with

$$\|F\|^2_{L^2_{\mathrm{od}}(\Omega)} := \iint_{(\mathbb{R}^n \times \mathbb{R}^n)\setminus(\Omega^c \times \Omega^c)} \frac{|F(x,y)|^2}{|x-y|^n} \, \mathrm{d}x\mathrm{d}y \,.$$

We endow $L^2_{\mathrm{od}}(\Omega)$ with the product operator $\odot : L^2_{\mathrm{od}}(\Omega) \times L^2_{\mathrm{od}}(\Omega) \times \to L^1(\Omega)$ given by

$$F \odot G(x) := \int_{\mathbb{R}^n} \frac{F(x,y)G(x,y)}{|x-y|^n} \, \mathrm{d}y \,.$$

Note that $\odot$ is a continuous bilinear operator thanks to Fubini's theorem, and it plays the role of "pointwise scalar product" between two s-vector fields. With this respect, we define the (pointwise) "squared modulus" of a s-vector field $F \in L^2_{\mathrm{od}}(\Omega)$ by

$$|F|^2 := F \odot F \in L^1(\Omega) \,. \tag{2.2.4}$$

The (fractional) s-gradient is defined in [77] as a linear operator from the space of scalar valued functions $\widehat{H}^s(\Omega)$ into the space of s-vector fields over Ω. More precisely, we define it as the continuous linear operator $\mathrm{d}_s : \widehat{H}^s(\Omega) \to L^2_{\mathrm{od}}(\Omega)$ given by

$$\mathrm{d}_s u(x,y) := \frac{\sqrt{\gamma_{n,s}}}{\sqrt{2}} \, \frac{u(x) - u(y)}{|x-y|^s} \,. \tag{2.2.5}$$

Obviously, one has

$$\|\mathrm{d}_s u\|^2_{L^2_{\mathrm{od}}(\Omega)} = 2\mathcal{E}_s(u, \Omega) \quad \text{and} \quad \big\| |\mathrm{d}_s u|^2 \big\|_{L^1(\Omega)} \leqslant 2\mathcal{E}_s(u, \Omega)$$

for every $u \in \widehat{H}^s(\Omega)$.

In turn, the (fractional) s-divergence, denoted by div_s, is defined by duality as the adjoint operator to the s-gradient operator restricted to $H^s_{00}(\Omega)$. To do so, the main observation is that for $F \in L^2_{\mathrm{od}}(\Omega)$, we have

$$F \odot \mathrm{d}_s\varphi \in L^1(\mathbb{R}^n) \quad \forall \varphi \in H^s_{00}(\Omega)\,,$$

with

$$\|F \odot \mathrm{d}_s\varphi\|_{L^1(\mathbb{R}^n)} \leqslant \|F\|_{L^2_{\mathrm{od}}(\Omega)}[\varphi]_{H^s(\mathbb{R}^n)}\,.$$

In this way, we can indeed define $\mathrm{div}_s : L^2_{\mathrm{od}}(\Omega) \to H^{-s}(\Omega)$ as the continuous linear operator given by

$$\langle \mathrm{div}_s F, \varphi\rangle_\Omega := \int_{\mathbb{R}^n} F \odot \mathrm{d}_s\varphi \,\mathrm{d}x \qquad \forall \varphi \in H^s_{00}(\Omega)\,,$$

which satisfies the estimate $\|\mathrm{div}_s F\|_{H^{-s}(\Omega)} \leqslant \|F\|_{L^2_{\mathrm{od}}(\Omega)}$ for all $F \in L^2_{\mathrm{od}}(\Omega)$.

From the definition of d_s and div_s, it readily follows that

Proposition 2.2.4. *We have $(-\Delta)^s = \mathrm{div}_s(\mathrm{d}_s)$, i.e.,*

$$\langle (-\Delta)^s u, \varphi\rangle_\Omega = \int_{\mathbb{R}^n} \mathrm{d}_s u \odot \mathrm{d}_s\varphi \,\mathrm{d}x$$

for every $u \in \widehat{H}^s(\Omega)$ and every $\varphi \in H^s_{00}(\Omega)$.

One of the main results in [77] is a compensated compactness result relative to the s-gradient and s-divergence operators in the spirit of the classical "div-curl" lemma [21]. To present this result, let us recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ is defined as the set of all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{D_r(y)} \fint_{D_r(y)} |u - (u)_{y,r}|\,\mathrm{d}x < +\infty\,,$$

where $(u)_{y,r}$ denotes the average of u over the ball $D_r(y)$. The following theorem corresponds to [77, Proposition 2.4].

Theorem 2.2.1. *Let $F \in L^2_{\mathrm{od}}(\Omega)$ be such that*

$$\mathrm{div}_s F = 0 \quad \text{in } H^{-s}(\Omega)\,.$$

There exist a universal $\Lambda > 1$ such that for every ball $D_r(x_0)$ satisfying $D_{\Lambda r}(x_0) \subseteq \Omega$,

$$\left|\int_{\mathbb{R}^n} (F \odot \mathrm{d}_s u)\varphi \,\mathrm{d}x\right| \leqslant C\|F\|_{L^2_{\mathrm{od}}(\Omega)}\sqrt{\mathcal{E}_s(u,\Omega)}\Big([\varphi]_{\mathrm{BMO}(\mathbb{R}^n)} + r^{-n}\|\varphi\|_{L^1(\mathbb{R}^n)}\Big)$$

for every $u \in \widehat{H}^s(\Omega)$ and $\varphi \in \mathscr{D}(D_r(x_0))$, and a constant $C = C(n,s)$.

Remark 2.2.5. In the statement of [77, Proposition 2.4], the s-vector field F is assumed to be s-divergence free in the whole $\mathbb{R}^n$ and $u \in H^s(\mathbb{R}^n)$. However, a careful reading of the proof reveals that only the assumptions in Theorem 2.2.1 on F and u are used.

2.2.3 Weighted Sobolev spaces

For an open set $G \subseteq \mathbb{R}^{n+1}$, we consider the weighted L^2-space

$$L^2(G, |z|^a \mathrm{d}\mathbf{x}) := \Big\{ v \in L^1_{\mathrm{loc}}(G) : |z|^{\frac{a}{2}} v \in L^2(G) \Big\} \quad \text{with } a := 1 - 2s \,,$$

normed by

$$\|v\|^2_{L^2(G,|z|^a \mathrm{d}\mathbf{x})} := \int_G |z|^a |v|^2 \, \mathrm{d}\mathbf{x} \,.$$

Accordingly, we introduce the weighted Sobolev space

$$H^1(G, |z|^a \mathrm{d}\mathbf{x}) := \Big\{ v \in L^2(G, |z|^a \mathrm{d}\mathbf{x}) : \nabla v \in L^2(G, |z|^a \mathrm{d}\mathbf{x}) \Big\} \,,$$

normed by

$$\|v\|_{H^1(G,|z|^a \mathrm{d}\mathbf{x})} := \|v\|_{L^2(G,|z|^a \mathrm{d}\mathbf{x})} + \|\nabla v\|_{L^2(G,|z|^a \mathrm{d}\mathbf{x})} \,.$$

Both $L^2(G, |z|^a \mathrm{d}\mathbf{x})$ and $H^1(G, |z|^a \mathrm{d}\mathbf{x})$ are separable Hilbert spaces when equipped with the scalar product induced by their respective Hilbertian norms.

On $H^1(G, |z|^a \mathrm{d}\mathbf{x})$, we define the *weighted Dirichlet energy* $\mathbf{E}_s(\cdot, G)$ by setting

$$\mathbf{E}_s(v, G) := \frac{\boldsymbol{\delta}_s}{2} \int_G |z|^a |\nabla v|^2 \, \mathrm{d}\mathbf{x} \quad \text{with } \boldsymbol{\delta}_s := 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} \,. \tag{2.2.6}$$

The relevance of the normalization constant $\boldsymbol{\delta}_s > 0$ will be revealed in Section 2.2.4 (see (2.2.16)).

Some relevant remarks about $H^1(G, |z|^a \mathrm{d}\mathbf{x})$ are in order. For a bounded admissible open set $G \subseteq \mathbb{R}^{n+1}_+$, if $s \in (0, 1/2)$, the space $L^2(G, |z|^a \mathrm{d}\mathbf{x})$ embeds continuously into $L^\gamma(G)$ for every $1 \leqslant \gamma < \frac{1}{1-s}$ by Hölder inequality, and if $s \in [1/2, 1)$, $L^2(G, |z|^a \mathrm{d}\mathbf{x})$ obviously embeds continuously into $L^2(G)$ since $a \leqslant 0$. In any case,

$$H^1(G, |z|^a \mathrm{d}\mathbf{x}) \hookrightarrow W^{1,\gamma}(G) \tag{2.2.7}$$

continuously for some $\gamma > 1$. As a first consequence, $H^1(G, |z|^a \mathrm{d}\mathbf{x}) \hookrightarrow L^1(G)$ with compact embedding. Secondly, for such a γ, the compact linear trace operator

$$v \in W^{1,\gamma}(G) \mapsto v_{|\partial^0 G} \in L^1(\partial^0 G) \tag{2.2.8}$$

induces a compact linear trace operator from $H^1(G, |z|^a \mathrm{d}\mathbf{x})$ into $L^1(\partial^0 G)$, extending the usual trace of smooth functions. We shall denote by $v_{|\partial^0 G}$ the trace of $v \in H^1(G, |z|^a \mathrm{d}\mathbf{x})$ on $\partial^0 G$, or simply by v if it is clear from the context. We may now recall the following Poincaré inequality, see e.g. [81, Lemma 2.5].

Lemma 2.2.6. *If $v \in H^1(B_r^+, |z|^a \mathrm{d}\mathbf{x})$, then*

$$\big\|v - (v)_r\big\|_{L^1(D_r)} \leqslant C r^{\frac{n+2s}{2}} \|\nabla v\|_{L^2(B_r^+,|z|^a \mathrm{d}\mathbf{x})} \,,$$

for a constant $C = C(n, s)$, where $(v)_r$ denotes the average of v over D_r.

The next lemma states that the trace $v_{|\partial^0 G}$ has actually H^s-regularity, at least locally.

Lemma 2.2.7. *If $v \in H^1(B_{2r}^+, |z|^a \mathrm{d}\mathbf{x})$, then the trace of v on $\partial^0 B_r^+ \simeq D_r$ belongs to $H^s(D_r)$, and*

$$[v]^2_{H^s(D_r)} \leqslant C \, \mathbf{E}_s(v, B_{2r}^+) \,,$$

for a constant $C = C(n, s)$.

Proof. The proof follows exactly the one in [82, Lemma 2.3] which is stated only in dimension $n = 1$. We reproduce the proof (in arbitrary dimension) for convenience of the reader, slightly anticipating a well-known identity presented in Section 2.2.4 (see (2.2.16)).

Rescaling variables, we can assume that $r = 1$. Moreover, we may assume without loss of generality that v has a vanishing average over the half-ball B_2^+. Let $\zeta \in C^\infty(B_2; [0, 1])$ be a cutoff function such that $\zeta(\mathbf{x}) = 1$ for $|\mathbf{x}| \leqslant 1$, $\zeta(\mathbf{x}) = 0$ for $|\mathbf{x}| \geqslant 3/2$. The function $v_* := \zeta v$ belongs to $H^1(\mathbb{R}^{n+1}_+, |z|^a \mathrm{d}\mathbf{x})$, and Poincaré inequality in $H^1(\mathbb{R}^{n+1}_+, |z|^a \mathrm{d}\mathbf{x})$ (see e.g. [40]) yields

$$\int_{\mathbb{R}^{n+1}_+} z^a |\nabla v_*|^2 \,\mathrm{d}\mathbf{x} \leqslant 2\mathbf{E}_s(v, B_2^+) + C \int_{B_2^+} z^a |v|^2 \,\mathrm{d}\mathbf{x} \leqslant C\mathbf{E}_s(v, B_2^+) \,, \tag{2.2.9}$$

for a constant $C = C(\zeta, n, s)$. On the other hand, it follows from (2.2.16) in Section 2.2.4 below that

$$\iint_{D_1\times D_1} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} \,\mathrm{d}x\mathrm{d}y \leqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x - y|^{n+2s}} \,\mathrm{d}x\mathrm{d}y \leqslant C\mathbf{E}_s(v_*, \mathbb{R}^{n+1}_+) \,. \tag{2.2.10}$$

Gathering (2.2.9) and (2.2.10) leads to the announced estimate. □

2.2.4 Fractional harmonic extension and the Dirichlet-to-Neumann operator

Let us consider the so-called fractional Poisson kernel $\mathbf{P}_{n,s} : \mathbb{R}^{n+1}_+ \to [0, \infty)$ defined by

$$\mathbf{P}_{n,s}(\mathbf{x}) := \sigma_{n,s} \frac{z^{2s}}{|\mathbf{x}|^{n+2s}} \qquad \text{with } \sigma_{n,s} := \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(s)} \,, \tag{2.2.11}$$

where $\mathbf{x} := (x, z) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. The choice of the constant $\sigma_{n,s}$ is made in such a way that $\int_{\mathbb{R}^n} \mathbf{P}_{n,s}(x, z) \,\mathrm{d}x = 1$ for every $z > 0$ (see e.g. the computation in Remark 2.7.11). As shown in [18] (see also [85]), the function $\mathbf{P}_{n,s}$ solves

$$\begin{cases} \operatorname{div}(z^a \nabla \mathbf{P}_{n,s}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \,, \\ \mathbf{P}_{n,s} = \delta_0 & \text{on } \partial\mathbb{R}^{n+1}_+ \,, \end{cases}$$

where δ_0 denotes the Dirac distribution at the origin.

From now on, for a measurable function u defined over $\mathbb{R}^n$, we shall denote by u^{e} its extension to the half-space $\mathbb{R}^{n+1}_+$ given by the convolution (in the x-variable) of u with $\mathbf{P}_{n,s}$, i.e.,

$$u^{\mathrm{e}}(x, z) := \sigma_{n,s} \int_{\mathbb{R}^n} \frac{z^{2s} u(y)}{(|x - y|^2 + z^2)^{\frac{n+2s}{2}}} \,\mathrm{d}y \,. \tag{2.2.12}$$

Notice that u^{e} is well defined if u belongs to the Lebesgue space L^1 over $\mathbb{R}^n$ with respect to the probability measure

$$\mathfrak{m}_s := \sigma_{n,s} (1 + |y|^2)^{-\frac{n+2s}{2}} \,\mathrm{d}y \,. \tag{2.2.13}$$

In particular, u^{e} can be defined whenever $u \in \widehat{H}^s(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$ by Lemma 2.2.1. Moreover, if $u \in L^\infty(\mathbb{R}^n)$, then $u^{\mathrm{e}} \in L^\infty(\mathbb{R}^{n+1}_+)$ and

$$\|u^{\mathrm{e}}\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leqslant \|u\|_{L^\infty(\mathbb{R}^n)} \,. \tag{2.2.14}$$

For a function $u \in L^1(\mathbb{R}^n, \mathfrak{m}_s)$, the extension $u^{\rm e}$ has a pointwise trace on $\partial\mathbb{R}^{n+1}_+ \simeq \mathbb{R}^n$ which is equal to u at every Lebesgue point. In addition, $u^{\rm e}$ solves the equation

$$\begin{cases} \operatorname{div}(z^a \nabla u^{\rm e}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \,, \\ u^{\rm e} = u & \text{on } \partial\mathbb{R}^{n+1}_+ \,. \end{cases} \tag{2.2.15}$$

By analogy with the standard case $s = 1/2$ (for which (2.2.15) reduces to the Laplace equation), the map $u^{\rm e}$ is referred to as the *fractional harmonic extension* of u.

It has been proved in [18] that $u^{\rm e}$ belongs to the weighted space $H^1(\mathbb{R}^{n+1}_+, |z|^a {\rm d}\mathbf{x})$ whenever $u \in H^s(\mathbb{R}^n)$. Extending a well-known identity for $s = 1/2$, the H^s-seminorm of u coincides up to a multiplicative constant with the weighted L^2-norm of $\nabla u^{\rm e}$, and $u^{\rm e}$ turns out to minimize the weighted Dirichlet energy among all possible extensions. In other words,

$$[u]^2_{H^s(\mathbb{R}^n)} = \mathbf{E}_s(u^{\rm e}, \mathbb{R}^{n+1}_+) = \inf\Big\{ \mathbf{E}_s(v, \mathbb{R}^{n+1}_+) : v \in H^1(\mathbb{R}^{n+1}_+, |z|^a {\rm d}\mathbf{x})\,,\ v = u \text{ on } \mathbb{R}^n \Big\} \tag{2.2.16}$$

for every $u \in H^s(\mathbb{R}^n)$ (thanks to the choice of the normalization factor $\boldsymbol{\delta}_s$ in (2.2.6)).

If $u \in \widehat{H}^s(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, we have the following estimates on $u^{\rm e}$, somehow extending the first equality in (2.2.16) to the localized setting.

Lemma 2.2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For every $u \in \widehat{H}^s(\Omega)$, the extension $u^{\rm e}$ given by (2.2.12) belongs to $H^1(G, |z|^a {\rm d}\mathbf{x}) \cap L^2_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}, |z|^a {\rm d}\mathbf{x})$ for every bounded admissible open set $G \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\partial^0 G} \subseteq \Omega$. In addition, for every point $\mathbf{x}_0 = (x_0, 0) \in \Omega \times \{0\}$ and $r > 0$ such that $D_{3r}(x_0) \subseteq \Omega$,*

$$\|u^{\rm e}\|^2_{L^2(B^+_r(\mathbf{x}_0), |z|^a {\rm d}\mathbf{x})} \leqslant C \left(r^2 \mathcal{E}_s(u, D_{2r}(x_0)) + r^{2-2s} \|u\|^2_{L^2(D_{2r}(x_0))} \right) , \tag{2.2.17}$$

and

$$\mathbf{E}_s(u^{\rm e}, B^+_r(\mathbf{x}_0)) \leqslant C \mathcal{E}_s(u, D_{2r}(x_0)) \,, \tag{2.2.18}$$

for a constant $C = C(n, s)$.

Proof. Translating and rescaling variables, we can assume that $x_0 = 0$ and $r = 1$. Then (2.2.17) follows from [81, Lemma 2.10] (which is stated for $s \in (0, 1/2)$, but the proof is in fact valid for any $s \in (0,1)$). Denote by $\bar{u}$ the average of u over D_2. Noticing that $(u - \bar{u})^{\rm e} = u^{\rm e} - \bar{u}$, and applying [81, Lemma 2.10] to $u - \bar{u}$ yields

$$\mathbf{E}_s(u^{\rm e}, B^+_1) \leqslant C\big(\mathcal{E}_s(u, D_2) + \|u - \bar{u}\|^2_{L^2(D_2)} \big) \,.$$

On the other hand, by Poincaré inequality in $H^s(D_2)$, we have

$$\|u - \bar{u}\|^2_{L^2(D_2)} \leqslant C [u]^2_{H^s(D_2)} \leqslant C \mathcal{E}_s(u, D_2) \,,$$

and (2.2.18) follows. □

Corollary 2.2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $G \subseteq \mathbb{R}^{n+1}_+$ a bounded admissible open set such that $\overline{\partial^0 G} \subseteq \Omega$. The extension operator $u \mapsto u^{\rm e}$ defines a continuous linear operator from $\widehat{H}^s(\Omega)$ into $H^1(G, |z|^a {\rm d}\mathbf{x})$.*

Proof. Set $\delta := \operatorname{dist}(\partial^0 G, \Omega^c)$, and

$$h_1 := \min\Big\{ \frac{\delta}{12} \,, \inf\big\{ \operatorname{dist}(\mathbf{x}, \partial\mathbb{R}^{n+1}_+) : \mathbf{x} = (x, z) \in G \,,\ \operatorname{dist}((x,0), \partial^0 G) \geqslant \delta/2 \big\} \Big\} > 0 \,,$$

$$h_2 := \sup\Big\{\mathrm{dist}(\mathbf{x}, \partial\mathbb{R}^{n+1}_+) : \mathbf{x} = (x,z) \in G\Big\} < +\infty\,.$$

We also consider a large radius $R > 0$ in such a way that $G \subseteq D_R \times \mathbb{R}$, and we define

$$\omega := \Big\{x \in \mathbb{R}^n : \mathrm{dist}((x,0), \partial^0 G) < \delta/2\ \Big\}\,,$$

and

$$G_* := (\omega \times (0,h_1]) \cup (D_R \times (h_1,h_2))\,.$$

By construction, G_* is a bounded admissible open set satisfying $\overline{\partial^0 G_*} \subseteq \Omega$ and $G \subseteq G_*$. Therefore, it is enough to show that the extension operator is continuous from $\widehat{H}^s(\Omega)$ into $H^1(G_*, |z|^a \mathrm{d}\mathbf{x})$. In other words, we can assume without loss of generality that $G = G_*$.

Covering $\omega \times (0,h_1]$ by finitely many half-balls $B^+_{\delta/6}(\mathbf{x_i})$ with $\mathbf{x}_i \in \omega \times \{0\}$, and applying Lemma 2.2.8 in those balls, we infer that $u^{\mathrm{e}} \in H^1(\omega \times (0,h_1), |z|^a \mathrm{d}\mathbf{x})$, and

$$\|u^{\mathrm{e}}\|^2_{H^1(\omega\times(0,h_1),|z|^a\mathrm{d}\mathbf{x})} \leqslant C_G\big(\mathcal{E}_s(u,\Omega) + \|u\|^2_{L^2(\Omega)}\big)\,,$$

for a constant $C_G = C_G(G,n,s)$.

On the other hand, one may derive from formula (2.2.12) and Jensen inequality that

$$|u^{\mathrm{e}}(\mathbf{x})|^2 + |\nabla u^{\mathrm{e}}(\mathbf{x})|^2 \leqslant C_G \int_{\mathbb{R}^n} \frac{|u(y)|^2}{(|x-y|^2 + h_1^2)}\,\mathrm{d}y \quad \forall \mathbf{x} = (x,z) \in D_R \times (h_1,h_2)\,.$$

It then follows from Lemma 2.2.1 that $u^{\mathrm{e}} \in H^1(D_R \times (h_1,h_2), |z|^a \mathrm{d}\mathbf{x})$ with

$$\|u^{\mathrm{e}}\|^2_{H^1(D_R\times(h_1,h_2),|z|^a\mathrm{d}\mathbf{x})} \leqslant C_G\big(\mathcal{E}_s(u,\Omega) + \|u\|^2_{L^2(\Omega)}\big)\,,$$

which completes the proof. □

Another useful fact about the extension by convolution with $\mathbf{P}_{n,s}$, is that it preserves some local Hölder continuity. It is very classical and follows from the explicit formula (and regularity) of $\mathbf{P}_{n,s}$. Details are left to the reader.

Lemma 2.2.10. *If $u \in L^\infty(\mathbb{R}^n) \cap C^{0,\beta}(D_R)$ for some $\beta \in (0, \min(1,2s))$, then $u^{\mathrm{e}} \in C^{0,\beta}(B^+_{R/4})$, and*

$$R^\beta [u^{\mathrm{e}}]_{C^{0,\beta}(B^+_{R/4})} \leqslant C_\beta\big(R^\beta [u]_{C^{0,\beta}(D_R)} + \|u\|_{L^\infty(\mathbb{R}^n)}\big)\,, \tag{2.2.19}$$

for a constant $C_\beta = C_\beta(\beta,n,s)$.

Let us now assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. If $u \in \widehat{H}^s(\Omega)$, the divergence free vector field $z^a \nabla u^{\mathrm{e}}$ admits a distributional normal trace on Ω, that we denote by $\mathbf{\Lambda}^{(2s)} u$. More precisely, we define $\mathbf{\Lambda}^{(2s)} u$ through its action on a test function $\varphi \in \mathscr{D}(\Omega)$ by setting

$$\Big\langle \mathbf{\Lambda}^{(2s)} u, \varphi \Big\rangle_\Omega := \int_{\mathbb{R}^{n+1}_+} z^a \nabla u^{\mathrm{e}} \cdot \nabla \Phi\, \mathrm{d}\mathbf{x}\,, \tag{2.2.20}$$

where Φ is any smooth extension of φ compactly supported in $\mathbb{R}^{n+1}_+ \cup \Omega$. Note that the right-hand side of (2.2.20) is well defined by Lemma 2.2.8. By the divergence theorem, it is routine to check that the integral in (2.2.20) does not depend on the choice of the extension Φ. It can be thought of as a *fractional Dirichlet-to-Neumann operator*. Indeed, whenever u is smooth, the distribution $\mathbf{\Lambda}^{(2s)}u$ is the pointwise-defined function given by

$$\mathbf{\Lambda}^{(2s)}u(x) = -\lim_{z\downarrow 0} z^a \partial_z u^{\mathrm{e}}(x,z) = 2s \lim_{z\downarrow 0} \frac{u^{\mathrm{e}}(x,0) - u^{\mathrm{e}}(x,z)}{z^{2s}}$$

at each point $x \in \Omega$.

In the case $\Omega = \mathbb{R}^n$, it has been proved in [18] that $\mathbf{\Lambda}^{(2s)}$ coincides with the distribution $(-\Delta)^s$, up to the multiplicative factor $\boldsymbol{\delta}_s$. In the localized setting, this identity still holds, see e.g. [81, Lemma 2.12] and [80, Lemma 2.9].

Lemma 2.2.11. *If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then*

$$(-\Delta)^s = \boldsymbol{\delta}_s \mathbf{\Lambda}^{(2s)} \text{ on } \widehat{H}^s(\Omega)\,.$$

One of the main consequences of Lemma 2.2.11 is a local counterpart of (2.2.16) concerning the minimality of u^{e}. This is the purpose of Corollary 2.2.12 below, inspired from [17, Lemma 7.2], and taken from [81, Corollary 2.13].

Corollary 2.2.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $G \subseteq \mathbb{R}^{n+1}_+$ an admissible bounded open set such that $\overline{\partial^0 G} \subseteq \Omega$. Let $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$, and let u^{e} be its fractional harmonic extension to $\mathbb{R}^{n+1}_+$ given by (2.2.12). Then,*

$$\mathbf{E}_s(v,G) - \mathbf{E}_s(u^{\mathrm{e}},G) \geqslant \mathcal{E}_s(v,\Omega) - \mathcal{E}_s(u,\Omega) \tag{2.2.21}$$

for all $v \in H^1(G;\mathbb{R}^d, |z|^a \mathrm{d}\mathbf{x})$ such that $v - u^{\mathrm{e}}$ is compactly supported in $G \cup \partial^0 G$. In the right-hand side of (2.2.21), the trace of v on $\partial^0 G$ is extended by u outside $\partial^0 G$.

2.2.5 Inner variations, monotonicity formula, and density functions

In this section, our main goal is to present the *monotonicity formula* satisfied by critical points of $\mathcal{E}_s(\cdot,\Omega)$ under *inner variations*, i.e., by stationary points. We start recalling the notion of first inner variation, and then give an explicit formula to represent it.

Definition 2.2.13. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Given a map $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$ and a vector field $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, the first (inner) variation of $\mathcal{E}_s(\cdot,\Omega)$ at u and evaluated at X is defined as

$$\delta\mathcal{E}_s(u,\Omega)[X] := \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u\circ\phi_{-t},\Omega)\right]_{|t=0}\,,$$

where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the integral flow on $\mathbb{R}^n$ generated by X, i.e., for every $x \in \mathbb{R}^n$, the map $t \mapsto \phi_t(x)$ is defined as the unique solution of the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = X(\phi_t(x))\,, \\ \phi_0(x) = x\,. \end{cases}$$

The following representation result for $\delta\mathcal{E}_s$ was obtained in [81, Corollary 2.14] as a direct consequence of Corollary 2.2.12. We reproduce here the proof for completeness.

Proposition 2.2.14. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $G \subseteq \mathbb{R}^{n+1}_+$ an admissible bounded open set such that $\overline{\partial^0 G} \subseteq \Omega$. For each $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$, and each $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in $\partial^0 G$, we have*

$$\delta\mathcal{E}_s(u,\Omega)[X] = \frac{\boldsymbol{\delta}_s}{2}\int_G z^a\Big(|\nabla u^{\mathrm{e}}|^2\mathrm{div}\mathbf{X} - 2\sum_{i,j=1}^{n+1}(\partial_i u^{\mathrm{e}}\cdot\partial_j u^{\mathrm{e}})\partial_j\mathbf{X}_i\Big)\,\mathrm{d}\mathbf{x} + \frac{\boldsymbol{\delta}_s a}{2}\int_G z^{a-1}|\nabla u^{\mathrm{e}}|^2\mathbf{X}_{n+1}\,\mathrm{d}\mathbf{x}\,, \quad (2.2.22)$$

where $\mathbf{X} = (\mathbf{X}_1,\dots,\mathbf{X}_{n+1}) \in C^1(\overline{G};\mathbb{R}^{n+1})$ is any vector field compactly supported in $G\cup\partial^0 G$, and satisfying $\mathbf{X} = (X,0)$ on $\partial^0 G$.

Proof. Let $\mathbf{X} \in C^1(\overline{G},\mathbb{R}^{n+1})$ be an arbitrary vector field compactly supported in $G\cup\partial^0 G$ and satisfying $\mathbf{X} = (X,0)$ on $\partial^0 G$. We consider a compactly supported C^1-extension of $\mathbf{X}$ to the whole space $\mathbb{R}^{n+1}$, still denoted by $\mathbf{X}$, such that $\mathbf{X} = (X,0)$ on $\mathbb{R}^n\times\{0\}\simeq\mathbb{R}^n$. We define $\{\mathbf{\Phi}_t\}_{t\in\mathbb{R}}$ as the integral flow on $\mathbb{R}^{n+1}$ generated by $\mathbf{X}$. Observe that $\mathbf{\Phi}_t = (\phi_t,0)$ on $\mathbb{R}^n$, and $\mathrm{spt}(\mathbf{\Phi}_t - \mathrm{id}_{\mathbb{R}^{n+1}})\cap\overline{\mathbb{R}^{n+1}_+} \subseteq G\cup\partial^0 G$. Then, $v_t := u^{\mathrm{e}}\circ\mathbf{\Phi}_{-t} \in H^1(G;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ and $\mathrm{spt}(v_t - u^{\mathrm{e}}) \subseteq G\cup\partial^0 G$. By Corollary 2.2.12, we have

$$\mathbf{E}_s(v_t,G) - \mathbf{E}_s(u^{\mathrm{e}},G) \geqslant \mathcal{E}_s(v_t,\Omega) - \mathcal{E}_s(u,\Omega) \quad \forall t\in\mathbb{R}\,. \quad (2.2.23)$$

Since $v_t = u\circ\phi_{-t}$ on $\mathbb{R}^n$, dividing both sides of (2.2.23) by $t\neq 0$, and letting $t\uparrow 0$ and $t\downarrow 0$ leads to

$$\delta\mathcal{E}_s(u,\Omega)[X] = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_s(u^{\mathrm{e}}\circ\mathbf{\Phi}_{-t},G)\right]_{|t=0}\,. \quad (2.2.24)$$

On the other hand, standard computations (see e.g. [107, Chapter 2.2]) show that the right-hand side of (2.2.24) is equal to the right-hand side of (2.2.22). □

Definition 2.2.15. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. A map $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$ is said to be *stationary* in Ω if $\delta\mathcal{E}_s(u,\Omega) = 0$.

As we shall see in the next sections, stationarity is a crucial ingredient in the partial regularity theory since it implies the aforementioned monotonicity formula. This is the purpose of the following proposition whose proof follows exactly [81, Proof of Lemma 4.2] using vector fields in (2.2.22) of the form $\mathbf{X} = \eta(|\mathbf{x}-\mathbf{x}_0|)(\mathbf{x}-\mathbf{x}_0)$ with $\eta(t)\sim\chi_{[0,r]}(t)$.

Proposition 2.2.16. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. If $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$ is stationary in Ω, then for every $\mathbf{x}_0 = (x_0,0) \in \Omega\times\{0\}$, the "density function"*

$$r \in (0,\mathrm{dist}(x_0,\Omega^c)) \mapsto \mathbf{\Theta}_s(u^{\mathrm{e}},\mathbf{x}_0,r) := \frac{1}{r^{n-2s}}\mathbf{E}_s(u^{\mathrm{e}},B_r^+(\mathbf{x}_0))$$

is nondecreasing. Moreover,

$$\mathbf{\Theta}_s(u^{\mathrm{e}},\mathbf{x}_0,r) - \mathbf{\Theta}_s(u^{\mathrm{e}},\mathbf{x}_0,\rho) = \boldsymbol{\delta}_s\int_{B_r^+(\mathbf{x}_0)\setminus B_\rho^+(\mathbf{x}_0)} z^a\frac{|(\mathbf{x}-\mathbf{x}_0)\cdot\nabla u^{\mathrm{e}}|^2}{|\mathbf{x}-\mathbf{x}_0|^{n+2-2s}}\,\mathrm{d}\mathbf{x}$$

for every $0<\rho<r<\mathrm{dist}(x_0,\Omega^c)$.

As a straightforward consequence, we have

Corollary 2.2.17. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. If $u \in \widehat{H}^s(\Omega;\mathbb{R}^d)$ is stationary in Ω, then for every $x_0 \in \Omega$, the limit*

$$\boldsymbol{\Xi}_s(u,x_0) := \lim_{r\to 0} \boldsymbol{\Theta}_s(u^{\rm e},(x_0,0),r) \tag{2.2.25}$$

exists, and the function $\boldsymbol{\Xi}_s(u,\cdot):\Omega\to[0,\infty)$ is upper semicontinuous. In addition, for every $\mathbf{x}_0=(x_0,0)\in\Omega\times\{0\}$,

$$\boldsymbol{\Theta}_s(u^{\rm e},\mathbf{x}_0,r) - \boldsymbol{\Xi}_s(u,x_0) = \boldsymbol{\delta}_s \int_{B_r^+(\mathbf{x}_0)} z^a \frac{|(\mathbf{x}-\mathbf{x}_0)\cdot\nabla u^{\rm e}|^2}{|\mathbf{x}-\mathbf{x}_0|^{n+2-2s}}\,{\rm d}\mathbf{x} \tag{2.2.26}$$

for every $0<r<\operatorname{dist}(x_0,\Omega^c)$.

Proof. The existence of the limit in (2.2.25) and (2.2.26) are direct consequences of the monotonicity formula established in Proposition 2.2.16. Then the function $\boldsymbol{\Xi}_s(u,\cdot)$ is upper semicontinuous as a pointwise limit of a decreasing family of continuous functions. □

As we previously said, the monotonicity of the density function $r\mapsto\boldsymbol{\Theta}_s(u^{\rm e},\mathbf{x}_0,r)$ is one of the most important ingredients to obtain partial regularity. We shall see in the next sections that the density function relative to the nonlocal energy $\mathcal{E}_s$ also plays a role. For $u\in\widehat{H}^s(\Omega;\mathbb{R}^d)$ and a point $x\in\Omega$, we define the density function $r\in(0,\operatorname{dist}(x,\Omega^c))\mapsto\boldsymbol{\theta}_s(u,x,r)$ by setting

$$\boldsymbol{\theta}_s(u,x_0,r) := \frac{1}{r^{n-2s}}\mathcal{E}_s(u,D_r(x_0))\,. \tag{2.2.27}$$

Now we aim to show that one density function is small if and only the other one is also small at a comparable scale. This is the purpose of the following lemma.

Lemma 2.2.18. *Let $\Omega\subseteq\mathbb{R}^n$ be an open set, and $u\in\widehat{H}^s(\Omega;\mathbb{R}^d)\cap L^\infty(\mathbb{R}^n)$ be such that $\|u\|_{L^\infty(\mathbb{R}^n)}\leqslant M$. For every $\varepsilon>0$, there exists $\delta=\delta(n,s,M,\varepsilon)>0$ and $\alpha=\alpha(n,s,M,\varepsilon)\in(0,1/4]$ such that*

$$\boldsymbol{\Theta}_s(u^{\rm e},\mathbf{x}_0,r)\leqslant\delta \quad\Longrightarrow\quad \boldsymbol{\theta}_s(u,x_0,\alpha r)\leqslant\varepsilon$$

for every $\mathbf{x}_0=(x_0,0)\in\Omega\times\{0\}$ and $r>0$ satisfying $\overline{D}_r(x_0)\subseteq\Omega$.

Proof. Without loss of generality, we can assume that $x_0=0$. We give ourselves $\varepsilon>0$, and we shall choose the parameter $\alpha\in(0,1/4]$ later on. Using Lemma 2.2.7, we first estimate

$$\begin{aligned}\mathcal{E}_s(u,D_{\alpha r}) &\leqslant \frac{\gamma_{n,s}}{4}\iint_{D_{r/2}\times D_{r/2}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y + \frac{\gamma_{n,s}}{2}\iint_{D_{\alpha r}\times D^c_{r/2}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y\\ &\leqslant C_1\mathbf{E}_s(u^{\rm e},B_r^+) + 2M^2\gamma_{n,s}\iint_{D_{\alpha r}\times D^c_{r/2}}\frac{{\rm d}x{\rm d}y}{|x-y|^{n+2s}}\,,\end{aligned}$$

where $C_1=C_1(n,s)>0$. Observe that for $(x,y)\in D_{\alpha r}\times D^c_{r/2}$, we have $|x-y|\geqslant|y|-\alpha r\geqslant\frac{1}{2}|y|$, so that

$$2\gamma_{n,s}\iint_{D_{\alpha r}\times D^c_{r/2}}\frac{{\rm d}x{\rm d}y}{|x-y|^{n+2s}}\leqslant 2^{n+2s+1}\gamma_{n,s}\iint_{D_{\alpha r}\times D^c_{r/2}}\frac{{\rm d}x{\rm d}y}{|y|^{n+2s}} = C_2\alpha^n r^{n-2s}\,,$$

where $C_2=C_2(n,s)>0$. Consequently,

$$\boldsymbol{\theta}_s(u,0,\alpha r)\leqslant\frac{C_1}{\alpha^{n-2s}}\boldsymbol{\Theta}_s(u^{\rm e},0,r)+C_2M^2\alpha^{2s}\,.$$

Choosing

$$\alpha = \min\Big\{1/4, \Big(\frac{\varepsilon}{2C_2M^2}\Big)^{1/2s}\Big\} \quad \text{and} \quad \delta := \frac{\alpha^{n-2s}\varepsilon}{2C_1}\,,$$

provides the desired conclusion. □

Corollary 2.2.19. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. If $u \in \widehat{H}^s(\Omega;\mathbb{R}^d) \cap L^\infty(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \boldsymbol{\theta}_s(u,x_0,r) = 0 \quad \Longleftrightarrow \quad \lim_{r\to 0} \boldsymbol{\Theta}_s(u^{\mathrm{e}},\mathbf{x}_0,r) = 0$$

for every $\mathbf{x}_0 = (x_0,0) \in \Omega \times \{0\}$.

Proof. By Lemma 2.2.8, we have

$$\boldsymbol{\Theta}_s(u^{\mathrm{e}},\mathbf{x}_0,r) \leqslant C\boldsymbol{\theta}_s(u,x_0,2r)\,,$$

for a constant $C > 0$ depending only on n and s, and implication $\Longrightarrow$ follows. The reverse implication is a straightforward application of Lemma 2.2.18. □

2.2.6 Energy monotonicity and mean oscillation estimates

In the light of Proposition 2.2.16, the purpose of this section is to show a mean oscillation estimate for maps having a nondecreasing density function at every point. For $v \in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$, a point $\mathbf{x} \in \partial^0 B_R^+$, and $r \in (0, R-|\mathbf{x}|)$, we keep the notation

$$\boldsymbol{\Theta}_s(v,\mathbf{x}_0,r) := \frac{1}{r^{n-2s}}\mathbf{E}_s(v,B_r^+(\mathbf{x}_0))\,.$$

The main estimate is the following.

Lemma 2.2.20. *Let $v \in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ and $\zeta \in \mathscr{D}(D_{5R/8})$ be such that $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ in $D_{R/2}$, and $|\nabla\zeta| \leqslant LR^{-1}$ for some constant $L > 0$. Assume that for every $\mathbf{x} \in \partial^0 B_R^+$, the density function $r \in (0,R-|\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(v,\mathbf{x},r)$ is nondecreasing. Then $(\zeta v)_{|\mathbb{R}^n}$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ and*

$$[\zeta v]^2_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C_L\big(\boldsymbol{\Theta}_s(v,0,R) + R^{2s-2-n}\|v\|^2_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}\big)$$

for a constant $C_L = C(L,n,s)$.

Before proving this lemma, let us recall that $u \in L^1(D_R)$ belongs to $\mathrm{BMO}(D_R)$ if

$$[u]_{\mathrm{BMO}(D_R)} := \sup_{D_r(y)\subseteq D_R} \fint_{D_r(y)} |u-(u)_{y,r}|\,\mathrm{d}x < +\infty\,,$$

where $(u)_{y,r}$ denotes the average of u over the ball $D_r(y)$. To prove Lemma 2.2.20, we shall make use of the well-known John-Nirenberg inequality, see e.g. [51, Section 6.3].

Lemma 2.2.21. *Let $u \in \mathrm{BMO}(D_R)$. For every $p \in [1,\infty)$, there exists a constant $C_p = C_p(n,p)$ such that*

$$[u]^p_{\mathrm{BMO}(D_R)} \leqslant \sup_{D_r(y)\subseteq D_R} \fint_{D_r(y)} |u-(u)_{y,r}|^p\,\mathrm{d}x \leqslant C_p[u]^p_{\mathrm{BMO}(D_R)}\,.$$

Proof of Lemma 2.2.20. Step 1. Rescaling variables, we may assume that $R = 1$. Let us fix an arbitrary ball $D_r(y) \subseteq D_1$ with $y \in D_{7/8}$ and $0 < r \leqslant 1/8$. Using the Poincaré inequality in Lemma 2.2.6 and the monotonicity assumption on $\boldsymbol{\Theta}_s(v, \mathbf{x}, \cdot)$, we estimate

$$\frac{1}{r^n}\int_{D_r(y)} |v - (v)_{y,r}|\,\mathrm{d}x \leqslant C\sqrt{\boldsymbol{\Theta}_s(v, \mathbf{y}, r)} \leqslant C\sqrt{\boldsymbol{\Theta}_s(v, \mathbf{y}, 1/8)} \leqslant C\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)}\,,$$

where $\mathbf{y} = (y, 0)$ and $C = C(n, s)$. In particular, $v_{|D_{7/8}}$ belongs to $\mathrm{BMO}(D_{7/8})$, and

$$[v]_{\mathrm{BMO}(D_{7/8})} \leqslant C\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)}\,. \tag{2.2.28}$$

By the John-Nirenberg inequality in Lemma 2.2.21, inequality (2.2.28), the continuity of the trace operator (see Section 2.2.3), and Hölder inequality, it follows that

$$\begin{aligned}\|v\|_{L^n(D_{7/8})} &\leqslant \big\|v - (v)_{0,7/8}\big\|_{L^n(D_{7/8})} + C\|v\|_{L^1(D_{7/8})} \\ &\leqslant C\Big([v]_{\mathrm{BMO}(D_{7/8})} + \|v\|_{L^1(D_1)}\Big) \leqslant C\Big(\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)} + \|v\|_{L^2(B_1^+, |z|^a\mathrm{d}\mathbf{x})}\Big)\,.\end{aligned} \tag{2.2.29}$$

Step 2. Let us now consider a ball $D_r(y) \subseteq D_{7/8}$ with $y \in D_{3/4}$ and $0 < r \leqslant 1/8$. Since

$$|\zeta v - (\zeta v)_{y,r}| \leqslant |\zeta v - \zeta(v)_{y,r}| + |\zeta(v)_{y,r} - (\zeta v)_{y,r}| \leqslant |v - (v)_{y,r}| + Lr\fint_{D_r(y)} |v|\,\mathrm{d}x \quad \text{on } D_{7/8}\,,$$

we can deduce from (2.2.28) and (2.2.29) that

$$\begin{aligned}\frac{1}{r^n}\int_{D_r(y)} |\zeta v - (\zeta v)_{y,r}|\,\mathrm{d}x &\leqslant C_L\Big(\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)} + r^{1-n}\|v\|_{L^1(D_r(y))}\Big) \\ &\leqslant C_L\Big(\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)} + \|v\|_{L^n(D_{7/8})}\Big) \leqslant C_L\Big(\sqrt{\boldsymbol{\Theta}_s(v, 0, 1)} + \|v\|_{L^2(B_1^+, |z|^a\mathrm{d}\mathbf{x})}\Big)\,,\end{aligned}$$

for a constant $C_L = C(L, n, s)$.

Next, for a ball $D_r(y)$ with $y \notin D_{3/4}$ and $0 < r \leqslant 1/8$, we have

$$\frac{1}{r^n}\int_{D_r(y)} |\zeta v - (\zeta v)_{y,r}|\,\mathrm{d}x = 0\,,$$

since ζ is supported in $D_{5/8}$.

Finally, for a ball $D_r(y)$ with $r > 1/8$, we estimate

$$\frac{1}{r^n}\int_{D_r(y)} |\zeta v - (\zeta v)_{y,r}|\,\mathrm{d}x \leqslant C\int_{D_1} |\zeta v|\,\mathrm{d}x \leqslant C\|v\|_{L^1(D_1)} \leqslant C\|v\|_{L^2(B_1^+, |z|^a\mathrm{d}\mathbf{x})}\,,$$

which completes the proof. □

Corollary 2.2.22. *Let $u \in \widehat{H}^s(D_{2R}; \mathbb{R}^d)$ and $\zeta \in \mathscr{D}(D_{5R/8})$ be as in Lemma 2.2.20. Assume that for every $\mathbf{x} \in \partial^0 B_R^+$, the density function $r \in (0, 2R - |\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}}, \mathbf{x}, r)$ is nondecreasing. Then ζu belongs to $\mathrm{BMO}(\mathbb{R}^n)$ and*

$$[\zeta u]^2_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C_L\big(\boldsymbol{\theta}_s(u, 0, 2R) + R^{-n}\|u\|^2_{L^2(D_{2R})}\big)\,,$$

for a constant $C_L = C(L, n, s) > 0$.

Proof. Apply Lemma 2.2.20 to u^{e} in B_R^+, and then conclude with the help of Lemma 2.2.8. □

2.3 Fractional harmonic maps and weighted harmonic maps with free boundary

In this section, our goal is to review in details the notion of weakly s-harmonic maps, the associated Euler-Lagrange equation, and more importantly to present its characterization in terms of fractional (nonlocal) conservation laws. We shall also prove at the end of this section that the fractional harmonic extension of an s-harmonic map satisfies a suitable (degenerate) partially free boundary condition, in the spirit of the classical harmonic map system with partially free boundary.

2.3.1 Fractional harmonic maps into spheres and conservation laws

Definition 2.3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. A map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is said to be *a weakly s-harmonic map* in Ω (with values in $\mathbb{S}^{d-1}$) if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s\Big(\frac{u+t\varphi}{|u+t\varphi|},\Omega\Big)\right]_{|t=0} = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega;\mathbb{R}^d)\,.$$

If u is also stationary in Ω (in the sense of Definition 2.2.15), we say that u is *a stationary weakly s-harmonic map* in Ω.

Definition 2.3.2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. A map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is said to be *a minimizing s-harmonic map* in Ω (with values in $\mathbb{S}^{d-1}$) if

$$\mathcal{E}_s(u,\Omega) \leqslant \mathcal{E}_s(w,\Omega)$$

for every $w \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ such that $\mathrm{spt}(u-w)$ is compactly included in Ω.

Remark 2.3.3. A minimizing s-harmonic map in Ω is obviously a critical point with respect to both inner and (constrained) outer variations of the energy. In other words, if u is a minimizing s-harmonic map in Ω, then u is also a stationary weakly s-harmonic map in Ω.

Remark 2.3.4. If $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is a weakly s-harmonic map in Ω (stationary, minimizing, respectively), then u is also weakly s-harmonic in Ω' (stationary, minimizing, respectively) for any open subset $\Omega' \subseteq \Omega$. It can be directly checked from the definitions, or one can rely on the Euler-Lagrange equation presented below and Remark 2.2.3.

Proposition 2.3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. A map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is weakly s-harmonic in Ω if and only if*

$$\big\langle(-\Delta)^s u,\varphi\big\rangle_\Omega = 0 \tag{2.3.1}$$

for every $\varphi \in H^s_{00}(\Omega;\mathbb{R}^d)$ such that $\mathrm{spt}(\varphi) \subseteq \Omega$ and $\varphi(x) \in \mathrm{Tan}(u(x),\mathbb{S}^{d-1})$ for a.e. $x \in \Omega$. Equivalently,

$$(-\Delta)^s u(x) = \Big(\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\Big)u(x) \quad \text{in } \mathscr{D}'(\Omega)\,. \tag{2.3.2}$$

Proof. Let $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$, fix $\varphi \in \mathscr{D}(\Omega;\mathbb{R}^d)$, and notice that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{u+t\varphi}{|u+t\varphi|}\Big)\right]_{|t=0} = \varphi - (u\cdot\varphi)u \in H^s_{00}(\Omega;\mathbb{R}^d)\,.$$

Hence,

$$\Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s\Big(\frac{u+t\varphi}{|u+t\varphi|},\Omega\Big)\Big]_{|t=0} = \langle(-\Delta)^s u,\varphi\rangle_\Omega - \langle(-\Delta)^s u,(u\cdot\varphi)u\rangle_\Omega\,.$$

On the other hand, since $|u|^2=1$, we have

$$\begin{aligned}(u(x)-u(y))\cdot((u(x)\cdot\varphi(x))u(x)-(u(y)\cdot\varphi(y))u(y))\\ =\frac{1}{2}|u(x)-u(y)|^2u(x)\cdot\varphi(x)+\frac{1}{2}|u(x)-u(y)|^2u(y)\cdot\varphi(y)\,,\end{aligned}$$

and it follows that

$$\langle(-\Delta)^s u,(u\cdot\varphi)u\rangle_\Omega = \int_\Omega\Big(\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\Big)u(x)\cdot\varphi(x)\,\mathrm{d}x\,. \tag{2.3.3}$$

Consequently, u is weakly s-harmonic in Ω if and only if (2.3.2) holds.

By approximation, (2.3.2) also holds for any test function $\varphi\in H^s_{00}(\Omega;\mathbb{R}^d)\cap L^\infty(\mathbb{R}^n)$ compactly supported in Ω. In view of of the right-hand side of (2.3.2), (2.3.1) clearly holds for every $\varphi\in H^s_{00}(\Omega;\mathbb{R}^d)\cap L^\infty(\mathbb{R}^n)$ compactly supported in Ω and satisfying $\varphi\cdot u=0$. By a standard truncation argument, it implies that (2.3.1) holds for every $\varphi\in H^s_{00}(\Omega;\mathbb{R}^d)$ compactly supported in Ω and satisfying $\varphi\cdot u=0$.

The other way around, if (2.3.1) holds, then the map $\varphi-(u\cdot\varphi)u$ with $\varphi\in\mathscr{D}(\Omega;\mathbb{R}^d)$ is admissible, and (2.3.1) combined with (2.3.3) shows that (2.3.2) holds, i.e., u is weakly s-harmonic in Ω. □

Remark 2.3.6. The variational equation (2.3.1) corresponds to the weak formulation of the implicit equation

$$(-\Delta)^s u\perp \mathrm{Tan}(u,\mathbb{S}^{d-1})\quad\text{in }\Omega\,,$$

and in equation (2.3.2), the Lagrange multiplier associated with the $\mathbb{S}^{d-1}$-constraint is made explicit.

Remark 2.3.7. A weakly s-harmonic map u in Ω which is smooth in Ω, is stationary in Ω. Indeed, if $X\in C^1(\Omega;\mathbb{R}^n)$ is compactly supported in Ω, the smoothness of u implies that

$$\delta\mathcal{E}_s(u,\Omega)[X]=\langle(-\Delta)^s u, X\cdot\nabla u\rangle_\Omega\,.$$

Since $|u|^2=1$, we have $(X\cdot\nabla u)\cdot u=0$, and thus $\delta\mathcal{E}_s(u,\Omega)[X]=0$.

Now we rewrite the Euler-Lagrange equation (2.3.2) in a more compact form using the fractional s-gradient $\mathrm{d}_s u$ defined in Section 2.2.2. More precisely, if $u=:(u^1,\dots,u^d)$, then

$$\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y=\sum_{j=1}^d\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|u^j(x)-u^j(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y=\sum_{j=1}^d|\mathrm{d}_s u^j|^2=:|\mathrm{d}_s u|^2\,,$$

according to (2.2.4) and (2.2.5). We can thus rephrase Proposition 2.3.5 as follows: $u\in\widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is weakly s-harmonic in Ω if and only if

$$(-\Delta)^s u(x)=|\mathrm{d}_s u|^2u\quad\text{in }\mathscr{D}'(\Omega)\,. \tag{2.3.4}$$

Our aim is to further rewrite equation (2.3.4), or more precisely its right-hand side, to reveal the fractional "div-curl structure" of Section 2.2.2 in the spirit of the well known div-curl structure hidden in the classical equation for harmonic maps into spheres [60]. Following [77], the starting point is to notice that for each $i, j \in \{1, \dots, d\}$,

$$\begin{aligned} |\mathrm{d}_s u^j|^2(x) u^i(x) &= \int_{\mathbb{R}^n} \frac{u^i(x)\mathrm{d}_s u^j(x,y)\mathrm{d}_s u^j(x,y)}{|x-y|^n}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \frac{u^i(x)\mathrm{d}_s u^j(x,y) - u^j(x)\mathrm{d}_s u^i(x,y)}{|x-y|^n}\,\mathrm{d}_s u^j(x,y)\,\mathrm{d}y \\ &\qquad + (\mathrm{d}_s u^i \odot \mathrm{d}_s u^j)(x) u^j(x)\,. \end{aligned} \tag{2.3.5}$$

Then, since $|u|^2 = 1$, we have

$$\begin{aligned} \sum_{j=1}^d (\mathrm{d}_s u^i \odot \mathrm{d}_s u^j)(x) u^j(x) &= \sum_{j=1}^d \frac{\gamma_{n,s}}{2} \int_{\mathbb{R}^n} \frac{(u^j(x)-u^j(y))u^j(x)}{|x-y|^{n+2s}} (u^i(x)-u^i(y))\,\mathrm{d}y \\ &= \frac{\gamma_{n,s}}{4} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} (u^i(x)-u^i(y))\,\mathrm{d}y\,. \end{aligned} \tag{2.3.6}$$

We can now introduce for $i, j \in \{1, \dots, d\}$,

$$\boldsymbol{\Omega}^{ij}(x,y) := u^i(x)\mathrm{d}_s u^j(x,y) - u^j(x)\mathrm{d}_s u^i(x,y) \in L^2_{\mathrm{od}}(\Omega)\,, \tag{2.3.7}$$

and

$$T^i(x) := \frac{\gamma_{n,s}}{4} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} (u^i(x)-u^i(y))\,\mathrm{d}y \in L^1(\Omega)\,, \tag{2.3.8}$$

to derive from (2.3.5) and (2.3.6) the following reformulation of equation (2.3.4).

Lemma 2.3.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. A map $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is weakly s-harmonic in Ω if and only if*

$$(-\Delta)^s u^i = \Big(\sum_{j=1}^d \boldsymbol{\Omega}^{ij} \odot \mathrm{d}_s u^j \Big) + T^i \quad \text{in } \mathscr{D}'(\Omega) \tag{2.3.9}$$

for every $i = 1, \dots, d$, where $\boldsymbol{\Omega}^{ij}$ and T^i are given by (2.3.7) and (2.3.8), respectively.

Remark 2.3.9. The presence of the extra term T^i in (2.3.9), compared with the classical harmonic map equation (see [60]), is essentially due to the fact that the s-gradient $\mathrm{d}_s u$ is not tangent to the target sphere.

The fundamental observation made in [77, Lemma 3.1] for $\Omega = \mathbb{R}$ and $s = 1/2$ is a characterization of the 1/2-harmonic map equation in terms of nonlocal conservation laws satisfied by the $\boldsymbol{\Omega}^{ij}$'s (thus extending [105] to the fractional setting). In the following proposition, we slightly generalize this result to a domain of arbitrary dimension and $s \in (0,1)$. The proof remains essentially the same, and we provide it for the reader's convenience.

Proposition 2.3.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. A map $u \in \widehat{H}^s(\Omega; \mathbb{S}^{d-1})$ is weakly s-harmonic in Ω if and only if*

$$\mathrm{div}_s\, \boldsymbol{\Omega}^{ij} = 0 \quad \text{in } H^{-s}(\Omega) \tag{2.3.10}$$

for each $i, j \in \{1, \dots, d\}$, where $\boldsymbol{\Omega}^{ij}$ is given by (2.3.7).

Proof. Step 1. Assume that u is a weakly s-harmonic in Ω, and let us compute $\operatorname{div}_s \boldsymbol{\Omega}^{ij}$. For $\varphi \in \mathscr{D}(\Omega)$, we have

$$\int_{\mathbb{R}^n} \boldsymbol{\Omega}^{ij} \odot \mathrm{d}_s\varphi \, \mathrm{d}x = \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \big(u^i(x)\mathrm{d}_s u^j(x,y)\mathrm{d}_s\varphi(x,y) - u^j(x)\mathrm{d}_s u^i(x,y)\mathrm{d}_s\varphi(x,y)\big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}\,.$$

An elementary computation shows

$$\begin{cases} u^i(x)\mathrm{d}_s\varphi(x,y) = \mathrm{d}_s(u^i\varphi)(x,y) - \varphi(y)\mathrm{d}_s u^i(x,y) \\ u^j(x)\mathrm{d}_s\varphi(x,y) = \mathrm{d}_s(u^j\varphi)(x,y) - \varphi(y)\mathrm{d}_s u^j(x,y) \end{cases},$$

so that

$$\int_{\mathbb{R}^n} \boldsymbol{\Omega}^{ij} \odot \mathrm{d}_s\varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{d}_s u^j \odot \mathrm{d}_s(u^i\varphi)\, \mathrm{d}x - \int_{\mathbb{R}^n} \mathrm{d}_s u^i \odot \mathrm{d}_s(u^j\varphi)\,\mathrm{d}x\,.$$

Since $u^j\varphi$ and $u^i\varphi$ belong to $H^s_{00}(\Omega)$, we infer from Proposition 2.2.4 and equation (2.3.4) that

$$\begin{aligned}\int_{\mathbb{R}^n} \boldsymbol{\Omega}^{ij} \odot \mathrm{d}_s\varphi \, \mathrm{d}x &= \langle(-\Delta)^s u^j, u^i\varphi\rangle_\Omega - \langle(-\Delta)^s u^i, u^j\varphi\rangle_\Omega \\ &= \int_\Omega |\mathrm{d}_s u|^2 u^j u^i \varphi\,\mathrm{d}x - \int_\Omega |\mathrm{d}_s u|^2 u^i u^j\varphi\,\mathrm{d}x = 0\,.\end{aligned} \tag{2.3.11}$$

Therefore $\operatorname{div}_s \boldsymbol{\Omega}^{ij} = 0$ in $\mathscr{D}'(\Omega)$, and by approximation also in $H^{-s}(\Omega)$ (see (2.2.2)).

Step 2. We assume that (2.3.10) holds, and we aim to prove that (2.3.4) holds. We fix $\varphi \in \mathscr{D}(\Omega)$, and we set $\psi := \varphi - (u\cdot\varphi)u \in H^s_{00}(\Omega;\mathbb{R}^d)$, which satisfies $\psi\cdot u = 0$ a.e. in $\mathbb{R}^n$. As in the proof of Proposition 2.3.5, proving (2.3.4) reduces to show that

$$\langle(-\Delta)^s u, \psi\rangle_\Omega = 0\,.$$

Using $|u|^2 = 1$, we first observe that

$$\langle(-\Delta)^s u, \psi\rangle_\Omega = \sum_{i=1}^d \langle(-\Delta)^s u^i, \psi^i\rangle_\Omega = \sum_{i,j=1}^d \langle(-\Delta)^s u^i, (\psi^i u^j)u^j\rangle_\Omega\,.$$

Since $\psi^i u^j \in H^s_{00}(\Omega)$, we obtain as in (2.3.11),

$$\begin{aligned}\langle(-\Delta)^s u^i, (\psi^i u^j)u^j\rangle_\Omega &= \langle(-\Delta)^s u^j, (\psi^i u^j)u^i\rangle_\Omega - \int_{\mathbb{R}^n} \boldsymbol{\Omega}^{ij}\odot \mathrm{d}_s(\psi^i u^j)\,\mathrm{d}x \\ &= \langle(-\Delta)^s u^j, (\psi^i u^j)u^i\rangle_\Omega\end{aligned}$$

for every $i,j \in \{1,\dots,d\}$, thanks to (2.3.10). Therefore,

$$\langle(-\Delta)^s u, \psi\rangle_\Omega = \sum_{i,j=1}^d \langle(-\Delta)^s u^j, (\psi^i u^j)u^i\rangle_\Omega = \sum_{j=1}^d \langle(-\Delta)^s u^j, (\psi\cdot u)u^j\rangle_\Omega = 0\,,$$

and the proof is complete. □

2.3.2 Weighted harmonic maps with free boundary

Definition 2.3.11. Let $G \subseteq \mathbb{R}^{n+1}_+$ be a bounded admissible open set. A map $v \in H^1(G;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ satisfying $v(\mathbf{x}) \in \mathbb{S}^{d-1}$ is said to be *a weighted weakly harmonic map in G with respect to the partially free boundary condition* $v(\partial^0 G) \subseteq \mathbb{S}^{d-1}$ if

$$\int_G z^a \nabla v \cdot \nabla \Phi \,\mathrm{d}\mathbf{x} = 0 \tag{2.3.12}$$

for every $\Phi \in H^1(G;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ such that $\Phi = 0$ on $\partial^+ G$ and $\Phi(\mathbf{x}) \in \mathrm{Tan}(v(\mathbf{x}),\mathbb{S}^{d-1})$ for a.e. $\mathbf{x} \in \partial^0 G$. In short, we shall say that v is a weighted weakly harmonic map with free boundary in G.

Remark 2.3.12. If $v \in H^1(G;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ is a weighted weakly harmonic map with free boundary in G, then (2.3.12) means that v satisfies in the weak sense

$$\begin{cases} \mathrm{div}(z^a\nabla v) = 0 & \text{in } G\,, \\ z^a \dfrac{\partial v}{\partial \nu} \perp \mathrm{Tan}(v,\mathbb{S}^{d-1}) & \text{on } \partial^0 G\,. \end{cases} \tag{2.3.13}$$

In particular, v is smooth in G by standard elliptic regularity.

In view of Remark 2.3.6, equation (2.3.13) above, and Lemma 2.2.11, it is clear that weighted weakly harmonic maps with free boundary and weakly s-harmonic maps are intimately related. This relation is made precise in the following proposition (see [80, Proposition 4.6]).

Proposition 2.3.13. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. If a map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ is a weakly s-harmonic map in Ω, then its extension u^{e} given by (2.2.12) is a weighted weakly harmonic map with free boundary in every bounded admissible open set $G \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\partial^0 G} \subseteq \Omega$.*

Proof. Let us first assume that u is a weakly s-harmonic map in Ω, and let $G \subseteq \mathbb{R}^{n+1}_+$ be bounded admissible open set such that $\overline{\partial^0 G} \subseteq \Omega$. Let $\Phi \in H^1(G;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ such that $\Phi = 0$ on $\partial^+ G$, and $\Phi \cdot u = 0$ on $\partial^0 G$. We extend Φ by 0 to the whole half-space $\mathbb{R}^{n+1}_+$, and the resulting map, still denoted by Φ, belongs to $H^1(\mathbb{R}^{n+1}_+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$. In view of (2.2.16), $\Phi_{|\mathbb{R}^n} \in H^s_{00}(\Omega;\mathbb{R}^d)$, and $\mathrm{spt}(\Phi_{|\mathbb{R}^n}) \subseteq \Omega$. Since $\Phi_{|\mathbb{R}^n} \cdot u = 0$, we conclude from Lemma 2.2.11 and Proposition 2.3.5 that

$$\int_G z^a \nabla u^{\mathrm{e}} \cdot \nabla \Phi \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{n+1}_+} z^a \nabla u^{\mathrm{e}} \cdot \nabla \Phi \,\mathrm{d}\mathbf{x} = \frac{1}{\boldsymbol{\delta}_s}\langle(-\Delta)^s u, \Phi_{|\mathbb{R}^n}\rangle_\Omega = 0\,.$$

Hence, u^{e} is indeed a weighted weakly harmonic map with free boundary in G. □

2.4 Small energy Hölder regularity

In this section, we present the main epsilon-regularity theorem asserting that under a certain smallness assumption of the energy in a ball, a weakly s-harmonic map satisfying the monotonicity formula is Hölder continuous in a smaller ball. Hölder regularity will be improved to Lipschitz regularity in the next section with an explicit control on the Lipschitz norm in terms of the energy.

Theorem 2.4.1. *There exist constants* $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0(n,s) > 0$ *and* $\beta_0 = \beta_0(n,s) \in (0,1)$ *such that the following holds. Let* $u \in \widehat{H}^s(D_R;\mathbb{S}^{d-1})$ *be a weakly* s*-harmonic map in* D_R *such that the function* $r \in (0, R-|\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}}, \mathbf{x}, r)$ *is nondecreasing for every* $\mathbf{x} \in \partial^0 B_R^+$*. If*

$$\boldsymbol{\theta}_s(u,0,R) \leqslant \boldsymbol{\varepsilon}_0\,, \tag{2.4.1}$$

then $u \in C^{0,\beta_0}(D_{R/2})$ *and*

$$R^{2\beta_0}[u]^2_{C^{0,\beta_0}(D_{R/2})} \leqslant C\boldsymbol{\theta}_s(u,0,R)\,, \tag{2.4.2}$$

for a constant $C = C(n,s)$*.*

For what follows, it is useful to translate the epsilon-regularity theorem above only in terms of the extension. This is the purpose of the following corollary.

Corollary 2.4.1. *There exist three constants* $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_1(n,s) > 0$*,* $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}_1(n,s) \in (0,1)$*,* $\beta_1 = \beta_1(n,s) \in (0,1)$ *such that the following holds. Let* $u \in \widehat{H}^s(D_{2R};\mathbb{S}^{d-1})$ *be a weakly* s*-harmonic map in* D_{2R} *such that the function* $r \in (0,2R-|\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}},\mathbf{x},r)$ *is nondecreasing for every* $\mathbf{x} \in \partial^0 B^+_{2R}$*. If*

$$\boldsymbol{\Theta}_s(u^{\mathrm{e}},0,R) \leqslant \boldsymbol{\varepsilon}_1\,, \tag{2.4.3}$$

then $u^{\mathrm{e}} \in C^{0,\beta_1}(B^+_{\boldsymbol{\kappa}_1 R})$ *and*

$$R^{2\beta_1}[u^{\mathrm{e}}]^2_{C^{0,\beta_1}(B^+_{\boldsymbol{\kappa}_1 R})} \leqslant C\,,$$

for a constant $C = C(n,s)$*.*

Proof. We consider the constant $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0(n,s) > 0$ given by Theorem 2.4.1. Since $|u| \equiv 1$, we obtain from Lemma 2.2.18 the existence of $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_1(n,s) > 0$ and $\alpha = \alpha(n,s) \in (0,1/4]$ such that the condition $\boldsymbol{\Theta}_s(u^{\mathrm{e}},0,R) \leqslant \boldsymbol{\varepsilon}_1$ implies $\boldsymbol{\theta}_s(u,0,\alpha R) \leqslant \boldsymbol{\varepsilon}_0$. In turn, Theorem 2.4.1 tells us that $u \in C^{0,\beta_0}(D_{\alpha R/2})$. Then Lemma 2.2.10 implies that $u^{\mathrm{e}} \in C^{0,\beta_1}(B^+_{\boldsymbol{\kappa}_1 R})$ with $\beta_1 := \min(\beta_0, s)$ and $\boldsymbol{\kappa}_1 := \alpha/8$. Moreover, combining (2.2.19) and (2.4.2) leads to

$$\begin{aligned} R^{2\beta_1}[u^{\mathrm{e}}]^2_{C^{0,\beta_1}(B^+_{\boldsymbol{\kappa}_1 R})} \leqslant C\big(R^{2\beta_1}[u]^2_{C^{0,\beta_1}(D_{\alpha R/2})}+1\big) \leqslant C\big(R^{2\beta_0}[u]^2_{C^{0,\beta_0}(D_{\alpha R/2})}+1\big) \\ \leqslant C\big(\boldsymbol{\theta}_s(u,0,\alpha R)+1\big) \leqslant C\,, \end{aligned}$$

and the proof is complete. □

Remark 2.4.2. In the case $n \leqslant 2s$, the function $r \in (0,R-|\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}},\mathbf{x},r)$ is nondecreasing for every $u \in \widehat{H}^s(D_R;\mathbb{R}^d)$. In other words, in the case $n \leqslant 2s$, Theorem 2.4.1 and Corollary 2.4.1 apply to arbitrary weakly s-harmonic maps. Moreover, in the case $n = 1$ and $s \in (0,1/2)$ (i.e., $n < 2s$), the conclusions of Theorem 2.4.1 and Corollary 2.4.1 apply even without the smallness assumptions (2.4.1) or (2.4.3), since it follows from the classical imbedding $H^s(\mathbb{R}) \hookrightarrow C^{0,s-1/2}(\mathbb{R})$. For our purposes, it is convenient to state it suitably. This is the object of the proposition below, whose proof is postponed to the end of Section 2.4.1.

Proposition 2.4.3. *Assume that* $n = 1$ *and* $s \in (1/2,1)$*. If* $u \in \widehat{H}^s(D_R;\mathbb{R}^d) \cap L^\infty(\mathbb{R}^n)$*, then* $u \in C^{0,s-1/2}(D_{R/2})$ *and*

$$R^{2s-1}[u]^2_{C^{0,s-1/2}(D_{R/2})} \leqslant C\boldsymbol{\theta}_s(u,0,R)\,, \tag{2.4.4}$$

for a constant $C = C(s)$*.*

2.4.1 Proof of Theorem 2.4.1 and Proposition 2.4.3

The key point to prove Theorem 2.4.1 is to obtain a geometric decay of the energy in small balls. Then Hölder continuity follows classically from Campanato's criterion. The purpose of the next proposition, very much inspired by [39, Proposition 3.1], is exactly to show such decay.

Proposition 2.4.4. *Assume that $n \geqslant 2s$. There exist two constants $\boldsymbol{\varepsilon}_* = \boldsymbol{\varepsilon}_*(n,s) > 0$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(n,s) \in (0,1/4)$ such that the following holds. Let $u \in \widehat{H}^s(D_1;\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_1 such that the function $r \in (0,1-|\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\rm e},\mathbf{x},r)$ is nondecreasing for every $\mathbf{x} \in \partial^0 B_1^+$. If*

$$\mathcal{E}_s(u,D_1) \leqslant \boldsymbol{\varepsilon}_*\,,$$

then

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(u,D_{\boldsymbol{\tau}}) \leqslant \frac{1}{2}\mathcal{E}_s(u,D_1)\,.$$

Proof. We fix the constant $\boldsymbol{\tau} \in (0,1/4)$ that will be specified later on. We proceed by contradiction assuming that there exists a sequence $\{u_k\}$ of stationary weakly s-harmonic maps in D_1 satisfying

$$\varepsilon_k^2 := \mathcal{E}_s(u_k,D_1) \underset{k\to\infty}{\longrightarrow} 0\,,$$

and

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(u_k,D_{\boldsymbol{\tau}}) > \frac{1}{2}\mathcal{E}_s(u_k,D_1)\,. \tag{2.4.5}$$

(Note that this later condition ensures that $\varepsilon_k > 0$.) Then we consider the (expanded) map

$$w_k := \frac{u_k - (u_k)_{0,1}}{\varepsilon_k} \in \widehat{H}^s(D_1;\mathbb{R}^d) \cap L^\infty(\mathbb{R}^n)\,,$$

which satisfies

$$\fint_{D_1} w_k\,{\rm d}x = 0 \quad \text{and} \quad \mathcal{E}_s(w_k,D_1) = 1\,.$$

Assumption (2.4.5) also rewrites

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w_k,D_{\boldsymbol{\tau}}) > \frac{1}{2}\,. \tag{2.4.6}$$

By Poincaré inequality in $H^s(D_1)$, we have

$$\|w_k\|^2_{L^2(D_1)} \leqslant C\mathcal{E}_s(w_k,D_1) \leqslant C\,.$$

Therefore $\{w_k\}$ is bounded in $\widehat{H}^s(D_1;\mathbb{R}^d)$, so that we can find a (not relabeled) subsequence and $w \in \widehat{H}^s(D_1;\mathbb{R}^d)$ such that $w_k \rightharpoonup w$ weakly in $\widehat{H}^s(D_1)$ and $w_k \to w$ strongly in $L^2(D_1)$ (see Remark 2.2.2). In particular, $\|w\|_{L^2(D_1)} \leqslant C$. By lower semicontinuity of the energy $\mathcal{E}_s(\cdot,D_1)$, we also have $\mathcal{E}_s(w,D_1) \leqslant 1$ (see again Remark 2.2.2).

Recalling that u_k satisfies

$$\big\langle(-\Delta)^s u_k,\varphi\big\rangle_{D_1} = \int_{D_1} |{\rm d}_s u_k|^2 u_k\cdot\varphi\,{\rm d}x \qquad \forall\varphi \in \mathscr{D}(D_1;\mathbb{R}^d)\,,$$

we obtain in terms of w_k,

$$\big\langle(-\Delta)^s w_k,\varphi\big\rangle_{D_1} = \varepsilon_k\int_{D_1} |{\rm d}_s w_k|^2 u_k\cdot\varphi\,{\rm d}x \qquad \forall\varphi \in \mathscr{D}(D_1;\mathbb{R}^d)\,. \tag{2.4.7}$$

Since $|u_k| \equiv 1$, it leads to

$$\left|\left\langle(-\Delta)^s w_k, \varphi\right\rangle_{D_1}\right| \leqslant \varepsilon_k \left\| |\mathrm{d}_s w_k|^2 \right\|_{L^1(D_1)} \|\varphi\|_{L^\infty(D_1)} \\ \leqslant 2\varepsilon_k \mathcal{E}_s(w_k, D_1) \|\varphi\|_{L^\infty(D_1)} = 2\varepsilon_k \|\varphi\|_{L^\infty(D_1)} \underset{k\to\infty}{\longrightarrow} 0$$

for every $\varphi \in \mathscr{D}(D_1; \mathbb{R}^d)$. On the other hand, the weak convergence in $\widehat{H}^s(D_1)$ of w_k towards w implies that

$$\left\langle(-\Delta)^s w_k, \varphi\right\rangle_{D_1} \underset{k\to\infty}{\longrightarrow} \left\langle(-\Delta)^s w, \varphi\right\rangle_{D_1} \qquad \forall \varphi \in \mathscr{D}(D_1; \mathbb{R}^d)\,.$$

As a consequence, w satisfies

$$(-\Delta)^s w = 0 \quad \text{in } H^{-s}(D_1)\,. \tag{2.4.8}$$

By Lemma 2.B.6 in Appendix 2.B, w is (locally) smooth in D_1, and we have the estimate

$$\|w\|^2_{L^\infty(D_{1/2})} + \|\nabla w\|^2_{L^\infty(D_{1/2})} \leqslant C\big(\mathcal{E}_s(w, D_1) + \|w\|^2_{L^2(D_1)}\big) \leqslant C\,. \tag{2.4.9}$$

In view of (2.4.9), we have

$$\iint_{D_{\boldsymbol{\tau}}\times D_{\boldsymbol{\tau}}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant C \iint_{D_{\boldsymbol{\tau}}\times D_{\boldsymbol{\tau}}} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+2s-2}} \leqslant C\boldsymbol{\tau}^{n+2-2s}\,. \tag{2.4.10}$$

Then, writing

$$\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\tau}}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y = \iint_{D_{\boldsymbol{\tau}}\times (D_{1/2}\setminus D_{\boldsymbol{\tau}})} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ + \iint_{D_{\boldsymbol{\tau}}\times D^c_{1/2}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,, \tag{2.4.11}$$

we first estimate, using (2.4.9),

$$\iint_{D_{\boldsymbol{\tau}}\times (D_{1/2}\setminus D_{\boldsymbol{\tau}})} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant C \iint_{D_{\boldsymbol{\tau}}\times (D_{1/2}\setminus D_{\boldsymbol{\tau}})} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+2s-2}} \leqslant C\boldsymbol{\tau}^n\,. \tag{2.4.12}$$

Next we infer from Lemma 2.2.1 and (2.4.9) that

$$\iint_{D_{\boldsymbol{\tau}}\times D^c_{1/2}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant 2 \iint_{D_{\boldsymbol{\tau}}\times D^c_{1/2}} \frac{|w(x)|^2+|w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ \leqslant C\Big(\int_{D_{\boldsymbol{\tau}}} |w(x)|^2\,\mathrm{d}x + \boldsymbol{\tau}^n \int_{D^c_{1/2}} \frac{|w(y)|^2}{(|y|+1)^{n+2s}}\,\mathrm{d}y\Big) \leqslant C\boldsymbol{\tau}^n\,. \tag{2.4.13}$$

Gathering (2.4.10), (2.4.11), (2.4.12), and (2.4.13) yields

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w, D_{\boldsymbol{\tau}}) \leqslant C\boldsymbol{\tau}^{2s}\,. \tag{2.4.14}$$

By Lemma 2.4.5 – which is postponed at the end of the proof – there exists a universal constant $\boldsymbol{\sigma} \in (0,1)$ such that

$$w_k \to w \text{ strongly in } H^s(D_{\boldsymbol{\sigma}})\,. \tag{2.4.15}$$

In view of (2.4.14), we can choose $\boldsymbol{\tau}$ (depending only on n and s) in such a way that

$$0<\boldsymbol{\tau}<\boldsymbol{\sigma}/2 \quad \text{and} \quad \frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w,D_{\boldsymbol{\tau}})\leqslant\frac{1}{4}\,. \tag{2.4.16}$$

From (2.4.10) and the strong convergence in (2.4.15), we first infer that for k large enough,

$$\iint_{D_{\boldsymbol{\tau}}\times D_{\boldsymbol{\tau}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\leqslant\iint_{D_{\boldsymbol{\tau}}\times D_{\boldsymbol{\tau}}}\frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y+\boldsymbol{\tau}^n\,. \tag{2.4.17}$$

In the same way, for k large enough, one obtains from (2.4.15),

$$\begin{aligned}\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\tau}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y&=\iint_{D_{\boldsymbol{\tau}}\times(D_{\boldsymbol{\sigma}}\setminus D_{\boldsymbol{\tau}})}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\&\qquad+\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\sigma}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\&\leqslant\boldsymbol{\tau}^n+\iint_{D_{\boldsymbol{\tau}}\times(D_{\boldsymbol{\sigma}}\setminus D_{\boldsymbol{\tau}})}\frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\&\qquad+\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\sigma}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,.\end{aligned} \tag{2.4.18}$$

Then we estimate by means of Lemma 2.2.1,

$$\begin{aligned}\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\sigma}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y&\leqslant 2\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\sigma}}}\frac{|w_k(x)|^2+|w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\ \leqslant C\Big(\int_{D_{\boldsymbol{\tau}}}|w_k(x)|^2\,\mathrm{d}x+\boldsymbol{\tau}^n\int_{D^c_{\boldsymbol{\sigma}}}\frac{|w_k(y)|^2}{(|y|+1)^{n+2s}}\,\mathrm{d}y\Big)&\leqslant C\Big(\int_{D_{\boldsymbol{\tau}}}|w_k(x)|^2\,\mathrm{d}x+\boldsymbol{\tau}^n\Big)\,.\end{aligned}$$

Since $w_k\to w$ strongly in $L^2(D_1)$ and in view of (2.4.9), we deduce that for k large enough,

$$\iint_{D_{\boldsymbol{\tau}}\times D^c_{\boldsymbol{\sigma}}}\frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\leqslant C\Big(\int_{D_{\boldsymbol{\tau}}}|w(x)|^2\,\mathrm{d}x+\boldsymbol{\tau}^n\Big)\leqslant C\boldsymbol{\tau}^n\,. \tag{2.4.19}$$

Combining (2.4.17), (2.4.18), and (2.4.19) together with (2.4.16), we conclude that for k large enough,

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w_k,D_{\boldsymbol{\tau}})\leqslant\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w,D_{\boldsymbol{\tau}})+C\boldsymbol{\tau}^{2s}\leqslant\frac{1}{4}+C\boldsymbol{\tau}^{2s}\,.$$

Hence, we can choose $\boldsymbol{\tau}\in(0,1/4)$ small enough (depending only on n and s) in such a way that $\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(w_k,D_{\boldsymbol{\tau}})\leqslant 1/2$ whenever k is large enough, contradicting (2.4.6). □

As it is transparent from the proof above, Proposition 2.4.4 crucially rests on the strong convergence stated in (2.4.15) that we now prove.

Lemma 2.4.5. *There exists a universal constant $\boldsymbol{\sigma}\in(0,1)$ such that the weakly converging subsequence $\{w_k\}$ (towards w) actually converges strongly in $H^s(D_{\boldsymbol{\sigma}})$.*

Proof. We choose the constant $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{\sigma}:=\min\Big\{\frac{4}{5\Lambda},\frac{1}{32}\Big\}\,,$$

where $\Lambda>1$ is the universal constant given by Theorem 2.2.1.

Step 1. Subtracting (2.4.8) from equation (2.4.7) leads to

$$\big\langle(-\Delta)^s(w_k-w),\varphi\big\rangle_{D_1}=\varepsilon_k\int_{D_1}|\mathrm{d}_s w_k|^2u_k\cdot\varphi\,\mathrm{d}x\qquad\forall\varphi\in\mathscr{D}(D_1;\mathbb{R}^d)\,. \tag{2.4.20}$$

By approximation (see (2.2.2)), this equation also holds for every $\varphi\in H^s_{00}(D_1;\mathbb{R}^d)\cap L^\infty(D_1)$ compactly supported in D_1. Let us now fix a smooth cutoff function $\zeta\in\mathscr{D}(D_{5\boldsymbol{\sigma}/4})$ such that $0\leqslant\zeta\leqslant1$, $\zeta=1$ in $D_{\boldsymbol{\sigma}}$. Using the test function $\varphi_k:=\zeta(w_k-w)\in H^s_{00}(D_1;\mathbb{R}^d)\cap L^\infty(D_1)$ in (2.4.20) yields

$$\big\langle(-\Delta)^s(w_k-w),\varphi_k\big\rangle_{D_1}=\varepsilon_k\int_{D_1}|\mathrm{d}_s w_k|^2u_k\cdot\varphi_k\,\mathrm{d}x\,. \tag{2.4.21}$$

Setting

$$L_k:=\big\langle(-\Delta)^s(w_k-w),\zeta(w_k-w)\big\rangle_{D_1}\quad\text{and}\quad R_k:=\varepsilon_k\int_{D_1}|\mathrm{d}_s w_k|^2u_k\cdot\varphi_k\,\mathrm{d}x\,,$$

we claim that

$$L_k\geqslant[w_k-w]^2_{H^s(D_{\boldsymbol{\sigma}})}+o(1)\quad\text{as }k\to\infty\,, \tag{2.4.22}$$

and

$$\lim_{k\to\infty}R_k=0\,. \tag{2.4.23}$$

Identity (2.4.21) rewrites $L_k=R_k$, and the two claims above will imply that $[w_k-w]^2_{H^s(D_{\boldsymbol{\sigma}})}\to0$ as $k\to\infty$, whence the conclusion.

Step 2. This step is devoted to the proof of (2.4.22). For simplicity, let us denote

$$\triangle_k:=w_k-w\,.$$

Since $\zeta=1$ in $D_{\boldsymbol{\sigma}}$, and $\zeta=0$ in $D^c_{2\boldsymbol{\sigma}}$, we have

$$L_k=[\triangle_k]^2_{H^s(D_{\boldsymbol{\sigma}})}+\frac{\gamma_{n,s}}{2}\big(L_k^{(1)}+L_k^{(2)}+L_k^{(3)}\big)\,, \tag{2.4.24}$$

with

$$L_k^{(1)}:=\iint_{(D_1\setminus D_{\boldsymbol{\sigma}})\times(D_1\setminus D_{\boldsymbol{\sigma}})}\frac{(\triangle_k(x)-\triangle_k(y))\cdot(\zeta(x)\triangle_k(x)-\zeta(y)\triangle_k(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,,$$

$$L_k^{(2)}:=2\iint_{D_{\boldsymbol{\sigma}}\times(D_1\setminus D_{\boldsymbol{\sigma}})}\frac{(\triangle_k(x)-\triangle_k(y))\cdot(\zeta(x)\triangle_k(x)-\zeta(y)\triangle_k(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,,$$

and

$$L_k^{(3)}:=2\iint_{D_{2\boldsymbol{\sigma}}\times D_1^c}\frac{(\triangle_k(x)-\triangle_k(y))\cdot\triangle_k(x)}{|x-y|^{n+2s}}\,\zeta(x)\,\mathrm{d}x\mathrm{d}y\,,$$

Concerning $L_k^{(1)}$, we first rewrite

$$\begin{aligned}L_k^{(1)}=&\iint_{(D_1\setminus D_{\boldsymbol{\sigma}})\times(D_1\setminus D_{\boldsymbol{\sigma}})}\frac{((\triangle_k(x)-\triangle_k(y))\cdot\triangle_k(x))(\zeta(x)-\zeta(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\&\qquad+\iint_{(D_1\setminus D_{\boldsymbol{\sigma}})\times(D_1\setminus D_{\boldsymbol{\sigma}})}\frac{|\triangle_k(x)-\triangle_k(y)|^2}{|x-y|^{n+2s}}\,\zeta(y)\,\mathrm{d}x\mathrm{d}y\\\geqslant&\iint_{(D_1\setminus D_{\boldsymbol{\sigma}})\times(D_1\setminus D_{\boldsymbol{\sigma}})}\frac{((\triangle_k(x)-\triangle_k(y))\cdot\triangle_k(x))(\zeta(x)-\zeta(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,.\end{aligned}$$

Recalling that

$$\mathcal{E}_s(\triangle_k, D_1) \leqslant 2\mathcal{E}_s(w_k, D_1) + 2\mathcal{E}_s(w, D_1) \leqslant 4\,,$$

we estimate by means of Hölder inequality,

$$\begin{aligned}\left|\iint_{(D_1\setminus D_\sigma)\times(D_1\setminus D_\sigma)} \frac{((\triangle_k(x)-\triangle_k(y))\cdot\triangle_k(x))(\zeta(x)-\zeta(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right| \\ \leqslant \sqrt{\mathcal{E}_s(\triangle_k, D_1)}\left(\iint_{(D_1\setminus D_\sigma)\times(D_1\setminus D_\sigma)} \frac{|\triangle_k(x)|^2|\zeta(x)-\zeta(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{1/2} \\ \leqslant C\left(\iint_{(D_1\setminus D_\sigma)\times(D_1\setminus D_\sigma)} \frac{|\triangle_k(x)|^2}{|x-y|^{n+2s-2}}\,\mathrm{d}x\mathrm{d}y\right)^{1/2} \leqslant C\|\triangle_k\|_{L^2(D_1)}\,.\end{aligned}$$

Since $\|\triangle_k\|_{L^2(D_1)} \to 0$, we conclude that

$$L_k^{(1)} \geqslant o(1) \quad \text{as } k\to\infty\,. \tag{2.4.25}$$

Exactly in the same way, one derives

$$L_k^{(2)} \geqslant o(1) \quad \text{as } k\to\infty\,. \tag{2.4.26}$$

For the last term $L_k^{(3)}$, we use again Hölder inequality to derive

$$|L_k^{(3)}| \leqslant 2\sqrt{\mathcal{E}_s(\triangle_k, D_1)}\left(\iint_{D_{2\sigma}\times D_1^c} \frac{|\triangle_k(x)|^2\zeta^2(x)}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{1/2} \leqslant C\|\triangle_k\|_{L^2(D_1)} = o(1) \tag{2.4.27}$$

as $k\to\infty$. Gathering now (2.4.24) with (2.4.25), (2.4.26), and (2.4.27) leads to (2.4.22).

Step 3. In order to prove (2.4.23), we need to rewrite R_k in a suitable form. First, we rewrite

$$R_k = \frac{1}{\varepsilon_k}\int_{D_1} |\mathrm{d}_s u_k|^2 u_k\cdot\varphi_k\,\mathrm{d}x\,,$$

and we recall from Lemma 2.3.8 that for each $i=1,\dots,d$,

$$|\mathrm{d}_s u_k|^2 u_k^i = \Big(\sum_{j=1}^n \boldsymbol{\Omega}_k^{ij}\odot\mathrm{d}_s u_k^j\Big) + T_k^i = \varepsilon_k\Big(\sum_{j=1}^n \boldsymbol{\Omega}_k^{ij}\odot\mathrm{d}_s w_k^j\Big) + T_k^i\,,$$

where $\boldsymbol{\Omega}_k^{ij} \in L^2_{\mathrm{od}}(D_1)$ is given by

$$\boldsymbol{\Omega}_k^{ij}(x,y) := u_k^i(x)\mathrm{d}_s u_k^j(x,y) - u_k^j(x)\mathrm{d}_s u_k^i(x,y)\,,$$

and

$$\begin{aligned}T_k^i(x) &:= \frac{\gamma_{n,s}}{4}\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}(u_k^i(x)-u_k^i(y))\,\mathrm{d}y \\ &= \frac{\gamma_{n,s}\varepsilon_k^3}{4}\int_{\mathbb{R}^n} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}(w_k^i(x)-w_k^i(y))\,\mathrm{d}y\,.\end{aligned}$$

Hence,

$$R_k = \Big(\sum_{i,j=1}^n \int_{D_1} (\boldsymbol{\Omega}_k^{ij}\odot\mathrm{d}_s w_k^j)\varphi_k^i\,\mathrm{d}x\Big) + \varepsilon_k^2\int_{D_1} \widetilde{T}_k\cdot\varphi_k\,\mathrm{d}x =: R_k^{(1)} + R_k^{(2)}\,,$$

where we have set

$$\widetilde{T}_k(x) := \frac{\gamma_{n,s}}{4}\int_{\mathbb{R}^n} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}(w_k(x)-w_k(y))\,\mathrm{d}y\,.$$

Step 4. We shall now prove that

$$\lim_{k\to\infty} R_k^{(1)} = 0\,. \tag{2.4.28}$$

First, notice that formula (2.2.12) shows that $u_k^{\mathrm{e}} = \varepsilon_k w_k^{\mathrm{e}} + (u_k)_{0,1}$, which implies that

$$\mathbf{\Theta}_s(u_k^{\mathrm{e}},\mathbf{x},r) = \varepsilon_k^2 \mathbf{\Theta}_s(w_k^{\mathrm{e}},\mathbf{x},r) \quad \text{for every } \mathbf{x}\in\partial^0 B_1^+ \text{ and } r\in(0,1-|\mathbf{x}|)\,.$$

As a consequence, our assumption on $\mathbf{\Theta}_s(u_k^{\mathrm{e}},\mathbf{x},r)$ tells us that $r\mapsto \mathbf{\Theta}_s(w_k^{\mathrm{e}},\mathbf{x},r)$ defined on $\in(0,1-|\mathbf{x}|)$ is nondecreasing for every $\mathbf{x}\in\partial^0B_1^+$.

Applying Corollary 2.2.22 (with $R=2\boldsymbol{\sigma}$), we deduce that

$$[\zeta w_k]_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\Big(\mathcal{E}_s(w_k,4\boldsymbol{\sigma}) + \|w_k\|^2_{L^2(D_{4\boldsymbol{\sigma}})}\Big)^{1/2} \leqslant C\,,$$

for some constant C depending only on n, s, and ζ. Since $w_k\to w$ strongly in $L^2(D_1)$ and ζ is supported in $D_{5\boldsymbol{\sigma}/4}$, we have $\zeta w_k\to\zeta w$ strongly in $L^1(\mathbb{R}^n)$ (in other words, $\|\varphi_k\|_{L^1(\mathbb{R}^n)}\to 0$). By lower semicontinuity of the BMO-seminorm with respect to the L^1-convergence, we deduce that $\zeta w\in\mathrm{BMO}(\mathbb{R}^n)$, and then, recalling $\varphi_k=\zeta(w_k-w)$,

$$[\varphi_k]_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant C\,.$$

Next, we recall from Proposition 2.3.10 that u_k being weakly s-harmonic in D_1 yields

$$\mathrm{div}_s\,\mathbf{\Omega}_k^{ij} = 0 \quad \text{in } H^{-s}(D_1)\,,$$

for each $i,j\in\{1,\dots,d\}$. Applying Theorem 2.2.1 (with $x_0=0$ and $r=5\boldsymbol{\sigma}/4$), we infer that

$$\begin{aligned}\left|\int_{D_1}(\mathbf{\Omega}_k^{ij}\odot \mathrm{d}_s w_k^j)\varphi_k^i\,\mathrm{d}x\right| &\leqslant C\|\mathbf{\Omega}_k^{ij}\|_{L^2_{\mathrm{od}}(D_1)}\sqrt{\mathcal{E}_s(w_k^j,D_1)}\Big([\varphi_k^i]_{\mathrm{BMO}(\mathbb{R}^n)} + \|\varphi_k^i\|_{L^1(\mathbb{R}^n)}\Big)\\ &\leqslant C\|\mathbf{\Omega}_k^{ij}\|_{L^2_{\mathrm{od}}(D_1)}\,.\end{aligned}$$

Since $|u_k|\equiv 1$, we have the pointwise estimate $|\mathbf{\Omega}_k^{ij}(x,y)|\leqslant|\mathrm{d}_s u_k^j(x,y)|+|\mathrm{d}_s u_k^i(x,y)|$ which leads to $\|\mathbf{\Omega}_k^{ij}\|^2_{L^2_{\mathrm{od}}(D_1)}\leqslant C\mathcal{E}_s(u_k,D_1)=O(\varepsilon_k^2)$ for each $i,j\in\{1,\dots,d\}$. Consequently,

$$R_k^{(1)} = O(\varepsilon_k)\,,$$

and (2.4.28) is proved.

Step 5. We complete the proof of (2.4.23) showing now that

$$\lim_{k\to\infty} R_k^{(2)} = 0\,. \tag{2.4.29}$$

Using the fact that φ_k is supported in $D_{5\boldsymbol{\sigma}/4}\subseteq D_{1/20}\subseteq D_{1/16}$, we first write

$$R_k^{(2)} = \varepsilon_k^2\int_{D_1}\widetilde{T}_k\cdot\varphi_k\,\mathrm{d}x = \frac{\gamma_{n,s}}{4}\varepsilon_k^2(I_k+II_k)\,, \tag{2.4.30}$$

with

$$I_k := \iint_{D_{1/16}\times D_{1/16}} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}(w_k(x)-w_k(y))\cdot\varphi_k(x)\,\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{2}\iint_{D_{1/16}\times D_{1/16}} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}(w_k(x)-w_k(y))\cdot(\varphi_k(x)-\varphi_k(y))\,\mathrm{d}x\mathrm{d}y\,,$$

and

$$II_k := \iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}(w_k(x)-w_k(y))\cdot\varphi_k(x)\,\mathrm{d}x\mathrm{d}y\,. \tag{2.4.31}$$

We shall estimate separately the two terms I_k and II_k.

Concerning I_k, we apply Hölder inequality to reach

$$|I_k| \leqslant \frac{1}{2}\iint_{D_{1/16}\times D_{1/16}} \frac{|w_k(x)-w_k(y)|^3|\varphi_k(x)-\varphi_k(y)|}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y$$
$$\leqslant C[w_k]^3_{W^{s/3,6}(D_{1/16})}[\varphi_k]_{H^s(D_{1/16})}\,, \tag{2.4.32}$$

where $[\cdot]^3_{W^{s/3,6}(D_{1/16})}$ denotes the $W^{s/3,6}(D_{1/16})$-seminorm (i.e., of the Sobolev-Slobodeckij space, see (2.C.9)). Recalling our notation $\triangle_k := w_k - w$ and the fact that $0\leqslant\zeta\leqslant 1$, we have

$$[\varphi_k]^2_{H^s(D_{1/16})} \leqslant C\left([\triangle_k]^2_{H^s(D_{1/16})} + \iint_{D_{1/16}\times D_{1/16}} \frac{|\zeta(x)-\zeta(y)|^2|\triangle_k(x)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)$$
$$\leqslant C\left([\triangle_k]^2_{H^s(D_{1/16})} + \iint_{D_{1/16}\times D_{1/16}} \frac{|\triangle_k(x)|^2}{|x-y|^{n+2s-2}}\,\mathrm{d}x\mathrm{d}y\right)$$
$$\leqslant C\Big(\mathcal{E}_s(\triangle_k, D_{1/16}) + \|\triangle_k\|^2_{L^2(D_{1/16})}\Big) \leqslant C\,. \tag{2.4.33}$$

To estimate $[w_k]_{W^{s/3,6}(D_{1/16})}$, we proceed as follows. First, we fix a further cutoff function $\eta\in\mathscr{D}(D_{1/8})$ satisfying $0\leqslant\eta\leqslant 1$, $\eta\equiv 1$ in $D_{1/16}$, and $|\nabla\eta|\leqslant C$. Then we apply Corollary 2.C.11 (in Appendix 2.C) to ηw_k to derive

$$[w_k]^2_{W^{s/3,6}(D_{1/16})} = [\eta w_k]^2_{W^{s/3,6}(D_{1/16})} \leqslant C\Big(\sup_{D_r(\bar{x})\subseteq\mathbb{R}^n}\frac{1}{r^{n-2s}}[\eta w_k]^2_{H^s(D_r(\bar{x}))}\Big)\,, \tag{2.4.34}$$

and it remains to estimate the right-hand side of (2.4.34). To this purpose, we need to distinguish different types of balls:

Case 1: $\bar{x}\in D_{3/16}$ and $0<r\leqslant 1/16$. Arguing as in (2.4.33), we obtain

$$[\eta w_k]^2_{H^s(D_r(\bar{x}))} \leqslant C\left([w_k]^2_{H^s(D_r(\bar{x}))} + \iint_{D_r(\bar{x})\times D_r(\bar{x})}\frac{|w_k(x)|^2}{|x-y|^{n+2s-2}}\,\mathrm{d}x\mathrm{d}y\right)$$
$$\leqslant C\Big([w_k]^2_{H^s(D_r(\bar{x}))} + r^{2-2s}\|w_k\|^2_{L^2(D_r(\bar{x}))}\Big)\,.$$

Applying Hölder inequality in the case $n\geqslant 3$, we obtain

$$[\eta w_k]^2_{H^s(D_r(\bar{x}))} \leqslant \begin{cases} C\Big([w_k]^2_{H^s(D_r(\bar{x}))} + r^{n-2s}\|w_k\|^2_{L^n(D_r(\bar{x}))}\Big) & \text{if } n\geqslant 3\\ C\Big([w_k]^2_{H^s(D_r(\bar{x}))} + r^{2-2s}\|w_k\|^2_{L^2(D_r(\bar{x}))}\Big) & \text{if } n\leqslant 2\,. \end{cases} \tag{2.4.35}$$

Let us now recall that $r \mapsto \mathbf{\Theta}_s(w_k^{\rm e}, \mathbf{x}, r)$ is nondecreasing for every $\mathbf{x} \in B_1^+$ (see Step 4). By the proof of Lemma 2.2.20, Step 1 (applied to $w_k^{\rm e}$), we have

$$[w_k]_{\mathrm{BMO}(D_{7/16})} \leqslant C\sqrt{\mathbf{E}_s(w_k^{\rm e}, B_{1/2}^+)} \leqslant C\sqrt{\mathcal{E}_s(w_k, D_1)} \leqslant C\,, \tag{2.4.36}$$

where we have used Lemma 2.2.8 in the last inequality. In case $n \geqslant 3$, we apply the John-Nirenberg inequality in Lemma 2.2.21 and use the fact that $D_r(\bar{x}) \subseteq D_{7/16}$, to derive

$$\begin{aligned}\|w_k\|_{L^n(D_r(\bar{x}))} \leqslant \|w_k\|_{L^n(D_{7/16})} &\leqslant \big\|w_k - (w_k)_{0,7/16}\big\|_{L^n(D_{7/16})} + C\|w_k\|_{L^1(D_{7/16})} \\ &\leqslant C\big([w_k]_{\mathrm{BMO}(D_{7/16})} + \|w_k\|_{L^2(D_{7/16})}\big) \leqslant C\,. \end{aligned}\tag{2.4.37}$$

Back to (2.4.35) and in view of Lemma 2.2.7, we have thus proven that

$$\begin{aligned}[\eta w_k]^2_{H^s(D_r(\bar{x}))} &\leqslant C\big([w_k]^2_{H^s(D_r(\bar{x}))} + r^{n-2s}\big) \\ &\leqslant C\Big(\mathbf{E}_s(w_k^{\rm e}, B_{2r}^+(\bar{\mathbf{x}})) + r^{n-2s}\Big) \leqslant Cr^{n-2s}\big(\mathbf{\Theta}_s(w_k^{\rm e}, \bar{\mathbf{x}}, 2r) + 1\big)\,,\end{aligned}$$

with $\bar{\mathbf{x}} := (\bar{x}, 0)$. Then the monotonicity of $r \mapsto \mathbf{\Theta}_s(w_k^{\rm e}, \bar{\mathbf{x}}, 2r)$ together with Lemma 2.2.8 yields

$$\begin{aligned}\frac{1}{r^{n-2s}}[\eta w_k]^2_{H^s(D_r(\bar{x}))} &\leqslant C\big(\mathbf{\Theta}_s(w_k^{\rm e}, \bar{\mathbf{x}}, 1/8) + 1\big) \\ &\leqslant C\big(\mathbf{E}_s(w_k^{\rm e}, B_{1/2}^+) + 1\big) \leqslant C\big(\mathcal{E}_s(w_k, D_1) + 1\big) \leqslant C\,.\end{aligned}$$

Case 2: $\bar{x} \notin D_{3/16}$ and $0 < r \leqslant 1/16$. This case is trivial since $\eta w_k \equiv 0$ in $D_r(\bar{x})$.

Case 3: $\bar{x} \in \mathbb{R}^n$ and $r > 1/16$. Since ηw_k is supported in $D_{1/8}$ and $0 \leqslant \eta \leqslant 1$, we have (recall that $n - 2s \geqslant 0$)

$$\begin{aligned}\frac{1}{r^{n-2s}}[\eta w_k]^2_{H^s(D_r(\bar{x}))} &\leqslant 16^{2s-n}[\eta w_k]^2_{H^s(\mathbb{R}^n)} \\ &\leqslant C\Big([\eta w_k]^2_{H^s(D_{1/4})} + \iint_{D_{1/8}\times D^c_{1/4}} \frac{|\eta(x)w_k(x)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y\Big) \\ &\leqslant C\big([\eta w_k]^2_{H^s(D_{1/4})} + \|w_k\|^2_{L^2(D_{1/8})}\big)\,.\end{aligned}$$

Arguing as in (2.4.33), we obtain

$$[\eta w_k]^2_{H^s(D_{1/4})} \leqslant C\big(\mathcal{E}_s(w_k, D_{1/4}) + \|w_k\|^2_{L^2(D_{1/4})}\big)\,,$$

and thus

$$\frac{1}{r^{n-2s}}[\eta w_k]^2_{H^s(D_r(\bar{x}))} \leqslant C\big(\mathcal{E}_s(w_k, D_1) + \|w_k\|^2_{L^2(D_1)}\big) \leqslant C\,. \tag{2.4.38}$$

Gathering Cases 1, 2, and 3 above, we have proved that the right-hand side of (2.4.34) remains bounded independently of k. We can now conclude that $[w_k]_{W^{s/3,6}(D_{1/16})} \leqslant C$ using (2.4.34). In view of (2.4.32) and (2.4.33), we have thus obtained that

$$|I_k| \leqslant C\,, \tag{2.4.39}$$

and it only remains to estimate the term II_k (defined in (2.4.31)).

First, we trivially have

$$\begin{aligned}|II_k| &\leqslant \iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(x)-w_k(y)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y\\ &\leqslant 4\iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(x)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y\\ &\qquad\qquad + 4\iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(y)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y\,.\end{aligned} \tag{2.4.40}$$

On the other hand,

$$\begin{aligned}\iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(x)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y &\leqslant C\int_{D_{1/20}} |w_k(x)|^3|\triangle_k(x)|\,\mathrm{d}x\\ &\leqslant C\|w_k\|^3_{L^6(D_{1/20})}\|\triangle_k\|_{L^2(D_1)}\,.\end{aligned}$$

Recalling from (2.4.36) that $\{w_k\}$ is bounded in $\mathrm{BMO}(D_{7/16})$, we can argue as in (2.4.37) to infer that $\{w_k\}$ is bounded in $L^6(D_{1/20})$. Hence,

$$\iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(x)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y \leqslant C\|\triangle_k\|_{L^2(D_1)}\,. \tag{2.4.41}$$

Since $|u_k|\equiv 1$, we have $|w_k|\leqslant 2/\varepsilon_k$, and consequently

$$\begin{aligned}\iint_{D_{1/20}\times D^c_{1/16}} \frac{|w_k(y)|^3}{|x-y|^{n+2s}}|\triangle_k(x)|\,\mathrm{d}x\mathrm{d}y &\leqslant \frac{2}{\varepsilon_k}\int_{D_{1/20}}\left(\int_{D^c_{1/16}} \frac{|w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\right)|\triangle_k(x)|\,\mathrm{d}x\\ &\leqslant \frac{C}{\varepsilon_k}\int_{D_{1/20}}\left(\int_{\mathbb{R}^n} \frac{|w_k(y)|^2}{(|y|+1)^{n+2s}}\,\mathrm{d}y\right)|\triangle_k(x)|\,\mathrm{d}x\\ &\leqslant \frac{C}{\varepsilon_k}\Big(\mathcal{E}_s(w_k,D_1)+\|w_k\|^2_{L^2(D_1)}\Big)\|\triangle_k\|_{L^2(D_1)}\,,\end{aligned} \tag{2.4.42}$$

where we have used Lemma 2.2.1 in the last inequality. Combining (2.4.40), (2.4.41), and (2.4.42), we obtain the estimate

$$|II_k| \leqslant C\varepsilon_k^{-1}\|\triangle_k\|_{L^2(D_1)} = o(\varepsilon_k^{-1})\,. \tag{2.4.43}$$

In view of (2.4.30), (2.4.39), and (2.4.43), we have thus proved that

$$R_k^{(2)} = o(\varepsilon_k)\,,$$

and thus (2.4.29) holds, which completes the whole proof. □

Proof of Theorem 2.4.1. Rescaling variables, we can assume that $R=2$. We need to distinguish the two cases $n\geqslant 2s$, and $n=1$ with $s\in(1/2,1)$.

Case 1: $n\geqslant 2s$. We choose $\boldsymbol{\varepsilon}_0 := 2^{2s-n}\boldsymbol{\varepsilon}_*$ where $\boldsymbol{\varepsilon}_*=\boldsymbol{\varepsilon}_*(n,s)>0$ is the constant provided by Proposition 2.4.4. We fix an arbitrary point $x_0\in D_1$, and we observe that condition (2.4.1) implies

$$\mathcal{E}_s(u,D_1(x_0)) \leqslant \mathcal{E}_s(u,D_2) = 2^{n-2s}\boldsymbol{\theta}_s(u,0,2) \leqslant \boldsymbol{\varepsilon}_*\,.$$

Setting $\mathbf{e} := \mathcal{E}_s(u, D_2)$, Proposition 2.4.4 then leads to

$$\frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(u, D_{\boldsymbol{\tau}}(x_0)) \leqslant \frac{1}{2}\mathcal{E}_s(u, D_1(x_0)) \leqslant \frac{1}{2}\mathbf{e}\,, \tag{2.4.44}$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}(n,s) \in (0, 1/4)$. Considering the rescaled map $u_{\boldsymbol{\tau}}(x) := u(\boldsymbol{\tau}x + x_0)$, one realizes from (2.4.44) that $u_{\boldsymbol{\tau}}$ satisfies $\mathcal{E}_s(u_{\boldsymbol{\tau}}, D_1) \leqslant \frac{1}{2}\boldsymbol{\varepsilon}_*$, and thus Proposition 2.4.4 applies. Unscaling variables, it yields

$$\begin{aligned}\frac{1}{(\boldsymbol{\tau}^{n-2s})^2}\mathcal{E}_s(u, D_{\boldsymbol{\tau}^2}(x_0)) &= \frac{1}{\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(u_{\boldsymbol{\tau}}, D_{\boldsymbol{\tau}}) \\ &\leqslant \frac{1}{2}\mathcal{E}_s(u_{\boldsymbol{\tau}}, D_1) = \frac{1}{2\boldsymbol{\tau}^{n-2s}}\mathcal{E}_s(u, D_{\boldsymbol{\tau}}(x_0)) \leqslant \frac{1}{4}\mathbf{e}\,.\end{aligned}$$

Arguing by induction, we infer that

$$\mathcal{E}_s(u, D_{\boldsymbol{\tau}^k}(x_0)) \leqslant \frac{\boldsymbol{\tau}^{k(n-2s)}}{2^k}\mathbf{e} \quad \text{for each } k = 0, 1, 2, 3, \ldots. \tag{2.4.45}$$

Let us now fix an arbitrary $r \in (0, 1)$, and consider the integer k such that $\boldsymbol{\tau}^{k+1} < r \leqslant \boldsymbol{\tau}^k$. From (2.4.45), we deduce that

$$\frac{1}{r^{n-2s}}\mathcal{E}_s(u, D_r(x_0)) \leqslant \frac{1}{r^{n-2s}}\mathcal{E}_s(u, D_{\boldsymbol{\tau}^k}(x_0)) \leqslant \frac{\boldsymbol{\tau}^{2s-n}}{2^k}\mathbf{e} \leqslant 2\boldsymbol{\tau}^{2s-n}\mathbf{e}\, r^{2\beta_0}\,,$$

with $2\beta_0 := \log(2)/\log(1/\boldsymbol{\tau})$. By the Poincaré inequality in $H^s(D_r(x_0))$, it yields

$$\frac{1}{r^n}\int_{D_r(x_0)} \left|u - (u)_{x_0,r}\right|^2 \mathrm{d}x \leqslant \frac{C}{r^{n-2s}}[u]^2_{H^s(D_r(x_0))} \leqslant \frac{C}{r^{n-2s}}\mathcal{E}_s(u, D_r(x_0)) \leqslant C\mathbf{e}\, r^{2\beta_0}\,.$$

In view of the arbitrariness of r and x_0, we can apply Campanato's criterion (see e.g. [73, Theorem I.6.1]), and it yields $u \in C^{0,\beta_0}(D_1)$ with

$$|u(x) - u(y)| \leqslant C\sqrt{\mathbf{e}}\,|x - y|^{\beta_0} \qquad \forall x, y \in D_1\,,$$

which completes the proof.

Case 2: $n = 1$ and $s \in (1/2, 1)$. In this case, we simply choose $\boldsymbol{\varepsilon}_0 := 1$, and we invoke Proposition 2.4.3 whose proof is given below. □

Proof of Proposition 2.4.3. Rescaling variables, we can assume that $R = 1$. Without loss of generality, we can also assume that u has a vanishing average over D_1. We consider a given cutoff function $\zeta \in \mathscr{D}(D_{3/4})$ such that $0 \leqslant \zeta \leqslant 1$ and $\zeta = 1$ in $D_{1/2}$. Arguing as (2.4.38), we obtain that $\zeta u \in H^s(\mathbb{R}; \mathbb{R}^d)$ with

$$[\zeta u]^2_{H^s(\mathbb{R})} \leqslant C\big(\mathcal{E}_s(u, D_1) + \|u\|^2_{L^2(D_1)}\big)\,. \tag{2.4.46}$$

On the other hand, by the continuous embedding $H^s(\mathbb{R}^n) \hookrightarrow C^{0,s-1/2}(\mathbb{R}^n)$ (see e.g. [57, Theorem 1.4.4.1]), we have

$$[\zeta u]^2_{C^{0,s-1/2}(\mathbb{R})} \leqslant C\big([\zeta u]^2_{H^s(\mathbb{R})} + \|\zeta u\|^2_{L^2(\mathbb{R})}\big) \leqslant C\big([\zeta u]^2_{H^s(\mathbb{R})} + \|u\|^2_{L^2(D_1)}\big)\,. \tag{2.4.47}$$

Combining (2.4.47) with (2.4.46) and applying Poincaré inequality in $H^s(D_1)$, we derive that

$$[u]^2_{C^{0,s-1/2}(D_{1/2})} \leqslant [\zeta u]^2_{C^{0,s-1/2}(\mathbb{R})} \leqslant C\big(\mathcal{E}_s(u, D_1) + \|u\|^2_{L^2(D_1)}\big) \leqslant C\mathcal{E}_s(u, D_1)\,,$$

which completes the proof of (2.4.4). □

2.5 Small energy Lipschitz regularity

In this section, our goal is to improve the conclusion of Theorem 2.4.1 to Lipschitz continuity, as stated in the following theorem. Higher order regularity will be the object of the next section.

Theorem 2.5.1. *Let $\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_1(n,s) > 0$ be the constant given by Corollary 2.4.1. There exists a constant $\boldsymbol{\kappa}_2 = \boldsymbol{\kappa}_2(n,s) \in (0,1)$ such that the following holds. Let $u \in \widehat{H}^s(D_{2R};\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_{2R} such that the function $r \in (0, 2R - |\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}}, \mathbf{x}, r)$ is nondecreasing for every $\mathbf{x} \in \partial^0 B^+_{2R}$. If*

$$\boldsymbol{\Theta}_s(u^{\mathrm{e}}, 0, R) \leqslant \boldsymbol{\varepsilon}_1 \,, \tag{2.5.1}$$

then $u \in C^{0,1}(D_{\boldsymbol{\kappa}_2 R})$ and

$$R^2 \|\nabla u\|^2_{L^\infty(D_{\boldsymbol{\kappa}_2 R})} \leqslant C \boldsymbol{\Theta}_s(u^{\mathrm{e}}, 0, R) \,,$$

for a constant $C = C(n,s)$.

The proof of Theorem 2.5.1 consists in considering the system satisfied by the $\mathbb{S}^{d-1}$-valued map $u^{\mathrm{e}}/|u^{\mathrm{e}}|$. By Corollary 2.4.1, u^{e} is Hölder continuous, and therefore $|u^{\mathrm{e}}| \geqslant 1/2$ in a smaller half-ball B^+_r. In particular, $v := u^{\mathrm{e}}/|u^{\mathrm{e}}|$ is well defined and Hölder continuous in B^+_r. We shall see that it satisfies in the weak sense the degenerate system with *homogeneous* Neumann boundary condition

$$\begin{cases} -\mathrm{div}(z^a \rho^2 \nabla v) = z^a \rho^2 |\nabla v|^2 v & \text{in } B^+_r \,, \\ z^a \rho^2 \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial^0 B^+_r \,, \end{cases} \tag{2.5.2}$$

with Hölder continuous weight $\rho^2 := |u^{\mathrm{e}}|^2$. Up to the extra weight term ρ^2, this system fits into the class of degenerate harmonic map systems with free boundary considered in [93]. Adjusting the arguments in [93] to take care of the extra weight ρ^2, we shall prove that v is Lipschitz continuous in an even smaller half-ball. Since $u^{\mathrm{e}} = v$ on $\partial^0 B^+_r$, the conclusion will follow straight away.

2.5.1 Proof of Theorem 2.5.1

The aforementioned Lipschitz estimate on the map $u^{\mathrm{e}}/|u^{\mathrm{e}}|$ is the object of the following proposition.

Proposition 2.5.1. *Let $u \in \widehat{H}^s(D_{2R};\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_{2R}. Assume that $u^{\mathrm{e}} \in C^{0,\beta}(B^+_R)$ for some exponent $\beta \in (0,1)$, and that $|u^{\mathrm{e}}| \geqslant 1/2$ in B^+_R. Setting $\eta := R^\beta [u^{\mathrm{e}}]_{C^{0,\beta}(B^+_R)}$, the map $u^{\mathrm{e}}/|u^{\mathrm{e}}|$ is Lipschitz continuous in $\overline{B}^+_{R/3}$, and*

$$R^2 \|\nabla(u^{\mathrm{e}}/|u^{\mathrm{e}}|)\|^2_{L^\infty(B^+_{R/3})} \leqslant C_{\eta,\beta} \boldsymbol{\Theta}_s(u^{\mathrm{e}}, 0, R) \,,$$

for a constant $C_{\eta,\beta} = C_{\eta,\beta}(\eta, \beta, n, s)$.

Before proving this proposition, we need to show that $u^{\mathrm{e}}/|u^{\mathrm{e}}|$ satisfies system (2.5.2) in the weak sense.

Lemma 2.5.2. *Let $u \in \widehat{H}^s(D_{2R};\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_{2R}. Assume that $\rho := |u^{\rm e}|$ satisfies $\rho \geqslant 1/2$ a.e. in B_R^+. Then the map $v := u^{\rm e}/\rho$ belongs to $H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ and it satisfies*

$$\int_{B_R^+} z^a\rho^2\nabla v\cdot\nabla\phi\,\mathrm{d}\mathbf{x} = \int_{B_R^+} z^a\rho^2|\nabla v|^2 v\cdot\phi\,\mathrm{d}\mathbf{x}$$

for every $\phi \in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R^+)$ such that $\phi = 0$ on ∂^+B_R.

Proof. First recall from (2.2.14) and Lemma 2.2.8 that $u^{\rm e}$ belongs to $H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(\mathbb{R}^{n+1}_+)$, and consequently, $\rho\in H^1(B_R^+,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(\mathbb{R}^{n+1}_+)$. By assumption $\rho\geqslant 1/2$, so that $1/\rho\in H^1(B_R^+,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(\mathbb{R}^{n+1}_+)$. The space $H^1(B_R^+,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(\mathbb{R}^{n+1}_+)$ being an algebra, it follows that $v\in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$, and by definition $|v|=1$ a.e. in B_R^+.

Let us now fix $\Phi\in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R^+)$ such that $\Phi=0$ on ∂^+B_R. Again, $H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R^+)$ being an algebra, $\psi:=\Phi-(\Phi\cdot v)v\in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R^+)$. It also satisfies $\psi=0$ on ∂^+B_R, and by construction, we have $v\cdot\psi=0$ a.e. in B_R^+. Now we consider $\xi:=\rho\psi\in H^1(B_R^+;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(\mathbb{R}^n)$, which still satisfies $\xi=0$ on ∂^+B_R, and $u^{\rm e}\cdot\xi=0$ in B_R^+. In particular, $u\cdot\xi=0$ on $\partial^0B_R^+$.

By Proposition 2.3.13, the map $u^{\rm e}$ is a weighted weakly harmonic map with free boundary in the half-ball B_R^+, i.e., it satisfies (2.3.12). Hence,

$$\int_{B_R^+} z^a\nabla u^{\rm e}\cdot\nabla\xi\,\mathrm{d}\mathbf{x} = 0\,. \tag{2.5.3}$$

On the other hand, $\partial_i u^{\rm e} = \partial_i\rho v+\rho\partial_i v$ and $\partial_i\xi=\partial_i\rho\psi+\rho\partial_i\psi$ in B_R^+ for $i=1,\dots,n+1$. Then we notice that $v\cdot\psi=0$ implies $v\cdot\partial_i\psi=-\partial_i v\cdot\psi$ in B_R^+ for $i=1,\dots,n+1$. In the same way, the fact that $|v|^2=1$ leads to $v\cdot\partial_i v=0$ in B_R^+ for $i=1,\dots,n+1$. As a consequence,

$$\partial_i u^{\rm e}\cdot\partial_i\xi = (\partial_i\rho v+\rho\partial_i v)\cdot(\partial_i\rho\psi+\rho\partial_i\psi) = \rho^2\partial_i v\cdot\partial_i\psi \quad \text{a.e. in } B_R^+\,,$$

for $i=1,\dots,n+1$. Inserting this identity in (2.5.3) yields

$$\int_{B_R^+} z^a\rho^2\nabla v\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = 0\,. \tag{2.5.4}$$

To conclude, we notice that

$$\partial_i v\cdot\partial_i\psi = \partial_i v\cdot\big(\partial_i\Phi-(v\cdot\Phi)\partial_i v-(\partial_i v\cdot\Phi+v\cdot\partial_i\Phi)v\big) = \partial_i v\cdot\partial_i\Phi-|\partial_i v|^2 v\cdot\Phi \quad \text{a.e. in } B_R^+\,,$$

for $i=1,\dots,n+1$. Using this last identity in (2.5.4) leads to the announced conclusion. □

As usual, to deal with homogeneous Neumann condition, we extend the equation to the whole ball by symmetry. In this way, proving estimates up to the boundary reduces to prove interior estimates.

Corollary 2.5.3. *Let $u\in\widehat{H}^s(D_{2R};\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_{2R}. Assume that $|u^{\rm e}|\geqslant 1/2$ a.e. in B_R^+. Then the function ρ and the map v defined by*

$$\rho(\mathbf{x}) := \begin{cases} |u^{\rm e}(x,z)| & \text{if } \mathbf{x}=(x,z)\in B_R^+ \\ |u^{\rm e}(x,-z)| & \text{if } \mathbf{x}=(x,z)\in B_R^- \end{cases} \tag{2.5.5}$$

and

$$v(\mathbf{x}) := \begin{cases} u^{\mathrm{e}}(x,z)/\rho(\mathbf{x}) & \text{if } \mathbf{x}=(x,z)\in B_R^+ \\ u^{\mathrm{e}}(x,-z)/\rho(\mathbf{x}) & \text{if } \mathbf{x}=(x,z)\in B_R^- \end{cases} \tag{2.5.6}$$

belong to $H^1(B_R,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ *and* $H^1(B_R;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ *respectively, and*

$$\int_{B_R}|z|^a\rho^2\nabla v\cdot\nabla\Phi\,\mathrm{d}\mathbf{x}=\int_{B_R}|z|^a\rho^2|\nabla v|^2 v\cdot\Phi\,\mathrm{d}\mathbf{x} \tag{2.5.7}$$

holds for every $\Phi\in H^1(B_R;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ *such that* $\Phi=0$ *on* ∂B_R.

Proof. The fact that ρ and v belong to $H^1(B_R,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ and $H^1(B_R;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ respectively follows from Lemma 2.5.2 together with the symmetry with respect to the hyperplane $\{z=0\}$.

We now consider an arbitrary $\Phi\in H^1(B_R;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ satisfying $\Phi=0$ on ∂B_R. We split Φ into its symmetric and anti-symmetric parts defined by

$$\Phi^s(x,z):=\frac{\Phi(x,z)+\Phi(x,-z)}{2}\quad\text{and}\quad\Phi^a(x,z):=\frac{\Phi(x,z)-\Phi(x,-z)}{2}\,.$$

Clearly, $\Phi^s,\Phi^a\in H^1(B_R;\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})\cap L^\infty(B_R)$ and $\Phi^s=\Phi^a=0$ on ∂B_R. By construction, we have $\Phi^s(x,-z)=\Phi^s(x,z)$ and $\Phi^a(x,-z)=-\Phi^a(x,z)$, so that $\partial_z\Phi^s(x,z)=-\partial_z\Phi^s(x,-z)$ and $\partial_z\Phi^a(x,z)=\partial_z\Phi^a(x,-z)$. The map v being symmetric with respect to $\{z=0\}$, it also satisfies $\partial_z v(x,z)=-\partial_z v(x,-z)$. Therefore,

$$(\nabla v\cdot\nabla\Phi^s)(x,z)=(\nabla v\cdot\nabla\Phi^s)(x,-z)\quad\text{and}\quad(\nabla v\cdot\nabla\Phi^a)(x,z)=-(\nabla v\cdot\nabla\Phi^a)(x,-z)\,.$$

As a first consequence,

$$\int_{B_R}|z|^a\rho^2\nabla v\cdot\nabla\Phi^a\,\mathrm{d}\mathbf{x}=0\,. \tag{2.5.8}$$

Since $(v\cdot\Phi^a)(x,-z)=-(v\cdot\Phi^a)(x,z)$, we also have

$$\int_{B_R}|z|^a\rho^2|\nabla v|^2 v\cdot\Phi^a\,\mathrm{d}\mathbf{x}=0\,. \tag{2.5.9}$$

Then we infer from Lemma 2.5.2 that

$$\begin{aligned}\int_{B_R}|z|^a\rho^2\nabla v\cdot\nabla\Phi^s\,\mathrm{d}\mathbf{x}&=2\int_{B_R^+}z^a\rho^2\nabla v\cdot\nabla\Phi^s\,\mathrm{d}\mathbf{x}\\&=2\int_{B_R^+}z^a\rho^2|\nabla v|^2v\cdot\Phi^s\,\mathrm{d}\mathbf{x}=\int_{B_R}|z|^a\rho^2|\nabla v|^2v\cdot\Phi^s\,\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{2.5.10}$$

Gathering (2.5.8), (2.5.9), and (2.5.10) leads to (2.5.7), and the proof is complete. □

Proof of Proposition 2.5.1. Rescaling variables, we can assume without loss of generality that $R=1$. Throughout the proof, we shall write for a measurable set $A\subseteq\mathbb{R}^{n+1}$,

$$|A|_a:=\int_A|z|^a\,\mathrm{d}\mathbf{x}\,,$$

and we notice that for $\mathbf{y}\in\mathbb{R}^n\times\{0\}$,

$$|B_r(\mathbf{y})|_a=|B_r|_a=|B_1|_a r^{n+2-2s}\,. \tag{2.5.11}$$

We start by applying Corollary 2.5.3 to consider the (symmetrized) modulus function ρ and the (symmetrized) phase map v defined by (2.5.5) and (2.5.6), respectively. Since $u^{\rm e}$ belongs to $C^{0,\beta}(B_R^+)$ and $|u^{\rm e}| \geqslant 1/2$ in B_R^+, it follows that $v \in C^{0,\beta}(B_R)$, and $\rho \in C^{0,\beta}(B_R)$ with $\rho \geqslant 1/2$ in B_R. By Corollary 2.5.3, v satisfies (2.5.7), and from this equation we shall obtain that $v \in C^{0,1}(B_{R/3})$. We proceed in several steps.

Step 1. Let us fix $\mathbf{y} \in D_{1/2} \times \{0\}$ and $r \in (0, 1/2]$. We consider the unique weak solution $w \in H^1(B_r(\mathbf{y}); \mathbb{R}^d, |z|^a \mathrm{d}\mathbf{x})$ of

$$\begin{cases} \operatorname{div}(|z|^a \nabla w) = 0 & \text{in } B_r(\mathbf{y})\,, \\ w = v & \text{on } \partial B_r(\mathbf{y})\,, \end{cases} \tag{2.5.12}$$

see Appendix 2.A. The map v being continuous in $\overline{B}_r(\mathbf{y})$, it follows from Lemma 2.A.3 that $w \in C^0(\overline{B}_r(\mathbf{y}))$. Moreover, since v is symmetric with respect to the hyperplane $\{z = 0\}$, Lemma 2.A.2 tells us that w is also symmetric with respect to $\{z = 0\}$.

Now we estimate through Minkowski inequality,

$$\begin{aligned} \left(\frac{1}{|B_{r/2}|_a} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} &\leqslant \left(\frac{1}{|B_{r/2}|_a} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} \\ &\quad + C \left(\frac{1}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla (v - w)|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} , \end{aligned} \tag{2.5.13}$$

and our first aim is to estimate the two terms in the right-hand side of this inequality.

From the definition of η and the fact that $0 \leqslant \rho \leqslant 1$, we have

$$|\rho^2(\mathbf{x}) - \rho^2(\mathbf{y})| \leqslant 2\eta |\mathbf{x} - \mathbf{y}|^\beta \leqslant C\eta r^\beta \quad \forall \mathbf{x} \in B_r(\mathbf{y})\,. \tag{2.5.14}$$

Consequently,

$$\begin{aligned} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \, \mathrm{d}\mathbf{x} &\leqslant \rho^2(\mathbf{y}) \int_{B_{r/2}(\mathbf{y})} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x} + \int_{B_{r/2}(\mathbf{y})} |z|^a |\rho^2 - \rho^2(\mathbf{y})| |\nabla w|^2 \, \mathrm{d}\mathbf{x} \\ &\leqslant (1 + C\eta r^\beta) \int_{B_{r/2}(\mathbf{y})} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x}\,. \end{aligned} \tag{2.5.15}$$

Since w is symmetric with respect to $\{z = 0\}$, we infer from Lemma 2.A.4 and (2.5.11) that the function

$$t \in (0, r] \mapsto \frac{1}{|B_t|_a} \int_{B_t(\mathbf{y})} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x}$$

is nondecreasing. Hence,

$$\begin{aligned} \frac{1}{|B_{r/2}|_a} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \, \mathrm{d}\mathbf{x} &\leqslant \frac{(1 + C\eta r^\beta)}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x} \\ &\leqslant \frac{(1 + C\eta r^\beta)}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a |\nabla v|^2 \, \mathrm{d}\mathbf{x}\,, \end{aligned}$$

where we have used the minimality of w stated in Lemma 2.A.1 in the last inequality. Using $\rho(\mathbf{y}) = 1$ and $\rho \geqslant 1/2$, we now estimate as above,

$$\begin{aligned} \int_{B_r(\mathbf{y})} |z|^a |\nabla v|^2 \, \mathrm{d}\mathbf{x} &\leqslant \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \, \mathrm{d}\mathbf{x} + \int_{B_r(\mathbf{y})} |z|^a |\rho^2 - \rho^2(\mathbf{y})| |\nabla v|^2 \, \mathrm{d}\mathbf{x} \\ &\leqslant (1 + C\eta r^\beta) \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \, \mathrm{d}\mathbf{x}\,, \end{aligned}$$

to reach

$$\frac{1}{|B_{r/2}|_a}\int_{B_{r/2}(\mathbf{y})}|z|^a\rho^2|\nabla w|^2\,\mathrm{d}\mathbf{x}\leqslant\frac{(1+C\eta r^\beta)^2}{|B_r|_a}\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla v|^2\,\mathrm{d}\mathbf{x}\,. \tag{2.5.16}$$

Next, we recall that $v-w\in H^1(B_r(\mathbf{y});\mathbb{R}^d,|z|^a\mathrm{d}\mathbf{x})$ satisfies $v-w=0$ on $\partial B_r(\mathbf{y})$. Hence, we can apply Corollary 2.5.3 to deduce that

$$\begin{aligned}\int_{B_r(\mathbf{y})}&|z|^a\rho^2|\nabla(v-w)|^2\,\mathrm{d}\mathbf{x}\\&=\int_{B_r(\mathbf{y})}|z|^a\rho^2\nabla v\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}-\int_{B_r(\mathbf{y})}|z|^a\rho^2\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\\&=\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla v|^2v\cdot(v-w)\,\mathrm{d}\mathbf{x}-\int_{B_r(\mathbf{y})}|z|^a\rho^2\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{2.5.17}$$

On the other hand, the equation (2.5.12) satisfied by w yields

$$\begin{aligned}\int_{B_r(\mathbf{y})}|z|^a\rho^2\nabla w\cdot\nabla(v-w)|\,\mathrm{d}\mathbf{x}&=\rho^2(\mathbf{y})\int_{B_r(\mathbf{y})}|z|^a\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\\&\qquad+\int_{B_r(\mathbf{y})}|z|^a(\rho^2-\rho^2(\mathbf{y}))\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\\&=\int_{B_r(\mathbf{y})}|z|^a(\rho^2-\rho^2(\mathbf{y}))\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{2.5.18}$$

By (2.5.14) and the minimality of w, we have

$$\begin{aligned}\left|\int_{B_r(\mathbf{y})}|z|^a(\rho^2-\rho^2(\mathbf{y}))\nabla w\cdot\nabla(v-w)\,\mathrm{d}\mathbf{x}\right|&\leqslant C\eta r^\beta\int_{B_r(\mathbf{y})}|z|^a|\nabla w||\nabla(v-w)|\,\mathrm{d}\mathbf{x}\\&\leqslant C\eta r^\beta\int_{B_r(\mathbf{y})}|z|^a\big(|\nabla w|^2+|\nabla v|^2\big)\,\mathrm{d}\mathbf{x}\\&\leqslant C\eta r^\beta\int_{B_r(\mathbf{y})}|z|^a|\nabla v|^2\,\mathrm{d}\mathbf{x}\\&\leqslant C\eta r^\beta\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla v|^2\,\mathrm{d}\mathbf{x}\,,\end{aligned} \tag{2.5.19}$$

where we have used that $\rho\geqslant 1/2$ in the last inequality. Combining (2.5.17), (2.5.18), (2.5.19), and using that $|v|=1$, we infer that

$$\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla(v-w)|^2\,\mathrm{d}\mathbf{x}\leqslant\big(\|v-w\|_{L^\infty(B_r(\mathbf{y}))}+C\eta r^\beta\big)\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla v|^2\,\mathrm{d}\mathbf{x}\,. \tag{2.5.20}$$

Let us now bound $\|v-w\|_{L^\infty(B_r(\mathbf{y}))}$. First, notice that for $\mathbf{x}\in B_r(\mathbf{y})$,

$$|v(\mathbf{x})-w(\mathbf{x})|\leqslant|v(\mathbf{x})-v(\mathbf{y})|+|w(\mathbf{x})-v(\mathbf{y})|\leqslant C\eta r^\beta+|w(\mathbf{x})-v(\mathbf{y})|\,. \tag{2.5.21}$$

Next we observe that for each $i=1,\dots,d$, the scalar function $\mathbf{x}\mapsto w^i(\mathbf{x})-v^i(\mathbf{y})$ in $H^1(B_r(\mathbf{y}),|z|^a\mathrm{d}\mathbf{x})$ satisfies in the weak sense

$$\begin{cases}\operatorname{div}(|z|^a\nabla(w^i-v^i(\mathbf{y})))=0 & \text{in } B_r(\mathbf{y})\,,\\ w^i-v^i(\mathbf{y})=v^i-v^i(\mathbf{y}) & \text{on } \partial B_r(\mathbf{y})\,.\end{cases}$$

It then follows from Lemma 2.A.3 that for each $i = 1, \dots, d$,

$$\|w^i - v^i(\mathbf{y})\|_{L^\infty(B_r(\mathbf{y}))} \leqslant \|v^i - v^i(\mathbf{y})\|_{L^\infty(\partial B_r(\mathbf{y}))} \leqslant \|v - v(\mathbf{y})\|_{L^\infty(\partial B_r(\mathbf{y}))} \leqslant C\eta r^\beta .$$

Back to (2.5.21), we have thus obtained

$$\|w - v\|_{L^\infty(B_r(\mathbf{y}))} \leqslant C\eta r^\beta .$$

Using this estimate in (2.5.20), we derive that

$$\int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla(v-w)|^2 \,\mathrm{d}\mathbf{x} \leqslant C\eta r^\beta \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} . \tag{2.5.22}$$

Now, inserting estimates (2.5.16) and (2.5.22) in (2.5.13), and then squaring both sides of the resulting inequality, we are led to

$$\frac{1}{|B_{r/2}|_a} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{(1 + C_\eta r^{\beta/2})}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} ,$$

for a constant $C_\eta = C_\eta(\eta, n, s)$. Iterating this inequality along dyadic radii $r_k := 2^{-k}$ with $k \geqslant 1$, we deduce that

$$\begin{aligned}\frac{1}{|B_{r_{k+1}}|_a} \int_{B_{r_{k+1}}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} &\leqslant \Big(\prod_{j=1}^{k} (1 + C_\eta 2^{-j\beta/2}) \Big) \frac{1}{|B_{1/2}|_a} \int_{B_{1/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \\ &\leqslant C_{\eta,\beta} \int_{B_1} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} , \end{aligned} \tag{2.5.23}$$

for a constant $C_{\eta,\beta} = C_{\eta,\beta}(\eta, \beta, n, s)$. Next, for an arbitrary radius $r \in (0, 1/2]$, we consider the integer $k \geqslant 1$ satisfying $r_{k+1} < r \leqslant r_k$, and estimate

$$\frac{1}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{2^{n+2-2s}}{|B_{r_k}|_a} \int_{B_{r_k}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} ,$$

to conclude from (2.5.23) and the symmetry of v and ρ with respect to $\{z = 0\}$ that

$$\frac{1}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant C_{\eta,\beta} \int_{B_1^+} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \quad \forall r \in (0, 1/2] .$$

Noticing that $|\nabla u^{\mathrm{e}}|^2 = |\nabla \rho|^2 + \rho^2 |\nabla v|^2$, and in view of the arbitrariness of $\mathbf{y}$, we have thus proved that

$$\frac{1}{|B_r|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant C_{\eta,\beta} \int_{B_1^+} |z|^a |\nabla u^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x} \quad \forall \mathbf{y} \in D_{1/2} \times \{0\} ,\ \forall r \in (0, 1/2] . \tag{2.5.24}$$

Step 2. Our main goal in this step is to obtain an estimate similar to (2.5.24) for balls which are not centered at points of $\{z = 0\}$. By symmetry of v and ρ with respect to $\{z = 0\}$, it is enough to consider balls centered at points of $\mathbb{R}^{n+1}_+$.

Let us fix an arbitrary point $\mathbf{y} = (y, t) \in B^+_{1/3}$, and notice that $\overline{B}_{t/2}(\mathbf{y}) \subseteq B_1^+$. We also consider an arbitrary radius $r \in (0, t/2]$ (so that $\overline{B}_r(\mathbf{y}) \subseteq B_1^+$). As in Step 1, we introduce the (weak) solution $w \in H^1(B_r(\mathbf{y}); \mathbb{R}^d, |z|^a \mathrm{d}\mathbf{x})$ of (2.5.12). Exactly as in (2.5.13), we have

$$\begin{aligned}\left(\Big(\frac{2}{r}\Big)^{n+1} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \right)^{1/2} &\leqslant \left(\Big(\frac{2}{r}\Big)^{n+1} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \,\mathrm{d}\mathbf{x} \right)^{1/2} \\ &\quad + C \left(\frac{1}{r^{n+1}} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla(v-w)|^2 \,\mathrm{d}\mathbf{x} \right)^{1/2} . \end{aligned} \tag{2.5.25}$$

Arguing precisely as in Step 1, we derive that (2.5.22) still holds. Then, we estimate as in (2.5.15),

$$\int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \,\mathrm{d}\mathbf{x} \leqslant \big(\rho^2(\mathbf{y}) + C\eta r^\beta\big) \int_{B_{r/2}(\mathbf{y})} |z|^a |\nabla w|^2 \,\mathrm{d}\mathbf{x}\,. \tag{2.5.26}$$

Applying Lemma 2.A.5 with $\theta = t/r$ and then the minimality of w, we obtain

$$\begin{aligned}\Big(\frac{2}{r}\Big)^{n+1} \int_{B_{r/2}(\mathbf{y})} |z|^a |\nabla w|^2 \,\mathrm{d}\mathbf{x} &\leqslant \Big(1 + \frac{Cr}{t}\Big) \frac{1}{r^{n+1}} \int_{B_r(\mathbf{y})} |z|^a |\nabla w|^2 \,\mathrm{d}\mathbf{x} \\ &\leqslant \Big(1 + \frac{Cr}{t}\Big) \frac{1}{r^{n+1}} \int_{B_r(\mathbf{y})} |z|^a |\nabla v|^2 \,\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{2.5.27}$$

Combining (2.5.26) with (2.5.27), and using again the Hölder continuity of ρ^2 (as in (2.5.14)) together with $1/2 \leqslant \rho \leqslant 1$, we deduce that

$$\Big(\frac{2}{r}\Big)^{n+1} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla w|^2 \,\mathrm{d}\mathbf{x} \leqslant \Big(1 + C(\eta r^\beta + r/t)\Big) \frac{1}{r^{n+1}} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x}\,. \tag{2.5.28}$$

Inserting (2.5.22) and (2.5.28) in (2.5.25), we infer that

$$\frac{1}{|B_{r/2}(\mathbf{y})|} \int_{B_{r/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{1 + C_\eta (r^{\beta/2} + r/t)}{|B_r(\mathbf{y})|} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x}\,,$$

for a constant $C_\eta = C_\eta(\eta, n, s)$. Arguing as Step 1 (using the dyadic radii $r_k := 2^{-k}t$), the arbitrariness of $r \in (0, t/2]$ in this latter estimate implies that

$$\frac{1}{|B_r(\mathbf{y})|} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{C_{\eta,\beta}}{|B_{t/2}(\mathbf{y})|} \int_{B_{t/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \quad \forall r \in (0, t/2]\,, \tag{2.5.29}$$

for a constant $C_{\eta,\beta} = C_{\eta,\beta}(\eta, \beta, n, s)$. Then, we notice that for every radius $r \in (0, t/2]$,

$$|B_r(\mathbf{y})|_a \leqslant \begin{cases} t^a (1 + r/t)^a |B_r(\mathbf{y})| & \text{if } s \leqslant 1/2\,, \\ t^a (1 - r/t)^a |B_r(\mathbf{y})| & \text{if } s > 1/2\,, \end{cases}$$

and

$$|B_r(\mathbf{y})|_a \geqslant \begin{cases} t^a (1 - r/t)^a |B_r(\mathbf{y})| & \text{if } s \leqslant 1/2\,, \\ t^a (1 + r/t)^a |B_r(\mathbf{y})| & \text{if } s > 1/2\,. \end{cases}$$

Consequently, dividing (2.5.29) by t^a, we obtain

$$\frac{1}{|B_r(\mathbf{y})|_a} \int_{B_r(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{C_{\eta,\beta}}{|B_{t/2}(\mathbf{y})|_a} \int_{B_{t/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \quad \forall r \in (0, t/2]\,. \tag{2.5.30}$$

Setting $\widetilde{\mathbf{y}} := (y, 0) \in D_{1/3} \times \{0\}$, we now observe that $B_{t/2}(\mathbf{y}) \subseteq B^+_{3t/2}(\widetilde{\mathbf{y}})$ and $3t/2 \leqslant 1/2$. Using the symmetry of v and ρ with respect to $\{z = 0\}$ and (2.5.24), we deduce that

$$\begin{aligned}\frac{1}{|B_{t/2}(\mathbf{y})|_a} \int_{B_{t/2}(\mathbf{y})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} &\leqslant \frac{C}{|B^+_{3t/2}(\widetilde{\mathbf{y}})|_a} \int_{B^+_{3t/2}(\widetilde{\mathbf{y}})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \\ &\leqslant \frac{C}{|B_{3t/2}(\widetilde{\mathbf{y}})|_a} \int_{B_{3t/2}(\widetilde{\mathbf{y}})} |z|^a \rho^2 |\nabla v|^2 \,\mathrm{d}\mathbf{x} \\ &\leqslant C_{\eta,\beta} \int_{B_1^+} |z|^a |\nabla u^{\mathrm{e}}|^2 \,\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{2.5.31}$$

Combining (2.5.30) and (2.5.31), and in view of the arbitrariness of $\mathbf{y}$, we infer that

$$\frac{1}{|B_r(\mathbf{y})|_a}\int_{B_r(\mathbf{y})}|z|^a\rho^2|\nabla v|^2\,\mathrm{d}\mathbf{x}\leqslant C_{\eta,\beta}\int_{B_1^+}|z|^a|\nabla u^{\mathrm{e}}|^2\,\mathrm{d}\mathbf{x},$$

for every $\mathbf{y}=(y,t)\in B_{1/3}^+$ and every $r\in(0,t/2]$. Still by symmetry of v and ρ, this estimate actually holds for every $\mathbf{y}=(y,t)\in B_{1/3}\setminus\{z=0\}$ and $r\in(0,|t|/2)$. By Lebesgue's differentiation theorem, we have thus proved that

$$\rho^2|\nabla v|^2\leqslant C_{\eta,\beta}\int_{B_1^+}|z|^a|\nabla u^{\mathrm{e}}|^2\,\mathrm{d}\mathbf{x}\quad\text{a.e. in } B_{1/3}\,,$$

and the conclusion follows from the fact that $\rho\geqslant 1/2$. □

Proof of Theorem 2.5.1. Once again, rescaling variables, we can assume that $R=1$. Under condition (2.5.1), Corollary 2.4.1 says that $u^{\mathrm{e}}\in C^{0,\beta_1}(B_{\boldsymbol{\kappa}_1}^+)$ and $[u^{\mathrm{e}}]_{C^{0,\beta_1}(B_{\boldsymbol{\kappa}_1}^+)}$ is bounded by a constant depending only on n and s. Since $|u^{\mathrm{e}}|=|u|=1$ on $\partial^0B_{\boldsymbol{\kappa}_1}^+$, we can thus find a constant $\boldsymbol{\kappa}_2=\boldsymbol{\kappa}_2(n,s)\in(0,1)$ such that $6\boldsymbol{\kappa}_2\leqslant\boldsymbol{\kappa}_1$ and $|u^{\mathrm{e}}|\geqslant 1/2$ in $B_{3\boldsymbol{\kappa}_2}^+$. Since $\beta_1=\beta_1(n,s)$, and $(3\boldsymbol{\kappa}_2)^{\beta_1}[u^{\mathrm{e}}]_{C^{0,\beta_1}(B_{3\boldsymbol{\kappa}_2}^+)}$ is bounded by a constant depending only on n and s, Proposition 2.5.1 implies that $v:=u^{\mathrm{e}}/|u^{\mathrm{e}}|$ is Lipschitz continuous in $\overline{B}_{\boldsymbol{\kappa}_2}^+$ with

$$|v(\mathbf{x})-v(\mathbf{y})|\leqslant C\boldsymbol{\Theta}_s(u^{\mathrm{e}},0,\boldsymbol{\kappa}_2)|\mathbf{x}-\mathbf{y}|\leqslant C\boldsymbol{\Theta}_s(u^{\mathrm{e}},0,1)|\mathbf{x}-\mathbf{y}|\quad\forall\mathbf{x},\mathbf{y}\in\overline{B}_{\boldsymbol{\kappa}_2}^+\,,$$

for a constant $C=C(n,s)$. Since $v(\mathbf{x})=u(x)$ for every $\mathbf{x}=(x,0)\in\partial^0B_{\boldsymbol{\kappa}_2}^+$, the conclusion follows. □

2.6 Higher order regularity

We have now reached the final stage of our small energy regularity result where it only remains to prove that a Lipschitz continuous s-harmonic map is of class C^∞. To reach this result, we shall apply (local) Schauder type estimates for $(-\Delta)^s$. We only refer to [94] for those estimates as it is best suited to our presentation (see also [106]).

Theorem 2.6.1. *Let $u\in\widehat{H}^s(D_1;\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_1. If u is Lipschitz continuous in D_1, then $u\in C^\infty(D_{1/2})$.*

Proof. The proof of Theorem 2.6.1 follows from a bootstrap procedure. The initiation of the induction consists in passing from Lipschitz regularity to $C^{1,\alpha}$-regularity, and it is the object of Proposition 2.6.1 in the following subsection. Then we shall prove in Proposition 2.6.5 that $C^{k,\alpha}$-regularity upgrades to $C^{k+1,\alpha}$-regularity for every integer $k\geqslant 1$. In applying this bootstrap argument, we first fix an arbitrary point $x_0\in D_{1/2}$ and an integer $k\geqslant 1$. We translate variables by x_0 and rescale suitably in order to apply Proposition 2.6.1 and Proposition 2.6.5, and then conclude that u is $C^{k,\alpha}$ in a neighborhood of x_0. □

2.6.1 Hölder continuity of first order derivatives

Proposition 2.6.1. *Let $u\in\widehat{H}^s(D_3;\mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_3. If u is Lipschitz continuous in D_3, then $u\in C^{1,\alpha}(D_{r_*})$ for every $\alpha\in(0,1)$ and some $r_*=r_*(n,s)\in(0,1/2)$.*

One of the main ingredients to obtain an improved regularity is the following elementary lemma.

Lemma 2.6.2. *Let $f : D_3 \to \mathbb{R}^d$ be a Lipschitz continuous function, $g : D_3 \to \mathbb{R}^d$ an Hölder continuous function, and $\zeta : D_1 \to [0,1]$ a measurable function. Assume that one of the following items holds:*

(i) $s \in (0, 1/2)$ and $g \in C^{0,\alpha}(D_3)$ for some $\alpha \in (2s, 1]$;

(ii) $s \in (0, 1/2)$ and $g \in C^{0,\alpha}(D_3)$ for every $\alpha \in (0, 2s)$;

(iii) $s \in [1/2, 1)$ and $g \in C^{0,\alpha}(D_3)$ for every $\alpha \in (0, 1)$.

Then the function

$$G : x \in D_1 \mapsto \int_{D_1} \frac{(f(x+y) - f(x)) \cdot (g(x+y) - g(x))}{|y|^{n+2s}} \zeta(y) \, \mathrm{d}y \tag{2.6.1}$$

belongs to

1. $C^{0,\alpha}(D_1)$ in case (i);

2. $C^{0,\alpha'}(D_1)$ for every $\alpha' \in (0, 2s)$ in case (ii);

3. $C^{0,\alpha'}(D_1)$ for every $\alpha' \in (0, 2 - 2s)$ in case (iii).

Proof. Step 1. We first claim that G is well defined in all cases. To simplify the notation, we write

$$\Gamma(x, y) := (f(x+y) - f(x)) \cdot (g(x+y) - g(x)) \,. \tag{2.6.2}$$

Observe that in all cases, we have $1 + \alpha > 2s$ (it holds for every $\alpha \in (0, 2s)$ in case (ii), and we can choose such $\alpha \in (0, 1)$ in case (iii)). Since $|\Gamma(x, y)| \leqslant C_{f,g,\alpha} |y|^{1+\alpha}$, we have

$$\int_{D_1} \frac{|\Gamma(x, y)|}{|y|^{n+2s}} \, \mathrm{d}y \leqslant C_{f,g,\alpha} \int_{D_1} \frac{\mathrm{d}y}{|y|^{n+2s-(1+\alpha)}} \leqslant C_{f,g,\alpha} \quad \forall x \in D_1 \,,$$

for a constant $C_{f,g,\alpha}$ depending only on f, g, α, n, and s.

Step 2, case (i). Fix arbitrary points $x, h \in D_1$. Since

$$|\Gamma(x+h, y) - \Gamma(x, y)| \leqslant C_{f,g,\alpha} |h|^\alpha |y|^\alpha \quad \forall y \in D_1 \,, \tag{2.6.3}$$

we have

$$|G(x+h) - G(x)| \leqslant C_{f,g,\alpha} |h|^\alpha \int_{D_1} \frac{1}{|y|^{n+2s-\alpha}} \, \mathrm{d}y \leqslant C_{f,g,\alpha} |h|^\alpha \,,$$

for a constant $C_{f,g,\alpha}$ depending only on f, g, α, n, and s.

Step 3, case (ii). Let us fix an arbitrary $\varepsilon \in (0, s)$. We set $\alpha := 2s - \varepsilon$ and $\beta := 1 - 2\varepsilon$. Since

$$|\Gamma(x+h, y) - \Gamma(x, y)| \leqslant |\Gamma(x+h, y)| + |\Gamma(x, y)| \leqslant C_{f,g,\varepsilon} |y|^{1+\alpha} \,,$$

we can use (2.6.3) to obtain

$$|\Gamma(x+h, y) - \Gamma(x, y)| \leqslant C_{f,g,\varepsilon} |y|^{(1+\alpha)(1-\beta)} |h|^{\alpha\beta} |y|^{\alpha\beta} = C_{f,g,\varepsilon} |y|^{2s+\varepsilon} |h|^{\alpha\beta} \quad \forall y \in D_1 \,. \tag{2.6.4}$$

Hence,

$$|G(x+h) - G(x)| \leqslant C_{f,g,\varepsilon} |h|^{\alpha\beta} \int_{D_1} \frac{1}{|y|^{n-\varepsilon}} \, \mathrm{d}y \leqslant C_{f,g,\varepsilon} |h|^{\alpha\beta} \,, \tag{2.6.5}$$

for a constant $C_{f,g,\varepsilon} > 0$ depending only on f, g, ε, n, and s.

Step 4, case (iii). Now we fix an arbitrary $\varepsilon \in (0, 1 - s)$, and we set $\alpha := 1 - \varepsilon$ and $\beta := 2 - 2s - 2\varepsilon$. Then (2.6.4) still holds, and consequently also (2.6.5). ☐

Proof of Proposition 2.6.1. Step 1. We start by fixing a radial cutoff function $\zeta \in \mathscr{D}(\mathbb{R}^n)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $D_{1/2}$, and $\zeta = 0$ in $\mathbb{R}^n \setminus D_{3/4}$. With ζ in hands, we rewrite for $x \in D_1$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y &= \int_{\mathbb{R}^n} \frac{|u(x+y)-u(x)|^2}{|y|^{n+2s}}\,\mathrm{d}y \\ &= \int_{D_1} \frac{|u(x+y)-u(x)|^2}{|y|^{n+2s}}\zeta(y)\,\mathrm{d}y + \int_{D^c_{1/2}} \frac{|u(x+y)-u(x)|^2}{|y|^{n+2s}}(1-\zeta(y))\,\mathrm{d}y\,,\end{aligned} \tag{2.6.6}$$

and we set

$$G_u(x) := \int_{D_1} \frac{|u(x+y)-u(x)|^2}{|y|^{n+2s}}\zeta(y)\,\mathrm{d}y\,. \tag{2.6.7}$$

By Lemma 2.6.2 (applied to $f = g = u$), the function G_u is Lipschitz continuous in D_1 for $s \in (0, 1/2)$, and it belongs to $C^{0,\alpha}(D_1)$ for every $\alpha \in (0, 2-2s)$ for $s \in [1/2, 1)$.

Concerning the second term in the right-hand side of (2.6.6), we use the identity $|u|^2 = 1$ to rewrite it as

$$\begin{aligned}\int_{D^c_{1/2}} \frac{|u(x+y)-u(x)|^2}{|y|^{n+2s}}(1-\zeta(y))\,\mathrm{d}y &= \int_{\mathbb{R}^n} \frac{2(1-\zeta(y))}{|y|^{n+2s}}\,\mathrm{d}y \\ &\quad - \bigg(\int_{\mathbb{R}^n} \frac{2(1-\zeta(y))}{|y|^{n+2s}} u(x+y)\,\mathrm{d}y\bigg)\cdot u(x)\,.\end{aligned} \tag{2.6.8}$$

In view of (2.6.8), it is convenient to introduce the constant $L_\zeta > 0$ and the function $Z \in C^\infty(\mathbb{R}^n)$ given by

$$L_\zeta := \int_{\mathbb{R}^n} \frac{2(1-\zeta(y))}{|y|^{n+2s}}\,\mathrm{d}y \quad \text{and} \quad Z(x) := \frac{2}{L_\zeta}\frac{(1-\zeta(x))}{|x|^{n+2s}}\,.$$

In this way, the right-hand side of (2.6.8) can be written as

$$H_u(x) := L_\zeta\big(1 - Z * u(x)\cdot u(x)\big) \quad \text{for } x \in D_1\,. \tag{2.6.9}$$

Notice that $Z * u \in C^\infty(\mathbb{R}^n)$, so that H_u is Lipschitz continuous in D_1.

Summarizing our manipulations in (2.6.6) and (2.6.8), we have obtained

$$\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y = G_u(x) + H_u(x) \qquad \forall x \in D_1\,.$$

Now we introduce the map $F_u : D_1 \to \mathbb{R}^d$ given by

$$F_u(x) := \frac{\gamma_{n,s}}{2}\big(G_u(x) + H_u(x)\big)u(x)\,. \tag{2.6.10}$$

Then $F_u \in C^{0,1}(D_1)$ for $s \in (0, 1/2)$, and $F_u \in C^{0,\alpha}(D_1)$ for every $\alpha \in (0, 2-2s)$ for $s \in [1/2, 1)$.

Step 2. We consider the map $u_0 : \mathbb{R}^n \to \mathbb{R}^d$ given by $u_0 := \zeta u$. Then $u_0 \in C^{0,1}(\mathbb{R}^n)$ and $u_0 = 0$ in $\mathbb{R}^n \setminus D_1$. In particular, $u_0 \in H^s_{00}(D_1; \mathbb{R}^d)$. A lengthy but straightforward computation shows that

$$(-\Delta)^s u_0 = \zeta(-\Delta)^s u + ((-\Delta)^s\zeta)u - \gamma_{n,s}\int_{\mathbb{R}^n} \frac{(\zeta(x)-\zeta(y))(u(x)-u(y))}{|x-y|^{n+2s}}\,\mathrm{d}y$$

in $H^{-s}(D_1;\mathbb{R}^d)$, i.e., in the sense of (2.2.3). Since u is a weakly s-harmonic map in D_3, it satisfies equation (2.3.2). In view of Step 1, we thus have

$$(-\Delta)^s u_0 = \zeta F_u + ((-\Delta)^s\zeta)u - \gamma_{n,s}\int_{\mathbb{R}^n}\frac{(\zeta(x)-\zeta(y))(u(x)-u(y))}{|x-y|^{n+2s}}\,\mathrm{d}y \quad \text{in } H^{-s}(D_1;\mathbb{R}^d)\,. \tag{2.6.11}$$

The function $(-\Delta)^s\zeta$ being smooth over $\mathbb{R}^n$, we infer from Step 1 that $\zeta F_u + \big((-\Delta)^s\zeta\big)u$ belongs to $C^{0,1}(D_1)$ for $s\in(0,1/2)$, and to $C^{0,\alpha}(D_1)$ for every $\alpha\in(0,2-2s)$ for $s\in[1/2,1)$. We now determine the regularity of the last term in the right-hand side of (2.6.11) arguing as in Step 1. We write it as

$$\int_{\mathbb{R}^n}\frac{(\zeta(x)-\zeta(y))(u(x)-u(y))}{|x-y|^{n+2s}}\,\mathrm{d}y =: I(x)+II(x)\,,$$

with

$$I(x) := \int_{D_1}\frac{(\zeta(x+y)-\zeta(x))(u(x+y)-u(x))}{|y|^{n+2s}}\zeta(y)\,\mathrm{d}y\,,$$

and

$$\begin{aligned} II(x) &:= \int_{\mathbb{R}^n}\frac{(\zeta(x+y)-\zeta(x))(u(x+y)-u(x))}{|y|^{n+2s}}(1-\zeta(y))\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\frac{(\zeta(x)-\zeta(y))(u(x)-u(y))}{|x-y|^{n+2s}}(1-\zeta(x-y))\,\mathrm{d}y\,. \end{aligned}$$

By Lemma 2.6.2, the term I belongs to $C^{0,1}(D_1)$ for $s\in(0,1/2)$, and to $C^{0,\alpha}(D_1)$ for every $\alpha\in(0,2-2s)$ for $s\in[1/2,1)$. On the other hand, the function ζ being smooth and equal to 1 in $D_{1/2}$, the term II has clearly the regularity of u in D_1, that is $C^{0,1}(D_1)$. Summarizing these considerations, we have shown that $u_0\in H^s(\mathbb{R}^n;\mathbb{R}^d)\cap L^\infty(\mathbb{R}^n)$ is a weak solution of

$$\begin{cases}(-\Delta)^s u_0 = F_0 & \text{in } D_1\,,\\ u_0 = 0 & \text{in } \mathbb{R}^n\setminus D_1\,,\end{cases}$$

for a right-hand side F_0 which belongs to $C^{0,1}(D_1)$ for $s\in(0,1/2)$, and to $C^{0,\alpha}(D_1)$ for every $\alpha\in(0,2-2s)$ for $s\in[1/2,1)$. From well-known (by now) regularity estimates for this equation (see e.g. [94, Section 2] or Theorem A.2.1), the map u_0 belongs to $C^{1,\alpha}(D_{1/2})$ for every $\alpha\in(0,2s)$ for $s\in(0,1/2)$, and to $C^{1,\alpha}(D_{1/2})$ for every $\alpha\in(0,1)$ for $s\in[1/2,1)$. Since $u_0=u$ in $D_{1/2}$, the proposition is proved in the case $s\in[1/2,1)$, and we obtained $u\in C^{1,\alpha}(D_{1/2})$ for every $\alpha\in(0,2s)$ for $s\in(0,1/2)$.

Step 3. We now assume that $s\in(0,1/2)$, and it remains to prove that u actually belongs to $C^{1,\alpha}(D_{r_*})$ for every $\alpha\in(0,1)$ and a radius $r_*\in(0,1/2)$ depending only on s. To this purpose, we rescale u by setting $\widetilde{u}(x):=u(x/6)$, and from Step 3, we infer that $\widetilde{u}\in C^{1,\alpha}(D_3)$ for every $\alpha\in(0,2s)$. We shall now make use of the following lemma.

Lemma 2.6.3. *Assume that $s\in(0,1/2)$. Let $f:D_3\to\mathbb{R}^d$ and $g:D_3\to\mathbb{R}^d$ be two C^1-functions, and $\zeta:D_1\to[0,1]$ a measurable function. Assume that one of the following items holds:*

(i) $f,g\in C^{1,\alpha}(D_3)$ for every $\alpha\in(0,2s)$;

(ii) $f,g\in C^{1,\alpha}(D_3)$ for some $\alpha\in(2s,1)$;

Then the function $G:D_1\to\mathbb{R}$ given by (2.6.1) belongs to

1. $C^{1,\alpha'}(D_1)$ *for every* $\alpha' \in (0,2s)$ *in case (i);*

2. $C^{1,\alpha}(D_1)$ *in case (ii);*

and for $x \in D_1$,

$$\partial_i G(x) = \int_{D_1} \frac{(\partial_i f(x+y) - \partial_i f(x)) \cdot (g(x+y) - g(x))}{|y|^{n+2s}} \zeta(y)\,\mathrm{d}y + \int_{D_1} \frac{(f(x+y) - f(x)) \cdot (\partial_i g(x+y) - \partial_i g(x))}{|y|^{n+2s}} \zeta(y)\,\mathrm{d}y\,, \quad (2.6.12)$$

for $i = 1, \ldots, n$.

Proof. We keep using notation (2.6.2). First we fix an arbitrary point $x \in D_1$ and we claim that G admits a partial derivative $\partial_i G$ at x. Indeed, for $t > 0$ small enough, we have

$$\big|\Gamma(x + te_i, y) - \Gamma(x,y)\big| \leqslant C_{f,g}|y|t \quad \forall y \in D_1\,,$$

since f and g are C^1 over D_3. Hence,

$$\frac{|\Gamma(x + te_i, y) - \Gamma(x,y)|}{|y|^{n+2s}t} \leqslant C_{f,g}|y|^{1-2s-n} \in L^1(D_1)\,,$$

and it follows from the dominated convergence theorem that G admits a partial derivative $\partial_i G$ at x given by formula (2.6.12).

Next we apply Lemma 2.6.2 to the right-hand side of (2.6.12) to deduce that $\partial_i G$ is Hölder continuous, and the conclusion follows. □

Proof of Proposition 2.6.1 completed. We consider the function $G_{\widetilde{u}} : D_1 \to \mathbb{R}$ as defined in (2.6.7) with $\tilde{u}$ in place of u. By Lemma 2.6.3 (applied to $f = g = \widetilde{u}$), $G_{\widetilde{u}} \in C^{1,\alpha}(D_1)$ for every $\alpha \in (0,2s)$. On the other hand, the function $H_{\widetilde{u}} : D_1 \to \mathbb{R}$ as defined in (2.6.9) clearly belongs to $C^{1,\alpha}(D_1)$ for every $\alpha \in (0,2s)$. Consequently, the map $F_{\widetilde{u}} : D_1 \to \mathbb{R}^d$ as defined in (2.6.10) also belongs to $C^{1,\alpha}(D_1)$ for every $\alpha \in (0,2s)$. Since $\widetilde{u}$ is a rescaling of u, it is also s-harmonic in D_1, and thus $(-\Delta)^s\widetilde{u} = F_{\widetilde{u}}$ in $\mathscr{D}'(D_1)$. Next, we keep arguing as in Step 2, and we consider the bounded map $\widetilde{u}_0 := \zeta\widetilde{u}$. Applying Lemma 2.6.3 again, we argue as in Step 2 to infer that $(-\Delta)^s\widetilde{u}_0 = \widetilde{F}_0$ in $H^{-s}(D_1;\mathbb{R}^d)$, for a right-hand side $\widetilde{F}_0 \in C^{1,\alpha}(D_1)$ for every $\alpha \in (0,2s)$. By the results in [94], we have $\widetilde{u}_0 \in C^{1,\alpha}(D_{1/2})$ for every $\alpha \in (0,4s)$ if $4s < 1$, and $\widetilde{u}_0 \in C^{1,\alpha}(D_{1/2})$ for every $\alpha \in (0,1)$ if $4s \geqslant 1$. Once again, since $\widetilde{u}_0 = \widetilde{u}$ in $D_{1/2}$, we have $\widetilde{u} \in C^{1,\alpha}(D_{1/2})$ for every $\alpha \in (0,4s)$ if $4s < 1$, and $\widetilde{u} \in C^{1,\alpha}(D_{1/2})$ for every $\alpha \in (0,1)$ if $4s \geqslant 1$.

In the case $s \in [1/4, 1/2)$, we have thus proved that $u \in C^{1,\alpha}(D_{1/12})$ for every $\alpha \in (0,1)$. Hence it remains to consider the case $s < 1/4$. In that case, we repeat the preceding argument considering the rescaling $\widehat{u}(x) := \widetilde{u}(x/6)$. Following the same notation as above, Lemma 2.6.3 tells us that $G_{\widehat{u}}$ belongs to $C^{1,\alpha}(D_1)$ for every $\alpha \in (0,4s)$, and hence also $F_{\widehat{u}}$. Then, applying the results of [94] to $\widehat{u}_0$, we conclude that $\widehat{u} \in C^{1,\alpha}(D_{1/2})$ for every $\alpha \in (0,6s)$ if $6s < 1$, and $\widehat{u} \in C^{1,\alpha}(D_{1/12})$ for every $\alpha \in (0,1)$ if $6s \geqslant 1$. Therefore, if $s \geqslant 1/6$, then $u \in C^{1,\alpha}(D_{1/72})$ for every $\alpha \in (0,1)$, which is the announced regularity. On the other hand, if $s \in (0,1/6)$, then we repeat the argument. It is now clear that repeating a finite number ℓ of times this argument, one reaches the conclusion that $u \in C^{1,\alpha}(D_{(6)^{-\ell}/2})$ for every $\alpha \in (0,1)$, and ℓ is essentially the integer part of $1/2s$. □

Before closing this subsection, we provide an analogue of Lemma 2.6.3 in the case $s \in [1/2, 1)$.

Lemma 2.6.4. *Assume that $s \in [1/2, 1)$. Let $f : D_3 \to \mathbb{R}^d$ and $g : D_3 \to \mathbb{R}^d$ be two C^1-functions, and $\zeta : D_1 \to [0, 1]$ a measurable function. If f and g belongs to $C^{1,\alpha}(D_3)$ for every $\alpha \in (0, 1)$, then the function $G : D_1 \to \mathbb{R}$ given by (2.6.1) belongs to $C^{1,\alpha'}(D_1)$ for every $\alpha' \in (0, 2-2s)$, and (2.6.12) holds.*

Proof. We proceed as in the proof of Lemma 2.6.3 using notation (2.6.2). We fix an arbitrary point $x \in D_1$ and we want to show that G admits a partial derivative $\partial_i G$ at x. For $t > 0$ small, we have

$$\Gamma(x+te_i, y) - \Gamma(x, y) = \Big(\int_0^t (\partial_i f(x+y+\rho e_i) - \partial_i f(x+\rho e_i))\,\mathrm{d}\rho\Big) \cdot (g(x+y+te_i) - g(x+te_i)) \\ + (f(x+y) - f(x)) \cdot \Big(\int_0^t (\partial_i g(x+y+\rho e_i) - \partial_i g(x+\rho e_i))\,\mathrm{d}\rho\Big)$$

for every $y \in D_1$. Fixing an exponent $\alpha \in (2s-1, 1)$, we deduce that

$$\big|\Gamma(x+te_i, y) - \Gamma(x, y)\big| \leqslant C_{f,g,\alpha}|y|^{1+\alpha} t \quad \forall y \in D_1 \,.$$

Consequently,

$$\frac{|\Gamma(x+te_i, y) - \Gamma(x, y)|}{|y|^{n+2s} t} \leqslant C_{f,g,\alpha}|y|^{n+2s-1-\alpha} \in L^1(D_1)\,.$$

As in the proof of Lemma 2.6.3, it now follows that G admits a partial derivative $\partial_i G$ at x given by (2.6.12), and the Hölder continuity of the partial derivatives of G is a consequence of Lemma 2.6.2. □

2.6.2 Hölder continuity of higher order derivatives

Proposition 2.6.5. *Let $u \in \widehat{H}^s(D_3; \mathbb{S}^{d-1})$ be a weakly s-harmonic map in D_3. If $u \in C^{k,\alpha}(D_3)$ for some integer $k \geqslant 1$ and every $\alpha \in (0, 1)$, then $u \in C^{k+1,\alpha}(D_{r_*})$ for every $\alpha \in (0, 1)$, where the radius $r_* \in (0, 1/2)$ is given by Proposition 2.6.1.*

Proof. We proceed as in Step 1 in the proof of Proposition 2.6.1, and we consider the function $G_u : D_1 \to \mathbb{R}$ given by (2.6.7). We claim that $G_u \in C^{k,\alpha}(D_1)$ for every $\alpha \in (0, 1)$ if $s \in (0, 1/2)$, and that $G_u \in C^{k,\alpha}(D_1)$ for every $\alpha \in (0, 2-2s)$ if $s \in [1/2, 1)$, together with the formula

$$\partial^\beta G_u(x) = \sum_{\nu \leqslant \beta} \binom{\beta}{\nu} \int_{D_1} \frac{(\partial^\nu u(x+y) - \partial^\nu u(x)) \cdot (\partial^{\beta-\nu} u(x+y) - \partial^{\beta-\nu} u(x))}{|y|^{n+2s}} \zeta(y)\,\mathrm{d}y \tag{2.6.13}$$

for every multi-index $\beta \in \mathbb{N}^n$ of length $|\beta| \leqslant k$. To prove this claim, we distinguish the case $s \in (0, 1/2)$ from the case $s \in [1/2, 1)$.

Case $s \in (0, 1/2)$. We proceed by induction. First notice that the fact that $G_u \in C^{1,\alpha}(D_1)$ for every $\alpha \in (0, 1)$ follows from Lemma 2.6.3, as well as (2.6.13) with $|\beta| = 1$. Next we assume that $G_u \in C^{\ell,\alpha}(D_1)$ for every $\alpha \in (0, 1)$ for some integer $\ell < k$, and that (2.6.13) holds for every multi-index β satisfying $|\beta| = \ell$. Applying Lemma 2.6.3 to each term in the right-hand side of (2.6.13), we infer that $\partial^\beta G_u \in C^{1,\alpha}(D_1)$ for every $\alpha \in (0, 1)$ and each β satisfying $|\beta| = \ell$, and that (2.6.13) holds for multi-indices β' in place of β of length $|\beta'| = |\beta| + 1$. The claim is thus proved for $s \in (0, 1/2)$.

Case $s \in [1/2, 1)$. We proceed exactly as in the previous case but using Lemma 2.6.4 instead of Lemma 2.6.3.

Now we consider the function $H_u : D_1 \to \mathbb{R}$ given by (2.6.9) which clearly belongs to $C^{k,\alpha}(D_1)$ for every $\alpha \in (0,1)$ by our assumption on u. Consequently, the map $F_u : D_1 \to \mathbb{R}^d$ belongs to $C^{k,\alpha}(D_1)$ for every $\alpha \in (0,1)$ if $s \in (0,1/2)$, and to $C^{k,\alpha}(D_1)$ for every $\alpha \in (0,2-2s)$ if $s \in [1/2,1)$. By the results in [94] (together with Lemmas 2.6.3 and 2.6.4), it implies that the map $u_0 := \zeta u$ as defined in Step 2, proof of Proposition 2.6.1, belongs to $C^{k+1,\alpha}(D_{1/2})$ for every $\alpha \in (0,2s)$ if $s \in (0,1/2)$, and to $C^{k+1,\alpha}(D_{1/2})$ for every $\alpha \in (0,1)$ if $s \in [1/2,1)$. Since $u_0 = u$ in $D_{1/2}$, the proof is thus complete for $s \in [1/2,1)$. In the case $s \in (0,1/2)$, we argue as in the proof of Proposition 2.6.1, Step 3, applying (inductively) Lemma 2.6.3 to formula (2.6.13) with $|\beta| = k$. It leads to the fact that $u \in C^{k+1,\alpha}(D_{r_*})$ for every $\alpha \in (0,1)$, and hence concludes the proof. □

2.7 Partial regularity for stationary and minimizing s-harmonic maps

In this section, we complete the proof of Theorems 2.1.1, 2.1.2 and 2.1.3. For $n > 2s$, we need to prove compactness of stationary / minimizing s-harmonic map to apply Federer's dimension reduction principle. This is the object of the first subsection.

2.7.1 Compactness properties of s-harmonic maps

Theorem 2.7.1. *Assume that $n > 2s$ and $s \neq 1/2$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\{u_k\} \subseteq \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ be a sequence of stationary weakly s-harmonic maps in Ω. Assume that $\sup_k \mathcal{E}_s(u_k,\Omega) < +\infty$, and that $u_k \to u$ a.e. in $\mathbb{R}^n$. Then $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$, $u_k \rightharpoonup u$ weakly in $\widehat{H}^s(\Omega;\mathbb{R}^d)$, and u is a stationary weakly s-harmonic map in Ω. In addition, for every open subset $\omega \subseteq \Omega$ and every admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\omega} \subseteq \Omega$ and $\overline{\partial^0 G} \subseteq \Omega$,*

(i) $u_k \to u$ strongly in $\widehat{H}^s(\omega;\mathbb{R}^d)$;

(ii) $u_k^{\rm e} \to u^{\rm e}$ strongly in $H^1(G;\mathbb{R}^d,|z|^a d\mathbf{x})$.

Theorem 2.7.2. *Assume that $s \in (0,1/2)$. In addition to Theorem 2.7.1, if each u_k is assumed to be a minimizing s-harmonic map in Ω, then the limit u is a minimizing s-harmonic map in Ω.*

Theorem 2.7.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\{u_k\} \subseteq \widehat{H}^{1/2}(\Omega;\mathbb{S}^{d-1})$ be a sequence of minimizing $1/2$-harmonic maps in Ω. Assume that $\sup_k \mathcal{E}_{\frac{1}{2}}(u_k,\Omega) < +\infty$, and that $u_k \to u$ a.e. in $\mathbb{R}^n$. Then the conclusion of Theorem 2.7.1 holds and the limit u is a minimizing $1/2$-harmonic map in Ω.*

Remark 2.7.1. In the case $s \in (1/2,1)$, we do not know if minimality of the sequence $\{u_k\}$ implies minimality of the limit. We believe this is indeed the case, but we won't need this fact.

Remark 2.7.2. In the case $n = 1$ and $s \in (1/2,1)$, sequences of (arbitrary) weakly s-harmonic maps with uniformly bounded energy are relatively compact, i.e., the conclusion of Theorem 2.7.1 holds. This fact is a consequence of the Lipschitz estimate established in Theorem 2.5.1 together with Remark 2.4.2. Since we shall not need this, we leave the details to the reader.

Remark 2.7.3. In the case $s = 1/2$, sequences of (stationary or not) 1/2-harmonic maps are not compact in general, see e.g. [23, 79, 84, 78]. The prototypical example is the following sequence of smooth 1/2-harmonic maps from $\mathbb{R}^n$ into $\mathbb{S}^1 \subset \mathbb{C}$ given by

$$u_k(x) = u_k(x_1) := \frac{kx_1 - i}{kx_1 + i}\,, \quad k \in \mathbb{N}\,,$$

which is converging weakly but not strongly to the constant map 1 in $\widehat{H}^{1/2}(D_r)$ for every $r > 0$. (Recall that u_k being smooth, it is stationary, see Remark 2.3.7.)

Proof of Theorem 2.7.1. Step 1. We fix two arbitrary admissible bounded open sets G, G' of $\mathbb{R}^{n+1}_+$ such that $\overline{G} \subseteq G' \cup \partial^0 G'$ and satisfying $\overline{\partial^0 G'} \subseteq \Omega$. Since $u_k \to u$ a.e. in $\mathbb{R}^n$ and $|u_k| = 1$, we first deduce that $|u| = 1$ and $u_k \to u$ strongly in $L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R}^d)$. It then follows from our assumption that $\{u_k\}$ is bounded in $\widehat{H}^s(\Omega;\mathbb{R}^d)$. Then we derive from Remark 2.2.2 that $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ and $u_k \rightharpoonup u$ weakly in $\widehat{H}^s(\Omega;\mathbb{R}^d)$. In view of Corollary 2.2.9, $u_k^{\rm e} \rightharpoonup u^{\rm e}$ weakly in $H^1(G';\mathbb{R}^d, |z|^a{\rm d}\mathbf{x})$. Since $|u_k| \leqslant 1$, we have $u_k^{\rm e}(\mathbf{x}) \to u^{\rm e}(\mathbf{x})$ for every $\mathbf{x} \in G'$ by dominated convergence. In turn, we have $|u_k^{\rm e} - u^{\rm e}| \leqslant 2$, and it follows by dominated convergence again that $u_k^{\rm e} \to u^{\rm e}$ strongly in $L^2(G';\mathbb{R}^d, |z|^a{\rm d}\mathbf{x})$. Recalling that ${\rm div}(z^a \nabla u_k^{\rm e}) = 0$ in G', we infer from standard elliptic regularity that $u_k^{\rm e} \to u^{\rm e}$ in $C^1_{\rm loc}(G')$. In particular,

$$u_k^{\rm e} \to u^{\rm e} \quad \text{strongly in } H^1_{\rm loc}(G';\mathbb{R}^d)\,. \tag{2.7.1}$$

We aim to show that $u_k^{\rm e} \to u^{\rm e}$ strongly in $H^1(G;\mathbb{R}^d, |z|^a{\rm d}\mathbf{x})$. To prove this strong convergence, we consider the finite measures on $G' \cup \partial^0 G'$ given by

$$\mu_k := \frac{\boldsymbol{\delta}_s}{2} z^a |\nabla u_k^{\rm e}|^2 \mathscr{L}^{n+1} \llcorner G'\,.$$

Since $\sup_k \mu_k(G' \cup \partial^0 G') < +\infty$, we can find a further (not relabeled) subsequence such that

$$\mu_k \rightharpoonup \frac{\boldsymbol{\delta}_s}{2} z^a |\nabla u^{\rm e}|^2 \mathscr{L}^{n+1} \llcorner G' + \mu_{\rm sing} \quad \text{as } k \to \infty\,, \tag{2.7.2}$$

weakly* as Radon measures on $G' \cup \partial^0 G'$ for some finite nonnegative measure $\mu_{\rm sing}$. In view of (2.7.1), the defect measure $\mu_{\rm sing}$ is supported by $\partial^0 G'$.

Since u_k is stationary in Ω, it satisfies the monotonicity formula in Proposition 2.2.16, and thus

$$\mu_k(B_\rho(\mathbf{x})) \leqslant \mu_k(B_r(\mathbf{x})) \tag{2.7.3}$$

for every $\mathbf{x} \in \partial^0 G'$ and $0 < \rho < r < {\rm dist}(\mathbf{x}, \partial^+ G')$. From the weak* convergence of μ_k towards μ, we then infer that

$$\mu(B_\rho(\mathbf{x})) \leqslant \mu(B_r(\mathbf{x}))$$

for every $\mathbf{x} \in \partial^0 G'$ and $0 < \rho < r < {\rm dist}(\mathbf{x}, \partial^+ G')$. As a consequence, the $(n - 2s)$-dimensional density

$$\Theta^{n-2s}(\mu, \mathbf{x}) := \lim_{r \to 0} \frac{\mu(B_r(\mathbf{x}))}{r^{n-2s}}$$

exists and is finite at every point $\mathbf{x} \in \partial^0 G'$. More precisely, (2.7.3) implies that

$$\Theta^{n-2s}(\mu, \mathbf{x}) \leqslant \big({\rm dist}(\mathbf{x}, \partial^+ G')\big)^{2s-n} \sup_k \mathbf{E}_s(u_k, G') < +\infty \quad \forall \mathbf{x} \in \partial^0 G'\,.$$

We now consider the "concentration set"

$$\Sigma := \Big\{\mathbf{x} \in \partial^0 G' : \inf_r \big\{ \liminf_{k \to \infty} r^{2s-n} \mu_k(B_r(\mathbf{x})) : 0 < r < {\rm dist}(\mathbf{x}, \partial^+ G')\big\} \geqslant \boldsymbol{\varepsilon}_1 \Big\}\,,$$

where the constant $\boldsymbol{\varepsilon}_1 > 0$ is given by Corollary 2.4.1. From the monotonicity of μ_k and μ together with (2.7.2), we deduce that

$$\Sigma = \Big\{\mathbf{x} \in \partial^0 G' : \lim_{r\to 0} \liminf_{k\to\infty} r^{2s-n}\mu_k(B_r(\mathbf{x})) \geqslant \boldsymbol{\varepsilon}_1\Big\} = \Big\{\mathbf{x} \in \partial^0 G' : \lim_{r\to 0} r^{2s-n}\mu(B_r(\mathbf{x})) \geqslant \boldsymbol{\varepsilon}_1\Big\},$$

that is

$$\Sigma = \Big\{\mathbf{x} \in \partial^0 G' : \Theta^{n-2s}(\mu,\mathbf{x}) \geqslant \boldsymbol{\varepsilon}_1\Big\}.$$

Observing that $\mathbf{x} \in \partial^0 G' \mapsto \Theta^{n-2s}(\mu,\mathbf{x})$ is upper semicontinuous, the set Σ is thus a relatively closed subset of $\partial^0 G'$.

We claim that $\operatorname{spt}(\mu_{\rm sing}) \subseteq \Sigma$. To prove this inclusion, we fix an arbitrary point $\mathbf{x}_0 = (x_0, 0) \in \partial G' \setminus \Sigma$. Then we can find a radius $0 < r < \operatorname{dist}(\mathbf{x}_0, \partial^+ G')$ such that $r^{2s-n}\mu(B_r(\mathbf{x}_0)) < \boldsymbol{\varepsilon}_1$ and $\mu(\partial B_r(\mathbf{x}_0)) = 0$. By (2.7.2) and our choice of r, we have $\lim_k \mu_k(B_r(\mathbf{x}_0)) = \mu(B_r(\mathbf{x}_0))$. Therefore, $r^{2s-n}\mu_k(B_r(\mathbf{x}_0)) < \boldsymbol{\varepsilon}_1$ for k large enough, and we derive from Theorem 2.5.1 that for k large enough, u_k is bounded in $C^{0,1}(D_{\boldsymbol{\kappa}_2 r}(x_0))$ (and $u \in C^{0,1}(D_{\boldsymbol{\kappa}_2 r}(x_0))$), where the constant $\boldsymbol{\kappa}_2 \in (0,1)$ only depends on n and s. It then follows by dominated convergence that

$$[u_k - u]^2_{H^s(D_{\boldsymbol{\kappa}_2 r}(x_0))} \underset{k\to\infty}{\longrightarrow} 0\,.$$

Setting $w_k := u_k - u$, we now estimate

$$\mathcal{E}_s(w_k, D_{2\boldsymbol{\kappa}_2 r/3}(x_0)) \leqslant C\Big([u_k - u]^2_{H^s(D_{\boldsymbol{\kappa}_2 r}(x_0))} + \iint_{D_{2\boldsymbol{\kappa}_2 r/3}(x_0)\times D^c_{\boldsymbol{\kappa}_2 r}(x_0)} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y\Big)\,.$$

Since $|w_k| \leqslant 2$ and $w_k \to 0$ a.e. in $\mathbb{R}^n$, by dominated convergence we have

$$\iint_{D_{2\boldsymbol{\kappa}_2 r/3}(x_0)\times D^c_{\boldsymbol{\kappa}_2 r}(x_0)} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y \underset{k\to\infty}{\longrightarrow} 0\,. \tag{2.7.4}$$

Hence $\mathcal{E}_s(w_k, D_{2\boldsymbol{\kappa}_2 r/3}(x_0)) \to 0$, and it follows from Lemma 2.2.8 that

$$\mathbf{E}_s(u^{\rm e}_k - u^{\rm e}, B^+_{\boldsymbol{\kappa}_2 r/3}(\mathbf{x}_0)) \leqslant \mathcal{E}_s(u_k - u, D_{2\boldsymbol{\kappa}_2 r/3}(x_0)) \to 0\,.$$

Hence, $u^{\rm e}_k \to u^{\rm e}$ strongly in $H^1(B^+_{\boldsymbol{\kappa}_2 r/3}(\mathbf{x}_0), |z|^a{\rm d}\mathbf{x})$, and thus $\mu_{\rm sing}(B_{\boldsymbol{\kappa}_2 r/3}(\mathbf{x}_0)) = 0$. This shows that $\mathbf{x}_0 \notin \operatorname{spt}(\mu_{\rm sing})$, and the claim is proved.

Next we claim that $\mu(\Sigma) = 0$. Indeed, assume by contradiction that $\mu(\Sigma) > 0$. Then the density $\Theta^{n-2s}(\mu,\mathbf{x})$ exists, it is positive (greater than $\boldsymbol{\varepsilon}_1$) and finite, at every point $\mathbf{x} \in \Sigma$. By Marstrand's theorem (see e.g. [76, Theorem 14.10]), it implies that $n - 2s$ is an integer, a contradiction.

Knowing that $\mu(\Sigma) = 0$, we then deduce that $\mu_{\rm sing}(\Sigma) = 0$. But $\mu_{\rm sing}$ being supported by Σ, it implies that $\mu_{\rm sing} \equiv 0$. As a consequence, $\mathbf{E}_s(u^{\rm e}_k, G) \to \mathbf{E}_s(u^{\rm e}, G)$, which combined with the weak convergence in $H^1(G;\mathbb{R}^d, |z|^a{\rm d}\mathbf{x})$ implies that $\mathbf{E}_s(u^{\rm e}_k - u^{\rm e}, G) \to 0$. We have thus proved that $u^{\rm e}_k \to u^{\rm e}$ strongly in $H^1(G;\mathbb{R}^d, |z|^a{\rm d}\mathbf{x})$.

Step 2. We consider in this step an open subset $\omega \subseteq \Omega$ such that $\overline{\omega} \subseteq \Omega$, and our goal is to prove that $u_k \to u$ strongly in $\widehat{H}^s(\omega;\mathbb{R}^d)$. Set $\delta := \frac{1}{8}\operatorname{dist}(\omega, \Omega^c)$, and consider a finite covering of ω by balls $(D_\delta(x_i))_{i\in I}$ with $x_i \in \overline{\omega}$. By Lemma 2.2.7 and Step 1, we have for each $i \in I$,

$$[u_k - u]^2_{H^s(D_{2\delta}(x_i))} \leqslant C\mathbf{E}_s(u_k^{\mathrm{e}} - u^{\mathrm{e}}, B^+_{4\delta}(\mathbf{x}_i)) \underset{k\to\infty}{\longrightarrow} 0\,, \tag{2.7.5}$$

where $\mathbf{x}_i := (x_i, 0)$. Writing again $w_k := u_k - u$, we now estimate

$$\begin{aligned}\mathcal{E}_s(w_k,\omega) &\leqslant C\iint_{\omega\times\mathbb{R}^n} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\leqslant C\sum_{i\in I}\iint_{D_\delta(x_i)\times\mathbb{R}^n} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &\leqslant C\sum_{i\in I}\Big([w_k]^2_{H^s(D_{2\delta}(x_i))} + \iint_{D_\delta(x_i)\times D^c_{2\delta}(x_i)} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\Big)\,. \end{aligned}\tag{2.7.6}$$

As in (2.7.4), by dominated convergence we have

$$\iint_{D_\delta(x_i)\times D^c_{2\delta}(x_i)} \frac{|w_k(x)-w_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \underset{k\to\infty}{\longrightarrow} 0 \quad \forall i\in I\,. \tag{2.7.7}$$

Combining (2.7.5), (2.7.6), and (2.7.7) leads to $\mathcal{E}_s(w_k,\omega)\to 0$, and thus $u_k \to u$ strongly in $\widehat{H}^s(\omega;\mathbb{R}^d)$.

Step 3. Our aim in this step is to show that u is a weakly s-harmonic map in Ω, i.e., u satisfies equation (2.3.2), or equivalently (2.3.4), by Proposition 2.3.5. To this purpose, we fix an arbitrary $\varphi \in \mathscr{D}(\Omega;\mathbb{R}^d)$, and we choose an open subset $\omega \subseteq \Omega$ such that $\mathrm{spt}(\varphi)\subseteq\omega$ and $\overline{\omega}\subseteq\Omega$. Writing again $w_k := u_k - u$, we have proved in Step 2 that $\mathcal{E}_s(w_k,\omega)\to 0$.

Recalling our notations from Section 2.2.2, we observe that

$$|\mathrm{d}_s u_k|^2 - |\mathrm{d}_s u|^2 = |\mathrm{d}_s w_k|^2 + 2\mathrm{d}_s w_k \odot \mathrm{d}_s u\,,$$

and then estimate

$$\begin{aligned}\big\||\mathrm{d}_s u_k|^2 - |\mathrm{d}_s u|^2\big\|_{L^1(\omega)} &\leqslant \big\||\mathrm{d}_s w_k|^2\big\|_{L^1(\omega)} + 2\|\mathrm{d}_s w_k\odot \mathrm{d}_s u\|_{L^1(\omega)} \\ &\leqslant 2\mathcal{E}_s(w_k,\omega) + 2\|\mathrm{d}_s w_k\|_{L^2_{\mathrm{od}}(\omega)}\|\mathrm{d}_s u\|_{L^2_{\mathrm{od}}(\omega)} \\ &\leqslant 2\mathcal{E}_s(w_k,\omega) + 2\sqrt{2}\|\mathrm{d}_s u\|_{L^2_{\mathrm{od}}(\omega)}\sqrt{\mathcal{E}_s(w_k,\omega)}\,.\end{aligned}$$

Therefore $|\mathrm{d}_s u_k|^2 \to |\mathrm{d}_s u|^2$ in $L^1(\omega)$, and we can find a further (not relabeled) subsequence and $h\in L^1(\omega)$ such that

$$|\mathrm{d}_s u_k|^2(x) \to |\mathrm{d}_s u|^2(x) \text{ for a.e. } x\in\omega, \text{ and } |\mathrm{d}_s u_k|^2(x) \leqslant h(x) \text{ for a.e. } x\in\omega\,.$$

Since $|u_k| = 1$ and $u_k \to u$ a.e. in ω, it follows by dominated convergence that $|\mathrm{d}_s u_k|^2 u_k$ converges to $|\mathrm{d}_s u|^2 u$ in $L^1(\omega)$. Consequently,

$$\int_\Omega |\mathrm{d}_s u_k|^2 u_k\cdot\varphi\,\mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_\Omega |\mathrm{d}_s u|^2 u\cdot\varphi\,\mathrm{d}x\,.$$

On the other hand, since u_k converges to u weakly in $\widehat{H}^s(\Omega;\mathbb{R}^d)$, we have $\langle(-\Delta)^s u_k,\varphi\rangle_\Omega \to \langle(-\Delta)^s u,\varphi\rangle_\Omega$. Hence,

$$\langle(-\Delta)^s u,\varphi\rangle_\Omega = \lim_{k\to\infty}\langle(-\Delta)^s u_k,\varphi\rangle_\Omega = \lim_{k\to\infty}\int_\Omega |\mathrm{d}_s u_k|^2 u_k\cdot\varphi\,\mathrm{d}x = \int_\Omega |\mathrm{d}_s u|^2 u\cdot\varphi\,\mathrm{d}x\,,$$

so that u is indeed weakly s-harmonic in Ω (see (2.3.4)).

Step 4. It now only remains to prove that u is stationary in Ω. This is in fact an easy consequence of the strong convergence of $u^{\rm e}$ established in Step 1. Indeed, let us fix an arbitrary vector field $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω. Combining the strong convergence of $u_k^{\rm e}$ established in Step 1 together with the representation of the first variation $\delta\mathcal{E}_s$ stated in Proposition 2.2.14, we obtain that $\delta\mathcal{E}_s(u_k,\Omega)[X] \to \delta\mathcal{E}_s(u,\Omega)[X]$, whence $\delta\mathcal{E}_s(u,\Omega) = 0$. □

Proof of Theorem 2.7.2. In view of Remark 2.3.3 and Theorem 2.7.1, it only remains to prove that the limiting map u is a minimizing s-harmonic map in Ω. We follow here the argument in [82, Theorem 4.1].

Let us now consider an arbitrary $\widetilde{u} \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ such that $\operatorname{spt}(u-\widetilde{u}) \subseteq \Omega$. We select an open subset $\omega \subseteq \Omega$ with Lipschitz boundary such that $\operatorname{spt}(u-\widetilde{u}) \subseteq \omega$ and $\overline{\omega} \subseteq \Omega$. Define

$$\widetilde{u}_k(x) := \begin{cases} \widetilde{u}(x) & \text{if } x \in \omega\,, \\ u_k(x) & \text{otherwise}\,. \end{cases}$$

Since $s \in (0,1/2)$ and $\partial\omega$ is Lipschitz regular, it turns out that $\widetilde{u}_k \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ (see e.g. [81, Section 2.1]), and $\operatorname{spt}(u_k - \widetilde{u}_k) \subseteq \Omega$. By minimality of u_k, we have $\mathcal{E}_s(u_k,\Omega) \leqslant \mathcal{E}_s(\widetilde{u}_k,\Omega)$. Since $\widetilde{u}_k = u_k$ in $\mathbb{R}^n \setminus \omega$, it reduces to

$$\begin{aligned}\mathcal{E}_s(u_k,\omega) &\leqslant \mathcal{E}_s(\widetilde{u}_k,\omega)\\ &= \frac{\gamma_{n,s}}{4}\iint_{\omega\times\omega} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y + \frac{\gamma_{n,s}}{2}\iint_{\omega\times\omega^c} \frac{|\widetilde{u}(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,{\rm d}x{\rm d}y\,.\end{aligned}$$

On the other hand,

$$\frac{|\widetilde{u}(x)-u_k(y)|^2}{|x-y|^{n+2s}} \leqslant \frac{4}{|x-y|^{n+2s}} \in L^1(\omega\times\omega^c)\,,$$

since ω has Lipschitz boundary. Hence, $\mathcal{E}_s(\widetilde{u}_k,\omega) \to \mathcal{E}_s(\widetilde{u},\omega)$ by dominated convergence and the fact that $\widetilde{u} = u$ in $\mathbb{R}^n\setminus\omega$. By Fatou's Lemma, we have $\liminf_k \mathcal{E}_s(u_k,\omega) \geqslant \mathcal{E}_s(u,\omega)$, and we reach the conclusion that $\mathcal{E}_s(u,\omega) \leqslant \mathcal{E}_s(\widetilde{u},\omega)$. Once again, the fact that $\widetilde{u} = u$ in $\mathbb{R}^n \setminus \omega$ then implies that $\mathcal{E}_s(u,\Omega) \leqslant \mathcal{E}_s(\widetilde{u},\Omega)$. By arbitrariness of $\widetilde{u}$, we conclude that u is indeed a minimizing s-harmonic map in Ω. □

We now close this subsection with an easy consequence of Theorem 2.7.1 and Theorem 2.7.3 in terms of the pointwise density function $\boldsymbol{\Xi}_s(u,\cdot)$ defined in (2.2.25).

Corollary 2.7.4. *Assume that $n > 2s$. In addition to Theorem 2.7.1 and Theorem 2.7.3, if $\{x_k\} \subseteq \Omega$ is a sequence converging to $x_* \in \Omega$, then*

$$\limsup_{k\to\infty} \boldsymbol{\Xi}_s(u_k,x_k) \leqslant \boldsymbol{\Xi}_s(u,x_*)\,.$$

Proof. Without loss of generality, we can assume that $x_* = 0$. Applying Corollary 2.2.17, we obtain for $r > 0$ small enough and $r_k := |x_k|$,

$$\boldsymbol{\Xi}_s(u_k,x_k) \leqslant \boldsymbol{\Theta}(u_k^{\rm e},\mathbf{x}_k,r) \leqslant \frac{1}{r^{n-2s}}\mathbf{E}_s(u_k^{\rm e},B^+_{r+r_k})\,, \tag{2.7.8}$$

where $\mathbf{x}_k := (x_k,0)$. By Theorem 2.7.1 (in the case $s \neq 1/2$) and Theorem 2.7.3 (in the case $s = 1/2$), $u_k^{\rm e} \to u^{\rm e}$ strongly in $H^1(B^+_{2r},|z|^a{\rm d}\mathbf{x})$. Since $r_k \to 0$, we deduce from (2.7.8) that

$$\limsup_{k\to\infty} \boldsymbol{\Xi}_s(u_k,x_k) \leqslant \boldsymbol{\Theta}(u^{\rm e},0,r)\,,$$

and the conclusion follows letting $r \to 0$. □

2.7.2 Tangent maps

We assume throughout this subsection that $s \in (0,1)$ and $n > 2s$. We consider a bounded open set $\Omega \subseteq \mathbb{R}^n$ and a map $u \in \widehat{H}^s(\Omega;\mathbb{S}^{d-1})$ that we assume to be

- a stationary weakly s-harmonic map in Ω for $s \neq 1/2$;
- a minimizing $1/2$-harmonic map in Ω for $s = 1/2$.

We shall apply the results of Section 2.7.1 to define the so-called *tangent maps* of u at a given point. To this purpose, we fix a point of study $x_0 \in \Omega$ and a reference radius $\rho_0 > 0$ such that $D_{2\rho_0}(x_0) \subseteq \Omega$. We introduce the rescaled function

$$u_{x_0,\rho}(x) := u(x_0 + \rho x)\,,$$

and we observe that $(u_{x_0,\rho})^{\rm e}(\mathbf{x}) = u^{\rm e}(\mathbf{x}_0 + \rho\mathbf{x}) = u^{\rm e}_{x_0,\rho}(\mathbf{x})$ with $\mathbf{x}_0 = (x_0, 0)$. Rescaling variables, $u_{x_0,\rho}$ is a stationary weakly s-harmonic map in $(\Omega - x_0)/\rho$ for $s \neq 1/2$, or a minimizing $1/2$-harmonic map in $(\Omega - x_0)/\rho$ for $s = 1/2$. In addition,

$$\boldsymbol{\Theta}_s(u^{\rm e}_{x_0,\rho}, 0, r) = \boldsymbol{\Theta}_s(u^{\rm e}, \mathbf{x}_0, \rho r) \quad \forall r \in (0, \rho_0/\rho]\,. \tag{2.7.9}$$

Together with the monotonicity formula in Proposition 2.2.16 and Lemma 2.2.8, this identity yields

$$\boldsymbol{\Theta}_s(u^{\rm e}_{x_0,\rho}, 0, r) \leqslant \boldsymbol{\Theta}_s(u^{\rm e}, \mathbf{x}_0, \rho_0) \leqslant C\rho_0^{2s-n}\mathcal{E}_s(u,\Omega) \quad \forall r \in (0, \rho_0/\rho]\,,$$

for a constant C depending only on n and s. In turn, Lemma 2.2.7 implies that

$$[u_{x_0,\rho}]^2_{H^s(D_{2r})} \leqslant C\rho_0^{2s-n} r^{n-2s}\mathcal{E}_s(u,\Omega) \quad \forall r \in (0, \rho_0/(4\rho)]\,.$$

Using $|u_{x_0,\rho}| = 1$, we can now estimate for $r \in (0, \rho_0/(4\rho)]$,

$$\mathcal{E}_s(u_{x_0,\rho}, D_r) \leqslant C\Big([u_{x_0,\rho}]^2_{H^s(D_{2r})} + \iint_{D_r\times D^c_{2r}} \frac{{\rm d}x{\rm d}y}{|x-y|^{n+2s}}\Big) \leqslant Cr^{n-2s}\big(\rho_0^{2s-n}\mathcal{E}_s(u,\Omega) + 1\big)\,.$$

Given a sequence $\rho_k \to 0$, we deduce from the above estimate that

$$\limsup_{k\to\infty} \mathcal{E}_s(u_{x_0,\rho_k}, D_r) < +\infty \quad \forall r > 0\,.$$

Applying Theorem 2.7.1, Theorem 2.7.2, and Theorem 2.7.3, we can now find a subsequence $\{\rho'_k\}$ and $\varphi \in H^s_{\rm loc}(\mathbb{R}^n;\mathbb{S}^{d-1})$ such that

$$u_{x_0,\rho'_k} \to \varphi \text{ strongly in } \widehat{H}^s(D_r)\,,$$

and

$$u^{\rm e}_{x_0,\rho'_k} \to \varphi^{\rm e} \text{ strongly in } H^1(B^+_r, |z|^a{\rm d}\mathbf{x}) \text{ for all } r > 0\,,$$

where

(i) *if $s \neq 1/2$:* φ is a stationary weakly s-harmonic map in D_r for all $r > 0$;

(ii) *if $s \leqslant 1/2$ and u minimizing:* φ is a minimizing s-harmonic map in D_r for all $r > 0$.

Definition 2.7.5. Every function φ obtained by this process will be referred to as a *tangent map to u at the point x_0*. The family of all tangent maps to u at x_0 is denoted by $T_{x_0}(u)$.

We now present some classical properties of tangent maps following e.g. [107] or [81, Section 6].

Lemma 2.7.6. *If $\varphi \in T_{x_0}(u)$, then*

$$\boldsymbol{\Theta}_s(\varphi^{\rm e},0,r) = \boldsymbol{\Xi}_s(\varphi,0) = \boldsymbol{\Xi}_s(u,x_0) \quad \forall r>0\,,$$

and φ is positively 0-homogeneous, i.e., $\varphi(\lambda x) = \varphi(x)$ for every $\lambda > 0$ and $x \in \mathbb{R}^n$. In particular,

$$\boldsymbol{\Xi}_s(\varphi,\lambda x) = \boldsymbol{\Xi}_s(\varphi,x) \quad \textit{for every } x \in \mathbb{R}^n \setminus \{0\} \textit{ and } \lambda > 0\,. \tag{2.7.10}$$

Proof. From the strong convergence of $u^{\rm e}_{x_0,\rho'_k}$ to $\varphi^{\rm e}$ in $H^1(B_r^+,|z|^a{\rm d}\mathbf{x})$ and (2.7.9), we first deduce that

$$\boldsymbol{\Theta}_s(\varphi^{\rm e},0,r) = \lim_{k\to\infty} \boldsymbol{\Theta}_s(u^{\rm e},\mathbf{x}_0,\rho'_k r) = \boldsymbol{\Xi}_s(u,x_0) \quad \forall r>0\,.$$

Then, the constancy of $r \mapsto \boldsymbol{\Theta}_s(\varphi^{\rm e},0,r)$ together with the monotonicity formula in Proposition 2.2.16 implies that $\mathbf{x}\cdot\nabla\varphi^{\rm e}(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^{n+1}_+$. Hence, $\varphi^{\rm e}$ is positively 0-homogeneous, and the homogeneity of φ follows. As a consequence, for $x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$,

$$\boldsymbol{\Theta}_s(\varphi^{\rm e},\lambda\mathbf{x},r) = \boldsymbol{\Theta}_s(\varphi^{\rm e},\mathbf{x},r/\lambda)\,,$$

where $\mathbf{x} := (x,0)$. Letting now $r \to 0$ yields (2.7.10). □

Lemma 2.7.7. *If $\varphi \in T_{x_0}(u)$, then*

$$\boldsymbol{\Xi}_s(\varphi,y) \leqslant \boldsymbol{\Xi}_s(\varphi,0) \quad \forall y \in \mathbb{R}^n\,.$$

In addition, the set

$$S(\varphi) := \Big\{ y \in \mathbb{R}^n : \boldsymbol{\Xi}_s(\varphi,y) = \boldsymbol{\Xi}_s(\varphi,0) \Big\}$$

is a linear subspace of $\mathbb{R}^n$, and $\varphi(x+y) = \varphi(x)$ for every $y \in S(\varphi)$ and every $x \in \mathbb{R}^n$.

Proof. Step 1. By Corollary 2.2.17, we have have for every $y \in \mathbb{R}^n$ and $\rho > 0$,

$$\boldsymbol{\Xi}_s(\varphi,y) + \boldsymbol{\delta}_s \int_{B^+_\rho(\mathbf{y})} z^a \frac{|(\mathbf{x}-\mathbf{y})\cdot\nabla\varphi^{\rm e}|^2}{|\mathbf{x}-\mathbf{y}|^{n+2-2s}}\,{\rm d}\mathbf{x} = \boldsymbol{\Theta}_s(\varphi^{\rm e},\mathbf{y},\rho)\,, \tag{2.7.11}$$

where $\mathbf{y} = (y,0)$. On the other hand, by homogeneity of φ,

$$\boldsymbol{\Theta}_s(\varphi^{\rm e},\mathbf{y},\rho) \leqslant \frac{(\rho+|y|)^{n-2s}}{\rho^{n-2s}} \boldsymbol{\Theta}_s(\varphi^{\rm e},0,\rho+|y|) = \frac{(\rho+|y|)^{n-2s}}{\rho^{n-2s}} \boldsymbol{\Xi}_s(\varphi,0)\,.$$

Combining this inequality with (2.7.11) and letting $\rho \to \infty$ yields

$$\boldsymbol{\Xi}_s(\varphi,y) + \boldsymbol{\delta}_s \int_{\mathbb{R}^{n+1}_+} z^a \frac{|(\mathbf{x}-\mathbf{y})\cdot\nabla\varphi^{\rm e}(\mathbf{x})|^2}{|\mathbf{x}-\mathbf{y}|^{n+2-2s}}\,{\rm d}\mathbf{x} \leqslant \boldsymbol{\Xi}_s(\varphi,0)\,.$$

Step 2. Next, assume that $\boldsymbol{\Xi}_s(\varphi,y) = \boldsymbol{\Xi}_s(\varphi,0)$ for some $y \neq 0$. Then $(\mathbf{x}-\mathbf{y})\cdot\nabla\varphi^{\rm e}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^{n+1}_+$. By 0-homogeneity of $\varphi^{\rm e}$, we then have $\mathbf{y}\cdot\nabla\varphi^{\rm e}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^{n+1}_+$, and thus

$$\varphi(x+y) = \varphi(x) \quad \forall x \in \mathbb{R}^n\,. \tag{2.7.12}$$

The other way around, if (2.7.12) holds for some $y \neq 0$, then $(\mathbf{x}-\mathbf{y})\cdot\nabla\varphi^{\rm e}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^{n+1}_+$ (again by homogeneity). We then infer from (2.7.11) and (2.7.12) that for $\rho > 0$,

$$\boldsymbol{\Xi}_s(\varphi,y) = \boldsymbol{\Theta}_s(\varphi^{\rm e},\mathbf{y},\rho) = \boldsymbol{\Theta}_s(\varphi^{\rm e},0,\rho) = \boldsymbol{\Xi}_s(\varphi,0)\,,$$

i.e., $y \in S(\varphi)$. Hence, (2.7.12) caracterizes $S(\varphi)$, and the linearity of $S(\varphi)$ follows. □

Remark 2.7.8. If there exists $\varphi \in T_{x_0}(u)$ such that $\dim S(\varphi) = n$, then φ is clearly constant, and thus $\boldsymbol{\Xi}_s(u, x_0) = \boldsymbol{\Xi}_s(\varphi, 0) = 0$. By Theorem 2.5.1, u is thus continuous in a neighborhood of x_0, so that $\varphi = u(x_0)$. In other words, $T_{x_0}(u) = \{u(x_0)\}$.

As a consequence, if on the contrary $\boldsymbol{\Xi}_s(u, x_0) > 0$, then all tangent maps $\varphi \in T_{x_0}(u)$ must be nonconstant, and hence satisfy $\dim S(\varphi) \leqslant n - 1$.

Lemma 2.7.9. *Assume that $s \in [1/2, 1)$. If $\varphi \in T_{x_0}(u)$ is not constant, then*

$$\dim S(\varphi) \leqslant n - 2\,.$$

Proof. We proceed by contradiction assuming that there exists a nonconstant tangent map $\varphi \in T_{x_0}(u)$ such that $\dim S(\varphi) = n - 1$. Rotating coordinates if necessary, we can assume that $S(\varphi) = \{0\} \times \mathbb{R}^{n-1}$. By Lemma 2.7.7, the map φ only depends on the x_1-variable, that is $\varphi(x) =: \psi(x_1)$ where $\psi \in H^s_{\rm loc}(\mathbb{R}; \mathbb{S}^{d-1})$. Since φ is positively 0-homogeneous and nonconstant, the map ψ is of the form

$$\psi(x_1) = \begin{cases} \mathrm{a} & \text{if } x_1 > 0 \\ \mathrm{b} & \text{if } x_1 < 0\,, \end{cases} \tag{2.7.13}$$

for some points $\mathrm{a}, \mathrm{b} \in \mathbb{S}^{d-1}$, $\mathrm{a} \neq \mathrm{b}$. However, the space $H^s_{\rm loc}(\mathbb{R})$ embeds into $C^{0,s-1/2}_{\rm loc}(\mathbb{R})$, which enforces $\mathrm{a} = \mathrm{b}$, a contradiction. □

Lemma 2.7.10. *Assume that $n \geqslant 2$, $s \in (0, 1/2)$, and that u is a minimizing s-harmonic map in Ω. If $\varphi \in T_{x_0}(u)$ is not constant, then*

$$\dim S(\varphi) \leqslant n - 2\,.$$

To prove Lemma 2.7.10, we shall make use of the following pleasant computation.

Remark 2.7.11. For $n \geqslant 2$, we have

$$\alpha_{n,s} := \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}x'}{(1 + |x'|^2)^{\frac{n+2s}{2}}} = \frac{\gamma_{1,s}}{\gamma_{n,s}}\,. \tag{2.7.14}$$

Indeed, we easily compute in polar coordinates, and then setting $t := r^2$,

$$\int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}x'}{(1 + |x'|^2)^{\frac{n+2s}{2}}} = |\mathbb{S}^{n-2}| \int_0^{+\infty} \frac{r^{n-2}}{(1 + r^2)^{\frac{n+2s}{2}}}\,\mathrm{d}r = \frac{|\mathbb{S}^{n-2}|}{2} \int_0^{+\infty} \frac{t^{\frac{n-1}{2}-1}}{(1 + t)^{\frac{n+2s}{2}}}\,\mathrm{d}t\,.$$

Recalling the value of $\gamma_{n,s}$ given in (2.2.1), we thus have

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}x'}{(1 + |x'|^2)^{\frac{n+2s}{2}}} &= \frac{|\mathbb{S}^{n-2}|}{2} \mathrm{B}\Big(\frac{n-1}{2}, \frac{1+2s}{2}\Big) \\ &= \frac{|\mathbb{S}^{n-2}|}{2} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+2s}{2})}{\Gamma(\frac{n+2s}{2})} = \frac{\gamma_{1,s}}{\gamma_{n,s}}\,, \end{aligned} \tag{2.7.15}$$

where $\mathrm{B}(\cdot, \cdot)$ denotes the Euler Beta function.

Proof of Lemma 2.7.10. Step 1. We proceed again by contradiction assuming that there exists a nonconstant tangent map $\varphi \in T_{x_0}(u)$ such that $\dim S(\varphi) = n - 1$. Rotating coordinates if necessary, we can proceed as in the proof of Lemma 2.7.9 to infer that $\varphi(x) =: \psi(x_1)$ where $\psi \in H^s_{\rm loc}(\mathbb{R}; \mathbb{S}^{d-1})$ is of the form (2.7.13) for some points $\mathrm{a}, \mathrm{b} \in \mathbb{S}^{d-1}$, $\mathrm{a} \neq \mathrm{b}$. We claim that ψ is a minimizing s-harmonic map in the interval $(-1, 1)$. Once the claim is proved (which is the object of the next step), we infer from the regularity result [82, Theorem 1.2] that ψ is continuous in $(-1,1)$, which again enforces $\mathrm{a}=\mathrm{b}$, a contradiction.

Step 2. We now prove that ψ is a minimizing s-harmonic map in $(-1,1)$. To this purpose, we fix an arbitrary competitor $v \in \widehat{H}^s((-1,1);\mathbb{S}^{d-1})$ such that $\mathrm{spt}(v-\psi) \subseteq (-1,1)$. Given $r>1$, we consider the open set $Q_r \subseteq \mathbb{R}^n$ defined by $Q_r := (-1,1)\times D'_r$ where D'_r denotes the open ball in $\mathbb{R}^{n-1}$ centered at the origin of radius r. We define a map $\widetilde{v}_r \in \widehat{H}^s(Q_r;\mathbb{S}^{d-1})$ by setting for $x=(x_1,x')\in\mathbb{R}^n$,

$$\widetilde{v}_r(x) := \begin{cases} v(x_1) & \text{if } |x'|<r\,, \\ \psi(x_1) & \text{if } |x'|\geqslant r\,. \end{cases}$$

Recalling that u is assumed to be minimizing, φ is minimizing in every ball. Since $\mathrm{spt}(\widetilde{v}_r-\varphi)\subseteq Q_{r+1}$, we thus have

$$\mathcal{E}_s(\varphi,Q_{r+1}) \leqslant \mathcal{E}_s(\widetilde{v}_r,Q_{r+1})\,.$$

Since $\widetilde{v}_r=\varphi$ in $\mathbb{R}^n\setminus Q_r$, it reduces to

$$\mathcal{E}_s(\varphi,Q_r) \leqslant \mathcal{E}_s(\widetilde{v}_r,Q_r)\,. \tag{2.7.16}$$

We claim that

$$\frac{1}{|D'_r|}\,\mathcal{E}_s(\widetilde{v}_r,Q_r) \underset{r\to\infty}{\longrightarrow} \mathcal{E}_s(v,(-1,1))\,, \tag{2.7.17}$$

where $|D'_r|$ denotes the volume of D'_r in $\mathbb{R}^{n-1}$. Since we could have taken v to be equal to ψ, (2.7.17) also holds with φ in place of $\widetilde{v}_r$ and ψ in place of v. Therefore, dividing both sides of (2.7.16) by $|D'_r|$ and letting $r\to\infty$ leads to

$$\mathcal{E}_s(\psi,(-1,1)) \leqslant \mathcal{E}_s(v,(-1,1))\,,$$

which proves that ψ is indeed minimizing in $(-1,1)$.

Let us now compute $\mathcal{E}_s(\widetilde{v}_r,Q_r)$ to prove (2.7.17). First, by Fubini's theorem we have

$$\begin{aligned}\iint_{Q_r\times Q_r} &\frac{|\widetilde{v}_r(x)-\widetilde{v}_r(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \\ &= \iint_{(-1,1)^2} |v(x_1)-v(y_1)|^2\Big(\iint_{D'_r\times D'_r}\frac{\mathrm{d}x'\mathrm{d}y'}{(|x_1-y_1|^2+|x'-y'|^2)^{\frac{n+2s}{2}}}\Big)\,\mathrm{d}x_1\mathrm{d}y_1\,.\end{aligned}$$

Then we observe that a change of variables yields

$$\begin{aligned}\iint_{D'_r\times D'_r} &\frac{\mathrm{d}x'\mathrm{d}y'}{(|x_1-y_1|^2+|x'-y'|^2)^{\frac{n+2s}{2}}} \\ &= \iint_{D'_r\times\mathbb{R}^n}\frac{\mathrm{d}x'\mathrm{d}y'}{(|x_1-y_1|^2+|x'-y'|^2)^{\frac{n+2s}{2}}} - A_r(|x_1-y_1|) \\ &= \frac{\alpha_{n,s}|D'_r|}{|x_1-y_1|^{1+2s}} - A_r(|x_1-y_1|)\,,\end{aligned}$$

where $\alpha_{n,s}$ is given by (2.7.14), and $A_r(t)$ is defined for $t>0$ by

$$A_r(t) := \iint_{D'_r\times(D'_r)^c}\frac{\mathrm{d}x'\mathrm{d}y'}{(t^2+|x'-y'|^2)^{\frac{n+2s}{2}}}\,.$$

Therefore,

$$\iint_{Q_r\times Q_r}\frac{|\widetilde{v}_r(x)-\widetilde{v}_r(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y=\alpha_{n,s}|D'_r|\iint_{(-1,1)^2}\frac{|v(x_1)-v(y_1)|^2}{|x_1-y_1|^2}\,\mathrm{d}x_1\mathrm{d}y_1 - \iint_{(-1,1)^2}|v(x_1)-v(y_1)|^2A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,. \tag{2.7.18}$$

Similarly, we compute

$$\begin{aligned}\iint_{Q_r\times(Q_r)^c}&\frac{|\widetilde{v}_r(x)-\widetilde{v}_r(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y\\ &=\iint_{(-1,1)\times(-1,1)^c}|v(x_1)-v(y_1)|^2\Big(\iint_{D'_r\times\mathbb{R}^n}\frac{\mathrm{d}x'\mathrm{d}y'}{(|x_1-y_1|^2+|x'-y'|^2)^{\frac{n+2s}{2}}}\Big)\,\mathrm{d}x_1\mathrm{d}y_1\\ &\qquad+\iint_{(-1,1)^2}|v(x_1)-\psi(y_1)|^2A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,,\end{aligned}$$

so that

$$\iint_{Q_r\times(Q_r)^c}\frac{|\widetilde{v}_r(x)-\widetilde{v}_r(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y=\alpha_{n,s}|D'_r|\iint_{(-1,1)\times(-1,1)^c}\frac{|v(x_1)-v(y_1)|^2}{|x_1-y_1|^2}\,\mathrm{d}x_1\mathrm{d}y_1 + \iint_{(-1,1)^2}|v(x_1)-\psi(y_1)|^2A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,. \tag{2.7.19}$$

Combining (2.7.18) and (2.7.19) leads to

$$\frac{1}{|D'_r|}\,\mathcal{E}_s(\widetilde{v}_r,Q_r)=\mathcal{E}_s(v,(-1,1))-I_r+II_r\,,$$

where

$$I_r:=\frac{\gamma_{1,s}}{4|D'_r|}\iint_{(-1,1)^2}|v(x_1)-v(y_1)|^2A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,,$$

and

$$II_r:=\frac{\gamma_{1,s}}{2|D'_r|}\iint_{(-1,1)^2}|v(x_1)-\psi(y_1)|^2A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,.$$

Since $|v|=|\psi|=1$, we have

$$I_r+II_r\leqslant Cr^{1-n}\iint_{(-1,1)^2}A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\,,$$

and using Fubini's theorem again, we estimate

$$\begin{aligned}\iint_{(-1,1)^2}&A_r(|x_1-y_1|)\,\mathrm{d}x_1\mathrm{d}y_1\\ &\leqslant\iint_{D'_r\times(D'_r)^c}\Big(\iint_{(-1,1)\times\mathbb{R}}\frac{\mathrm{d}x_1\mathrm{d}y_1}{(|x_1-y_1|^2+|x'-y'|^2)^{\frac{n+2s}{2}}}\Big)\,\mathrm{d}x'\mathrm{d}y'\\ &\leqslant C\iint_{D'_r\times(D'_r)^c}\frac{\mathrm{d}x'\mathrm{d}y'}{|x'-y'|^{n-1+2s}}\\ &\leqslant Cr^{n-1-2s}\,.\end{aligned}$$

Therefore,

$$\frac{1}{|D'_r|}\,\mathcal{E}_s(\widetilde{v}_r,Q_r)=\mathcal{E}_s(v,(-1,1))+O(r^{-2s})\,,$$

and the proof is complete. □

2.7.3 Proof of Theorems 2.1.1, 2.1.2 and 2.1.3

Proof of Theorem 2.1.1. Let us fix an arbitrary point $x_0 \in \Omega$, and set $r_0 := \frac{1}{2}\mathrm{dist}(x_0, \Omega^c)$. Without loss of generality, we can assume that $x_0 = 0$, so that our aim is to show that u is smooth in a neighborhood of $x_0 = 0$. As noticed in Remark 2.4.2, the function $r \in (0, 2r_0 - |\mathbf{x}|) \mapsto \boldsymbol{\Theta}_s(u^{\mathrm{e}}, \mathbf{x}, r)$ is nondecreasing for every $\mathbf{x} \in \partial^0 B^+_{2r_0}$. Moreover, since $2s - n = 2s - 1 \geqslant 0$, we have

$$\lim_{r \to 0} \boldsymbol{\theta}_s(u, 0, r) = 0 \,.$$

Then we deduce from Corollary 2.2.19 that

$$\lim_{r \to 0} \boldsymbol{\Theta}_s(u^{\mathrm{e}}, 0, r) = 0 \,.$$

As a consequence, we can find $r_1 \in (0, r_0)$ such that $\boldsymbol{\Theta}(u^{\mathrm{e}}, 0, r_1) \leqslant \boldsymbol{\varepsilon}_1$, where the constant $\boldsymbol{\varepsilon}_1$ is given by Corollary 2.4.1. From Theorem 2.5.1, we infer that $u \in C^{0,1}(D_{\boldsymbol{\kappa}_2} r_1)$ for a constant $\boldsymbol{\kappa}_2 \in (0,1)$ depending only on s. In turn, Theorem 2.6.1 tells us that $u \in C^\infty(D_{\boldsymbol{\kappa}_2 r_1/2})$. □

Proof of Theorem 2.1.2, case $s = 1/2$. Considering the constant $\boldsymbol{\varepsilon}_1 > 0$ given by Corollary 2.4.1, we define

$$\Sigma := \Big\{ x \in \Omega : \boldsymbol{\Xi}_s(u, x) \geqslant \boldsymbol{\varepsilon}_1 \Big\} \,. \tag{2.7.20}$$

By Corollary 2.2.17, Σ is a relatively closed subset of Ω. On the other hand, it is well known that $\mathscr{H}^{n-1}(\Sigma) = 0$, see e.g. [116, Corollary 3.2.3].

We claim that $u \in C^\infty(\Omega \setminus \Sigma)$. Indeed, if $x_0 \in \Omega \setminus \Sigma$, then we can find a radius $r \in (0, \frac{1}{2}\mathrm{dist}(x_0, \Omega^c))$ such that $\boldsymbol{\Theta}(u^{\mathrm{e}}, 0, r) \leqslant \boldsymbol{\varepsilon}_1$. Applying Theorem 2.5.1 and Theorem 2.6.1, we conclude that $u \in C^\infty(D_{\boldsymbol{\kappa}_2 r/2})$, and the claim is proved.

Obviously, $\mathrm{sing}(u) \subseteq \Sigma$, and it now only remains to show that $\mathrm{sing}(u) = \Sigma$. This is in fact a direct consequence of the regularity result in [64, Theorem 4.1]. Indeed, assume by contradiction that there is a point $x_0 \in \Sigma \setminus \mathrm{sing}(u)$. Since $\mathrm{sing}(u)$ is relatively closed subset of Ω, we can find $r > 0$ such that $D_{2r}(x_0) \subseteq \Omega \setminus \mathrm{sing}(u)$, i.e., u is continuous in $D_{2r}(x_0)$. Consequently, u^{e} is continuous in $B^+_r(\mathbf{x}_0) \cup \partial^0 B^+_r(\mathbf{x}_0)$, where $\mathbf{x}_0 = (x_0, 0)$. However, by Proposition 2.3.13 (with $s = 1/2$), $u^{\mathrm{e}} \in H^1(B^+_r(\mathbf{x}_0); \mathbb{R}^d)$ also solves

$$\int_{B_r(x_0)} \nabla u^{\mathrm{e}} \cdot \nabla \Phi \, \mathrm{d}\mathbf{x} = 0$$

for every $\Phi \in H^1(B_r(\mathbf{x}_0); \mathbb{R}^d)$ such that $\Phi = 0$ on $\partial^+ B_r(\mathbf{x}_0)$ and $u \cdot \Phi = 0$ on $\partial^0 B_r(\mathbf{x}_0)$. Then [64, Theorem 4.1] tells us that $u^{\mathrm{e}} \in C^{1,\alpha}(B^+_{r/2}(x_0))$ for every $\alpha \in (0,1)$. Consequently, $\boldsymbol{\Xi}_s(u, x_0) = 0$, i.e., $x_0 \notin \Sigma$, a contradiction. □

Proof of Theorem 2.1.2, case $s \neq 1/2$. We still consider the relatively closed subset Σ of Ω defined in (2.7.20). As in the case $s = 1/2$, it follows from Theorem 2.5.1 and Theorem 2.6.1 that $u \in C^\infty(\Omega \setminus \Sigma)$. In particular, $\mathrm{sing}(u) \subseteq \Sigma$. On the other hand, if u is continuous in a neighborhood of a point $x_0 \in \Omega$, then $T_{x_0}(u) = \{u(x_0)\}$, and thus $\boldsymbol{\Xi}_s(u, x_0) = 0$. Hence, $x_0 \notin \Sigma$, and we conclude that $\mathrm{sing}(u) = \Sigma$. In view of Remark 2.7.8 and Lemma 2.7.9, we have

$$\Sigma = \begin{cases} \{x \in \Omega : \dim S(\varphi) \leqslant n - 1 \;\; \forall \varphi \in T_x(u)\} & \text{if } s \in (0, 1/2)\,; \\ \{x \in \Omega : \dim S(\varphi) \leqslant n - 2 \;\; \forall \varphi \in T_x(u)\} & \text{if } s \in (1/2, 1)\,. \end{cases}$$

We can now apply e.g. [107, Chapter 3.4, proof of Lemma 1] (which only relies on the upper semicontinuity of $\boldsymbol{\Xi}_s$ stated in Corollary 2.7.4, the strong convergence of blowups to tangent maps, and the structure results on tangent maps established in Section 2.7.2) to conclude that $\dim_{\mathscr{H}}\Sigma \leqslant n-1$ for $s \in (0,1/2)$, $\dim_{\mathscr{H}}\Sigma \leqslant n-2$ for $s \in (1/2,1)$, and that Σ is locally finite in Ω if $n=1$ with $s \in (0,1/2)$ or $n=2$ with $s \in (1/2,1)$. □

Proof of Theorem 2.1.3. For $s \in (1/2,1)$, we simply apply Theorem 2.1.2 (recalling minimality implies stationarity). We thus assume that $s \in (0,1/2]$. Since u is minimizing in Ω, the results in Section 2.7.2 apply. Hence, we can repeat the proof of Theorem 2.1.2 to derive that $u \in C^\infty(\Omega \setminus \Sigma)$, $\mathrm{sing}(u) = \Sigma$, where Σ is still given by (2.7.20). In view of Lemma 2.7.9 and Lemma 2.7.10, we now have

$$\Sigma = \big\{x \in \Omega : \dim S(\varphi) \leqslant n-2 \ \ \forall \varphi \in T_x(u)\big\}\,.$$

Once again, [107, Chapter 3.4, proof of Lemma 1] shows that $\dim_{\mathscr{H}}\Sigma \leqslant n-2$, and that Σ is locally finite in Ω if $n=2$. □

Appendices

2.A On the degenerate Laplace equation

In this first appendix, our aim is to recall some of the properties satisfied by weak solutions of the (scalar) degenerate linear elliptic equation

$$\mathrm{div}(|z|^a\nabla w) = 0 \quad \text{in } B_R(\mathbf{x}_0)\,, \tag{2.A.1}$$

with $\mathbf{x}_0 = (x_0, z_0) \in \mathbb{R}^{n+1}$. Those properties are essentially taken from [93], and we reproduce here the statements for convenience of the reader. The notion of weak solution to this equation corresponds to the variational formulation. In other words, we say that $w \in H^1(B_R(\mathbf{x}_0), |z|^a\mathrm{d}\mathbf{x})$ is a weak solution of (2.A.1) if

$$\int_{B_R} |z|^a \nabla w \cdot \nabla \Phi \,\mathrm{d}\mathbf{x} = 0$$

for every $\Phi \in H^1(B_R(\mathbf{x}_0), |z|^a\mathrm{d}\mathbf{x})$ such that $\Phi = 0$ on $\partial B_R(\mathbf{x}_0)$.

One may complement (2.A.1) with a boundary condition of the form $w = v$ on $\partial B_R(\mathbf{x}_0)$ for a given $v \in H^1(B_R(\mathbf{x}_0), |z|^a\mathrm{d}\mathbf{x})$. This boundary condition is thus interpreted in the sense of traces. Classically, such a boundary condition uniquely determines the solution of (2.A.1) which can be characterized by energy minimality.

Lemma 2.A.1. *Let $v \in H^1(B_R(\mathbf{x}_0), |z|^a\mathrm{d}\mathbf{x})$. The equation*

$$\begin{cases} \mathrm{div}(|z|^a\nabla w) = 0 & \text{in } B_R(\mathbf{x_0})\,,\\ w = v & \text{on } \partial B_R(\mathbf{x}_0)\,, \end{cases} \tag{2.A.2}$$

admits a unique weak solution which is characterized by

$$\int_{B_R(\mathbf{x}_0)} |z|^a|\nabla w|^2\,\mathrm{d}\mathbf{x} \leqslant \int_{B_R(\mathbf{x}_0)} |z|^a|\nabla \Phi|^2\,\mathrm{d}\mathbf{x}$$

for every $\Phi \in H^1(B_R(\mathbf{x}_0), |z|^a\mathrm{d}\mathbf{x})$ satisfying $\Phi = v$ on $\partial B_R(\mathbf{x_0})$.

As for the usual Laplace equation, energy minimality can be used to prove that w inherits symmetries from the boundary condition. In our case, we make use of the following lemma.

Lemma 2.A.2. *Let $\mathbf{x}_0 \in \mathbb{R}^n \times \{0\}$ and $v \in H^1(B_R, |z|^a \mathrm{d}\mathbf{x})$. If v is symmetric with respect to $\{z = 0\}$, then the weak solution w of (2.A.2) is also symmetric with respect to $\{z = 0\}$.*

Concerning interior regularity of weak solutions, the issue is of course near the hyperplane $\{z = 0\}$. Indeed, if the ball $B_R(\mathbf{x}_0)$ is away from $\{z = 0\}$, then the operator becomes uniformly elliptic with smooth coefficients, and the classical elliptic theory tells us that weak solutions are C^∞ in the interior. For an arbitrary ball, the general results of [40] about degenerate elliptic equations apply, and they provide at least local Hölder continuity in the interior. Using the invariance of the equation with respect to the x-variables, the regularity can be further improved (see e.g. [93, Corollary 2.13]). Some boundary regularity and related maximum principles are also known from the general theory in [59]. We reproduce here the statement in [93, Lemma 2.18].

Lemma 2.A.3. *Let $v \in H^1(B_R(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x}) \cap C^0(\overline{B}_R(\mathbf{x}_0))$. The weak solution w of (2.A.2) belongs to $C^0(\overline{B}_R(\mathbf{x}_0))$. Moreover,*

$$\min_{\overline{B}_R(\mathbf{x}_0)} w = \min_{\partial B_R(\mathbf{x}_0)} v \quad \text{and} \quad \max_{\overline{B}_R(\mathbf{x}_0)} w = \max_{\partial B_R(\mathbf{x}_0)} v\,.$$

A further fundamental property of weak solutions of (2.A.1) is an energy monotonicity in which one has to distinguish balls centered at a point of $\{z = 0\}$ from balls lying away from $\{z = 0\}$. The two following lemmas are taken from [93, Lemma 2.8] and [93, Lemma 2.17], respectively.

Lemma 2.A.4. *Let $\mathbf{x}_0 \in \mathbb{R}^n \times \{0\}$ and $w \in H^1(B_R(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$ be a weak solution of (2.A.1). Assume that either $s \geqslant 1/2$, or that $s < 1/2$ and w is symmetric with respect to the hyperplane $\{z = 0\}$. Then,*

$$\frac{1}{\rho^{n+2-2s}} \int_{B_\rho(\mathbf{x}_0)} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x} \leqslant \frac{1}{r^{n+2-2s}} \int_{B_r(\mathbf{x}_0)} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x}$$

for every $0 < \rho \leqslant r \leqslant R$.

Lemma 2.A.5. *Let $w \in H^1(B_R(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$ be a weak solution of (2.A.1). If $\mathbf{x}_0 = (x_0, z_0) \in \mathbb{R}^{n+1}_+$ and $R > 0$ are such that $B_R(\mathbf{x}_0) \subseteq \mathbb{R}^{n+1}_+$ and $z_0 \geqslant \theta R$ for some $\theta \geqslant 2$, then*

$$\Big(\frac{2}{R}\Big)^{n+1} \int_{B_{R/2}(\mathbf{x}_0)} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x} \leqslant \Big(1 + \frac{C}{\theta - 1}\Big) \frac{1}{R^{n+1}} \int_{B_R(\mathbf{x}_0)} |z|^a |\nabla w|^2 \, \mathrm{d}\mathbf{x}\,,$$

for a constant $C = C(n)$.

2.B A Lipschitz estimate for s-harmonic functions

The purpose of this appendix is to provide an interior Lipschitz estimate for weak solutions $w \in \widehat{H}^s(D_1)$ of the fractional Laplace equation

$$(-\Delta)^s w = 0 \quad \text{in } H^{-s}(D_1)\,. \tag{2.B.3}$$

The notion of weak solution is understood here according to the weak formulation of the s-Laplacian operator, see (2.2.3). Interior regularity for weak solutions is known, and it tells us that w is locally C^∞ in D_1. The following estimate is probably also well known, but we give a proof for convenience of the reader.

Lemma 2.B.6. *If $w \in \widehat{H}^s(D_1)$ is a weak solution of (2.B.3), then $w \in C^\infty(D_{1/2})$, and*

$$\|w\|^2_{L^\infty(D_{1/2})} + \|\nabla w\|^2_{L^\infty(D_{1/2})} \leqslant C\big(\mathcal{E}_s(w, D_1) + \|w\|^2_{L^2(D_1)}\big)\,, \tag{2.B.4}$$

for a constant $C = C(n, s)$.

Proof. As we already mentioned, the regularity theory is already known, and we take advantage of this to only derive estimate (2.B.4). Let us fix an arbitrary point $x_0 \in D_{1/2}$. We consider the extension $w^{\rm e}$ which belongs to $H^1(B^+_{1/4}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$ with $\mathbf{x}_0 := (x_0, 0)$ by Lemma 2.2.8. In view of Lemma 2.2.11, it satisfies

$$\int_{B^+_{1/4}(\mathbf{x}_0)} z^a \nabla w^{\rm e} \cdot \nabla \Phi \,\mathrm{d}x = 0$$

for every $\Phi \in H^1(B^+_{1/4}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$ such that $\Phi = 0$ on $\partial^+ B^+_{1/4}(\mathbf{x}_0)$. Then we consider the even extension of $w^{\rm e}$ to the whole ball $B_{1/4}(\mathbf{x}_0)$ that we still denote by $w^{\rm e}$ (i.e. $w^{\rm e}(x, z) = w^{\rm e}(x, -z)$). Then $w^{\rm e} \in H^1(B_{1/4}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$, and arguing as in the proof of Corollary 2.5.3, we infer that $w^{\rm e}$ is a weak solution of (2.A.1) with $R = 1/4$. According to [93, Corollary 2.13], the weak derivatives $\partial_i w^{\rm e}$ belongs to $H^1(B_{1/8}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})$ for $i = 1, \ldots, n$, and they are weak solutions of (2.A.1) with $R = 1/8$. Now, applying [40, Theorem 2.3.12] to $w^{\rm e}$ and $\nabla_x w^{\rm e}$, we infer that $w^{\rm e} \in C^{1,\alpha}(B_{1/16}(\mathbf{x}_0))$ for some exponent $\alpha = \alpha(n, s) \in (0, 1)$,

$$[w^{\rm e}]_{C^{0,\alpha}(B_{1/16}(\mathbf{x}_0))} \leqslant C\|w^{\rm e}\|_{L^2(B_{1/8}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})}\,, \tag{2.B.5}$$

and

$$[\nabla_x w^{\rm e}]_{C^{0,\alpha}(B_{1/16}(\mathbf{x}_0))} \leqslant C\|\nabla_x w^{\rm e}\|_{L^2(B_{1/8}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})}\,, \tag{2.B.6}$$

for a constant $C = C(n, s)$.

On the other hand, for every $\mathbf{x} \in B_{1/16}(\mathbf{x}_0)$, we have (recall our notation in (2.5.11))

$$\begin{aligned}|w^{\rm e}(\mathbf{x})| \leqslant \Big|w^{\rm e}(\mathbf{x}) - \frac{1}{|B_{1/16}|_a}\int_{B_{1/16}(x_0)} |z|^a w^{\rm e}(\mathbf{y})\mathrm{d}\mathbf{y}\Big| + \frac{1}{|B_{1/16}|_a}\int_{B_{1/16}(x_0)} |z|^a |w^{\rm e}(\mathbf{y})|\mathrm{d}\mathbf{y} \\ \leqslant C\big([w^{\rm e}]_{C^{0,\alpha}(B_{1/16}(\mathbf{x}_0))} + \|w^{\rm e}\|_{L^2(B_{1/16}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})}\big)\,.\end{aligned}$$

Combining this estimate with (2.B.5) and Lemma 2.2.8 leads to

$$\|w^{\rm e}\|^2_{L^\infty(B_{1/16}(x_0))} \leqslant C\big(\mathcal{E}_s(w, D_1) + \|w\|^2_{L^2(D_1)}\big)\,.$$

The same argument applied to $\nabla_x w^{\rm e}$ and using (2.B.6) instead of (2.B.5) yields

$$\|\nabla_x w^{\rm e}\|^2_{L^\infty(B_{1/16}(x_0))} \leqslant C\|\nabla_x w^{\rm e}\|^2_{L^2(B_{1/8}(\mathbf{x}_0), |z|^a \mathrm{d}\mathbf{x})} \leqslant C\mathcal{E}_s(w, D_1)\,,$$

thanks to Lemma 2.2.8 again. Now the conclusion follows from the fact that $w^{\rm e} = w$ and $\nabla_x w^{\rm e} = \nabla w$ on $\partial^0 B^+_{1/16}(\mathbf{x}_0)$. □

2.C An embedding theorem between generalized $\mathcal{Q}_\alpha$-spaces

In this appendix, our goal is to prove one of the crucial estimates used in the proof of Theorem 2.4.1, Corollary 2.C.11 below. In turns out that this estimate does not explicitly appear in the existing literature (to the best of our knowledge), but it can be shortly derived from recent results in harmonic analysis. The purpose of this appendix is thus to explain how to combine those results to reach our goal. First, we need to recall some definitions and notations.

The space $\mathscr{S}_\infty(\mathbb{R}^n)$ can be defined as the topological subspace of the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ made of all functions φ such that the seminorm

$$\|\varphi\|_M := \sup_{|\gamma|\leqslant M} \sup_{\xi\in\mathbb{R}^n} |\partial^\gamma \widehat{\varphi}(\xi)|(|\xi|^M + |\xi|^{-M})$$

is finite for every $M \in \mathbb{N}$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, $|\gamma| := \gamma_1 + \ldots + \gamma_n$, and $\partial^\gamma := \partial_1^{\gamma_1} \ldots \partial_n^{\gamma_n}$. Its topological dual is denoted by $\mathscr{S}'_\infty(\mathbb{R}^n)$, and it is endowed with the weak $*$-topology, see e.g. [112, 114].

The following $\mathcal{Q}_p^{\alpha,q}$-spaces were introduced in [22, 114], generalizing the notion of $\mathcal{Q}_\alpha$-space (see [96, Section 1.2.4] and references therein), in the sense that $\mathcal{Q}_\alpha(\mathbb{R}^n) = \mathcal{Q}^{\alpha,2}_{n/\alpha}(\mathbb{R}^n)$.

Definition 2.C.7 ([22, 114])**.** Given $\alpha \in (0,1)$, $p \in (0,\infty]$ and $q \in [1,\infty)$, define $\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)$ as the space made of elements $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that $f(x)-f(y)$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\|f\|_{\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)} := \sup_Q |Q|^{1/p-1/q} \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}} \,\mathrm{d}x\mathrm{d}y \right)^{1/q} < +\infty\,,$$

where Q ranges over all cubes of dyadic edge lengths in $\mathbb{R}^n$.

Remark 2.C.8. Endowed with $\|\cdot\|_{\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)}$, the space $\mathcal{Q}_p^{\alpha,q}(\mathbb{R}^n)$ is a seminormed vector space, and

$$N_{\alpha,p,q}(f) := \sup_{D_r(x_0)\subseteq\mathbb{R}^n} r^{n/p-n/q} \left(\iint_{D_r(x_0)\times D_r(x_0)} \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}} \,\mathrm{d}x\mathrm{d}y \right)^{1/q}$$

provides an equivalent seminorm.

The following embeddings between $\mathcal{Q}_p^{\alpha,q}$-spaces hold.

Theorem 2.C.1. *If $0 < \alpha_1 < \alpha_2 < 1$, $1 \leqslant q_2 < q_1 < \infty$, and $0 < \lambda \leqslant n$ are such that*

$$\alpha_1 - \frac{\lambda}{q_1} = \alpha_2 - \frac{\lambda}{q_2}\,, \tag{2.C.7}$$

then $\mathcal{Q}^{\alpha_2,q_2}_{\frac{nq_2}{\lambda}}(\mathbb{R}^n) \hookrightarrow \mathcal{Q}^{\alpha_1,q_1}_{\frac{nq_1}{\lambda}}(\mathbb{R}^n)$ continuously.

As we briefly mentioned at the beginning of this appendix, this theorem actually follows quite directly from a more general embedding result between some homogeneous Triebel-Lizorkin-Morrey-Lorentz spaces [63] together with an identification result between various definitions of homogeneous Triebel-Lizorkin-Morrey type spaces [96], and a characterization of the $\mathcal{Q}_p^{\alpha,q}$-spaces within this scale of spaces [114]. We refer to the monograph [96] for what concerns the spaces involved here, and we limit ourselves to their basic definition. To this purpose, we consider a reference bump function $\psi \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\operatorname{spt}\widehat{\psi} \subseteq \left\{\xi\in\mathbb{R}^n : \frac{1}{2} \leqslant |\xi| \leqslant 2\right\} \quad \text{and} \quad |\widehat{\psi}(\xi)| \geqslant C > 0 \text{ for } \frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3}\,.$$

(In particular, $\psi \in \mathscr{S}_\infty(\mathbb{R}^n)$.) For $j \in \mathbb{Z}$, we denote by ψ_j the function defined by

$$\psi_j(x) := 2^{jn}\psi(2^j x)\,.$$

Definition 2.C.9. Given $p, q \in (0, \infty)$, $s \in \mathbb{R}$, and $\tau \in [0, \infty)$, the homogeneous Triebel-Lizorkin space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|^\tau} \left(\int_Q \left(\sum_{j=j_Q}^{\infty} (2^{js}|\psi_j * f(x)|)^q \right)^{p/q} \mathrm{d}x \right)^{1/p} < +\infty \, ,$$

where Q ranges over all cubes of dyadic edge lengths in $\mathbb{R}^n$, and $j_Q := -\log_2 \ell(Q)$ with $\ell(Q)$ the edge length of Q.

Definition 2.C.10. Given $0 < p \leqslant u < \infty$, $0 < q < \infty$, and $s \in \mathbb{R}$, the homogeneous Triebel-Lizorkin-Morrey space $\dot{\mathcal{E}}^s_{p,q,u}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,u}(\mathbb{R}^n)} := \sup_Q |Q|^{1/u-1/p} \left(\int_Q \left(\sum_{j\in\mathbb{Z}} (2^{js}|\psi_j * f(x)|)^p \right)^{q/p} \mathrm{d}x \right)^{1/q} < +\infty \, ,$$

where Q ranges over all cubes of dyadic edge lengths in $\mathbb{R}^n$.

Proof of Theorem 2.C.1. In [63], the author introduced a more refined scale of homogeneous Triebel-Lizorkin spaces of Morrey-Lorentz type, denoted by $\dot{F}^{s,u}_{M_{p,q,\lambda}}(\mathbb{R}^n)$. In the case $u = p = q$, those spaces coincide with the homogeneous Triebel-Lizorkin-Morrey spaces above, namely

$$\dot{F}^{s,p}_{M_{p,p,\lambda}}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{p,p,\frac{np}{\lambda}}(\mathbb{R}^n)$$

for every $p \in (0, \infty)$, $\lambda \in (0, n]$, and $s \in \mathbb{R}$. More precisely, their defining seminorms are equivalent (in one case the supremum is taken over all dyadic cubes, while in the other it is taken over balls). By [63, Theorem 4.1], under condition (2.C.7) the space $\dot{F}^{\alpha_2,q_2}_{M_{q_2,q_2,\lambda}}(\mathbb{R}^n)$ embeds continuously into $\dot{F}^{\alpha_1,q_1}_{M_{q_1,q_1,\lambda}}(\mathbb{R}^n)$. In other words,

$$\dot{\mathcal{E}}^{\alpha_2}_{q_2,q_2,\frac{nq_2}{\lambda}}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}^{\alpha_1}_{q_1,q_1,\frac{nq_1}{\lambda}}(\mathbb{R}^n) \tag{2.C.8}$$

continuously. On the other hand, [96, Theorem 1.1] tells us that

$$\dot{\mathcal{E}}^{\alpha_1}_{q_1,q_1,\frac{nq_1}{\lambda}}(\mathbb{R}^n) = \dot{F}^{\alpha_1,\frac{n-\lambda}{nq_1}}_{q_1,q_1}(\mathbb{R}^n) \quad \text{and} \quad \dot{\mathcal{E}}^{\alpha_2}_{q_2,q_2,\frac{nq_2}{\lambda}}(\mathbb{R}^n) = \dot{F}^{\alpha_2,\frac{n-\lambda}{nq_2}}_{q_2,q_2}(\mathbb{R}^n) \, ,$$

with equivalent seminorms. Finally, by [114, Theorem 3.1] we have

$$\dot{F}^{\alpha_1,\frac{n-\lambda}{nq_1}}_{q_1,q_1}(\mathbb{R}^n) = \mathcal{Q}^{\alpha_1,q_1}_{\frac{nq_1}{\lambda}}(\mathbb{R}^n) \quad \text{and} \quad \dot{F}^{\alpha_2,\frac{n-\lambda}{nq_2}}_{q_2,q_2}(\mathbb{R}^n) = \mathcal{Q}^{\alpha_2,q_2}_{\frac{nq_2}{\lambda}}(\mathbb{R}^n) \, ,$$

with equivalent seminorms. Hence, the conclusion follows from (2.C.8). □

We are now ready to state the important corollary of Theorem 2.C.1 used in the proof of Theorem 2.4.1. Given $s \in (0, 1)$, $p \in [1, \infty)$, and an open set $\Omega \subseteq \mathbb{R}^n$, we recall that the Sobolev-Slobodeckij $W^{s,p}(\Omega)$-seminorm of a measurable function f is given by

$$[f]^p_{W^{s,p}(\Omega)} := \iint_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, \mathrm{d}x\mathrm{d}y \, . \tag{2.C.9}$$

Corollary 2.C.11. *Let $s \in (0,1)$ and $f \in L^1(\mathbb{R}^n)$ with compact support. If*

$$\sup_{D_r(x) \subseteq \mathbb{R}^n} r^{2s-n} [f]^2_{H^s(D_r(x))} < +\infty \,, \tag{2.C.10}$$

then,

$$\sup_{D_r(x) \subseteq \mathbb{R}^n} r^{\frac{2s-n}{3}} [f]^2_{W^{s/3,6}(D_r(x))} \leqslant C \sup_{D_r(x) \subseteq \mathbb{R}^n} r^{2s-n} [f]^2_{H^s(D_r(x))} \,,$$

for a constant $C = C(n,s)$.

Proof. Since $f \in L^1(\mathbb{R}^n)$ has compact support, it clearly belongs to $\mathscr{S}'_\infty(\mathbb{R}^n)$. Then, condition (2.C.10) implies that $f \in \mathcal{Q}^{s,2}_{n/s}(\mathbb{R}^n)$. On the other hand, $\mathcal{Q}^{s,2}_{n/s}(\mathbb{R}^n) \hookrightarrow \mathcal{Q}^{s/3,6}_{3n/s}(\mathbb{R}^n)$ continuously by Theorem 2.C.1. Then the conclusion follows from the definition of $\mathcal{Q}^{s/3,6}_{3n/s}(\mathbb{R}^n)$ together with Remark 2.C.8. $\square$

Chapter 3

Large mass minimizers for an isoperimetric problem with a repulsive integrable potential

3.1 Introduction

We study a variant of Gamow's liquid drop model for the atomic nucleus, in which the repulsive term is given by a general nonnegative and radially nonincreasing kernel. More precisely, given $K : \mathbb{R}^n \to [0, +\infty)$ a measurable and nonnegative function, $n \geqslant 2$, we consider the minimization problem

$$\min \left\{ P(E) + \iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \;:\; |E| = m \right\}, \tag{P1}$$

where the minimum is taken over all sets of finite perimeter of volume m and $P(E)$ denotes the perimeter of E. Except in Section 3.2, we shall always assume that K satisfies the following general hypotheses:

(H1) $K \in L^1(\mathbb{R}^n) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, and

$$\int_{\mathbb{R}^n} |x| K(x)\,\mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n} |\nabla K(x)||x|\,\mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n} |\nabla K(x)||x|^2\,\mathrm{d}x < +\infty;$$

(H2) there exists a nonnegative and nonincreasing function $k : (0, +\infty) \to \mathbb{R}$ such that $K(x) = k(|x|)$ for $\mathscr{L}^n - a.e.\ x \in \mathbb{R}^n$;

(H3) k' is continuous on $(0, +\infty)$.

Starting from Section 3.5, we add the extra assumption

(H4) $K(x) = o(|x|^{-(n+1)})$ at infinity, and $K(x) = o(|x|^{\alpha-n})$ near the origin, for some $\alpha > 0$.

As we will show, the so-called Bessel kernels are natural examples of kernels satisfying these assumptions. Let us remark that Problem (P1) is a minimization problem where the two terms compete with each other: the local perimeter term constrains the set E to concentrate as much as possible, while the nonlocal term acts as a repulsive term, forcing E to spread. Indeed, it is known that the perimeter is *minimized* by balls under volume constraint, while the nonlocal term is *maximized* by balls (by Riesz' symmetric rearrangement, using [71, Chapter 3.7] and the fact that K is equal to its symmetric rearrangement).

We are interested in the asymptotic behavior of this minimization problem for large masses, and give answers to several natural questions: does Problem (P1) admit minimizers? If so, what do those minimizers look like, are they regular? Can the ball be a minimizer?

To state our main results, we define

$$I_K^{l,p} := \int_{\mathbb{R}^n} |x|^p |\partial_r^l K(x)| \,\mathrm{d}x = n|B_1| \int_0^\infty |k^{(l)}(r)| r^{p+n-1} \,\mathrm{d}r$$

for $l \in \{0,1\}$ and $p \in \{0,1,2\}$, where $\partial_r K$ is the radial derivative of K, and

$$\mathbf{K}_{p,n} := \fint_{\mathbb{S}^{n-1}} |e \cdot x|^p \, d\mathscr{H}_x^{n-1}, \tag{3.1.1}$$

which does not depend on $e \in \mathbb{S}^{n-1}$ by symmetry. We prove that if $I_K^{0,1}$ is small enough, then there exists a critical mass above which Problem (P1) admits a minimizer, and that up to translations and rescaling, any sequence of minimizers converges to the unit ball as the mass goes to infinity.

Theorem 3.1.1. *Assume $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. Then there exists $m_e = m_e(n, K)$ such that, for any $m > m_e$, Problem (P1) admits a minimizer, and any minimizer E is, up to a translation, included in $4[B]_m$ up to a set of vanishing Lebesgue measure, where $[B]_m$ denotes the ball of volume m centered at the origin.*

Theorem 3.1.2. *Assume $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers going to infinity, and for all $k \in \mathbb{N}$, let E_k be a minimizer of Problem (P1) of mass m_k such that*

$$\int_{E_k} x \,\mathrm{d}x = 0.$$

Then letting $F_k := \left(\frac{|B_1|}{m_k}\right)^{\frac{1}{n}} E_k$, the sequence $(F_k)_{k\in\mathbb{N}}$ of sets of finite perimeter of volume $|B|$ converges to the unit ball B centered at the origin w.r.t. to the L^1 norm, i.e.,

$$|F_k \triangle B| \xrightarrow{k\to\infty} 0.$$

The main obstacle for proving the existence with the direct method in the calculus of variations is the possibility for a minimizing sequence to have some mass escape at infinity. We solve this problem of lack of compactness by showing that for large masses a minimizing sequence may be constrained inside a ball via a truncation lemma. One of the keys to do so, and to study Problem (P1) in general, is to use the integrability of the kernel K on $\mathbb{R}^n$, and to rewrite the nonlocal term as

$$\iint_{E\times E} K(x-y) \,\mathrm{d}x \,\mathrm{d}y = m I_K^{0,0} - \iint_{E\times E^c} K(x-y) \,\mathrm{d}x \,\mathrm{d}y,$$

so that Problem (P1) is equivalent to

$$\min \left\{ P(E) - \mathrm{Per}_K(E) \;:\; |E| = m \right\},$$

where

$$\mathrm{Per}_K(E) := \iint_{E\times E^c} K(x-y) \,\mathrm{d}x \,\mathrm{d}y.$$

The functional Per_K should be considered as a "nonlocal perimeter", which behaves in many ways as a (standard) perimeter term rather than a volume term. The convergence of rescaled minimizers follows from the computation of the Γ-limit of the rescaled functional of Problem (P1). Applying results from [88], when $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, we obtain that minimizers have a $C^{1,\frac{1}{2}}$ reduced boundary, and we show that they are necessarily connected whenever K is not compactly supported.

Then we recall a well-suited notion of stability (see Section 3.5.1) and show that if $I_K^{0,1}$ is above the threshold of Theorems 3.1.1 and 3.1.2, then for large masses, balls are not stable, and thus cannot be minimizers of Problem (P1).

Theorem 3.1.3. *If $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ and K satisfies (H4) as well, there exists $m_u = m_u(n, K)$ such that for any $m > m_u$ the ball $[B]_m$ is not stable for the functional of Problem (P1).*

The proof for the instability of large balls relies essentially on the study of the Jacobi operator associated with the minimized functional, and on a result similar to the one by J. Bourgain, H. Brezis, and P. Mironescu in [12] for $W^{1,2}(\mathbb{S}^{n-1})$, i.e., computation of the limit,

$$\lim_{\varepsilon\to 0} \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1},$$

where $(\eta_\varepsilon)_{\varepsilon>0}$ is a family of "$(n-1)$-dimensional mollifiers", and $f \in C^2(\partial B)$.

Outline of the chapter

This chapter is organized as follows. In Section 3.2 we discuss a few variants of Gamow's liquid drops model that have already been studied in the literature and motivate the choice for our assumptions (H1) to (H4). We also recall some well-known results on isoperimetric inequalities. In Section 3.3 we establish basic prerequisites on nonlocal perimeters, essentially that they behave to some extent similarly to the classical perimeter, and recall the definition and basic properties of Bessel kernels, which are the typical kernels for which our results apply. Then in Section 3.4 we prove existence of large mass minimizers, i.e. Theorem 3.1.1. We then compute the Γ-limit of the rescaled functional of Problem (P1), and show that if $I_K^{0,1}$ is small enough, the Γ-limit is a positive multiple of the perimeter, which implies Theorem 3.1.2, a straightforward consequence of Corollary 3.4.8. We conclude this section by establishing $C^{1,\frac{1}{2}}$ regularity (applying directly results in [88]) and connectedness of minimizers, and by giving some ideas and conjectures for deriving uniform density estimates (w.r.t. the volume) for minimizers of the rescaled problem, which have yet to be obtained. Eventually, in Section 3.5, we study the stability of the ball, showing that there is a threshold for $I_K^{0,1}$ above which large balls are unstable, i.e. Theorem 3.1.3, a direct consequence of Theorem 3.5.6.

Notation

For any set $E \in \mathbb{R}^n$, we define $E^c := \mathbb{R}^n \setminus E$, and we write $|E|$ for its volume. Given two sets E and F, we note $E\triangle F := (E \setminus F) \sqcup (F \setminus E)$ their symmetric difference. We say that two sets E and F are equivalent if $|E\triangle F| = 0$. We denote by $B_r(x)$ the n-dimensional open ball of radius r centered at x. For simplicity we write B_r when x is the origin, and B when $r = 1$. The volume of B is $\omega_n := |B| = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$, and the area of the unit sphere $\mathbb{S}^{n-1}$ is $\mathscr{H}^{n-1}(\partial B) = n\omega_n$. We also denote by $\mathbb{S}^k$ the k-dimensional unit sphere, and simply $|\mathbb{S}^k|$ its area. For any $m > 0$ and $x \in \mathbb{R}^n$ we let $[B]_{x,m}$ be the open ball of volume m

centered at x, or simply $[B]_m$ if $x = 0$. For any open set $\Omega \subseteq \mathbb{R}^n$, we denote by $\mathrm{BV}(\Omega)$ the space of functions with bounded variation in Ω, and for any $f \in \mathrm{BV}(\Omega)$ we let $|Df|$ be its total variation measure, and $[f]_{\mathrm{BV}(\Omega)} := \int_\Omega |Df|$. For a set of finite perimeter E in Ω, we let $\chi_E \in \mathrm{BV}(\Omega)$ be its characteristic function (i.e., $\chi_E(x) = 1$ if $x \in E$ and 0 otherwise), and define its perimeter in Ω by $P(E;\Omega) := \int_\Omega |D\chi_E|$. If $\Omega = \mathbb{R}^n$ we simply write $P(E) := P(E;\mathbb{R}^n)$. We denote by μ_E the Gauss–Green measure associated with the set of finite perimeter E and $\nu_E(x)$ the outer unit normal of $\partial^* E$ at x, where $\partial^* E$ stands for the reduced boundary of E.

3.2 Motivation and context

3.2.1 No repulsion: the Classical Isoperimetric Problem

Let us first say a few words about the simplest case for Problem (P1), that is, when $K \equiv 0$. In that case, (P1) is the classical isoperimetric problem which consists in minimizing the perimeter under a volume constraint. It is known that the unique minimizer, up to translations, is the ball, which gives the classical isoperimetric inequality

$$P(E) \geqslant P([B]_m),$$

for any set of finite perimeter E with volume m, and can be rewritten

$$P(E) \geqslant n\omega_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}}. \tag{3.2.1}$$

We also have a relative version of this isoperimetric inequality: there exists $C = C(n)$ such that, for any set of finite perimeter E in $\mathbb{R}^n$, then for every ball $B_r(x)$ we have

$$\min(|E \cap B_r(x)|, |E^{\mathrm{c}} \cap B_r(x)|)^{1-\frac{1}{n}} \leqslant CP(E; B_r(x)).$$

Besides the fact that balls are solutions to the classical isoperimetric problem, we have the following related question: if the perimeter of a set E of volume m is close to $P([B]_m)$, is E close to the ball $[B]_m$ in some sense, and if so, is it possible to quantify it? This question has been answered in [50] (see also [49] for a refinement), and we recall the given answer below. Given E such that $|E| = m$, we define the *isoperimetric deficit* of E by

$$D(E) := \frac{P(E) - P([B]_m)}{P([B]_m)},$$

and its *Fraenkel asymmetry* by

$$\alpha(E) := \min\left\{ \frac{|E \triangle [B]_{y,m}|}{m} \ : \ y \in \mathbb{R}^n \right\}.$$

The sharp quantitative isoperimetric inequality proven in [50] then states that there exists $C = C(n)$ such that

$$\alpha(E) \leqslant C\sqrt{D(E)}, \tag{3.2.2}$$

and that the $\frac{1}{2}$ exponent over $D(E)$ is sharp. In addition to their intrinsic interest, isoperimetric inequalities are a very useful tool to study related isoperimetric problems, and we shall often rely on them in this chapter.

3.2.2 Slow decay at infinity: Riesz potentials

Problems such as (P1) are essentially inspired by a model for the atomic nucleus introduced by George Gamow in the late 1920s, which is now referred to as Gamow's liquid drop model for the atomic nucleus. This denomination is due to the fact that in this simple model (then refined by Heisenberg, von Wiezsäcker and Bohr in the 1930s), the protons and neutrons inside the atomic nucleus are treated as an incompressible and uniformly charged fluid. In this model, the atomic nucleus is represented by a set $\Omega \subseteq \mathbb{R}^3$ of volume m (which we also call the mass), and its energy is given by

$$P(\Omega) + \frac{1}{8\pi} \iint_{\Omega\times\Omega} \frac{1}{|x-y|}\, \mathrm{d}x\, \mathrm{d}y.$$

The perimeter term represents the energy associated with the attractive short-range nuclear force, while the Coulombic repulsive term is due to the positively charged protons pushing themselves away from each other. This model successfully explained the phenomenon of nuclear fission: indeed, there are two critical masses $0 < m_1 \leqslant m_2 < \infty$ such that, below m_1, the problem admits a minimizer (no fission), and above m_2, there is no minimizer (fission). In fact, there exists another threshold $0 < m_0 \leqslant m_1$ such that, below it, the ball is the unique minimizer (up to translations) These results were first rigorously proven in [66] (see also [67] for the planar case). Many variants and generalizations of this model have been proposed since then, one of the most natural being to replace the Newton potential $\frac{1}{|x|^{n-2}}$ in dimension 3 by Riesz potentials in arbitrary dimension $n \geqslant 2$, that is

$$K(x) = \frac{1}{|x|^{n-\alpha}}, \quad \alpha \in (0, n).$$

The Newton case $\alpha = 2$ in dimension $n \geqslant 3$ was treated e.g. in [65], the Riesz cases with $\alpha \in (0, n-1)$ in [10], and the complete Riesz case $\alpha \in (0, n)$ in any dimension in [45], where the perimeter $P(E)$ can also be replaced by fractional perimeters $P_s(E)$, $s \in (0, 1)$.

Let us sum up some of what is known in the Riesz case in the following theorems.

Theorem 3.2.1 ([67, 66, 65, 10, 45])**.** *Given $n \geqslant 2$ and $\alpha \in (0, n)$, then there exists $m_0 = m_0(n, \alpha)$ such that for any $m < m_0$ the ball $[B]_m$ is the unique minimizer, up to translations, of Problem (P1) for $K(x) = |x|^{-(n-\alpha)}$.*

There are also some nonexistence results.

Theorem 3.2.2 ([10, 67, 66, 72])**.** *Given $n \geqslant 2$ and $\alpha \in (n-2, n)$, then there exists $m_1 = m_1(n, \alpha)$ such that for any $m > m_1$, Problem (P1) admits no minimizer for $K(x) = |x|^{-(n-\alpha)}$.*

These nonexistence results for large masses are in a sense not surprising. Indeed, on the one hand, note that without the perimeter term the problem

$$\min \left\{ \iint_{E\times E} \frac{1}{|x-y|^{n-\alpha}}\, \mathrm{d}x\, \mathrm{d}y \ : \ |E| = m \right\}$$

admits no minimizer, since it is always better (Riesz kernels being stricly radially decreasing) to split a set E into infinitely many pieces and send them farther from each other at infinity. On the other hand, the relatively slow decay at infinity of the Riesz kernels make them nonintegrable, which would explain why the repulsive potential takes over the perimeter term in (P1) for large masses, resulting in the nonexistence of minimizers.

As for the thresholds m_0, m_1, and m_2, physical evidence indicate that in dimension $n = 3$ at least, they should be equal, but this has yet to be proven.

3.2.3 Compactly supported potentials

An interesting case is when the potential K has compact support. Recalling our informal discussion on nonexistence for Riesz potentials, we see that in the compact case, sending disjoint pieces of a set E at infinity does not decrease the energy of the nonlocal term: when the pieces are far enough, they simply have no interaction between each other. Thus we may imagine that we should be able to build a minimizing sequence lying in a fixed ball, hence get some compactness and prove the existence of minimizers. In dimension $n = 2$ this strategy can be implemented quite easily (the advantage being that sets of finite perimeter are essentially bounded, i.e., included in a ball), but in higher dimension it is not that simple.

Fortunately, using the link between minimizers of (P1) and "almost-minimizers" of the perimeter (see Section 3.4.4), that case was successfully treated by S. Rigot in [88], yielding the following result.

Theorem 3.2.3 ([88]). *If K is compactly supported, then Problem (P1) always admits minimizers. In addition, for any minimizer E, $\partial^* E$ is a $C^{1,\frac{1}{2}}$-hypersurface, and, up to a renormalization, E has a finite number of connected components N, where N can be bounded depending only on K, n and m.*

In our case, since the kernels considered are radially symmetric, one can wonder if minimizers in Theorem 3.2.3 are simply balls. In fact by Theorem 3.1.3, if $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, then for large masses this is not the case, and we have the following symmetry-breaking result.

Corollary 3.2.4. *In addition to Theorem 3.2.3, if $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, none of the minimizers for Problem (P1) is a ball for m large enough.*

3.2.4 In-between: Bessel potentials

Between Riesz kernels, which are slowly decreasing kernels, and compactly supported kernels, it is natural to wonder what happens in the intermediate case of rapidly decreasing kernels such as Bessel kernels. Bessel kernels are given by the operators $(I - \Delta)^{-\frac{\alpha}{2}}$ for $\alpha \in (0, n)$, i.e. the Bessel kernel of order α is the fundamental solution of

$$(I - \Delta)^{\frac{\alpha}{2}} f = \delta_0,$$

where δ_0 is the Dirac distribution at the origin. In fact, we can consider the "generalized" Bessel-type potentials given by $(I - \kappa\Delta)^{-\frac{\alpha}{2}}$, where $\alpha, \kappa \in (0, +\infty)$. As far as we know there is little literature on Problem (P1) when K is a Bessel kernel, and especially on the asymptotic behavior for large masses. Compared with Riesz kernels (which are associated with the operators $(-\Delta)^{-\frac{\alpha}{2}}$), Bessel kernels are generally not explicit, in the sense that they only have an integral representation, and they do not behave as nicely as Riesz kernels under scaling. Near the origin, Riesz and Bessel kernels of the same order α behave similarly, however at infinity Bessel kernels decay much faster. Their decay at infinity is *exponential* (in particular, they are integrable), making them an intermediate case between Riesz kernels and compactly supported kernels. Note that even though Bessel kernels decay exponentially, the situation is very different from the compact support case: here, there is always a little interaction between pieces of E, no matter how far they are to one another, thus we cannot use the strategy implemented in [88] to get compactness of minimizing sequences, even in dimension $n = 2$.

For small masses, the similarity between Riesz and Bessel kernels near the origin suggests that Problem (P1) presents the same kind of behavior whether K is a Riesz or a Bessel kernel of order α, that is, there exists a critical mass below which, up to translations, the ball of volume m is the unique minimizer. In this "small volume" case we believe the approach for the Riesz case in [45] can be adapted without major difficulties, but this is not the subject of this chapter.

We are more interested in the case of large volumes. For Riesz kernels of order $\alpha \in (n-2, n)$, it is known that above a critical mass, Problem (P1) admits no minimizer. Here, the better integrability of the Bessel kernels changes the asymptotic behavior when the mass goes to infinity: if κ is small enough, Problem (P1) admits large mass minimizers, and up to translations, any sequence of normalized (to unit mass) minimizers converges to the unit ball as the mass goes to infinity. We end this introductory discussion on Bessel kernels here, leaving the more technical reminders for Section 3.3.2.

3.3 Preliminaries

3.3.1 Nonlocal perimeters, reformulation of the problem

First, let us make a few remarks on assumptions (H1) to (H4) made on K, and some of their immediate consequences.

Remark 3.3.1. By (H1) and (H2), k is absolutely continuous on $(0, +\infty)$. In addition, since K is radial and radially nonincreasing, $\nabla K(x) \cdot x = k'(|x|)|x| = -|\nabla K(x)||x|$, and

$$|\mathbb{S}^{n-1}| \int_0^\infty k(r) r^{n-1} \, \mathrm{d}r = \int_{\mathbb{R}^n} K(x) \, \mathrm{d}x < \infty,$$
$$|\mathbb{S}^{n-1}| \int_0^\infty |k'(r)| r^{n} \, \mathrm{d}r = \int_{\mathbb{R}^n} |\nabla K(x)||x| \, \mathrm{d}x < \infty,$$
$$|\mathbb{S}^{n-1}| \int_0^\infty |k'(r)| r^{n+1} \, \mathrm{d}r = \int_{\mathbb{R}^n} |\nabla K(x)||x|^2 \, \mathrm{d}x < \infty.$$

By (H2) and (H3), ∇K is also continuous in $\mathbb{R}^n \setminus \{0\}$. Furthermore (H1) implies that $K(x) \leqslant C|x|^{-(n+1)}$ at infinity for some $C \geqslant 0$. Indeed, we know that if $f \in L^1_{\mathrm{loc}}(0, +\infty)$ and $f' \in L^1(0, +\infty)$, then f has a limit at infinity. Applying this to $f(r) := r^{n+1} k(r)$, we have $f' \in L^1(0, +\infty)$ in view of (H1), thus there exists $l \geqslant 0$ such that $r^{n+1} k(r) \to l$ which shows that $k(r) \leqslant 2l r^{-(n+1)}$ in a neighborhood of infinity. The same reasoning applied to $f(r) = r^n k(r)$ shows that $r^n k(r)$ has a limit in 0^+, and for $r \mapsto r^{n-1} k(r)$ to be integrable near the origin, this limit must vanish, thus $k(r) = o(r^{-n})$ near 0. In addition, we have the relation

$$I_K^{1,1} = n I_K^{0,0}. \tag{3.3.1}$$

Indeed, integrating $(k(r) r^n)' = k'(r) r^n + n k(r) r^{n-1}$ between r and R, we find

$$k(R) R^n - k(r) r^n = \int_r^R k'(s) s^n \, \mathrm{d}s + n \int_r^R k(s) s^{n-1} \, \mathrm{d}s,$$

thus using the fact that $k(r) = o(r^{-n})$ near the origin and at infinity, letting r go to 0 and R go to infinity yields

$$-\int_0^\infty k'(s) s^n \, \mathrm{d}s = n \int_0^\infty k(s)^{n-1} \, \mathrm{d}s,$$

hence (3.3.1), since $k' \leqslant 0$.

Remark 3.3.2. The bounds at the origin and at infinity of K given by (H4) are in fact required only for the study of the stability of the ball conducted in Section 3.5. The bound at the origin is required in order to be able to use directly the formula for the second variation of the nonlocal term computed in [45] (and it may actually be unnecessary), while the bound at infinity is to ensure that the family of $(n-1)$-dimensional mollifiers given by k satisfies (3.A.5) (which can actually be dropped in dimension larger than 2).

As we mentioned in the introduction, the rest of our study relies on the crucial observation that Problem (P1) is in fact equivalent to

$$\min\left(P(E) - \mathrm{Per}_K(E) \;:\; |E| = m\right), \tag{P2}$$

where

$$\mathrm{Per}_K(E) := \iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

is sometimes called the *nonlocal K-perimeter of E* (see e.g. [7, 20]). Indeed, for any E such that $|E| = m$, using the integrability of the kernel K on $\mathbb{R}^n$, we have

$$\begin{aligned}\iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{E\times\mathbb{R}^n} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= m\|K\|_{L^1(\mathbb{R}^n)} - \mathrm{Per}_K(E).\end{aligned} \tag{3.3.2}$$

Since the ball $[B]_m$ of volume m maximizes $\iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$, this also shows that $[B]_m$ minimizes $\mathrm{Per}_K(E)$ among sets of volume m. From now on, we set

$$\mathcal{F}_K := P - \mathrm{Per}_K\,. \tag{3.3.3}$$

One of the reasons why Per_K can be thought of as a perimeter appears if one imagines that the kernel K goes to infinity at the origin, and decreases quickly away from it. Heuristically in that case the part in $\mathrm{Per}_K(E)$ that prevails would be when $|x-y| < \varepsilon$ where ε is small, i.e.

$$\mathrm{Per}_K(E) \simeq \iint\limits_{\substack{E\times E^c\\ |x-y|<\varepsilon}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

and $(E\times E^c)\cap\{|x-y| < \varepsilon\} \subseteq (\varepsilon\text{-neighb}(\partial E))^2$, where ε-neighb(∂E) denotes an ε-neighborhood of ∂E, whenever E has a smooth boundary. In our case K may not be singular at the origin, but we still assume that it is radially decreasing, so that for $(x,y)\in E\times E^c$, the closer x and y are to each other (and thus to ∂E), the more $K(x-y)$ increases. In addition, $\mathrm{Per}_K(E)$ can be controlled by the classical perimeter if $I_K^{0,1} = \int_{\mathbb{R}^n}|x|K(x)\,\mathrm{d}x$ is well defined, using that

$$\|f(h+\cdot) - f\|_{L^1(\mathbb{R}^n)} \leqslant |h|[f]_{\mathrm{BV}(\mathbb{R}^n)}, \quad \forall f\in \mathrm{BV}(\mathbb{R}^n),\ \forall h\in\mathbb{R}^n.$$

Indeed, by Fubini's theorem we find

$$\begin{aligned}\mathrm{Per}_K(E) = \iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)|K(h)\,\mathrm{d}x\,\mathrm{d}h \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\|\chi_E(h+\cdot) - \chi_E\|_{L^1(\mathbb{R}^n)}K(h)\,\mathrm{d}h \\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^n}|h|P(E)K(h)\,\mathrm{d}h = \frac{I_K^{0,1}}{2}P(E).\end{aligned}$$

We can even refine the constant in this inequality using the following proposition, inspired by [12, Theorem 2] (see also [32, Lemma 3]).

Proposition 3.3.3. *Let $f \in \mathrm{BV}(B_R(x_0))$ for some $x_0 \in \mathbb{R}^n$ and $R > 0$, and let $\rho : (0,+\infty) \to [0,+\infty)$ a measurable function such that $\int_{\mathbb{R}^n} \rho(|x|)\,\mathrm{d}x = 1$. Then*

$$\iint_{B_R(x_0)\times B_R(x_0)} \frac{|f(x)-f(y)|}{|x-y|}\rho(x-y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \mathbf{K}_{1,n}\int_{B_R(x_0)}|Df|, \tag{3.3.4}$$

where $\mathbf{K}_{1,n}$ is defined by (3.1.1). As a consequence, if $f \in \mathrm{BV}(\mathbb{R}^n)$ we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|}\rho(x-y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \mathbf{K}_{1,n}\int_{\mathbb{R}^n}|Df|. \tag{3.3.5}$$

Proof. The proof is similar to [32, Proof of Lemma 3], with the difference that we want to obtain an upper bound involving only the L^1 norm of the gradient on $B_R(x_0)$. Up to a translation, without loss of generality we may assume that $x_0 = 0$. By density of $C^\infty(B_R)\cap W^{1,1}(B_R)$ in $\mathrm{BV}(B_R)$ we shall assume that $f \in C^\infty(B_R)\cap W^{1,1}(B_R)$. Integrating on lines we have

$$f(x+h) - f(x) = \int_0^1 \nabla f(x+th)\cdot h\,\mathrm{d}t,$$

thus, making the change of variables $h = x-y$, and noting that $h \in B_{2R}$, using Fubini's theorem we find,

$$\begin{aligned}
&\iint_{B_R\times B_R} \frac{|f(x)-f(y)|}{|x-y|}\rho(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \int_{B_R}\left(\int_{B_{2R}} \chi_{B_R(x)}(h)\frac{|f(x)-f(x-h)|}{|h|}\rho(|h|)\,\mathrm{d}h\right)\mathrm{d}x \\
&\qquad = \int_{B_{2R}}\left(\int_{B_R(h)\cap B_R} |f(x-h)-f(x)|\,\mathrm{d}x\right)\frac{\rho(|h|)}{|h|}\,\mathrm{d}h \\
&\qquad = \int_{B_{2R}}\left(\int_{B_R(h)\cap B_R}\left|\int_0^1 \nabla f(x-th)\cdot h\,\mathrm{d}t\right|\mathrm{d}x\right)\frac{\rho(|h|)}{|h|}\,\mathrm{d}h.
\end{aligned} \tag{3.3.6}$$

Applying the coarea formula to (3.3.6) and then Cauchy-Schwarz inequality and Fubini's theorem gives

$$\begin{aligned}
&\iint_{B_R\times B_R} \frac{|f(x)-f(y)|}{|x-y|}\rho(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_0^{2R}\int_{\mathbb{S}^{n-1}}\left(\int_{B_R(r\sigma)\cap B_R}\left|\int_0^1 \nabla f(x-tr\sigma)\cdot\sigma\,\mathrm{d}t\right|\mathrm{d}x\right)\rho(r)r^{n-1}\,d\mathscr{H}^{n-1}_\sigma\,\mathrm{d}r \\
&\quad \leqslant \int_0^{2R}\int_{\mathbb{S}^{n-1}}\left(\int_0^1\int_{B_R(r\sigma)\cap B_R} |\nabla f(x-tr\sigma)\cdot\sigma|\,\mathrm{d}x\,\mathrm{d}t\right)\rho(r)r^{n-1}\,d\mathscr{H}^{n-1}_\sigma\,\mathrm{d}r.
\end{aligned}$$

Since $x - tr\sigma \in B_R$ whenever $x \in B_R(r\sigma)\cap B_R$ and $t \in [0,1]$ (indeed $|x-tr\sigma| = |t(x-r\sigma)+(1-t)x| \leqslant t|x-r\sigma| + (1-t)|x| < R$), a change of variables $y = x + tr\sigma$ yields

$$\begin{aligned}
&\iint_{B_R\times B_R} \frac{|f(x)-f(y)|}{|x-y|}\rho(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \int_0^{2R}\int_{\mathbb{S}^{n-1}}\left(\int_0^1\int_{B_R} |\nabla f(y)\cdot\sigma|\,\mathrm{d}y\,\mathrm{d}t\right)\rho(r)r^{n-1}\,d\mathscr{H}^{n-1}_\sigma\,\mathrm{d}r \\
&\quad = \int_0^{2R}\int_{\mathbb{S}^{n-1}}\int_{B_R} |\nabla f(y)\cdot\sigma|\rho(r)r^{n-1}\,\mathrm{d}y\,d\mathscr{H}^{n-1}_\sigma\,\mathrm{d}r.
\end{aligned} \tag{3.3.7}$$

Using Fubini's theorem once again and the equality

$$\fint_{\mathbb{S}^{n-1}} |\nabla f(y) \cdot \sigma| \, d\mathscr{H}_\sigma^{n-1} = \mathbf{K}_{1,n} |\nabla f(y)|,$$

from (3.3.7) we obtain

$$\begin{aligned}
&\iint_{B_R \times B_R} \frac{|f(x) - f(y)|}{|x-y|} \rho(|x-y|) \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_0^{2R} \int_{B_R} \int_{\mathbb{S}^{n-1}} |\nabla f(y) \cdot \sigma| \rho(r) r^{n-1} \, d\mathscr{H}_\sigma^{n-1} \, \mathrm{d}y \, \mathrm{d}r \\
&\quad = |\mathbb{S}^{n-1}| \mathbf{K}_{1,n} \int_0^{2R} \int_{B_R} |\nabla f(y)| \rho(r) r^{n-1} \, \mathrm{d}y \, \mathrm{d}r \\
&\quad = \mathbf{K}_{1,n} \left(\int_{B_R} |\nabla f(y)| \, \mathrm{d}y \right) \left(|\mathbb{S}^{n-1}| \int_0^{2R} \rho(r) r^{n-1} \, \mathrm{d}r \right) \\
&\quad \leqslant \mathbf{K}_{1,n} \left(\int_{B_R} |\nabla f(y)| \, \mathrm{d}y \right) \left(\int_{B_{2R}} \rho(|x|) \, \mathrm{d}x \right),
\end{aligned}$$

hence (3.3.4), since

$$\int_{\mathbb{R}^n} \rho(|x|) \, \mathrm{d}x = 1.$$

We deduce (3.3.5) by letting R go to infinity in (3.3.4). ☐

Remark 3.3.4. Note that by Proposition 3.3.10 below, the constant $\mathbf{K}_{1,n}$ is optimal for (3.3.5).

Corollary 3.3.5. *For any E set of finite perimeter, we have*

$$\mathrm{Per}_K(E) \leqslant \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2} P(E).$$

Proof. Setting $\rho_K(x) := \frac{|x| K(x)}{I_K^{0,1}}$, one needs only rewrite $\mathrm{Per}_K(E)$ as

$$\begin{aligned}
\mathrm{Per}_K(E) &= \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\chi_E(x) - \chi_E(y)| K(x-y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{I_K^{0,1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|} \rho_K(x-y) \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

and apply Proposition 3.3.3 to conclude. ☐

Recall that the classical perimeter is lower semicontinuous with respect to the classical topology of $L^1(\mathbb{R}^n)$. Here the nonlocal perimeter is in fact continuous for the L^1 convergence, as is shown in the following lemma.

Lemma 3.3.6. *For any sets E and F with finite Lebesgue measure, we have*

$$|\mathrm{Per}_K(E) - \mathrm{Per}_K(F)| \leqslant I_K^{0,0} |E \triangle F|.$$

In particular if $(E_k)_{k \in \mathbb{N}}$ is a sequence of sets with finite measure converging to E in $L^1(\mathbb{R}^n)$, then $\mathrm{Per}_K(E_k) \to \mathrm{Per}_K(E)$.

Proof. Let E and F be sets with finite (possibly different) Lebesgue measure. Using

$$\mathrm{Per}_K(E) = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |\chi_E(x)-\chi_E(y)|K(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

we have

$$\mathrm{Per}_K(E)-\mathrm{Per}_K(F) = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(|\chi_E(x)-\chi_E(y)|-|\chi_F(x)-\chi_F(y)|\right)K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Thus by the reversed triangle inequality,

$$\begin{aligned}
|\mathrm{Per}_K(E)-\mathrm{Per}_K(F)| &\leqslant \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(|\chi_E(x)-\chi_F(x)|+|\chi_E(y)-\chi_F(y)|\right)K(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\chi_{E\triangle F}(x)+\chi_{E\triangle F}(y)\right)K(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&= I_K^{0,0}|E\triangle F|.
\end{aligned}$$

The claim of continuity w.r.t. the L^1 convergence is clear since $\|\chi_E-\chi_F\|_{L^1(\mathbb{R}^n)} = |E\triangle F|$. □

One of the nice properties of the classical perimeter is its behavior under scaling. For any set of finite perimeter E and any $\lambda>0$, we obviously have $P(\lambda E)=\lambda^{n-1}P(E)$. This is unfortunately not the case for such nonlocal perimeters. However just like $|P(\lambda E)-P(E)|\leqslant C|1-\lambda|P(E)$ for some C not depending of E, when λ is close to 1, one can show that the difference between $\mathrm{Per}_K(\lambda E)$ and $\mathrm{Per}_K(E)$ is controlled by a perimeter term of the form $C|1-\lambda|P(E)$.

Lemma 3.3.7. *For any set of finite perimeter E, $t\mapsto \mathrm{Per}_K(tE)\in C^1(0,+\infty)$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Per}_K(tE) = \frac{2n}{t}\mathrm{Per}_K(tE)+\frac{1}{t}\iint_{(tE)\times(tE)^{\mathrm{c}}} \nabla K(x-y)\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.3.8}$$

In particular, for any $t\in(\frac{1}{2},2)$, we have

$$|\mathrm{Per}_K(tE)-\mathrm{Per}_K(E)| \leqslant C_1|1-t|P(E), \tag{3.3.9}$$

for some $C_1 = C_1(n, I_K^{1,0}, I_K^{1,2})$.

Proof. Up to considering $F := E^{\mathrm{c}}$, we can assume without loss of generality that $|E|<+\infty$. By scaling we have

$$\mathrm{Per}_K(tE) = t^{2n}\iint_{E\times E^{\mathrm{c}}} K(t(x-y))\,\mathrm{d}x\,\mathrm{d}y \tag{3.3.10}$$

thus $t\mapsto \mathrm{Per}_K(t(E)\in C^1(0,+\infty)$ if and only if $f : t\mapsto \iint_{E\times E^{\mathrm{c}}} K(t(x-y))\,\mathrm{d}x\,\mathrm{d}y \in C^1(0,+\infty)$, and we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Per}_K(tE) &= 2nt^{2n-1}\iint_{E\times E^{\mathrm{c}}} K(t(x-y))\,\mathrm{d}x\,\mathrm{d}y + t^{2n}f'(t)\\
&= \frac{2n}{t}\mathrm{Per}_K(tE)+t^{2n}f'(t).
\end{aligned}$$

Let us show that $f \in C^0(0,+\infty)$. Since K is C^0 away from the origin by Remark 3.3.1, then for $\mathscr{L}^{2n}$-a.e. $(x,y) \in E \times E^c$, $t \mapsto K(t(x-y)) \in C^0(0,+\infty)$. In addition, since K is positive and radially nonincreasing, for any $t > t_0 > 0$, we have

$$|K(t(x-y))| \leqslant K(t_0(x-y))$$

and $(x,y) \mapsto K(t_0(x-y))$ is integrable on $E \times E^c$, thus by the theorem of continuity under the integral, f is continuous on $(t_0,+\infty)$, for all $t_0 > 0$, hence $f \in C^0(0,+\infty)$.
Given $0 < t_0 < t_1$ we show that $f \in C^1(t_0,t_1)$ and compute its derivative. Let $t \in (t_0,t_1)$, and $h_0 > 0$ so that $t_0 < t-h_0 < t+h_0 < t_1$. For any $h \neq 0$ such that $|h| < h_0$, integrating on lines, we write

$$\frac{f(t+h)-f(t)}{h} = \iint_{E\times E^c} \int_0^1 \nabla K((t+sh)(x-y)) \cdot (x-y) \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y. \tag{3.3.11}$$

Given $\varepsilon > 0$, we claim that there exists $R > 0$ such that for any $h \in [-h_0,h_0] \setminus \{0\}$, we have

$$\iint\limits_{\substack{E\times E^c \\ |x-y|<\frac{1}{R} \text{ or } |x-y|>R}} \int_0^1 |\nabla K((t+sh)(x-y)) \cdot (x-y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon \tag{3.3.12}$$

and

$$\iint\limits_{\substack{E\times E^c \\ |x-y|<\frac{1}{R} \text{ or } |x-y|>R}} |\nabla K(t(x-y)) \cdot (x-y)| \,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon. \tag{3.3.13}$$

Indeed, on the one hand, changing variables we have

$$\begin{aligned} &\iint\limits_{\substack{E\times E^c \\ |x-y|>R}} \int_0^1 |\nabla K((t+sh)(x-y)) \cdot (x-y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_0^1 (t+sh)^{-(2n+1)} \iint\limits_{\substack{((t+sh)E)\times((t+sh)E)^c \\ |x-y|>R/(t+sh)}} |\nabla K(x-y) \cdot (x-y)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant \int_0^1 (t+sh)^{-(2n+1)} \int_{(t+sh)E} \int_{|x-y|>R/(t+sh)} |\nabla K(x-y) \cdot (x-y)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant |E| t_1^n t_0^{-(2n+1)} \int_{B^c_{\frac{R}{t_1}}} |\nabla K(x) \cdot x| \,\mathrm{d}x \leqslant \varepsilon, \end{aligned} \tag{3.3.14}$$

for any R large enough independently of h, since $\nabla K(x) \cdot x$ belongs to $L^1(\mathbb{R}^n)$. On the other hand, proceeding similarly, we find

$$\begin{aligned} &\iint\limits_{\substack{E\times E^c \\ |x-y|<\frac{1}{R}}} \int_0^1 |\nabla K((t+sh)(x-y)) \cdot (x-y)| \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \leqslant |E| t_1^n t_0^{-(2n+1)} \int_{B_{\frac{1}{Rt_0}}} |\nabla K(x) \cdot x| \,\mathrm{d}x \leqslant \varepsilon, \end{aligned} \tag{3.3.15}$$

for R large enough as well, hence (3.3.12). Then (3.3.13) follows the same way. Combining (3.3.11), (3.3.12) and (3.3.13) yields

$$\begin{aligned}
&\Big|\frac{f(t+h)-f(t)}{h} - \iint\limits_{E\times E^{c}} \nabla K(t(x-y))\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\
&\quad\leqslant \Big| \iint\limits_{\substack{E\times E^{c}\\ \frac{1}{R}\leqslant|x-y|\leqslant R}} \int_0^1 \nabla K((t+sh)(x-y))\cdot(x-y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \iint\limits_{\substack{E\times E^{c}\\ 1/R\leqslant|x-y|\leqslant R}} \nabla K(t(x-y))\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y\Big| + 2\varepsilon,
\end{aligned} \tag{3.3.16}$$

for some large R independent of h. Since $\nabla K(x)\cdot x$ is continuous away from the origin by (H3), there exists $C>0$ such that

$$|\nabla K((t+sh)(x-y))\cdot(x-y)| \leqslant C, \qquad \text{whenever } 1/R\leqslant|x-y|\leqslant R.$$

In addition

$$\int_0^1 \nabla K((t+sh)(x-y))\cdot(x-y)\,\mathrm{d}s \xrightarrow{h\to 0} \nabla K(t(x-y))\cdot(x-y)$$

for $\mathscr{L}^{2n}$-a.e. $(x,y)\in E\times E^{c}$, and $\mathscr{L}^{2n}\big((E\times E^{c})\cap\{\frac{1}{R}\leqslant|x-y|\leqslant R\}\big)<+\infty$, thus by dominated convergence

$$\begin{aligned}
&\iint\limits_{\substack{E\times E^{c}\\ \frac{1}{R}\leqslant|x-y|\leqslant R}} \int_0^1 |\nabla K((t+sh)(x-y))\cdot(x-y)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \xrightarrow{h\to 0} \iint\limits_{\substack{E\times E^{c}\\ \frac{1}{R}\leqslant|x-y|\leqslant R}} |\nabla K(t(x-y))\cdot(x-y)|\,\mathrm{d}x\,\mathrm{d}y\Big|.
\end{aligned} \tag{3.3.17}$$

Combining (3.3.16) and (3.3.17) eventually yields

$$f'(t) = \iint_{E\times E^{c}} \nabla K(t(x-y))\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

for any $t\in(t_0,t_1)$, by the arbitrariness ε. Then we show the continuity of f' in (t_0,t_1). For $t_0<t<t_1$ and h_0 as before, for any $h\in[-h_0,h_0]\setminus\{0\}$ we write

$$\begin{aligned}
&|f'(t+h)-f'(t)| \\
&\quad\leqslant \iint\limits_{\substack{E\times E^{c}\\ \frac{1}{R}\leqslant|x-y|\leqslant R}} |\nabla K((t+h)(x-y))-\nabla K(t(x-y))||x-y|\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \iint\limits_{\substack{E\times E^{c}\\ |x-y|<\frac{1}{R} \text{ or } |x-y|>R}} |\nabla K((t+h)(x-y))-\nabla K(t(x-y))||x-y|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.3.18}$$

As before, given $\varepsilon > 0$, for R large enough we have

$$\iint\limits_{\substack{E\times E^c \\ |x-y|<\frac{1}{R} \text{ or } |x-y|>R}} |\nabla K((t+h)(x-y)) - \nabla K(t(x-y))||x-y| \,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon, \tag{3.3.19}$$

for any $h \in [-h_0, h_0] \setminus \{0\}$, and using the continuity of ∇K (in particular its boundedness) away from the origin, by dominated convergence we find

$$\iint\limits_{\substack{E\times E^c \\ \frac{1}{R}\leqslant|x-y|\leqslant R}} |\nabla K((t+h)(x-y)) - \nabla K(t(x-y))||x-y| \,\mathrm{d}x\,\mathrm{d}y \xrightarrow{h\to 0} 0. \tag{3.3.20}$$

Combining (3.3.18), (3.3.19) and (3.3.20) gives the continuity of f' in (t_0, t_1). Hence by arbitrariness of t_0 and t_1 and (3.3.10), $t \mapsto \mathrm{Per}_K(tE) \in C^1(0, +\infty)$ with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Per}_K(tE) &= \frac{2n}{t}\mathrm{Per}_K(tE) + t^{2n}\iint_{E\times E^c} \nabla K(t(x-y))\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2n}{t}\mathrm{Per}_K(tE) + \frac{1}{t}\iint_{(tE)\times(tE)^c} \nabla K(x-y)\cdot(x-y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which gives (3.3.8). There remains to show (3.3.9). Let $t \in (\frac{1}{2}, 2)$. Integrating in t and using (3.3.8), we find

$$\begin{aligned} |\mathrm{Per}_K(tE) - \mathrm{Per}_K(E)| &\leqslant \int_{[1,t]} \left|\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Per}_K(sE)\right| \mathrm{d}s \\ &\leqslant 4n\int_{[1,t]} \mathrm{Per}_K(sE)\,\mathrm{d}s \\ &\qquad + 2\int_{[1,t]}\iint_{(sE)\times(sE)^c} |\nabla K(x-y)||x-y|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{3.3.21}$$

By Corollary 3.3.5, we have the estimates

$$\mathrm{Per}_K(sE) \leqslant \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}P(sE)$$

and

$$\iint_{(sE)\times(sE)^c} |\nabla K(x-y)||x-y|\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{I_K^{1,2}\mathbf{K}_{1,n}}{2}P(sE),$$

thus with (3.3.21) it follows

$$\begin{aligned} |\mathrm{Per}_K(tE) - \mathrm{Per}_K(E)| &\leqslant \left(2nI_K^{0,1} + I_K^{1,2}\right)P(E)\int_{[1,t]} s^{n-1}\,\mathrm{d}s \\ &\leqslant C|1-t^n|P(E) \\ &\leqslant C|1-t|P(E) \end{aligned}$$

for some $C = C(n, I_K^{0,1}, I_K^{1,2})$. □

In Problem (P2), we may want to work with sets of fixed volume $|B|$ instead of m, to make the volume constraint appear as a parameter in the functional instead. Thus we look at the equivalent rescaled problem, defined in the following proposition.

Proposition 3.3.8 (Rescaling)**.** *Given $m > 0$, let $\lambda := \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$, let us define the functions $\rho_K, \rho_{K,1/\lambda} : (0,+\infty) \to [0,+\infty)$ by $\rho_K(r) := \frac{rk(r)}{I_K^{0,1}}$, and $\rho_{K,1/\lambda}(r) := \lambda^n \rho_K(\lambda^n r)$. Then (P2) is equivalent to*

$$\min\left\{P(F) - \mathcal{V}_{K,\lambda}(F) \ : \ |F| = |B|\right\}, \tag{P2$'$}$$

where

$$\begin{aligned}\mathcal{V}_{K,\lambda}(F) :&= \iint_{F\times F^c} \lambda^{n+1} K(\lambda(x-y))\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{I_K^{0,1}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\chi_F(x)-\chi_F(y)|}{|x-y|}\rho_{K,1/\lambda}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

in the sense that (P2) admits a minimizer if and only if (P2$'$) does, and E is a minimizer of (P2) if and only if $F := \lambda^{-1}E$ is a minimizer of (P2$'$).

Proof. Given E such that $|E| = m$, $F = \lambda^{-1}E$ with $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$, we have $|F| = \omega_n = |B|$. Making the change of variables $x = \lambda x'$, $y = \lambda y'$ we find

$$\begin{aligned}&P(E) - \mathrm{Per}_K(E) \\ &\quad= \lambda^{n-1}\left(P(F) - \iint_{F\times F^c} \lambda^{n+1}K(\lambda|x-y|)\,\mathrm{d}x\,\mathrm{d}y\right) \\ &\quad= \lambda^{n-1}\left(P(F) - \frac{I_K^{0,1}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\chi_F(x)-\chi_F(y)|}{|x-y|}\rho_{K,1/\lambda}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\right),\end{aligned}$$

which gives the result. □

To study Problems (P1) and (P2) for large volumes, we can thus look instead at Problem (P2$'$) and its asymptotic behavior when λ goes to infinity. Letting $\varepsilon = 1/\lambda$, we readily check that the family of functions $\rho_{K,\lambda^{-1}} = \rho_{K,\varepsilon}$ defined in Proposition 3.3.8 for all $\varepsilon > 0$ is a family of *n-dimensional mollifiers*, which we define just below.

Definition 3.3.9. For any $n \geqslant 1$ we say that a family of measurable functions $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of n-dimensional mollifiers if, for all $\varepsilon > 0$,

- $\rho_\varepsilon : (0,+\infty) \to [0,+\infty)$;
- $|\mathbb{S}^{n-1}|\displaystyle\int_0^\infty \rho_\varepsilon(r) r^{n-1}\,\mathrm{d}r = 1$;
- $\displaystyle\lim_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(r)r^{n-1}\,\mathrm{d}r = 0, \quad \forall\delta > 0.$

The following proposition will help us explain the behavior of Problem (P2$'$) when λ goes to infinity.

Proposition 3.3.10 ([32])**.** *Let $f \in \mathrm{BV}(\mathbb{R}^n)$, and $(\rho_\varepsilon)_{\varepsilon>0}$, be a family of n-dimensional mollifiers, then*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \xrightarrow{\varepsilon\to 0} \mathbf{K}_{1,n}\int_{\mathbb{R}^n}|Df|,$$

where $\mathbf{K}_{1,n}$ is defined in Proposition 3.3.3.

Given a set of finite perimeter E, taking $f = \chi_E$, this shows that the functional minimized in Problem (P2$'$) converges pointwise to a multiple of the classical perimeter when λ goes to infinity. Thus we may guess that if this multiple is positive, Problem (P2$'$) will reduce to minimizing the classical perimeter under volume constraint when λ is large.

3.3.2 Bessel kernels

For any κ and α in $(0,+\infty)$ we denote by $\mathcal{B}_{\kappa,\alpha}$ the generalized Bessel kernel of order α in $\mathbb{R}^n$, i.e., the fundamental solution of

$$(I-\kappa\Delta)^{\frac{\alpha}{2}} f = \delta_0$$

in $\mathbb{R}^n$; or simply $\mathcal{B}_\alpha$ when $\kappa = 1$. The following proposition sums up some useful properties on $\mathcal{B}_\alpha$ (see e.g. [55, Chapter I.2.2], [109, Chapter V.3] and [5, Chapter II.3]).

Proposition 3.3.11. *The Bessel kernel of order $\alpha \in (0,+\infty)$ in $\mathbb{R}^n$ is given by*

$$\mathcal{B}_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi}} t^{\frac{\alpha-n}{2}}\frac{\mathrm{d}t}{t}. \tag{3.3.22}$$

The kernel $\mathcal{B}_\alpha$ is radial, radially nonincreasing, and C^1 away from the origin. In addition

$$I^{0,0}_{\mathcal{B}_\alpha} = 1, \quad I^{0,1}_{\mathcal{B}_\alpha} = n\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}, \quad I^{1,1}_{\mathcal{B}_\alpha} = n, \quad \text{and} \quad I^{1,2}_{\mathcal{B}_\alpha} < +\infty.$$

The asymptotic behavior of $\mathcal{B}_\alpha$ is

$$\mathcal{B}_\alpha(x) \underset{0}{\sim} \begin{cases} \dfrac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\dfrac{1}{|x|^{n-\alpha}} & \text{if } 0<\alpha<n \\ \dfrac{-\log(|x|)}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} & \text{if } \alpha = n \\ \dfrac{\Gamma\left(\frac{\alpha-n}{2}\right)}{2^n\pi^{\frac{n}{2}}} & \text{if } n<\alpha, \end{cases}$$

and

$$\mathcal{B}_\alpha(x) \underset{\infty}{\sim} \frac{1}{2^{\frac{n+\alpha-1}{2}}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\frac{\alpha-n-1}{2}}e^{-|x|}.$$

By scaling, the generalized Bessel kernel $\mathcal{B}_{\kappa,\alpha}$ satisfies

$$\mathcal{B}_{\kappa,\alpha}(x) = \frac{1}{\kappa^{\frac{n}{2}}}\mathcal{B}_\alpha\left(\frac{x}{\sqrt{\kappa}}\right), \quad \text{for a.e. } x\in\mathbb{R}^n, \tag{3.3.23}$$

thus

$$I^{0,0}_{\mathcal{B}_{\kappa,\alpha}} = 1, \quad I^{0,1}_{\mathcal{B}_{\kappa,\alpha}} = \kappa^{\frac{1}{2}} I^{0,1}_{\mathcal{B}_\alpha}, \quad \text{and} \quad I^{1,1}_{\mathcal{B}_{\kappa,\alpha}} = n.$$

Proof. The integral representation (3.3.22) and the asymptotics can be found respectively in [55] and [5], and the fact that $I^{0,0}_{\mathcal{B}_\alpha} = \|\mathcal{B}_\alpha\|_{L^1(\mathbb{R}^n)} = 1$ is well known, so we detail only the computations of $\int_{\mathbb{R}^n}|x|\mathcal{B}_\alpha(x)\,\mathrm{d}x$, $\int_{\mathbb{R}^n}|x||\nabla\mathcal{B}_\alpha(x)|\,\mathrm{d}x$, and the fact that $\int_{\mathbb{R}^n}|x|^2|\nabla\mathcal{B}_\alpha(x)|\,\mathrm{d}x$ is finite. By (3.3.22), using Fubini's theorem, we find

$$I^{0,1}_{\mathcal{B}_\alpha} = \int_{\mathbb{R}^n}|x|\mathcal{B}_\alpha(x)\,\mathrm{d}x = \frac{1}{(4\pi)^{\frac{\alpha}{2}}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\left(\int_{\mathbb{R}^n}|x|e^{-\frac{\pi|x|^2}{t}}\,\mathrm{d}x\right)\frac{\mathrm{d}t}{t}. \tag{3.3.24}$$

Changing variables, we compute

$$\begin{aligned}\int_{\mathbb{R}^n}|x|e^{-\frac{\pi|x|^2}{t}}\,\mathrm{d}x &= n\omega_n\left(\frac{t}{\pi}\right)^{\frac{n+1}{2}}\int_0^\infty r^n e^{-r^2}\,\mathrm{d}r \\ &= n\omega_n\left(\frac{t}{\pi}\right)^{\frac{n+1}{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{2} = \frac{nt^{\frac{n+1}{2}}}{\sqrt{4\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}.\end{aligned} \tag{3.3.25}$$

Injecting (3.3.25) into (3.3.24) yields

$$\begin{aligned} I^{0,1}_{\mathcal{B}_\alpha} &= \frac{1}{(4\pi)^{\frac{1+\alpha}{2}}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{n\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}\int_0^\infty e^{-\frac{t}{4\pi}}t^{\frac{\alpha+1}{2}}\frac{\mathrm{d}t}{t} \\ &= \frac{1}{(4\pi)^{\frac{1+\alpha}{2}}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{n\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}(4\pi)^{\frac{1+\alpha}{2}}\Gamma\left(\frac{1+\alpha}{2}\right) \\ &= n\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}. \end{aligned}$$

Now we show that $\mathcal{B}_\alpha$ is C^1 away from the origin, and compute its gradient. Note that the integrand of (3.3.22) is C^1 in x for almost every $t \in (0,+\infty)$. Let $r > 0$. For every $x \in \mathbb{R}^n \setminus \overline{B_r}$ we have

$$\left| e^{-\frac{\pi|x|^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\frac{1}{t}\right| \leqslant e^{-\frac{\pi r^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}-1} \in L^1(0,+\infty),$$

thus $\mathcal{B}_\alpha \in C^0(\mathbb{R}^n\setminus\overline{B_r})$ by the theorem of continuity under the integral. By arbitrariness of R, $\mathcal{B}_\alpha$ is then continuous away from the origin. Now let $0 < r < R$. For every $x \in B_R\setminus\overline{B_r}$, we have

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{\pi|x|^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\frac{1}{t}\right)\right| &= \left|-\frac{2\pi x}{t}e^{-\frac{\pi|x|^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\frac{1}{t}\right| \\ &\leqslant 2\pi R e^{-\frac{\pi r^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}-2} \in L^1(0,+\infty), \end{aligned}$$

thus by the theorem of derivation under the integral, $\mathcal{B}_\alpha \in C^1(B_R \setminus \overline{B_r})$, and for every $x \in B_R \setminus \overline{B_r}$,

$$\nabla\mathcal{B}_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}}\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty -\frac{2\pi x}{t}e^{-\frac{\pi|x|^2}{t}}e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\frac{\mathrm{d}t}{t}. \tag{3.3.26}$$

By arbitrariness of r and R, $\mathcal{B}_\alpha$ is C^1 away from the origin, and (3.3.26) holds for any $x \in \mathbb{R}^n\setminus\{0\}$. In view of (3.3.1), if $I^{1,1}_{\mathcal{B}_\alpha} = \int_{\mathbb{R}^n}|\partial_r\mathcal{B}_\alpha(x)||x|\,\mathrm{d}x$ and $I^{1,2}_{\mathcal{B}_\alpha} = \int_{\mathbb{R}^n}|\partial_r\mathcal{B}_\alpha(x)||x|^2\,\mathrm{d}x$ are finite, then $\mathcal{B}_\alpha$ satisfies (H1) and (H2) and we automatically have

$$I^{1,1}_{\mathcal{B}_\alpha} = nI^{0,0}_{\mathcal{B}_\alpha} = n,$$

thus we need only prove that the moments $I^{1,1}_{\mathcal{B}_\alpha}$ and $I^{1,2}_{\mathcal{B}_\alpha}$ are finite, without having to compute them. For the rest of the proof C denotes a constant, possibly changing from line to line, depending only on n and α. By Fubini's theorem, we have

$$I^{1,1}_{\mathcal{B}_\alpha} \leqslant C\int_0^\infty e^{-\frac{t}{4\pi}}t^{\frac{\alpha-n}{2}}\frac{2\pi}{t}\left(\int_{\mathbb{R}^n}|x|^2 e^{-\frac{\pi|x|^2}{t}}\,\mathrm{d}x\right)\frac{\mathrm{d}t}{t}. \tag{3.3.27}$$

Changing variables, we compute

$$\int_{\mathbb{R}^n}|x|^2e^{-\frac{\pi|x|^2}{t}}\,\mathrm{d}x = Ct^{\frac{n+2}{2}}\int_0^\infty r^{n+1}e^{-r^2}\,\mathrm{d}r = Ct^{1+\frac{n}{2}}, \tag{3.3.28}$$

thus injecting (3.3.28) into (3.3.27) yields

$$I^{1,1}_{\mathcal{B}_\alpha} \leqslant C\int_0^\infty e^{-\frac{t}{4\pi}}t^{\frac{\alpha}{2}-1}\,\mathrm{d}t < +\infty.$$

There remains to show that $I^{1,2}_{\mathcal{B}_\alpha}$ is finite. Once again, a use of Fubini's theorem gives

$$I^{1,2}_{\mathcal{B}_\alpha} \leqslant \frac{1}{(4\pi)^{\frac{\alpha}{2}}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-\frac{t}{4\pi}} t^{\frac{\alpha-n}{2}} \frac{2\pi}{t} \left(\int_{\mathbb{R}^n} |x|^3 e^{-\frac{\pi|x|^2}{t}} \,\mathrm{d}x \right) \frac{\mathrm{d}t}{t},$$

and a change of variables shows

$$\int_{\mathbb{R}^n} |x|^3 e^{-\frac{\pi|x|^2}{t}} \,\mathrm{d}x = C t^{\frac{n+3}{2}} \int_0^\infty r^{n+2} e^{-r^2} \,\mathrm{d}r \leqslant C t^{\frac{n+3}{2}},$$

hence

$$I^{1,2}_{\mathcal{B}_\alpha} \leqslant C \int_0^\infty e^{-\frac{t}{4\pi}} t^{\frac{\alpha-n}{2}} t^{\frac{n+1}{2}} \frac{\mathrm{d}t}{t} = C \int_0^\infty e^{-\frac{t}{4\pi}} t^{\frac{\alpha-1}{2}} \,\mathrm{d}t < +\infty,$$

which concludes the proof. □

Remark 3.3.12. The $\mathcal{B}_\alpha$ kernel can also be expressed in terms of the modified Bessel functions of the third kind $\mathbf{K}_\nu : (0,+\infty) \to (0,+\infty)$, defined for any $\nu \in \mathbb{R}$ by

$$\mathbf{K}_\nu(r) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{r^\nu e^{-r}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty e^{-rt} \left(t + \frac{t^2}{2}\right)^{\nu-\frac{1}{2}} \mathrm{d}t, \quad \text{if } \nu > -\frac{1}{2}, \tag{3.3.29}$$

and the relation $\mathbf{K}_\nu = \mathbf{K}_{-\nu}$ (see [5, Chapter II.3]). Then by [5, Chapter II.4], $\mathcal{B}_\alpha$ is given by

$$\mathcal{B}_\alpha(x) = \frac{1}{2^{\frac{n+\alpha-2}{2}} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \frac{\mathbf{K}_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{3.3.30}$$

From (3.3.29) and (3.3.30) it is easy to see that when $\alpha = n-1$, $\mathcal{B}_\alpha$ takes the explicit form

$$B_{n-1} = \frac{1}{(4\pi)^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \frac{e^{-|x|}}{|x|}.$$

In particular, when $n = 3$ and $\alpha = 2$, $\mathcal{B}_\alpha(x) = \frac{1}{4\pi}\frac{e^{-|x|}}{|x|}$. When $\alpha = n+1$, changing variables in (3.3.22), one can compute $\mathcal{B}_\alpha$ explicitly as well. Indeed, in that case,

$$B_{n+1}(x) = \frac{1}{(4\pi)^{\frac{n+1}{2}}} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} 2\sqrt{4\pi} \int_0^\infty e^{-t^2 - \frac{|x|^2}{4t^2}} \,\mathrm{d}t = \frac{2\pi}{(4\pi)^{\frac{n+1}{2}}} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} e^{-|x|}.$$

We have the following straightforward corollary of Proposition 3.3.11.

Corollary 3.3.13. *For any $\kappa \in (0,+\infty)$ and any $\alpha \in (0,+\infty)$, the kernel $\mathcal{B}_{\kappa,\alpha}$ satisfies assumptions (H1) to (H4).*

We can express the constants $\mathbf{K}_{p,n}$ in terms of the Gamma function as follows, in order to make the assumptions on $I^{0,1}_K$ in Theorems 3.1.1 to 3.1.3 more explicit.

Lemma 3.3.14. *For any $n \in \mathbb{N}$ and $p > 0$, we have*

$$\mathbf{K}_{p,n} = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{n+p}{2}\right)},$$

where $\mathbf{K}_{p,n}$ is given by (3.1.1). In particular

$$\mathbf{K}_{1,n} = \frac{2\Gamma\left(1+\frac{n}{2}\right)}{n\sqrt{\pi}\, \Gamma\left(1+\frac{n+1}{2}\right)} \quad \text{and } \mathbf{K}_{2,n-1} = \frac{1}{n}.$$

Proof. We may assume $e = (0, \dots, 0, 1)$ so that, $e \cdot x = x_n$. Recall that for every nonnegative $\mathscr{H}^{n-1}$-measurable function g on $\mathbb{S}^{n-1}$, we have (see e.g. [6, Corollary A.6])

$$\int_{\mathbb{S}^{n-1}} g \, d\mathscr{H}^{n-1} = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} g(\sqrt{1-t^2}x, t) \, d\mathscr{H}_x^{n-2} \, \mathrm{d}t. \tag{3.3.31}$$

This way we compute

$$\int_{\mathbb{S}^{n-1}} |x_n|^p \, d\mathscr{H}^{n-1} = |\mathbb{S}^{n-2}| \int_{-1}^{1} |t|^p (1-t^2)^{\frac{n-3}{2}} \, \mathrm{d}t = |\mathbb{S}^{n-2}| \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)},$$

thus

$$\mathbf{K}_{p,n} = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}.$$

□

Eventually, Bessel kernels satisfy all the hypotheses of Theorems 3.1.1 to 3.1.3, and using Lemma 3.3.14 we can translate the conditions $I^{0,1}_{\mathcal{B}_{\kappa,\alpha}} < \frac{2}{\mathbf{K}_{1,n}}$ and $I^{0,1}_{\mathcal{B}_{\kappa,\alpha}} > \frac{2}{\mathbf{K}_{1,n}}$ explicitly in terms of κ and α.

Proposition 3.3.15. *Let $\alpha > 0$ and $\kappa > 0$. Then defining*

$$\boldsymbol{\kappa}_{0,\alpha} := \pi \left(\frac{(n+1)\Gamma\left(\frac{\alpha}{2}\right)}{2\Gamma\left(\frac{1+\alpha}{2}\right)} \right)^2,$$

we have $I^{0,1}_{\mathcal{B}_{\kappa,\alpha}} < \frac{2}{\mathbf{K}_{1,n}} \iff \kappa < \boldsymbol{\kappa}_{0,\alpha}$, and $I^{0,1}_{\mathcal{B}_{\kappa,\alpha}} > \frac{2}{\mathbf{K}_{1,n}} \iff \kappa > \boldsymbol{\kappa}_{0,\alpha}$. In addition, since the Bessel kernels $\mathcal{B}_{\kappa,\alpha}$ satisfy (H1) to (H4), Theorems 3.1.1 to 3.1.3 directly apply.

3.4 Existence and convergence of large mass minimizers

3.4.1 Existence of large mass minimizers

In order to prove the existence of minimizers for large masses, we want to use the direct method in the calculus of variations, starting from a minimizing sequence. When $I^{0,1}_K$ is small enough, we will see that any minimizing sequence is bounded in $\mathrm{BV}(\mathbb{R}^n)$, but in order to get compactness in $L^1(\mathbb{R}^n)$ and pass to the limit, we need to show that no mass escapes at infinity. To do so, we will need to establish a few lemmas. First we show that for large masses, if the energy $\mathcal{F}_K(E)$ of some set E is smaller than that of a ball of same mass, then E is actually close to a ball.

Lemma 3.4.1. *Assume that $I^{0,1}_K < \frac{2}{\mathbf{K}_{1,n}}$. Then for any set E of volume $m = \omega_n \lambda^n$ such that*

$$\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m),$$

we have

$$|E \triangle [B]_{y,m}| \leqslant m\eta(\lambda),$$

for some ball $y \in \mathbb{R}^n$, where

$$\eta(\lambda) := C \left[\frac{I^{0,1}_K \mathbf{K}_{1,n}}{2} P(B) - \mathcal{V}_{K,\lambda}(B) \right]^{\frac{1}{2}},$$

and $C = C(n, I^{0,1}_K)$. Notice that $\eta(\lambda)$ goes to 0 as λ goes to infinity by Proposition 3.3.10.

Proof. The inequality
$$\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m)$$
rewrites
$$P(E) - P([B]_m) \leqslant \mathrm{Per}_K(E) - \mathrm{Per}_K([B]_m).$$
Scaling this inequality with $F := \lambda^{-1}E$ yields
$$P(F) - P(B) \leqslant \mathcal{V}_{K,\lambda}(F) - \mathcal{V}_{K,\lambda}(B),$$
where $\mathcal{V}_{K,\lambda}$ is defined in Proposition 3.3.8. By Corollary 3.3.5, this implies
$$\begin{aligned} P(F) - P(B) &\leqslant \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2} P(F) - \mathcal{V}_{K,\lambda}(B) \\ &= \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\left(P(F) - P(B)\right) + \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2} P(B) - \mathcal{V}_{K,\lambda}(B), \end{aligned}$$
thus
$$P(F) - P(B) \leqslant C_1 f(\lambda) \tag{3.4.1}$$
where $C_1 := \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right)^{-1}$ depends only on n and $I_K^{0,1}$, and
$$f(\lambda) = \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2} P(B) - \mathcal{V}_{K,\lambda}(B). \tag{3.4.2}$$
Using the quantitative isoperimetric inequality (3.2.2) and (3.4.1), we find
$$\alpha(F) \leqslant C_2\sqrt{f(\lambda)}$$
where $C_2 = C_2(n, I_K^{0,1})$. Hence there exists $B_1(y)$ such that
$$|F \triangle B_1(y)| \leqslant C_2\sqrt{f(\lambda)},$$
which gives the result by (3.4.2), and recalling that $E = \lambda F$. □

We also need a truncation lemma akin to [73, Lemma 29.12] or [45, Lemma 4.5] to quantify by how much the energy decreases when cutting a set which is already close to a ball.

Lemma 3.4.2. *Assume $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. There exist C_1, $C_2 \in (0,+\infty)$ depending only on n and $I_K^{0,1}$ such that the following holds. If E is a set of finite perimeter satisfying $|E \setminus B_{r_0}| \leqslant \eta$, for some positive constants η and r_0, then there exists $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ such that*
$$\mathcal{F}_K(E \cap B_r) \leqslant \mathcal{F}_K(E) - \frac{|E \setminus B_r|}{C_2\eta^{\frac{1}{n}}}. \tag{3.4.3}$$

Proof. Let $C_1, C_2 > 0$ to be fixed later, and E be a set of finite perimeter such that $|E \setminus B_{r_0}| \leqslant \eta$. We define $u(r) := |E \setminus B_r|$, and for now we assume that $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$. Since u is nonincreasing, we have $u(r) > 0$, for all $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$. Notice that u is absolutely continuous, and $u'(r) = -\mathscr{H}^{n-1}(E \cap \partial B_r)$ for $\mathscr{L}^1$-almost every $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$. By contradiction, let us assume that
$$P(E) - \mathrm{Per}_K(E) < P(E \cap B_r) - \mathrm{Per}_K(E \cap B_r) + \frac{|E \setminus B_r|}{C_2\eta^{\frac{1}{n}}}, \tag{3.4.4}$$

for all $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$. Recall that for almost every $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ we have $\mathscr{H}^{n-1}(\partial^* E \cap \partial B_r) = 0$ (see e.g. [73, Proposition 2.16]). Given such an r, note that $P(E) = P(E; B_r) + P(E; \overline{B_r}^{\mathrm{c}})$, and $P(E \cap B_r) = P(E; B_r) + \mathscr{H}^{n-1}(E \cap \partial B_r)$ (see e.g. [73, Lemma 15.12]). Thus

$$\begin{aligned} P(E) - P(E \cap B_r) &= P(E; B_r) + P(E; \overline{B_r}^{\mathrm{c}}) - P(E; B_r) - \mathscr{H}^{n-1}(E \cap \partial B_r) \\ &= P(E; \overline{B_r}^{\mathrm{c}}) - \mathscr{H}^{n-1}(E \cap \partial B_r) \\ &= P(E \setminus B_r) - 2\mathscr{H}^{n-1}(E \cap \partial B_r) = P(E \setminus B_r) + 2u'(r), \end{aligned} \tag{3.4.5}$$

where we also used $P(E \setminus B_r) = P(E; \overline{B_r}^{\mathrm{c}}) + \mathscr{H}^{n-1}(E \cap \partial B_r)$. On the other hand, noticing that

$$\begin{aligned} \mathrm{Per}_K(E) &= \iint_{(E \setminus B_r) \times E^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{(E \cap B_r) \times E^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{(E \setminus B_r) \times (E \setminus B_r)^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{(E \setminus B_r) \times (E \cap B_r)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \iint_{(E \cap B_r) \times E^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \mathrm{Per}_K(E \setminus B_r) - \iint_{(E \setminus B_r) \times (E \cap B_r)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \iint_{(E \cap B_r) \times E^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

and

$$\mathrm{Per}_K(E \cap B_r) = \iint_{(E \cap B_r) \times E^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{(E \cap B_r) \times (E \setminus B_r)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

we find

$$\mathrm{Per}_K(E) - \mathrm{Per}_K(E \cap B_r) = \mathrm{Per}_K(E \setminus B_r) - 2\iint_{E \cap B_r \times E \cap B_r^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.4.6}$$

Injecting (3.4.5) and (3.4.6) into (3.4.4), one gets

$$\begin{aligned} P(E \setminus B_r) - \mathrm{Per}_K(E \setminus B_r) &< -2u'(r) - 2\iint_{E \cap B_r \times E \cap B_r^{\mathrm{c}}} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \frac{u(r)}{C_2\eta^{\frac{1}{n}}} \\ &\leqslant -2u'(r) + \frac{u(r)}{C_2\eta^{\frac{1}{n}}}, \end{aligned} \tag{3.4.7}$$

for almost every $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$. Note that by Corollary 3.3.5, we know that $\mathrm{Per}_K(E \setminus B_r) \leqslant \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2} P(E \setminus B_r)$, thus

$$\left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) P(E \setminus B_r) < -2u'(r) + \frac{u(r)}{C_2\eta^{\frac{1}{n}}}, \quad \text{for a.e. } r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]. \tag{3.4.8}$$

Now by the isoperimetric inequality (3.2.2) we have

$$P(E \setminus B_r) \geqslant n\omega_n^{\frac{1}{n}} u(r)^{\frac{n-1}{n}}. \tag{3.4.9}$$

Since u is nonincreasing and $u(r_0) = |E \setminus B_{r_0}| \leqslant \eta$, we also know that

$$u(r) \leqslant u(r_0)^{\frac{1}{n}} u(r)^{\frac{n-1}{n}} \leqslant \eta^{\frac{1}{n}} u(r)^{\frac{n-1}{n}}. \tag{3.4.10}$$

Plugging (3.4.9) and (3.4.10) into (3.4.8), we deduce that for a.e. $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$,

$$n\omega_n^{\frac{1}{n}} \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) u(r)^{\frac{n-1}{n}} < -2u'(r) + \frac{u(r)^{\frac{n-1}{n}}}{C_2},$$

thus

$$C_3 n u(r)^{\frac{n-1}{n}} < -u'(r), \tag{3.4.11}$$

where

$$C_3 := \frac{1}{2}\left[\omega_n^{\frac{1}{n}}\left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) - \frac{1}{nC_2}\right].$$

We can choose C_2 large enough, depending only on $I_K^{0,1}$ and n, so that $C_3 > 0$. Then (3.4.11) can be rewritten

$$\left(u(r)^{\frac{1}{n}}\right)' = \frac{1}{n}u'(r)u(r)^{\frac{1}{n}-1} < -C_3, \qquad \text{for a.e. } r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}],$$

thus integrating between r_0 and $r_0 + C_1\eta^{\frac{1}{n}}$, one gets

$$u(r_0 + C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} \leqslant u(r_0)^{\frac{1}{n}} - C_3C_1\eta^{\frac{1}{n}} \leqslant (1 - C_1C_3)\eta^{\frac{1}{n}}. \tag{3.4.12}$$

Choosing C_1 large enough, depending only on n and $I_K^{0,1}$, we get $u(r_0+C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} < 0$, which is a contradiction. Recall that we assumed $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$, so there are two cases: for this C_1 large enough, *either* $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$, and then as we have seen there exists $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ such that (3.4.3) holds for some $C_2 = C_2(n, I_K^{0,1})$, *or* $u(r_0 + C_1\eta^{\frac{1}{n}}) = 0$, and then (3.4.3) holds for any C_2 with $r = r_0 + C_1\eta^{\frac{1}{n}}$. In any case, (3.4.3) holds. □

We are now able to prove the existence of large mass minimizers.

Proof of Theorem 3.1.1. By (3.3.2), we shall equivalently look at (P2).
Step 1. Let us show that there exists $m_e > 0$ depending only on K and n such that the following holds. For any set of finite perimeter E of mass $m > m_e$ satisfying $\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m)$, up to a translation, there exists F of mass m satisfying

$$\mathcal{F}_K(F) \leqslant \mathcal{F}_K(E) - \frac{C}{m^{\frac{1}{n}}}|E \setminus (4[B]_m)| \quad \text{ and } \quad F \subseteq 4[B]_m,$$

for some $C > 0$ depending only on n. Let $m_e > 0$ to be fixed later, and $\lambda_e := \left(\frac{m_e}{\omega_n}\right)^{\frac{1}{n}}$. By Lemma 3.4.1, if E satisfies $|E| = m > m_e$ and $\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m)$, then up to translating E, we have

$$|E \triangle B_\lambda| = |E \triangle [B]_m| \leqslant m\eta(\lambda),$$

where $m = \omega_n\lambda^n$ (and in particular, $\lambda > \lambda_e$). Using Lemma 3.4.2 with $m\eta(\lambda)$ in place of η, we can find $r \leqslant \lambda + C_1\lambda\eta(\lambda)^{\frac{1}{n}}$ such that

$$\mathcal{F}_K(E \cap B_r) \leqslant \mathcal{F}_K(E) - \frac{|E \setminus B_r|}{C_2 m^{\frac{1}{n}}\eta(\lambda)^{\frac{1}{n}}}. \tag{3.4.13}$$

We are going to show that, provided m_e (equivalently λ_e) is large enough, then there exists $F \subseteq B_{4\lambda}$ such that $|F| = |E|$ and $\mathcal{F}_K(F) < \mathcal{F}_K(E)$. Let $u := \frac{|E \setminus B_r|}{m}$, so that (3.4.13) can be rewritten

$$\mathcal{F}_K(E \cap B_r) \leqslant \mathcal{F}_K(E) - \frac{\omega_n^{\frac{n-1}{n}}}{C_2} \frac{\lambda^{n-1} u}{\eta(\lambda)^{\frac{1}{n}}}. \tag{3.4.14}$$

Let us define the rescaled set $F := \mu(E \cap B_r)$, where $\mu > 0$ is such that $|F| = m$, that is, $\mu = (1-u)^{-\frac{1}{n}}$, then we have

$$P(F) - \operatorname{Per}_K(F) = \mu^{n-1} P(E \cap B_r) - \operatorname{Per}_K\left(\mu(E \cap B_r)\right). \tag{3.4.15}$$

Note that $u \leqslant \frac{|E \setminus B_\lambda|}{m} \leqslant \eta(\lambda) \leqslant 1 - 2^{-n}$ for λ_e (equivalently m_e) large enough, depending only on K and n, since $\lambda > \lambda_e$ and $\eta(\lambda)$ goes to zero at λ goes to infinity. This implies $1 \leqslant \mu \leqslant 2$. Now by Lemma 3.3.7 we have

$$\operatorname{Per}_K(\mu(E \cap B_r)) \geqslant \operatorname{Per}_K(E \cap B_r) - C_3(\mu - 1) P(E \cap B_r) \tag{3.4.16}$$

for some positive constant C_3 that depends only on n, $I_K^{0,1}$ and $I_K^{1,2}$. Injecting (3.4.16) into (3.4.15), we find

$$\begin{aligned}
\mathcal{F}_K(F) &\leqslant \mu^{n-1} P(E \cap B_r) - \operatorname{Per}_K(E \cap B_r) + C_3(\mu - 1) P(E \cap B_r) \\
&\leqslant (1 + C_5 u) P(E \cap B_r) - \operatorname{Per}_K(E \cap B_r) + C_4 u P(E \cap B_r) \\
&= (1 + C_5 u)(P(E \cap B_r) - \operatorname{Per}_K(E \cap B_r)) + C_5 u \operatorname{Per}_K(E \cap B_r) \\
&\qquad + C_4 u P(E \cap B_r) \\
&= (1 + C_5 u) \mathcal{F}_K(E \cap B_r) + C_5 u \operatorname{Per}_K(E \cap B_r) + C_4 u P(E \cap B_r).
\end{aligned} \tag{3.4.17}$$

where we used the fact that $u = (1 - \mu^{-n})$, and introduced some positive constants C_4 and C_5 that depend only on n, $I_K^{0,1}$ and $I_K^{1,2}$, since $\mu \in [1,2]$. Injecting (3.4.14) into (3.4.17), and using the fact that $\operatorname{Per}_K(E \cap B_R) \leqslant \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2} P(E \cap B_r)$ by Corollary 3.3.5, we get

$$\begin{aligned}
\mathcal{F}_K(F) &\leqslant (1 + C_5 u) \left(\mathcal{F}_K(E) - \frac{\omega_n^{\frac{n-1}{n}}}{C_2} \frac{\lambda^{n-1} u}{\eta(\lambda)^{\frac{1}{n}}} \right) + C_6 u P(E \cap B_r) \\
&\leqslant \mathcal{F}_K(E) + u \left(C_5 P(E) + C_6 P(E \cap B_r) - \frac{\omega_n^{\frac{n-1}{n}}}{C_2} \frac{\lambda^{n-1}}{\eta(\lambda)^{\frac{1}{n}}} \right) \\
&\leqslant \mathcal{F}_K(E) + u \left((C_5 + C_6) P(E) + C_6 P(B_r) - \frac{\omega_n^{\frac{n-1}{n}}}{C_2} \frac{\lambda^{n-1}}{\eta(\lambda)^{\frac{1}{n}}} \right),
\end{aligned} \tag{3.4.18}$$

where C_6 depends only on n, $I_K^{0,1}$ and $I_K^{1,2}$. Recall that $\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m)$, thus by Corollary 3.3.5,

$$\left(1 - \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2}\right) P(E) \leqslant P([B]_m) = \lambda^{n-1} P(B), \tag{3.4.19}$$

which implies $P(E) \leqslant C_7 \lambda^{n-1}$ where $C_7 = C_7(n, I_K^{0,1})$. Since $0 \leqslant r \leqslant \lambda + C_1 \lambda \eta(\lambda)^{\frac{1}{n}}$ and $\eta(\lambda)$ vanishes at infinity, we can choose λ_e (recall that $\lambda \geqslant \lambda_e$) such that

$$r \leqslant 2\lambda, \tag{3.4.20}$$

thus

$$P(B_r) \leqslant C_8 \lambda^{n-1}, \tag{3.4.21}$$

for some $C_8 = C_8(n, I_K^{0,1})$. Plugging (3.4.19) and (3.4.21) into (3.4.18), we reach

$$\mathcal{F}_K(F) \leqslant \mathcal{F}_K(E) + u\lambda^{n-1}\left((C_5 + C_6)C_7 + C_6C_8 - \frac{\omega_n^{\frac{n-1}{n}}}{C_2\eta(\lambda)^{\frac{1}{n}}}\right). \tag{3.4.22}$$

Since $\eta(\lambda)$ vanishes at infinity, we can choose λ_e (i.e. m_e) even larger depending only on n, $I_K^{0,1}$ and $I_K^{1,2}$ such that

$$\mathcal{F}_K(F) < \mathcal{F}_K(E) - \frac{C}{\lambda}|E \setminus B_r| \leqslant \mathcal{F}_K(E) - \frac{C}{\lambda}|E \setminus B_{2\lambda}| \tag{3.4.23}$$

for some $C > 0$ depending only on n, where we also used the facts that $r \leqslant 2\lambda$ by (3.4.20) and $u = \frac{|E\setminus B_r|}{\omega_n\lambda^n}$. Recall that $F \subseteq B_{\mu r} \subseteq B_{2r} \subseteq B_{4\lambda}$, which concludes this step.

Step 2. We prove the existence of minimizers. For $m \geqslant m_e$, $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$, consider a minimizing sequence $(E_k)_{k\in\mathbb{N}}$ for Problem (P2). There are two cases: either $[B]_m$ is a minimizer of $\mathcal{F}_k$, and we are done, or $[B]_m$ is not a minimizer of $\mathcal{F}_K$, and up to a subsequence (not relabeled), $\mathcal{F}_K(E_k) \leqslant \mathcal{F}_K([B]_m)$. In the latter case, by Step 1 we can build another minimizing sequence $(F_k)_{k\in\mathbb{N}}$ of sets in $B_{4\lambda}$ such that $\mathcal{F}_K(F_k) < \mathcal{F}_K(E_k)$, for all $k \in \mathbb{N}$. Now $(\chi_{F_k})_{k\in\mathbb{N}}$ is also bounded in $\mathrm{BV}(\mathbb{R}^n)$. Indeed $\mathcal{F}_K(F_k) \leqslant \mathcal{F}_K([B]_m)$ implies

$$\left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) P(F_k) \leqslant P([B]_m),$$

thus $[\chi_{F_k}]_{\mathrm{BV}(\mathbb{R}^n)}$ is bounded, and $\|\chi_{F_k}\|_{L^1(\mathbb{R}^n)} = m$. By compactness of $\mathrm{BV}(\mathbb{R}^n)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the fact that $F_k \subseteq B_{4\lambda}$, up to the extraction of a subsequence (still not relabeled), χ_{F_k} converges to some function $f \in \mathrm{BV}(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ and almost everywhere. The almost everywhere convergence implies that f is the indicator function of some set of finite perimeter F, and the L^1 convergence ensures $|F| = m$. Now by lower semicontinuity of the perimeter w.r.t. the L^1 convergence, we have

$$P(F) \leqslant \liminf_k P(F_k),$$

and by continuity of the nonlocal perimeter in $L^1(\mathbb{R}^n)$ shown in Lemma 3.3.6, it follows

$$\mathrm{Per}_K(F_k) \to \mathrm{Per}_K(F).$$

Hence

$$\mathcal{F}_K(F) \leqslant \liminf_k \mathcal{F}_K(F_k),$$

which shows that F is a minimizer of Problem (P2), since F_k is a minimizing sequence.

Step 3. We show that for λ_e chosen as before, for any $\lambda \geqslant \lambda_e$ and $m = \omega_n\lambda^n$, a minimizer of volume m for Problem (P2) is (up to a Lebesgue negligible set) included in $B_{4\lambda}$. Consider E such a minimizer, then by minimality, we have $\mathcal{F}_K(E) \leqslant \mathcal{F}_K([B]_m)$, thus applying *Step 1* there exists a set F of mass m such that

$$\mathcal{F}_K(F) \leqslant \mathcal{F}_K(E) - C|E \setminus B_{4\lambda}|,$$

where $C > 0$. By minimality of E, necessarily $|E \setminus B_{4\lambda}| = 0$, hence the result. □

3.4.2 Indecomposability of minimizers

We recall here a measure theoretic notion of connectedness for sets of finite perimeter (see e.g. [3]), and show that if K is not compactly supported, then minimizers of Problem (P1) are *indecomposable*, which means that they are connected in a measure theoretic sense.

Definition 3.4.3. We say that a set of finite perimeter E is decomposable if there exist two sets of finite perimeter E_1 and E_2 such that $E = E_1 \sqcup E_2$, $|E_1| > 0$, $|E_2| > 0$ and $P(E) = P(E_1) + P(E_2)$. Naturally, we say that a set of finite perimeter is indecomposable if it is not decomposable.

As with the usual topological notion of connectedness, it is possible to partition a set of finite perimeter E into indecomposable sets (see [3, Theorem 1]) in a unique way (up to sets of vanishing Lebesgue measure). We call the sets composing this partition the $\mathcal{M}$-connected components of E. We have the following result establishing a link between the $\mathcal{M}$-connected components of a set of finite perimeter and the topological connected components.

Theorem 3.4.4 ([3, Theorem 2])**.** *If E is an open set of finite perimeter such that $\mathscr{H}^{n-1}(\partial E) = \mathscr{H}^{n-1}(\partial^* E)$, then the $\mathcal{M}$-connected components of E coincide with its topological connected components.*

Proposition 3.4.5. *If K is not compactly supported, then any minimizer E of Problem (P1) which is included in a ball B_R for some $R > 0$ is indecomposable.*

Proof. We proceed by contradiction. Assume that there exists a minimizer $E \subseteq B_R$ and two sets of finite E_1 and E_2 such that $E = E_1 \sqcup E_2$, $|E_1| > 0$, $|E_2| > 0$ and $P(E) = P(E_1) + P(E_2)$. Defining

$$G_K(F) := \iint_{F\times F} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

for any measurable set F, let us write

$$P(E) + \iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y = P(E_1) + P(E_2) + G_K(E_1) + G_K(E_2) + 2\iint_{E_1\times E_2} K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let $M > 0$ and $h \in \mathbb{R}^n$ such that $|h| > 2R + M$. Since E_1 and E_2 are included in B_R, then for any $(x, y) \in E_1 \times (E_2 + h)$ we have $|x - y| \geqslant M$. Obviously $P(E_2) = P(E_2 + h)$ and $G_K(E_2) = G_K(E_2 + h)$ by a change of variables. Now let us define the competitor $F_h := E_1 \sqcup (E_2 + h)$, which satisfies $|F_h| = |E_1| + |E_2| = |E|$. Since $|x - y| \geqslant M$ whenever $(x, y) \in E_1 \times (E_2 + h)$, $P(F_h) = P(E_1) + P(E_2 + h)$, thus we compute

$$\begin{aligned}
P(F_h) &+ \iint_{F_h\times F_h} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= P(E_1) + P(E_2+h) + G_K(E_1) + G_K(E_2+h) + 2\iint_{E_1\times(E_2+h)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= P(E_1) + P(E_2) + G_K(E_1) + G_K(E_2) + 2\iint_{E_1\times(E_2+h)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= P(E) + G_K(E) + 2\iint_{E_1\times(E_2+h)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - 2\iint_{E_1\times E_2} K(x-y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{3.4.24}$$

where we used the fact that $P(E) = P(E_1)+P(E_2)$ for the last equality. Since K is radial, positive and radially nonincreasing, we have

$$\int_A K(x)\,\mathrm{d}x > 0,$$

for any set A such that $|A| > 0$. In particular

$$\iint_{E_1\times E_2} K(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{E_1}\left(\int_{x-E_2} K(y)\,\mathrm{d}y\right)\mathrm{d}x > 0 \tag{3.4.25}$$

since $|E_1| > 0$ and $|E_2| > 0$. By a change of variables, we also have

$$\iint_{E_1\times(E_2+h)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{E_1}\left(\int_{B_M^c} K(y)\,\mathrm{d}y\right)\mathrm{d}x \leqslant |E_1|\int_{B_M^c} K(y)\,\mathrm{d}y. \tag{3.4.26}$$

Since $K \in L^1(\mathbb{R}^n)$, $\int_{B_M^c} K(y)\,\mathrm{d}y$ goes to zero as M goes to infinity, thus by (3.4.26) and (3.4.25) we can find some M large enough such that

$$\iint_{E_1\times(E_2+h)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - 2\iint_{E_1\times E_2} K(x-y)\,\mathrm{d}x\,\mathrm{d}y < 0$$

which yields, with (3.4.24),

$$P(F_h) + \iint_{F_h\times F_h} K(x-y)\,\mathrm{d}x\,\mathrm{d}y < P(E) + \iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

and contradicts the minimality of E. □

Remark 3.4.6. In particular, this shows that if $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, then any minimizer of Problem (P1) with $m > m_e$ is indecomposable (where m_e is given by Theorem 3.1.1), since it is included in the ball $4[B]_m$. In fact, we may even drop the assumption $E \subseteq B_R$ for some $R > 0$ in Proposition 3.4.5, since it turns out to always be the case, as is recalled in Section 3.4.4.

3.4.3 Γ-convergence to the classical perimeter

Using the results from Section 3.3.1, we establish a Γ-convergence result for the functional of Problem (P2′), and deduce that (rescaled) large mass minimizers converge, up to translations, to the ball. In view of Proposition 3.3.8, for any $\lambda \in (0, +\infty)$, let us define on $L^1(\mathbb{R}^n)$ the functional

$$\mathcal{F}_{K,\lambda}(f) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} |Df| - \frac{I_K^{0,1}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|}\rho_{K,1/\lambda}(x-y) & \\ \qquad\qquad \text{if } f \in \mathrm{BV}(\mathbb{R}^n) \text{ and } \displaystyle\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B|, \\ +\infty \qquad\qquad \text{otherwise,} \end{cases} \tag{3.4.27}$$

which is well defined and finite whenever $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B|$ and $f \in \mathrm{BV}(\mathbb{R}^n)$ by Proposition 3.3.3. $\mathcal{F}_{K,\lambda}$ is obviously defined so that it coincides with $P - \mathcal{V}_{K,\lambda}$ when f is a characteristic function of a set of volume $|B|$.

Proposition 3.4.7. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers going to infinity. If $I_K^{0,1} < 2/\mathbf{K}_{1,n}$, then the functionals $\mathcal{F}_{K,\lambda_k}$ defined by (3.4.27) Γ-converge w.r.t. the usual L^1 topology to the functional*

$$\mathcal{F}_{K,\infty}(f) := \begin{cases} \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right)\int_{\mathbb{R}^n}|Df| & \text{if } f\in \mathrm{BV}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B| \\ +\infty & \text{otherwise.}\end{cases} \tag{3.4.28}$$

Proof. We shall prove, in that order, that

$$\Gamma - \limsup \mathcal{F}_{K,\lambda_k}(f) \leqslant \mathcal{F}_{K,\infty}(f), \qquad \text{and} \qquad \mathcal{F}_{K,\infty}(f) \leqslant \Gamma - \liminf \mathcal{F}_{K,\lambda_k}(f),$$

where

$$\Gamma - \limsup \mathcal{F}_{K,\lambda_k}(f) := \min\left\{\limsup_k \mathcal{F}_{K,\lambda_k}(f_k) : f_k \xrightarrow{L^1(\mathbb{R}^n)} f\right\}.$$

and

$$\Gamma - \liminf \mathcal{F}_{K,\lambda_k}(f) := \min\left\{\liminf_k \mathcal{F}_{K,\lambda_k}(f_k) : f_k \xrightarrow{L^1(\mathbb{R}^n)} f\right\}.$$

Step 1. Let $f \in L^1(\mathbb{R}^n)$. If $f \notin \mathrm{BV}(\mathbb{R}^n)$ or $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \neq |B|$, $\mathcal{F}_{K,\infty}(f) = +\infty$ so the inequality is trivial. Let us assume $f \in \mathrm{BV}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B|$, and consider the constant sequence $f_k \equiv f$. Then by Proposition 3.3.10 we have

$$\lim_k \mathcal{F}_{K,\lambda_k}(f) = \mathcal{F}_{K,\infty}(f),$$

thus $\Gamma - \limsup \mathcal{F}_{K,\lambda_k}(f) \leqslant \mathcal{F}_{K,\infty}(f)$.

Step 2. Given $f \in L^1(\mathbb{R}^n)$, consider a sequence $f_k \in L^1(\mathbb{R}^n)$ such that $f_k \xrightarrow{L^1} f$. If $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \neq |B|$, then by L^1 convergence we have $\int_{\mathbb{R}^n} f_k(x)\,\mathrm{d}x \neq |B|$ for any k large enough, so that $\mathcal{F}_{K,\lambda_k}(f_k) = +\infty$, in which case

$$\liminf_k \mathcal{F}_{K,\lambda_k}(f_k) \geqslant \mathcal{F}_{K,\infty}(f)$$

is trivial. Thus we may now assume that $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B|$. By Proposition 3.3.3, we have

$$\mathcal{F}_{K,\lambda_k}(f_k) \geqslant \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right)\int_{\mathbb{R}^n}|Df_k|,$$

which holds even if $f_k \notin \mathrm{BV}(\mathbb{R}^n)$ or $\int_{\mathbb{R}^n} f_k(x)\,\mathrm{d}x \neq |B|$. Since the BV seminorm is lower semicontinuous with respect to the usual L^1 topology, and $(1 - (I_K^{0,1}\mathbf{K}_{1,n})/2) > 0$, we find

$$\liminf_k \mathcal{F}_{K,\lambda_k}(f_k) \geqslant \left(1 - \frac{I_K^{1,0}\mathbf{K}_{1,n}}{2}\right)\int_{\mathbb{R}^n}|Df| = \mathcal{F}_{K,\infty}(f),$$

where we used the fact that $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |B|$ for the last equality. This shows that $\mathcal{F}_{K,\infty}(f) \leqslant \Gamma - \liminf \mathcal{F}_{K,\lambda_k}(f)$. ☐

As usual, the Γ-convergence tells us that any converging sequence of minimizers of the functionals $\mathcal{F}_{K,\lambda_k}$, where $\lambda_k \to \infty$, necessarily converges to a minimizer of the Γ-limit, which gives the following corollary.

Corollary 3.4.8. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers going to infinity. If $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, any sequence of minimizers $(F_k)_{k\in\mathbb{N}}$ of Problem (P2′) for $\lambda = \lambda_k$ satisfying $\int_{F_k} x\,\mathrm{d}x = 0$ converges to B for the L^1 norm, i.e.,*

$$|F_k \triangle B| \xrightarrow{k\to\infty} 0.$$

Proof. By minimality of F_k and Proposition 3.3.3, we have

$$\left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) P(F_k) \leqslant P(F_k) - \mathcal{V}_{K,\lambda_k}(F_k) \leqslant P(B),$$

which yields

$$P(F_k) \leqslant \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right)^{-1} P(B), \quad \forall k \in \mathbb{N},$$

since $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$. Let $E_k := \lambda_k F_k$, and $m_k := \omega_n \lambda_k^n$. Then by Proposition 3.3.8, $(E_k)_{k\in\mathbb{N}}$ is a sequence of minimizers of Problem (P2) under the constraint $|E_k| = m_k$, where m_k goes to infinity. For any k large enough, $m_k > m_e$, where m_e is given by Theorem 3.1.1, thus $|E_k \setminus (4[B]_{m_k})| = 0$, i.e. up to subtracting a negligible set we can assume $E_k \subseteq 4[B]_{m_k} = B_{4\lambda_k}$, hence $F_k \subseteq B_4$ for all $k \in \mathbb{N}$. By compact embedding of $\mathrm{BV}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, there exists $f \in \mathrm{BV}(\mathbb{R}^n)$ and a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ such that $(\chi_{F_{n_k}})_{k\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^n)$ and almost everywhere to $f \in \mathrm{BV}(\mathbb{R}^n)$. The almost-everywhere convergence shows that $f = \chi_F$ for some set of finite perimeter $F \subseteq B_4$, and the L^1 convergence shows that $|F| = |B|$. In addition, still by L^1 convergence, we have $\int_F x\,\mathrm{d}x = 0$. In view of the Γ-convergence result given by Proposition 3.4.7, χ_F is a minimizer of the functional $\mathcal{F}_{K,\infty}$, which implies that F is a minimizer of the perimeter functional under the constraint $|F| = |B|$. Since the open unit ball centered at the origin is the unique minimizer (up to a translation) of the perimeter under volume constraint, the facts that $|F| = |B|$ and $\int_F x\,\mathrm{d}x = 0$ imply $F = B$. The L^1 convergence of $\chi_{F_{n_k}}$ to χ_B simply rewrites

$$|F_{n_k} \triangle B| \xrightarrow{k\to 0} 0.$$

Since we could have done the same reasoning for any subsequence of $(F_k)_{k\in\mathbb{N}}$ from the start, the whole sequence $(F_k)_{k\in\mathbb{N}}$ actually converges to B. □

Note that Theorem 3.1.2 follows immediately from Corollary 3.4.8, by the equivalence with Problems (P1) and (P2) due to (3.3.2) and Proposition 3.3.8.

3.4.4 Regularity of minimizers

We address here the question of regularity of minimizers of Problem (P1) (equivalently (P2)). Applying the extensive uniform regularity theory of *volume-constrained almost-minimizers* for the perimeter developed in [88] (here we prefer to adopt a terminology similar to the one in [45, 49], reserving the denomination of *quasi-minimizers* to another kind of minimality that we recall below as well), we readily obtain uniform $C^{1,\alpha}$-regularity of the boundary of any minimizer of Problem (P1) up to a singular set of Hausdorff dimension at most $(n-8)$ (the singular set being empty for $n < 7$).

Since sets of finite perimeter are defined up to a Lebesgue negligible set, we shall specify which sense we are giving to the boundary. Here we are not referring to the reduced boundary of E (which in fact does not have singular points if E is a minimizer of Problem (P1)), but rather to the support of the Gauss-Green measure of E, which is given by

$$\operatorname{spt} \nu_E = \{x \in \mathbb{R}^n \ : \ 0 < |E \cap B_r(x)| < |B_r(x)|, \forall r > 0\}.$$

It is known (see e.g. [73, Proposition 12.19]) that for any set of finite perimeter E, there exists E_0 an equivalent representative of E (that is, $|E \triangle E_0| = 0$) such that the topological boundary of E_0 agrees with $\operatorname{spt}\nu_E = \operatorname{spt}\nu_{E_0}$. Some authors simply denote $\operatorname{spt}\nu_E$ by ∂E, but we will refrain from doing so here.

Let us first elaborate on a few notions of minimality and quasi-minimality for the perimeter found in the literature, and recall some of the most remarkable related partial regularity results.

Definition 3.4.9. We say that a set of finite perimeter $E \subseteq \mathbb{R}^n$ is a local minimizer of the perimeter if for every ball $B_r(x) \subseteq \mathbb{R}^n$ and for every set of finite perimeter $F \subseteq \mathbb{R}^n$ such that $E \triangle F \subset\subset B_r(x)$, we have

$$P(E; B_r(x)) \leqslant P(F; B_r(x)).$$

It has been shown in [42], through the framework of area minimizing currents, that for any local minimizer E of the perimeter, $\operatorname{spt} \nu_E$ is locally a $C^{1,\alpha}$-hypersurface outside a set of at most $(n-8)$ Hausdorff dimension (it was already proven before in [83], following [33], that the reduced boundary $\partial^* E$ of any local minimizer of the perimeter is a $C^{1,\alpha}$-hypersurface, and that $\mathscr{H}^{n-1}(\operatorname{spt} \nu_E \setminus \partial^* E) = 0$). Similar partial regularity results were then obtained for sets of finite perimeter with prescribed mean curvature, in a general sense (see [74, 75]), possibly with a volume constraint (see [52, 53]). They were also extended to quasi-minimizers of the perimeter, that is, sets of finite perimeter E satisfying

$$P(E; B_r(x)) \leqslant (1 + \omega(r)) P(F; B_r(x)),$$

for every ball $B_r(x) \subseteq \mathbb{R}^n$ and every set of finite perimeter F such that $E \triangle F \subset\subset B_r(x)$, where $\omega : (0, +\infty) \to (0, +\infty]$ is an increasing function vanishing in 0.

It was proven in [111] that if $\omega(r) = Cr^{2\alpha}$ for some $C > 0$ and $\alpha \in (0, \frac{1}{2}]$, then the reduced boundary $\partial^* E$ of any quasi-minimizer of the perimeter E w.r.t. ω is locally a $(n-1)$-dimensional graph of class $C^{1,\alpha}$, and the Hausdorff dimension of $\operatorname{spt} \nu_E \setminus \partial^* E$ is at most $(n-8)$ (if $n \geqslant 7$, the singular set is empty). With no assumption on the function ω other than the fact that it vanishes at 0, it was then proven in [4] that outside a singular set of dimension at most $(n-8)$, $\operatorname{spt} \nu_E$ is a $C^{0,\alpha}$-hypersurface for every $\alpha \in (0, 1)$, and that the singular set is still empty for $n \leqslant 7$. This result however says nothing about the $C^{0,\alpha}$ constant of a quasi-minimizer E, nor does it say at which scales around a point (outside the singular set) the boundary of E is the graph of a $C^{0,\alpha}$ map. Finer, uniform versions of the results of [4] were then obtained by S. Rigot in [89]. It is in particular shown that if $\omega(r) = Cr^{2\alpha}$ for some $\alpha \in (0, \frac{1}{2}]$, for any quasi-minimizer E w.r.t. ω, $\operatorname{spt} \nu_E$ is locally a $(n-1)$-dimensional graph of class $C^{1,\alpha}$ outside a singular set of Hausdorff dimension at most $(n-8)$, and its $C^{1,\alpha}$-regularity constant can be bounded depending only on n and ω. In addition, when $n \leqslant 7$ there exists r_0 depending only on n and ω such that for every $x \in \operatorname{spt} \nu_E$, $\operatorname{spt} \nu_E \cap B_r(x)$ is a $(n-1)$-dimensional graph of class $C^{1,\alpha}$ for some $r > r_0$. S. Rigot then studied in [88] regularity of volume-constrained almost-minimizers of the perimeter (defined just below), and as an application obtained regularity results for minimizers of Problem (P1). While in [88] the kernel K of Problem (P1) is assumed to be compactly supported, this assumption is only used to get existence of a minimizer, and the regularity results essentially rely on the integrability of the kernel on $\mathbb{R}^n$. Let us now recall just a few of those results.

Definition 3.4.10 (Volume-constrained almost-minimizers)**.** Let $g : [0, +\infty) \to [0, +\infty)$ be such that $g(r) = o(r^{\frac{n-1}{n}})$ near 0. We say that a set of finite perimeter E of volume m is an *almost-minimizer of the perimeter w.r.t.* g *under the volume constraint* m if for every ball $B_r(x) \subseteq \mathbb{R}^n$ and every set of finite perimeter F of volume m such that $E \triangle F \subset\subset B_r(x)$, we have

$$P(E; B_r(x)) \leqslant P(F; B_r(x)) + g(|E \triangle F|).$$

It is shown in [88, Lemma 5.2.1] that any minimizer of Problem (P1) is a volume-constrained almost-minimizer of the perimeter w.r.t. $g(r) = I_K^{0,0} r$. It is also a direct consequence of Lemma 3.3.6. Indeed, we see that if a set E of volume m solves Problem (P1), then for any set of finite perimeter F such that $|F| = m$, then

$$P(E) \leqslant P(F) + \mathrm{Per}_K(E) - \mathrm{Per}_K(F) \leqslant P(F) + I_K^{0,0}|E\triangle F|.$$

It is then proven in [88, Proposition 4.3.1] that we can drop the volume-constraint in the almost-minimality condition, and obtain that any volume-constrained almost-minimizer of the perimeter w.r.t. $g(r) = Cr^p$, for $p > \frac{n-1}{n}$, is a local quasi-minimizer of the perimeter, in the following sense.

Definition 3.4.11 (r_0-quasi-minimizers of the perimeter)**.** Given $r_0 > 0$ and $\omega : (0, r_0) \to (0, +\infty]$ an increasing function such that $\omega(r) \xrightarrow{r\to 0} 0$, we say that a set of finite perimeter is a r_0-*quasi minimizer of the perimeter w.r.t.* ω if for every ball $B_r(x) \subseteq \mathbb{R}^n$ such that $r \in (0, r_0)$ and every set of finite perimeter F satisfying $E\triangle F \subset\subset B_r(x)$, we have

$$P(E; B_r(x)) \leqslant (1 + \omega(r))P(F; B_r(x)).$$

Proposition 3.4.12 ([88, Proposition 4.3.1])**.** *If $g(r) = C_1 r^p$ for some $p > \frac{n-1}{n}$, and E is a volume-constrained almost-minimizer of the perimeter of volume m w.r.t. g, then there exist $r_0 = r_0(n, C_1, p, m)$ and $C_2 = C_2(n, C_1, p, m)$ such that E is a r_0-quasi-minimizer of the perimeter w.r.t. to $\omega(r) := C_2 r^{2\alpha}$, where $\alpha := \min(\frac{np-(n-1)}{2}, \frac{1}{2})$.*

Remark 3.4.13. We could also apply directly results from [113] to get rid of the volume constraint. However, the approach differs from the one in [88] and does not give any control on the radius r_0 of quasi-minimality, which may depend on the minimizer E considered.

From this are deduced partial $C^{1,\alpha}$-regularity results when $n \geqslant 8$ and $C^{1,\alpha}$-regularity everywhere when $n \geqslant 7$ (see [88, Theorems 1.4.8 & 1.4.9], which are consequences of [89, Theorems 2.6 & 6.4]; see also [4, Theorems 4.7 & 4.10]), with regularity constants depending only on n, m and g. In our case, $p = 1$ and we obtain $C^{1,\frac{1}{2}}$ regularity. Let us sum up some of the regularity results we obtain in the end for minimizers of Problem (P1) or (P2′) in the following theorem.

Theorem 3.4.14. *Let E be a minimizer* [1] *of Problem (P1) or (P2′). Then $\partial^* E$ is locally a $(n-1)$-dimensional graph of class $C^{1,\frac{1}{2}}$. In addition, defining*

$$E_0 := \{x \in \mathbb{R}^n \ : \ \text{there exists } r > 0 \text{ s.t. } |B_r(x) \cap E| = |B_r(x)|\},$$

E_0 is an open set equivalent to E whose topological connected components coincide with the $\mathcal{M}$-connected components of E, and it is included in some ball B_R, where R depends only on n, m (or λ) and $I_K^{0,0}$. If $n \leqslant 7$, then $\partial E_0 = \partial^* E$, making the topological boundary of E_0 a $C^{1,\frac{1}{2}}$-hypersurface, and if $n \geqslant 8$, then $\dim_{\mathscr{H}}(\partial E_0 \setminus \partial^* E) \leqslant n - 8$. If we assume that K does not have a compact support, then E_0 is connected.

Proof. The fact that E_0 is an open set equivalent to E such that $\partial E_0 = \mathrm{spt}\,\nu_E$ is due to [88, Proposition 2.2.1] (see also [89, Lemma 3.6]). By Proposition 3.4.12 with $g(r) = I_K^{0,0} r$ (i.e. $p = 1$), E is a r_0-quasi-minimizer of the perimeter w.r.t. to $\omega(r) = Cr$ for some C and r_0 depending only on n, m and $I_K^{0,0}$. Using [111, Theorem 1] (see also [73, Theorem 28.1]), we know that $\partial^* E$ is a $C^{1,\frac{1}{2}}$-hypersurface, with $\dim_{\mathscr{H}}(\partial E_0 \setminus \partial^* E) \leqslant n - 8$ whenever $n > 8$,

[1] If one exists, no matter the value of $I_K^{0,1}$ compared to the threshold $\frac{2}{\mathbf{K}_{1,n}}$.

and $\partial E_0 = \partial^* E$ whenever $n \leqslant 7$. By the results of [88, 89] we know that there exists a singular set $\Sigma(E)$ (which is defined by a condition of mean-flatness, see [89, Section 6] or [4, Definition 4.6]) such that $\partial E_0 \setminus \Sigma(E)$ is locally made of $(n-1)$-dimensional graphs of class $C^{1,\frac{1}{2}}$, where $\dim_{\mathscr{H}}(\Sigma(E)) \leqslant n-8$ for $n > 8$ and $\Sigma(E) = \emptyset$ for $n < 7$ (see [4, Theorem 4.10]). In addition, a look at the proofs in [4] or at [89] shows that the $C^{1,\frac{1}{2}}$-regularity constants can be bounded depending only on n, m and $I_K^{0,0}$. By definition of this singular set and $C^{1,\frac{1}{2}}$ regularity of $\partial^* E$, we know that $\Sigma(E) \subseteq \partial E_0 \setminus \partial^* E$, so that the constant of $C^{1,\frac{1}{2}}$ regularity of $\partial^* E$ is obviously bounded depending only on n, m and $I_K^{0,0}$ as well. Since E_0 is open and $\mathscr{H}^{n-1}(\partial E_0) = \mathscr{H}^{n-1}(\partial^* E)$, [3, Theorem 2] implies that the $\mathcal{M}$-connected components of E coincide with the topological connected components of E_0. The fact that E_0 is included in a ball B_R such that $R = R(n, m, I_K^{0,0})$ comes from the density estimate

$$|E \cap B_r(x)| \geqslant c|B_r(x)|, \qquad \text{for } \mathscr{L}^n\text{-a.e. } x \in E \text{ and every } 0 < r < r_1, \tag{3.4.29}$$

where $c = c(n, m, I_K^{0,0})$, $r_1 = r_1(n, m, I_K^{0,0})$, which is a consequence of the results in [88]. More precisely, a uniform version of [88, Lemma 2.1.3] is obtained in [88, Section 4.1] (see in particular paragraph 4.1.3 therein), which readily implies (3.4.29). Eventually, if K does not have a compact support, by Proposition 3.4.5, E has only one $\mathcal{M}$-connected component, thus E_0 is connected. □

3.4.5 Towards uniform density estimates

In this section we present some results and some ideas which are for now only in the state of conjectures, the aim being to obtain uniform (in terms of λ) volume and perimeter density estimates for minimizers of Problem (P2′). As was already mentioned, by [88], for any minimizer E of Problem (P1) or (P2′), we can find lower density estimates, that is, there exist some c and r_1 depending only on n, λ and $I_K^{0,0}$ such that $|E \cap B_r(x)| \geqslant c|B_r(x)|$ for a.e. $x \in E$ and every $r \in (0, r_1)$. While these estimates do not depend on the minimizer, they still depend on the mass m (for (P1)) or the λ parameter (for the rescaled problem (P2′)): it is then possible for the constant c to vanish as λ goes to infinity, preventing us from saying e.g. that all minimizers of Problem (P2′) lie in some ball B_R independently of λ (however, we already know this fact by Theorem 3.1.1, albeit only for large masses when $I_K^{0,1} < \frac{\mathbf{K}_{1,n}}{2}$). Still, such uniform density estimates would be particularly useful to obtain Hausdorff convergence of the boundary of rescaled minimizers to ∂B as λ goes to infinity.

While S. Rigot does not procede that way to obtain density estimates, those can also be derived from the r_0-quasi-minimality of E (as in [73, Theorem 21.11]). By scaling, a minimizer E of Problem (P2′) is a volume-constrained almost-minimizer of the perimeter w.r.t. $g(r) = \lambda I_K^{0,0} r$. Unfortunately, the fact that the constant of quasi-minimality $\lambda I_K^{0,0}$ goes to infinity as λ goes to infinity implies that the density lower bound c we deduce vanishes as λ goes to infinity and prevents us from using this type of almost-minimality if we want to study the asymptotic behavior of rescaled minimizers.

Instead, we show that any minimizer of Problem (P2) is some type of quasi-minimizer of the perimeter, where the constant of quasi-minimality is invariant by scaling.

Definition 3.4.15 (Volume-constrained Λ-quasi-minimizers)**.** Let $\Lambda > 0$. We say that a set of finite perimeter E of volume m is a *volume-constrained* Λ*-quasi-minimizer of the perimeter* if, for every ball $B_r(x) \subseteq \mathbb{R}^n$ and every set of finite perimeter F such that $|F| = |E|$ and $E \triangle F \subset\subset B_r(x)$, we have

$$P(E; B_r(x)) \leqslant \Lambda P(F; B_r(x)).$$

We define the following unconstrained analogue.

Definition 3.4.16 ((Λ, r_0)-quasi minimizers)**.** Let $\Lambda > 0$ and $r_0 > 0$. We say that a set of finite perimeter E is a *(Λ, r_0)-quasi-minimizer for the perimeter* if, for every ball $B_r(x)$ such that $r \in (0, r_0)$ and for every set of finite perimeter F satisfying $E \triangle F \subset\subset B_r(x)$, we have

$$P(E; B_r(x)) \leqslant \Lambda P(F; B_r(x)).$$

Note that this would actually correspond to the quasi-minimality defined in Definition 3.4.11 with $g(r) \equiv C$ (this is however not allowed by the definition, which requires $g(r) = o(1)$ near 0). We can show that minimizers of Problem (P2) are volume-constrained Λ-quasi-minimizers for the perimeter, where $\Lambda = \Lambda(n, I_K^{0,1})$.

Proposition 3.4.17. *Let E be a minimizer of Problem (P2) of mass m. Then for every ball $B_r(x)$ and every set of finite perimeter F such that $E \triangle F \subset\subset B_r(x)$ and $|F| = m$, we have*

$$\left(1 - \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2}\right) P(E; B_r(x)) \leqslant \left(1 + \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2}\right) P(F; B_r(x)).$$

In particular, if $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, then E is a volume-constrained Λ-quasi-minimizer of the perimeter with $\Lambda := \left(1 + \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2}\right)\left(1 - \frac{I_K^{0,1} \mathbf{K}_{1,n}}{2}\right)^{-1}$.

Proof. Without loss of generality, up to a translation we may assume $x = 0$.
Let E be a minimizer of Problem (P2), and F be a set of finite perimeter such that $|E| = |F|$ and $E \triangle F \subset\subset B_r$. By minimality of E, we have

$$P(E) - \iint_{E \times E^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y \leqslant P(F) - \iint_{F \times F^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y. \tag{3.4.30}$$

Recall that

$$\begin{aligned} P(E) = \mathscr{H}^{n-1}(\partial^* E) &= \mathscr{H}^{n-1}(\partial^* E \cap B_r) + \mathscr{H}^{n-1}(\partial^* E \cap B_r^c) \\ &= P(E; B_r) + \mathscr{H}^{n-1}(\partial^* E \cap B_r^c). \end{aligned}$$

Note that the same holds for F, and $\partial^* F \cap B_r^c = \partial^* E \cap B_r^c$ since $E \triangle F \subset\subset B_r$. Hence (3.4.30) becomes

$$P(E; B_r) \leqslant P(F; B_r) + \iint_{E \times E^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y - \iint_{F \times F^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y. \tag{3.4.31}$$

Let us write

$$\begin{aligned} &\iint_{E \times E^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y - \iint_{F \times F^c} K(x-y)\, \mathrm{d}x\, \mathrm{d}y \\ &\quad = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (\chi_E(x)(1 - \chi_E(y)) - \chi_F(x)(1 - \chi_F(y)))K(x-y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{3.4.32}$$

Enumerating the differents cases $x, y \in (E \setminus F), (F \setminus E), (E \cap F)$ and $(E^c \cap F^c)$, we infer

$$\begin{aligned} &\chi_E(x)(1 - \chi_E(y)) - \chi_F(x)(1 - \chi_F(y)) \\ &= \begin{cases} 1 & \text{if } (x,y) \in ((E \setminus F) \times (F \setminus E)) \sqcup ((E \setminus F) \times (E^c \cap F^c)) \\ & \qquad \sqcup ((E \cap F) \times (F \setminus E)) \\ -1 & \text{if } (x,y) \in ((F \setminus E) \times (E \setminus F)) \sqcup ((F \setminus E) \times (E^c \cap F^c)) \\ & \qquad \sqcup ((E \cap F) \times (E \setminus F)) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.4.33}$$

Since K is symmetric, we have

$$\iint_{(E\setminus F)\times(F\setminus E)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{(F\setminus E)\times(E\setminus F)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

thus the integrals on $(E \setminus F) \times (F \setminus E)$ and on $(E \setminus F) \times (F \setminus E)$ cancel each other, and (3.4.32) and (3.4.33) yield

$$\begin{aligned}
&\iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{F\times F^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \iint_{(E\setminus F)\times(E^c\cap F^c)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{(E\cap F)\times(F\setminus E)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \iint_{(F\setminus E)\times(E^c\cap F^c)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{(E\cap F)\times(E\setminus F)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.4.34}$$

Dismissing the nonpositive terms in (3.4.34) and using that $E \setminus F \subseteq E\triangle F$ and $F \setminus E \subseteq E\triangle F$, we have

$$\begin{aligned}
&\iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{F\times F^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant \iint_{(E\triangle F)\times(E^c\cap F^c)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{(E\cap F)\times(E\triangle F)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \iint_{(E\triangle F)\times(E^c\cap F^c)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{(E\triangle F)\times(E\cap F)} K(x-y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{3.4.35}$$

where we used the symmetry of K again for the last equality. Noticing that $(E\triangle F)^c = (E \cap F) \sqcup (E^c \cap F^c)$, from (3.4.35) and Corollary 3.3.5 it follows

$$\begin{aligned}
&\iint_{E\times E^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{F\times F^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant \iint_{(E\triangle F)\times(E\triangle F)^c} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2} P(E\triangle F).
\end{aligned} \tag{3.4.36}$$

Now recall that for any sets of finite perimeter A and B, and any open set U, we have

$$P(A \cup B; U) + P(A \cap B; U) \leqslant P(A; U) + P(B; U)$$

and

$$P(A \setminus B; U) \leqslant P(A; U) + P(B; U),$$

thus, using that $E\triangle F = (E \cup F) \setminus (E \cap F)$ and $E\triangle F \subset\subset B_r$, we find

$$P(E\triangle F) = P(E\triangle F; B_r) \leqslant P(E\cup F; B_r)+P(E\cap F; B_r) \leqslant P(E; B_r)+P(F; B_r). \tag{3.4.37}$$

From (3.4.31), (3.4.36) and (3.4.37) we reach

$$\left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) P(E; B_r) \leqslant \left(1 + \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) P(F; B_r)$$

which concludes the proof. □

By scaling, we have the following immediate corollary.

Corollary 3.4.18. *If $I_K^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$, then every minimizer F of Problem (P2′) is a volume-constrained Λ-quasi-minimizer of the perimeter, where Λ is given by Proposition 3.4.17.*

Without the volume constraint, we can actually obtain scale-invariant density estimates for (local) quasi-minimizers of the perimeter, but we have yet to deal with the volume constraint, which will be tackled in a future work.

Proposition 3.4.19 (Density estimates)**.** *For any (Λ, r_0)-quasi-minimizer of the perimeter, we have*

$$\frac{1}{(1+\Lambda)^n} \leqslant \frac{|E \cap B_r(x)|}{|B_r(x)|} \leqslant 1 - \frac{1}{(1+\Lambda)^n}, \quad \forall r \in (0, r_0) \text{ and } \mathscr{L}^n\text{-a.e. } x \in E, \tag{3.4.38}$$

and

$$c \leqslant \frac{P(E; B_r(x))}{\mathscr{H}^{n-1}(\partial B_r(x))} \leqslant \Lambda, \quad \forall r \in (0, r_0) \text{ and } \mathscr{H}^{n-1}\text{-a.e. } x \in \partial^* E, \tag{3.4.39}$$

where c depends only on n.

Proof. Let E be a (Λ, r_0)-quasi-minimizer of the perimeter, and let

$$\overline{E}^{\mathcal{M}} := \left\{ x \in \mathbb{R}^n \ : \ \limsup_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} > 0 \right\}$$

be the essential closure of E. We know that $|\overline{E}^{\mathcal{M}} \triangle E| = 0$, and also that $\mathscr{H}^{n-1}(\partial^* E \setminus \overline{E}^{\mathcal{M}}) = 0$, thus we may focus on points in $\overline{E}^{\mathcal{M}}$. Let $x \in \overline{E}^{\mathcal{M}}$, and define $m : (0, r_0) \to [0, +\infty)$ by $m(s) := |E \cap B_s(x)|$. By definition of $\overline{E}^{\mathcal{M}}$ it is clear that $m(r) > 0$ for all $r \in (0, r_0)$. Since E is a set of finite perimeter, m is absolutely continuous, and for $\mathscr{L}^1$-almost every $s \in (0, r_0)$, we have $m'(s) = \mathscr{H}^{n-1}(E \cap \partial B_s(x))$. We know by [73, Proposition 2.16 & Example 2.17] and [73, Lemma 15.12] that for $\mathscr{L}^1$-a.e. $r \in (0, r_0)$, we have

$$\mathscr{H}^{n-1}(\partial B_r(x) \cap \partial^* E) = 0 \tag{3.4.40}$$

and

$$P(E \cap B_r(x)) = \mathscr{H}^{n-1}(E \cap \partial B_r(x)) + P(E; B_r(x)). \tag{3.4.41}$$

Let us consider a radius $r \in (0, r_0)$ such that (3.4.40) and (3.4.41) hold, and $s \in (r, r_0)$. Consider the competitor $F = E \setminus B_r(x)$, so that $E \triangle F \subseteq B_r(x) \subset\subset B_s(x)$. Thus by (Λ, r_0)-quasi minimality of E, we have

$$P(E; B_s(x)) \leqslant \Lambda P(F; B_s(x)). \tag{3.4.42}$$

Note that

$$P(F; B_s(x)) = P(E; B_s(x) \setminus \overline{B_r(x)}) + \mathscr{H}^{n-1}(E \cap \partial B_r(x))$$

thus by (3.4.42) we find

$$P(E; B_s(x)) \leqslant \Lambda \left(P(E; B_s(x) \setminus \overline{B_r(x)}) + \mathscr{H}^{n-1}(E \cap \partial B_r(x)) \right). \tag{3.4.43}$$

Observe that

$$\begin{aligned} P(E; B_s(x) \setminus \overline{B_r(x)}) = |\nu_E|(B_s(x) \setminus \overline{B_r(x)}) &\xrightarrow{s \to r^+} |\nu_E|(\partial B_r(x)) \\ &= \mathscr{H}^{n-1}(\partial^* E \cap \partial B_r(x)) = 0 \end{aligned} \tag{3.4.44}$$

by our choice of r, thus with (3.4.44), letting s go to r^+ in (3.4.43) yields

$$P(E; B_r(x)) \leqslant \Lambda \mathscr{H}^{n-1}(E \cap \partial B_r(x)). \tag{3.4.45}$$

In particular we find the upper bound in (3.4.39) for $\mathscr{L}^1$-a.e. $r \in (0, r_0)$, and hence for every $r \in (0, r_0)$ by left-continuity of $r \mapsto P(E; B_r(x))$. Adding $\mathscr{H}^{n-1}(E \cap \partial B_r(x))$ to both sides of (3.4.45) and using (3.4.41) we obtain

$$P(E \cap B_r(x)) \leqslant (1 + \Lambda)\mathscr{H}^{n-1}(E \cap \partial B_r(x)). \tag{3.4.46}$$

Using the classical isoperimetric inequality (3.2.1), we have

$$P(E \cap B_r(x)) \geqslant n\omega_n^{\frac{1}{n}} |E \cap B_r(x)|^{1-\frac{1}{n}} = n\omega_n^{\frac{1}{n}} m(r)^{1-\frac{1}{n}}$$

thus with (3.4.46) it follows

$$n\omega_n^{\frac{1}{n}} m(r)^{1-\frac{1}{n}} \leqslant (\Lambda + 1)m'(r).$$

Since $m(r) > 0$, this implies that for $\mathscr{L}^1$-a.e. $r \in (0, r_0)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[m(r)^{\frac{1}{n}}\right] \geqslant \frac{\omega_n^{1/n}}{1 + \Lambda},$$

and integrating between 0 and r gives

$$m(r) \geqslant \frac{|B_r|}{(1 + \Lambda)^n},$$

which is the lower bound of (3.4.38). Since E^{c} is a (Λ, r_0)-quasi-minimizer of the perimeter whenever E is a (Λ, r_0)-quasi-minimizer of the perimeter, by symmetry we get the corresponding upper bound of (3.4.38) as well. Finally, by the relative isoperimetric inequality we have

$$P(E; B_r(x)) \geqslant C \min(|E \cap B_r(x)|, |E^{\mathrm{c}} \cap B_r(x)|)^{\frac{n-1}{n}},$$

for some $C = C(n)$, hence the lower bound of (3.4.39) by (3.4.38). □

Now let us make some conjectures. We believe that the volume-constraint could be dropped in such a way that minimizers of Problem (P1) are local (Λ, r_0)-quasi-minimizers of the perimeter for some r_0 and Λ depending only on n and K, although it is not clear how to do so without adding a volume term, which we would like to avoid in order to get estimates uniform in λ for the rescaled problem. Then looking at the rescaled problem (P2′), one would obtain that minimizers are (Λ, r_0)-quasi-minimizers of the perimeter, where Λ and r_0 depend only on n and K, and use the uniform density estimates of Proposition 3.4.19 to prove the following: for any sequence of minimizers $(E_{\lambda_k})_{k\in\mathbb{N}}$ of Problem (P2′) such that $\lambda_k \to \infty$, the boundaries $\partial^* E_{\lambda_k}$ converge Hausdorff to ∂B.

3.5 Stability of the ball

3.5.1 First and second variations of perimeters

In this subsection we recall formulas for the first and second variations of the classical and non local perimeters, which can be found e.g. in [45, Section 6]. In all this subsection

E denotes an open set of finite parameter such that ∂E is a C^2 hypersurface. First we define some terminology.

Given a vector field $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, we define the flow induced by X as the solution in $t \in \mathbb{R}$ of the ODEs

$$\begin{cases} \partial_t \Phi_t(x) = X(\Phi_t(x)) \\ \Phi_0(x) = x. \end{cases}$$

It is well known that $\Phi_t(x)$ is well defined for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and that $(\Phi_t)_{t\in\mathbb{R}}$ is a one-parameter group of smooth diffeomorphisms on $\mathbb{R}^n$, i.e. $\Phi_t \circ \Phi_s = \Phi_{s+t}$ for all $s,t \in \mathbb{R}$, and $\Phi_0 = \mathrm{id}_{|\mathbb{R}^n}$. Given Φ_t a flow induced by X, we define $E_t := \Phi_t(E)$.

Definition 3.5.1. A vector field $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ induces a volume-preserving flow on E if there exists $\delta > 0$ such that $|E_t| = |E|$ for all $|t| < \delta$.

Given a functional $\mathcal{F}$ on sets of finite perimeter such that $t \mapsto \mathcal{F}(E_t) \in C^2(-\delta,\delta)$ for some $\delta > 0$, we define the first and second variations of $\mathcal{F}$ at E in the direction $X \in C_c^\infty(\mathbb{R}^n)$ by

$$\delta\mathcal{F}(E)[X] := \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(E_t)\right]_{|t=0}, \quad \delta^2\mathcal{F}(E)[X] := \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)\right]_{|t=0}.$$

Then we define the notion of volume-constrained stationary sets for a functional.

Definition 3.5.2. We say that E is a volume-constrained stationary set for the functional $\mathcal{F}$ if $\delta\mathcal{F}(E)[X] = 0$ for every $X \in C_c^\infty(\mathbb{R}^n)$ inducing a volume-preserving flow on E.

We are interested in the variations of the classical perimeter P and of the nonlocal perimeter Per_K, which we will deduce from the variations of the nonlocal term

$$G_K(E) := \iint_{E\times E} K(x-y)\,\mathrm{d}x\,\mathrm{d}y = |E| I_K^{0,1} - \mathrm{Per}_K(E).$$

For the classical perimeter, it is known that $t \mapsto P(E_t)$ is smooth in $(-\delta,\delta)$ whenever E is a set of finite perimeter, and if ∂E is a C^2-hypersurface, the first variation is

$$\delta P(E)[X] = \int_{\partial E} H_{\partial E}\,\zeta\,d\mathscr{H}^{n-1},$$

where ν_E is the outer unit normal to E, $\zeta := X \cdot \nu_E$, and $H_{\partial E}$ is the scalar mean curvature of ∂E. The second variation is given by

$$\begin{aligned}\delta^2 P(E)[X] = &\int_{\partial E} |\nabla_\tau \zeta|^2 - c_{\partial E}^2\,\zeta^2\,d\mathscr{H}^{n-1} \\ &+ \int_{\partial E} H_{\partial E}\big((\operatorname{div} X)\zeta - \operatorname{div}_\tau(\zeta X_\tau)\big)\,d\mathscr{H}^{n-1},\end{aligned}$$

where $c_{\partial E}^2$ is the sum of the squares of the principal curvatures of ∂E, $X_\tau := X - \zeta\nu_E$ and ∇_τ and div_τ denote respectively the tangential gradient and divergence on ∂E. In addition, if E is a volume-constrained stationary set for the perimeter, and X induces a volume-preserving flow on E, then the second variation of the perimeter takes the simpler form

$$\delta^2 P(E)[X] = \int_{\partial E} |\nabla_\tau\,\zeta|^2 - c_{\partial E}^2\,\zeta^2\,d\mathscr{H}^{n-1}. \tag{3.5.1}$$

Indeed, the fact that $t \mapsto |E_t|$ is constant in a neighborhood of 0 implies

$$0 = \left[\frac{\mathrm{d}}{\mathrm{d}t}|E_t|\right]_{|t=0} = \int_{\partial E} \zeta \, d\mathscr{H}^{n-1} \quad \text{and} \quad 0 = \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}|E_t|\right]_{|t=0} = \int_{\partial E} (\operatorname{div} X)\zeta \, d\mathscr{H}^{n-1}.$$

As for the second variation of G_K, note that $K \in C^1(\mathbb{R}^n \setminus \{0\})$, $K(x) = o(|x|^{\alpha-n})$ near the origin for some $\alpha > 0$ by (H4), and $K(x) = o(|x|^{-(n+1)})$ at infinity by Remark 3.3.1, thus K satisfies the assumption of the map G in [45, (6.7)], and we can apply [45, Theorem 6.1], to get

$$\begin{aligned}
\delta G_K(E)[X] &= \int_{\partial E} H^*_{K,\partial E}\, \zeta \, d\mathscr{H}^{n-1}, \\
\delta^2 G_K(E)[X] &= -\iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2 \, d\mathscr{H}^{n-1}_x \, d\mathscr{H}^{n-1}_y + \int_{\partial E} c^2_{K,\partial E}\, \zeta^2 \, d\mathscr{H}^{n-1} \\
&\quad + \int_{\partial E} H^*_{K,\partial E}\big((\operatorname{div} X)\zeta - \operatorname{div}_\tau(\zeta X_\tau)\big) \, d\mathscr{H}^{n-1},
\end{aligned} \tag{3.5.2}$$

since E is an open set such that ∂E is a C^2-hypersurface and $|E| < +\infty$, where

$$c^2_{K,\partial E}(x) := \int_{\partial E} K(x-y)|\nu_E(x)-\nu_E(y)|^2 \, d\mathscr{H}^{n-1}_y, \quad \forall x \in \partial E,$$

and $H^*_{K,\partial E}$, which plays the role of the mean curvature for the nonlocal perimeter Per_K, is defined by

$$H^*_{K,\partial E}(x) := 2\int_E K(x-y)\,\mathrm{d}y, \quad \forall x \in \partial E. \tag{3.5.3}$$

Note that all the integrals of (3.5.2) are finite whenever ∂E is a C^2-hypersurface. Indeed X_τ, ζ and ν_E are C^1 functions, and if φ is C^1, we have $K(x-y)|\varphi(x)-\varphi(y)|^2 \leqslant C\|\varphi\|_{C^1(\partial B)}|x-y|^2 K(x-y)$ which ensures that the integrals converge since $r \mapsto r^n k(r) \in L^1(0,+\infty)$ by (H1). Similarly to the perimeter functional, if E is a volume-constrained stationary set for G_K and X induces a volume-preserving flow on E, the fact that $t \mapsto |E_t|$ is constant in a neighborhood of 0 implies that the second variation of G_K is simply given by

$$\begin{aligned}
\delta^2 G_K(E)[X] = &-\iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2 \, d\mathscr{H}^{n-1}_x \, d\mathscr{H}^{n-1}_y \\
&+ \int_{\partial E} c^2_{K,\partial E}\, \zeta^2 \, d\mathscr{H}^{n-1}.
\end{aligned}$$

Observe that

$$\mathrm{Per}_K(E_t) = I^{0,1}_K |E_t| - G_K(E_t), \qquad \forall t \in (-\delta, \delta),$$

thus E is a volume-constrained stationary set for G_K if and only if it is one for Per_K, and in that case, if X induces a volume-preserving flow, $|E_t| = |E|$, thus the second variation of Per_K is given by

$$\begin{aligned}
\delta^2 \,\mathrm{Per}_K(E)[X] &= -\delta^2 G_K(E)[X] \\
&= \iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2 \, d\mathscr{H}^{n-1}_x \, d\mathscr{H}^{n-1}_y \\
&\quad - \int_{\partial E} c^2_{K,\partial E}\, \zeta^2 \, d\mathscr{H}^{n-1}.
\end{aligned} \tag{3.5.4}$$

We end this section by recalling the definition of stability in that setting.

Definition 3.5.3. We say that E is a volume-constrained stable set for a function $\mathcal{F}$ if E is a volume-constrained stationary set for $\mathcal{F}$ and

$$\delta^2\mathcal{F}(E)[X] \geqslant 0$$

for any vector field $X \in C_c^\infty(\mathbb{R}^n)$ inducing a volume-preserving flow on E.

3.5.2 Instability threshold for large balls

We are interested in the stability of the ball $[B]_m$ for (P1). As before, let $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$.

Proposition 3.5.4. *The ball $[B]_m$ is a volume-constrained stationary set for $P + G_K$ (and equivalently $P - \mathrm{Per}_K$), and the unit ball B is a volume-constrained stationary set for $\mathcal{F}_{K,\lambda} = P - \mathcal{V}_{K,\lambda}$. In addition, $[B]_m$ is a volume-constrained stable set for $P + G_K$ (equivalently $P - \mathrm{Per}_K$) iff B is stable for $\mathcal{F}_{K,\lambda}$. For $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ inducing a volume-preserving flow on B, the second variation of $\mathcal{F}_{K,\lambda}$ is given by*

$$\begin{aligned}
&\delta^2\mathcal{F}_{K,\lambda}(B)[X] \\
&\quad= \int_{\partial B} |\nabla_\tau \zeta|^2 \, d\mathscr{H}^{n-1} - \frac{I_K^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \iint_{\partial B\times\partial B} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^2}\eta_{K,\lambda}(|x-y|)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1} \\
&\qquad + \left(\frac{I_K^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} c^2_{K,\lambda,\partial B} - c^2_{\partial B}\right) \int_{\partial B} \zeta^2 \, d\mathscr{H}^{n-1}.
\end{aligned}$$

where $c^2_{\partial B} = n-1$ is the sum of the squares of the principal curvatures of ∂B, and where for all $r \in (0,+\infty)$, we defined

$$\eta_K(r) := \frac{|\mathbb{S}^{n-1}|}{I_K^{0,1}|\mathbb{S}^{n-2}|} r^2 k(r), \qquad \eta_{K,\lambda}(r) := \lambda^{n-1}\eta_K(\lambda r),$$

and

$$c^2_{K,\lambda,\partial B} := \int_{\partial B} \eta_{K,\lambda}(|x-y|)\, d\mathscr{H}_y^{n-1}.$$

Proof. Since balls minimize the perimeter under volume constraint, they are volume-constrained stationary sets for P. Recall that G_K is maximized by balls under volume constraint, while Per_K is minimized by balls under volume constraint, thus balls are stationary sets for G_K and Per_K[2]. By scaling of Per_K, the unit ball minimizes $\mathcal{V}_{K,\lambda}$ under volume constraint as well. In the end, $[B]_m$ is a volume-constrained stationary set for $P + G_K$ and $P - \mathrm{Per}_K$, and B is a stationary set for $\mathcal{F}_{K,\lambda}$. By (3.5.4) we already know that $[B]_m$ is a volume-constrained stable set for $P + G_K$ iff it is a stable set for $P - \mathrm{Per}_K$. Applying (3.5.4) to $\mathcal{V}_{K,\lambda} = \mathrm{Per}_{K_\lambda}$ where $K_\lambda(x) := \lambda^{n+1}K(\lambda x)$, we find

$$\begin{aligned}
\delta^2\mathcal{V}_{K,\lambda}(B)[X] = &\iint_{\partial B\times\partial B} \lambda^{n+1}K(\lambda(x-y))|\zeta(x)-\zeta(y)|^2\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1} \\
&- \iint_{\partial B\times\partial B} \lambda^{n+1}K(\lambda(x-y))|x-y|^2|\zeta(x)|^2\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1}
\end{aligned} \tag{3.5.5}$$

[2]One can also notice from (3.5.3) that $H^*_{K,\partial B}$ is constant by symmetry, which directly gives the stationarity of B in view of the expression of the first variation of G_K given by (3.5.2).

for every $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ inducing a volume-preserving flow on B. Scaling back, we see that

$$\begin{aligned}
&\lambda^{n-1}\delta^2\mathcal{V}_{K,\lambda}(B)[X]\\
&= \iint_{\partial B_\lambda\times\partial B_\lambda} K(x-y)|\lambda X(\lambda^{-1}x)\cdot\nu_{\partial B_\lambda}(x)-\lambda X(\lambda^{-1}y)\cdot\nu_{\partial B_\lambda}(y)|^2\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&\qquad-\iint_{\partial B_\lambda\times\partial B_\lambda} K(x-y)|\nu_{\partial B_\lambda}(x)-\nu_{\partial B_\lambda}(y)|^2\big(\lambda X(\lambda^{-1}x)\cdot\nu_{\partial B_\lambda}(x)\big)^2\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&= \iint_{\partial B_\lambda\times\partial B_\lambda} K(x-y)|(X_\lambda\cdot\nu_{\partial B_\lambda})(x)-(X_\lambda\cdot\nu_{\partial B_\lambda})(y)|^2\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&\qquad-\iint_{\partial B_\lambda\times\partial B_\lambda} K(x-y)|\nu_{\partial B_\lambda}(x)-\nu_{\partial B_\lambda}(y)|^2(X_\lambda\cdot\nu_{\partial B_\lambda})^2(x)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&= \delta^2 G_K([B]_m)[X_\lambda],
\end{aligned}$$

where X_λ is the vector field defined by $X_\lambda(x)=\lambda X(\lambda^{-1}x)$. Obviously we have

$$\lambda^{n-1}\delta^2P(B)[X]=\delta^2P([B]_m)[X_\lambda],$$

thus

$$\delta^2(P-\mathrm{Per}_K)([B]_m)[X_\lambda]=\lambda^{n-1}\delta^2\mathcal{F}_{K,\lambda}(B)[X]. \tag{3.5.6}$$

Observe that X is a volume-preserving flow on B if and only if X_λ is a volume-preserving flow on $B_\lambda=[B]_m$. Indeed, the flow induced by X_λ denoted by Φ_λ is given by $\Phi_{\lambda,t}(x)=\lambda\Phi_t(\lambda^{-1}x)$, where Φ is the flow induced by X, and it is then easy to see that $\Phi_{\lambda,t}(\lambda B)=\lambda\Phi_t(B)$. Hence with (3.5.6) we see that B is a volume-constrained stable set for $\mathcal{F}_{K,\lambda}$ if and only if $[B]_m$ is stable for $P-\mathrm{Per}_K$. In addition, we can rewrite (3.5.5) in terms of $\eta_{K,\lambda}$ by

$$\begin{aligned}
\delta^2\mathcal{V}_{K,\lambda}(B)[X] &= \iint_{\partial B\times\partial B}\frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^2}\lambda^{n-1}\,(\lambda|x-y|)^2\,k(\lambda|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&\qquad-\iint_{\partial B\times\partial B}\lambda^{n-1}\,(\lambda|x-y|)^2\,k(\lambda|x-y|)\zeta^2(x)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&= \frac{I_K^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\Bigg(\iint_{\partial B\times\partial B}\frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^2}\eta_{K,\lambda}(x-y)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&\qquad-\iint_{\partial B\times\partial B}\eta_{K,\lambda}(x-y)\zeta^2(x)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\Bigg).
\end{aligned}$$

Note that

$$c^2_{K,\lambda,\partial B}=\int_{\partial B}\eta_{K,\lambda}(x-y)\,d\mathscr{H}^{n-1}_y$$

does not depend on $x\in\partial B$, since $\eta_{K,\lambda}$ is invariant by rotations, thus by Fubini's theorem, we find

$$\begin{aligned}
\delta^2\mathcal{V}_{K,\lambda}(B)[X]=\frac{I_K^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\Bigg(&\iint_{\partial B\times\partial B}\frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^2}\eta_{K,\lambda}(x-y)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\\
&-c^2_{K,\lambda,\partial B}\int_{\partial B}\zeta^2\,d\mathscr{H}^{n-1}\Bigg).
\end{aligned} \tag{3.5.7}$$

Combining (3.5.5) and (3.5.7) gives the result. □

Remark 3.5.5. In view of the expressions of $\delta^2 P(B)[X]$ and $\delta^2\mathcal{V}_{K,\lambda}[X]$, it would be natural to define the quadratic functionals

$$\mathcal{Q}P(u) := \int_{\partial B} |\nabla_\tau u|^2 \, d\mathscr{H}^{n-1} - (n-1)\int_{\partial B} |u|^2 \, d\mathscr{H}^{n-1}$$

and

$$\begin{aligned}\mathcal{Q}\mathcal{V}_{K,\lambda}(u) := \iint_{\partial B\times\partial B} & \lambda^{n+1} K(\lambda(x-y))|u(x)-u(y)|^2 \, d\mathscr{H}^{n-1}_x \, d\mathscr{H}^{n-1}_y \\ & - \iint_{\partial B\times\partial B} \lambda^{n+1} K(\lambda(x-y))|x-y|^2 |u(x)|^2 \, d\mathscr{H}^{n-1}_x \, d\mathscr{H}^{n-1}_y,\end{aligned}$$

on the vector space of functions $u \in H^1(\partial B)$ such that $\int_{\partial B} u \, d\mathscr{H}^{n-1} = 0$, since $\zeta = X\cdot\nu_B$ is null-averaged on ∂B whenever X induces a volume-preserving flow on B. Instead of defining the stability of the ball by the nonnegativity of the quantity $\delta^2(P-\mathcal{V}_{K,\lambda})(B)[X]$ for every X inducing a volume-preserving flow on B, we could have defined it by the nonnegativity of the quadratic functional $\mathcal{Q}P - \mathcal{Q}\mathcal{V}_{K,\lambda}$ on null-averaged functions in $H^1(\partial B)$. In fact, it is interesting to remark that, by the proof of [45, Theorem 7.1], those two notions of stability coincide.

Let us point out that $\eta_{K,\lambda}$ is chosen in such a say that

$$|\mathbb{S}^{n-2}| \int_0^\infty \eta_{K,\lambda}(r) r^{n-2} \, \mathrm{d}r = \frac{|\mathbb{S}^{n-1}|}{I_K^{0,1}} \int_0^\infty r k(r) r^{n-1} \, \mathrm{d}r = 1.$$

Thus, similarly to Section 3.3.1, we may see that, setting $\eta_{K,\varepsilon} := \eta_{K,1/\lambda}$, $(\eta_{K,\varepsilon})_{\varepsilon>0}$ is a family of $(n-1)$-dimensional mollifiers. We wish to pass to the limit, however here we integrate on the product $\partial B\times\partial B$, so we cannot use Proposition 3.3.10. In [69, Theorem 1.1], an equivalent to Proposition 3.3.10 is given for smooth Riemannian manifolds, unfortunately the assumptions on the family of mollifiers are too strong to be applicable here (in particular, the monotonicity of η_ε, for all ε, which we do not assume). In Section 3.A we prove the required counterpart to Proposition 3.3.10 on spheres, which allows us to prove that large balls are unstable in some cases.

Theorem 3.5.6. *If $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, there exists m_u such that for any $m > m_u$ the ball $[B]_m$ is not a volume-constrained stable set for the functional $P - \mathrm{Per}_K$. Equivalently, defining $\lambda_u := \left(\frac{m_u}{\omega_n}\right)^{\frac{1}{n}}$, the unit ball is not stable for $\mathcal{F}_{K,\lambda}$ for any $\lambda > \lambda_u$.*

Proof. As we have seen in Proposition 3.5.4, given $m, \lambda > 0$ such that $\omega_n\lambda^n = m$, the ball $[B]_m$ is stable for $P - \mathrm{Per}_K$ if and only if the unit ball B is stable for $P - \mathcal{V}_{K,\lambda}$. Assume $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$, and consider a vector field $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ inducing a volume-preserving flow on B such that

$$\delta^2 P(B)[X] > 0.$$

Let $\zeta = X\cdot\nu_B$. Let us check that the family $(\eta_{K,\varepsilon})_{\varepsilon>0}$ defined by setting $\varepsilon = 1/\lambda$ is a family of $(n-1)$-dimensional mollifiers. Since k is nonnegative, we have $\eta_{K,\varepsilon} \geqslant 0$, and we have already checked

$$|\mathbb{S}^{n-2}| \int_0^\infty \eta_{K,\varepsilon}(r) r^{n-2} \, \mathrm{d}r = 1.$$

In addition, for any $R > 0$ we have

$$\int_R^\infty \eta_{K,\varepsilon}(r) r^{n-2} \, \mathrm{d}r = C \int_{\frac{R}{\varepsilon}}^\infty k(r) r^n \, \mathrm{d}r \xrightarrow{\varepsilon\to 0} 0,$$

since $r \mapsto r^n k(r) \in L^1(0,\infty)$ by (H1) and (H2). This shows that $(\eta_{K,\varepsilon})_{\varepsilon>0}$ is a family of $(n-1)$-dimensional mollifiers. Let us check that it also satisfies assumption (3.A.5). Let $\mathcal{K}$ be a compact subset of $(0,+\infty)$, and $a,b>0$ such that $\mathcal{K} \subseteq (a,b)$. Then for every $s \in K$, using the monotonicity of k, we have

$$\eta_{K,\varepsilon}(s) = \varepsilon^{-(n+1)} s^2 k(\varepsilon^{-1}s) \leqslant \varepsilon^{-(n+1)} b^2 k(\varepsilon^{-1}a) = a^{-(n+1)} b^2 \left(\frac{a}{\varepsilon}\right)^{n+1} k\left(\frac{a}{\varepsilon}\right)$$

Recalling that $k(s) = o(s^{-(n+1)})$ at infinity by (H4), we then see that $\sup_{s\in\mathcal{K}} \eta_\varepsilon(s)$ goes to 0 as ε goes to 0, i.e. the family of mollifiers satisfies (3.A.5). Thus we can apply Proposition 3.A.2, which gives

$$\lim_{\lambda\to\infty} \iint_{\partial B\times\partial B} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^2} \eta_{K,\lambda}(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1} = \mathbf{K}_{2,n-1} \int_{\partial B} |\nabla_\tau \zeta|^2\, d\mathscr{H}^{n-1}. \tag{3.5.8}$$

On the other hand, we compute as well

$$\begin{aligned} c^2_{K,\lambda,\partial B} = \int_{\partial B} \eta_{K,\lambda}(x-y)\, d\mathscr{H}_y^{n-1} &= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\partial B\times\partial B} \eta_{K,\lambda}(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1} \\ &= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\partial B\times\partial B} \frac{|x-y|^2}{|x-y|^2} \eta_{K,\lambda}(x-y)\, d\mathscr{H}_x^{n-1}\, d\mathscr{H}_y^{n-1} \\ &\xrightarrow{\lambda\to\infty} \frac{\mathbf{K}_{2,n-1}}{|\mathbb{S}^{n-1}|} \int_{\partial B} |\nabla\tau\, x|^2\, d\mathscr{H}_x^{n-1} = \mathbf{K}_{2,n-1}\, c^2_{\partial B}. \end{aligned} \tag{3.5.9}$$

Combining (3.5.8) and (3.5.9), with Proposition 3.5.4 we find

$$\begin{aligned} \lim_{\lambda\to\infty} \delta^2 \mathcal{F}_{K,\lambda}(B)[X] &= \left(1 - I_K^{0,1} \frac{\mathbf{K}_{2,n-1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\right) \left(\int_{\partial E} |\nabla_\tau \zeta|^2 - c^2_{\partial E}\, \zeta^2\, d\mathscr{H}^{n-1}\right) \\ &= \left(1 - I_K^{0,1} \frac{\mathbf{K}_{2,n-1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\right) \delta^2 P(B)[X], \end{aligned}$$

where we used (3.5.1) for the last equality. Now by Lemma 3.3.14 we see that in fact

$$\frac{\mathbf{K}_{2,n-1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{\mathbf{K}_{1,n}}{2},$$

thus

$$\lim_{\lambda\to\infty} \delta^2 \mathcal{F}_{K,\lambda}(B)[X] = \left(1 - \frac{I_K^{0,1}\mathbf{K}_{1,n}}{2}\right) \delta^2 P(B)[X] < 0,$$

since $I_K^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ and $\delta^2 P(B)[X] > 0$. This shows that there exists $\lambda_u > 0$ such that for any $\lambda \geqslant \lambda_u$, the unit ball is unstable, which concludes the proof. □

Appendix

3.A On fractional Sobolev norms and $W^{1,2}$ maps on the sphere

We prove a result similar to Proposition 3.3.10 in the case where B is replaced with the $(n-1)$-dimensional sphere. In [69] the case of a general Riemannian manifold is considered, yet the monotonicity of the mollifiers is required, which is too strong to be applied in our case.

We will often use the following to integrate $\mathbb{S}^{n-1}$ on slices (see e.g. [6, Corollary A.6]): a $\mathscr{H}^{n-1}$-measurable function g is integrable on $\mathbb{S}^{n-1}$ if and only if $(x,t) \mapsto (1-t^2)^{\frac{n-3}{2}} g(\sqrt{1-t^2}x, t)$ is integrable on $\mathbb{S}^{n-2} \times (-1,1)$, and in that case we have

$$\int_{\mathbb{S}^{n-1}} g \, d\mathscr{H}^{n-1} = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} g(\sqrt{1-t^2}x, t) \, d\mathscr{H}_x^{n-2} \, \mathrm{d}t. \tag{3.A.1}$$

We will need the following basic lemma.

Lemma 3.A.1. *Let f be a continuous nonnegative function from $(0,+\infty)$ such that*

$$\int_0^\infty f(r) r^{n-2} \, \mathrm{d}r < +\infty.$$

Then for every $R > 0$ and every $x \in \partial B_R$, the map $F : y \mapsto f(|x-y|)$ belongs to $L^1(\partial B_R)$ for any $R > 0$, and we have

$$\int_{\partial B_R} f(|x-y|) \, d\mathscr{H}_y^{n-1} = |\mathbb{S}^{n-2}| R \int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s) \, \mathrm{d}s.$$

In addition, if $n \geqslant 3$, we have

$$\int_{\partial B_R} f(|x-y|) \, d\mathscr{H}_y^{n-1} \leqslant |\mathbb{S}^{n-2}| R \int_0^{\sqrt{4R}} s^{n-2} f(s) \, \mathrm{d}s,$$

and if $n = 2$,

$$\int_{\partial B_R} f(|x-y|) \, d\mathscr{H}_y^{n-1} \leqslant |\mathbb{S}^{n-2}| R \left(\frac{1}{\sqrt{2}} \int_0^{\sqrt{2R}} f(s) s^{n-2} \, \mathrm{d}s + \frac{\pi\sqrt{R}}{2} \|f\|_{L^\infty(\sqrt{2R}, \sqrt{4R})} \right).$$

Proof. Up to a change of variables, we can assume that $x = e = (0,\dots,0,1)$ is the "north pole". Our computations will show that $y \mapsto f(|Re-y|) \in L^1(\partial B_R)$. Applying (3.A.1) to $y \mapsto f(R|e-y|)$ we find

$$\begin{aligned}
\int_{\partial B_R} f(|Re-y|) \, d\mathscr{H}_y^{n-1} &= R^{n-1} \int_{\partial B} f(R|e-y|) \, d\mathscr{H}_y^{n-1} \\
&= R^{n-1} \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(R|e - (\sqrt{1-t^2}y, t)|) \, d\mathscr{H}_y^{n-2} \, \mathrm{d}t \\
&= R^{n-1} \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(\sqrt{2R(1-t)}) \, d\mathscr{H}_y^{n-2} \, \mathrm{d}t \\
&= |\mathbb{S}^{n-2}| R^{n-1} \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} f(\sqrt{2R(1-t)}) \, \mathrm{d}t.
\end{aligned}$$

Changing variables with $s = \sqrt{2R(1-t)}$, it follows

$$\begin{aligned}
&\int_{\partial B_R} f(|Re-y|) \, d\mathscr{H}_y^{n-1} \\
&\quad = |\mathbb{S}^{n-2}| R^{n-1} \int_0^{\sqrt{4R}} \left(\left(\frac{s}{R}\right)^2 \left(R - \frac{s^2}{4}\right) \right)^{\frac{n-3}{2}} f(s) \frac{s \, \mathrm{d}s}{R} \\
&\quad = |\mathbb{S}^{n-2}| R \int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s) \, \mathrm{d}s
\end{aligned}$$

There remains to show that the integral on the right-hand side is finite, with the estimates. When $n \geqslant 3$, we have

$$\left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} \leqslant 1, \quad \forall s \in (0, \sqrt{4R}),$$

which gives the required estimate and shows that the integral is finite, since

$$\int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s)\,\mathrm{d}s \leqslant \int_0^{\infty} s^{n-2} k(s)\,\mathrm{d}s < +\infty.$$

When $n = 2$, let us split the integral into two parts

$$\begin{aligned}\int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s)\,\mathrm{d}s = &\int_0^{\sqrt{2R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s)\,\mathrm{d}s \\ &+ \int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s)\,\mathrm{d}s.\end{aligned} \tag{3.A.2}$$

On the one hand, we have

$$\int_0^{\sqrt{2R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s)\,\mathrm{d}s \leqslant \frac{1}{\sqrt{2}} \int_0^{\sqrt{2R}} f(s)\,\mathrm{d}s, \tag{3.A.3}$$

and on the other hand

$$\begin{aligned}\int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s)\,\mathrm{d}s &\leqslant \|f\|_{L^\infty(\sqrt{2R},\sqrt{4R})} \int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} \mathrm{d}s \\ &= \|f\|_{L^\infty(\sqrt{2R},\sqrt{4R})} \sqrt{4R} \int_{\frac{1}{\sqrt{2}}}^{1} \left(1 - s^2\right)^{-\frac{1}{2}} \mathrm{d}s \\ &= \|f\|_{L^\infty(\sqrt{2R},\sqrt{4R})} \frac{\pi R}{2}.\end{aligned} \tag{3.A.4}$$

hence the required estimate by combining (3.A.2) to (3.A.4). ☐

We can show a counterpart to Proposition 3.3.10 on the sphere.

Proposition 3.A.2. *Let $f \in C^2(\partial B)$ and let $(\eta_\varepsilon)_{\varepsilon>0}$ be a family of $(n-1)$-dimensional mollifiers satisfying also*

$$\sup_K \eta_\varepsilon \xrightarrow{\varepsilon \to 0} 0, \quad \textit{for every compact set } \mathcal{K} \subseteq (0, +\infty). \tag{3.A.5}$$

Then we have

$$\lim_{\varepsilon\to 0} \iint_{\partial B\times\partial B} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y = \mathbf{K}_{2,n-1} \int_{\partial B} |\nabla_\tau f|^2\,d\mathscr{H}^{n-1}.$$

Proof. Let $0 < r < \frac{1}{2}$ to be fixed later. Let us split into two parts the integral

$$\begin{aligned}\iint_{\partial B\times\partial B} &\frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ &= \iint_{\substack{\partial B\times\partial B \\ |x-y|<r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ &\quad + \iint_{\substack{\partial B\times\partial B \\ |x-y|\geqslant r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y.\end{aligned} \tag{3.A.6}$$

Changing variables and using the fact that $f \in C^2(\partial B)$, we have

$$\iint\limits_{\substack{\partial B\times\partial B\\|x-y|\geqslant r}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \leqslant C\int_{\partial B\cap\{|x-e|\geqslant r\}} \eta_\varepsilon(|e-y|)\,d\mathscr{H}^{n-1}_y \tag{3.A.7}$$

for some $C>0$ not depending on ε, where $e=(0,\dots,0,1)\in\mathbb{S}^{n-1}$. Using Lemma 3.A.1 with $f=\chi_{[r,+\infty)}\eta_\varepsilon$, we find

$$\int_{\partial B\cap\{|x-e|\geqslant r\}} \eta_\varepsilon(|e-y|)\,d\mathscr{H}^{n-1}_y = |\mathbb{S}^{n-1}|\int_r^2\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}} s^{n-2}\eta_\varepsilon(s)\,\mathrm{d}s. \tag{3.A.8}$$

If $n\geqslant 3$, we have $\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\leqslant 1$, thus (3.A.7) and (3.A.8) give

$$\iint\limits_{\substack{\partial B\times\partial B\\|x-y|\geqslant r}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \leqslant C\int_r^\infty s^{n-2}\eta_\varepsilon(s)\,\mathrm{d}s,$$

which goes to 0 as ε goes to 0 since η_ε is a family $(n-1)$-dimensional mollifiers. If $n=2$, by Lemma 3.A.1 we have the estimate

$$\int_r^2\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\eta_\varepsilon(s)\,\mathrm{d}s \leqslant \left(\frac{1}{\sqrt{2}}\int_r^{\sqrt{2}}\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s+\frac{\pi}{2}\|\eta_\varepsilon\|_{L^\infty(\sqrt{2},2)}\right) \tag{3.A.9}$$

which also goes to 0 as ε goes to 0 by assumption (3.A.5). Thus for any $r\in(0,1)$ we have

$$\lim_{\varepsilon\to 0}\iint\limits_{\substack{\partial B\times\partial B\\|x-y|\geqslant r}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y = 0. \tag{3.A.10}$$

Since $f\in C^2(\partial B)$, we have

$$|f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y)|\leqslant C|x-y|^2,\quad \forall x,y\in\partial B, \tag{3.A.11}$$

for some C depends only $\|f\|_{C^2(\partial B)}$. Let us write

$$\begin{aligned}|f(x)-f(y)|^2 &= |f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y)+\nabla_\tau f(x)\cdot(x-y)|^2\\ &= |\nabla_\tau f(x)\cdot(x-y)|^2+|f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y)|^2\\ &\qquad +2(\nabla_\tau f(x)\cdot(x-y))(f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y)).\end{aligned} \tag{3.A.12}$$

Since $|f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y)|^2\leqslant C|x-y|^4$ and $|(\nabla_\tau f(x)\cdot(x-y))(f(x)-f(y)-\nabla_\tau f(x)\cdot(x-y))|\leqslant C|x-y|^3$ for every $x,y\in\partial B$ by (3.A.11), (3.A.12) gives

$$|\nabla_\tau f(x)\cdot(x-y)|^2-C|x-y|^3\leqslant|f(x)-f(y)|^2\leqslant|\nabla_\tau f(x)\cdot(x-y)|^2+C|x-y|^3,$$

for every $x, y \in \partial B$. Thus

$$\begin{aligned}\Bigg| \iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ - \iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} \left|\nabla_\tau f(x)\cdot\frac{x-y}{|x-y|}\right|^2\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\Bigg| \\ \leqslant \iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} |x-y|\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y.\end{aligned} \tag{3.A.13}$$

Observe that

$$\iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} |x-y|\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \leqslant Cr\int_{\partial B\cap\{|e-y|<r\}}\eta_\varepsilon(|e-y|)\,d\mathscr{H}^{n-1}_y,$$

where C depends only on n and $\|f\|_{C^2(\partial B)}$, thus integrating on slices with Lemma 3.A.1 yields

$$\begin{aligned}\iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} |x-y|\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ \leqslant Cr\int_0^r\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\,d\mathscr{H}^{n-1}_y \\ \leqslant Cr\int_0^\infty \eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\,d\mathscr{H}^{n-1}_y \leqslant Cr\end{aligned} \tag{3.A.14}$$

where we used the fact that $\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}} \leqslant 2$ for all $n \geqslant 2$ and $s \leqslant r < 1$, and where C denotes a constant depending only on n and $\|f\|_{C^2(\partial B)}$. Once again, let us make a change of variables and integrate on slices using (3.A.1)

$$\begin{aligned}\iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}} \left|\nabla_\tau f(x)\cdot\frac{x-y}{|x-y|}\right|^2\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ = \int_{\partial B\cap\{|e-y|<r\}}\eta_\varepsilon(|e-y|)\int_{\partial B}\left|\nabla_\tau f(x)\cdot\frac{e-y}{|e-y|}\right|^2 d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ = \int_{\{\sqrt{2(1-s)}<r\}}\int_{\mathbb{S}^{n-2}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\eta_\varepsilon\big(\sqrt{2(1-s)}\big) \\ \int_{\partial B}\left|\nabla_\tau f(x)\cdot\frac{e-(\sqrt{1-s^2}y,s)}{\sqrt{2(1-s)}}\right|^2 d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-2}_y\,\mathrm{d}s.\end{aligned} \tag{3.A.15}$$

Writing

$$\frac{e-(\sqrt{1-s^2}y,s)}{\sqrt{2(1-s)}} = \frac{1}{\sqrt{2}}(\sqrt{1+s}y,\sqrt{1-s}) = \sqrt{\frac{1+s}{2}}(y,0)+\sqrt{\frac{1-s}{2}}(0_{\mathbb{R}^{n-1}},1),$$

we find

$$\left|\nabla_\tau f(x)\cdot\frac{e-(\sqrt{1-s^2}y,s)}{\sqrt{2(1-s)}}\right|^2 = \frac{1+s}{2}|\nabla_\tau f(x)\cdot(y,0)|^2+\frac{1-s}{2}|\nabla_\tau f(x)\cdot(0_{\mathbb{R}^{n-1}},1)|^2$$
$$+\sqrt{1-s^2}(\nabla_\tau f(x)\cdot(y,0))(\nabla_\tau f(x)\cdot(0_{\mathbb{R}^{n-1}},1))$$
$$=: \frac{1+s}{2}|\nabla_\tau f(x)\cdot(y,0)|^2+T(x,s) \tag{3.A.16}$$

where $T:\partial B\times(1-\frac{r^2}{2},1)\to\mathbb{R}^n$ is a continuous function such that

$$|T(x,s)|\leqslant C\sqrt{1-s}\leqslant Cr,\quad \forall s\in(1-\tfrac{r^2}{2},1), \tag{3.A.17}$$

for some C depending only on n and $\|f\|_{C^2(\partial B)}$, since $f\in C^2(\partial B)$ and $\sqrt{2(1-s)}\leqslant r$. In view of (3.A.15) to (3.A.17), we have

$$\iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}}\left|\nabla_\tau f(x)\cdot\frac{x-y}{|x-y|}\right|^2\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y = I_\varepsilon+II_\varepsilon \tag{3.A.18}$$

where

$$I_\varepsilon := \int_{\left\{\sqrt{2(1-s)}<r\right\}}\int_{\mathbb{S}^{n-2}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\left(\frac{1+s}{2}\right)\eta_\varepsilon(\sqrt{2(1-s)})$$
$$\int_{\partial B}|\nabla_\tau f(x)\cdot(y,0)|^2\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-2}_y\,\mathrm{d}s. \tag{3.A.19}$$

and

$$|II_\varepsilon|\leqslant Cr\int_{\left\{\sqrt{2(1-s)}<r\right\}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\eta_\varepsilon(\sqrt{2(1-s)})$$
$$\leqslant Cr\int_0^r\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s,$$

for some C depending only on n and $\|f\|_{C^2(\partial B)}$. Since $r<1$, we have $\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}<2$ for any $n\geqslant 2$, thus

$$|II_\varepsilon|\leqslant Cr\int_0^r\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\leqslant Cr,$$

where we used the fact that $|\mathbb{S}^{n-2}|\int_0^\infty\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s=1$. Hence given $\delta>0$, with (3.A.14) we can choose r small enough such that

$$|II_\varepsilon|\leqslant\delta\quad\text{and}\quad\iint\limits_{\substack{\partial B\times\partial B\\|x-y|<r}}|x-y|\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\leqslant\delta \tag{3.A.20}$$

for every $\varepsilon>0$. Recalling (3.A.10), we can then choose $\varepsilon_0>0$ such that, for all $\varepsilon<\varepsilon_0$, we have

$$\iint\limits_{\substack{\partial B\times\partial B\\|x-y|\geqslant r}}\frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y\leqslant\delta. \tag{3.A.21}$$

Combining (3.A.6), (3.A.13) and (3.A.18) to (3.A.21), we find

$$\Big| \iint_{\partial B\times\partial B} \frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y - I_\varepsilon\Big| \leqslant 3\delta, \quad \forall\varepsilon<\varepsilon_0, \tag{3.A.22}$$

for our choice of r. Then we compute

$$\begin{aligned} I_\varepsilon &= \int_{\{\sqrt{2(1-s)}<r\}}\int_{\mathbb{S}^{n-2}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\left(\frac{1+s}{2}\right)\eta_\varepsilon(\sqrt{2(1-s)}) \\ &\qquad\qquad \int_{\partial B}|\nabla_\tau f(x)\cdot(y,0)|\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-2}_y\,\mathrm{d}s \\ &= \int_{\{\sqrt{2(1-s)}<r\}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\left(\frac{1+s}{2}\right)\eta_\varepsilon(\sqrt{2(1-s)}) \\ &\qquad\qquad \int_{\partial B}\int_{\mathbb{S}^{n-2}}|\nabla_\tau f(x)\cdot(y,0)|\,d\mathscr{H}^{n-2}_y\,d\mathscr{H}^{n-1}_x\,\mathrm{d}s \\ &= \mathbf{K}_{2,n-1}|\mathbb{S}^{n-2}|\int_{\{\sqrt{2(1-s)}<r\}}\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}}\left(\frac{1+s}{2}\right)\eta_\varepsilon(\sqrt{2(1-s)}) \\ &\qquad\qquad \int_{\partial B}|\nabla_\tau f(x)|^2\,d\mathscr{H}^{n-1}_x\,\mathrm{d}s, \end{aligned}$$

where we used Fubini's theorem and the definition of $\mathbf{K}_{2,n-1}$. Changing variables, it follows

$$I_\varepsilon = \mathbf{K}_{2,n-1}\|\nabla_\tau f\|^2_{L^2(\partial B)}|\mathbb{S}^{n-2}|\int_0^r\left(1-\frac{s^2}{4}\right)^{\frac{n-3}{2}+1}\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s.$$

Now we could have chosen r small enough such that we have as well

$$0\leqslant \mathbf{K}_{2,n-1}\|\nabla_\tau f\|^2_{L^2(\partial B)}|\mathbb{S}^{n-2}|\left(1-\left(1-\frac{r^2}{4}\right)^{\frac{n-3}{2}+1}\right)\leqslant\delta,$$

since $n-3+2\geqslant 0$ for any $n\geqslant 2$, thus

$$\left|I_\varepsilon - \mathbf{K}_{2,n-1}\|\nabla_\tau f\|^2_{L^2(\partial B)}|\mathbb{S}^{n-2}|\int_0^r\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\right|\leqslant\delta, \tag{3.A.23}$$

where we used the fact $|\mathbb{S}^{n-2}|\int_0^1\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s=1$. Hence from (3.A.22) and (3.A.23), it follows

$$\begin{aligned} \Big|\iint_{\partial B\times\partial B}&\frac{|f(x)-f(y)|^2}{|x-y|^2}\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y \\ &-\mathbf{K}_{2,n-1}\|\nabla_\tau f\|^2_{L^2(\partial B)}|\mathbb{S}^{n-2}|\int_0^r\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\Big|\leqslant 4\delta, \quad\forall\varepsilon<\varepsilon_0. \end{aligned}\tag{3.A.24}$$

Notice that for every $r\in(0,1)$,

$$|\mathbb{S}^{n-2}|\int_0^r\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s = 1-|\mathbb{S}^{n-1}|\int_r^\infty\eta_\varepsilon(s)s^{n-2}\,\mathrm{d}s\xrightarrow{\varepsilon\to0}1,$$

so that letting ε go to 0 in (3.A.24), the arbitrariness of δ implies

$$\lim_{\varepsilon\to0}\iint_{\partial B\times\partial B}\frac{|f(x)-f(y)|^2}{|x-y|^2}\eta_\varepsilon(|x-y|)\,d\mathscr{H}^{n-1}_x\,d\mathscr{H}^{n-1}_y = \mathbf{K}_{2,n-1}\int_{\partial B}|\nabla_\tau f|^2\,d\mathscr{H}^{n-1}.$$

□

Regularity of s-harmonic functions and distributions

The main subject of this appendix is to give proofs of full regularity for s-harmonic functions, i.e., functions satisfying $(-\Delta)^s u = 0$ in a broad sense, and to derive Caccioppoli-type estimates, that is, L^∞ bounds on all the derivatives of u in terms on the L^2 norm of u. While the regularity of s-harmonic functions is well known, there seems to be a lack of good reference treating it a broad sense. For example, given $u \in \widehat{H}^s(\Omega)$, the s-Laplacian of u is defined in Chapters 1 and 2 by duality as

$$\langle (-\Delta)^s u, \varphi \rangle_{\widehat{H}^{-s}, \widehat{H}^s} := \frac{\gamma_{n,s}}{2} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{(u(x) - u(y)) \cdot (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y,$$

for all $\varphi \in \widehat{H}^s(\Omega)$, so that the equation $(-\Delta)^s u = 0$ in Ω is to be understood as

$$\langle (-\Delta)^s u, \varphi \rangle_{\widehat{H}^{-s}, \widehat{H}^s} = 0, \quad \forall \varphi \in H^s_{00}(\Omega).$$

In that functional setting, we ask the following question: is u smooth in Ω? We can find several regularity results in the literature, in several functional settings, but, as far we know, they do not apply here. For example, [15, Theorem 2.10] implies that, if $u \in \widehat{H}^s(\mathbb{R}^n)$ satisfies

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_r, \\ u = g & \text{in } \mathbb{R}^n \setminus \overline{B_r}, \end{cases}$$

then $u \in C^\infty(B_r)$ whenever $g \in C(\mathbb{R}^n)$ is integrable with respect to the measure $(1 + |x|)^{-(n+2s)} \, \mathrm{d}x$. Unfortunately this does not answer our question. This is not entirely a trivial question: in [16], where regularity estimates involving the fractional Laplacian in $\widehat{H}^s$ are obtained, the authors make sure to prove that s-harmonic functions are Hölder-continuous. In [106], where it is morally shown that the solution u of $(-\Delta)^s u = f$ gains a $2s$ "fractional derivatives" compared to f, the proofs use in particular the fact that if some φ satisfies weakly $(-\Delta)^s \varphi = 0$ in a ball, then it is smooth inside that ball (e.g. in the proof of Proposition 2.8), but this is not justified.

Outline of the appendix

In this appendix, we implement two different strategies to obtain full regularity of s-harmonic functions. First, in Section A.1, we define the fractional Laplace operator on a large subspace of tempered distributions, and using this distributional setting and the fact that the inverse of the s-Laplacian is given by convolution with the Riesz kernel of order $2s$, we prove elliptic regularity results; that is, we prove that distributions T satisfying $(-\Delta)^s T = f$ in a ball are morally "$2s$-times" more differentiable than f in that ball. We recover smoothness of s-harmonic distributions by taking $f = 0$. The caveat of this strategy is that, for technical reasons that we explain further below (and which can likely be dealt with), we exclude the case $n = 1$ and $s \in [\frac{1}{2}, 1)$.

The second strategy we implement, in Section A.3 to obtain smoothness of weakly s-harmonic functions is by using the Caffarelli-Silvestre extension to recover a degenerate but local elliptic equation in $\mathbb{R}^{n+1}$. The caveat here is that the setting is a bit less general, in the sense that we consider functions instead of distributions, using the functional setting of Chapters 1 and 2, and we only consider a vanishing source term $f = 0$. However, we do not exclude the case $n = 1$, $s \in [\frac{1}{2}, 1)$, and we derive the Cacciopoli-type estimates mentioned above.

Notation

In this appendix we shall denote by $\mathscr{D}(\Omega)$ the space of smooth compactly supported functions in Ω, by $\mathscr{S}$ the Schwartz space of rapidly decreasing functions in $\mathbb{R}^n$, and we let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$, defined by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2i\pi x\cdot\xi}\, \mathrm{d}x, \quad \forall f \in L^1(\mathbb{R}^n).$$

For $k \in \mathbb{N}$ and $\alpha \in (0,1)$, We denote by $C^{k,\alpha}(\mathbb{R}^n)$ those functions f which are k-times differentiable, whose derivatives are all bounded in $L^\infty(\mathbb{R}^n)$, and such that

$$\|f\|_{C^{k,\alpha}(\mathbb{R}^n)} := \sum_{|\gamma|<k} \|\partial^\gamma f\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma|=k} \left(\sup_{x,y\in\mathbb{R}^n} \frac{|\partial^\gamma u(x) - \partial^\gamma u(y)|}{|x-y|^\alpha} \right) < +\infty,$$

where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ is called a multi-index and $|\gamma| := \gamma_1 + \ldots + \gamma_n$. Beware that $C^{k,\alpha}(\mathbb{R}^n)$ is not made of all functions which are k-times differentiable and whose partial derivatives are α-Hölder continuous, since we also require those partial derivatives to be bounded in L^∞.
For $k \in (0, +\infty)$ and $p \in (1, +\infty)$, we denote by $W^{k,p}(\Omega)$ the usual (possibly fractional) Sobolev-Slobodeckij space whose norm is given by

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\gamma|\leqslant k} \|\partial^\gamma f\|_{L^p(\Omega)}$$

when k is an integer, and

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\gamma|\leqslant\lfloor k\rfloor} \|\partial^\gamma f\|_{L^p(\Omega)} + \sum_{|\gamma|=\lfloor k\rfloor} \left(\iint_{\Omega\times\Omega} \frac{|\partial^\gamma f(x) - \partial^\gamma f(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{p}},$$

when k is not integer, where $\alpha := k - \lfloor k \rfloor \in (0,1)$. In Sections A.1 and A.2, we denote by $B_R(x)$ the open ball of radius R and centered at x in $\mathbb{R}^n$, and if $x = 0$ we write simply B_R. In Section A.3 we adopt another convention for balls. We reserve the notation $B_R(\mathbf{x})$ for the open ball of radius R centered at $\mathbf{x} = (x, 0) \in \mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$, and $D_R(x)$ the open ball of radius R centered at x in $\mathbb{R}^n$. We let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$, and for $\mathbf{x} = (x, 0) \in \mathbb{R}^{n+1}$, we define $B_R^+(\mathbf{x}) := B_R(\mathbf{x}) \cap \mathbb{R}^{n+1}_+$ and $\partial^0 B_R^+(\mathbf{x}) := D_R \times \{0\}$.

A.1 The distributional s-Laplacian

The aim of this section is to give a general defininition of the s-Laplacian on a large class of tempered distributions. We recall a few definitions and statements from [106] (see also [70, 55, 15]).

Definition A.1.1. For $s \in (0,1)$ and $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we define $(-\Delta)^s u$ by

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n} \frac{2u(x) - u(x+h) - u(x-h)}{|h|^{n+2s}}\,\mathrm{d}h \\ &= \gamma_{n,s}\,\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,\end{aligned} \tag{A.1.1}$$

and for $s \in (0, \frac{n}{2})$ and $u \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$,

$$(-\Delta)^{-s} u(x) = \gamma_{n,-s}\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-2s}}\,\mathrm{d}y = u * I_{2s}(x), \tag{A.1.2}$$

for every $x \in \mathbb{R}^n$, where I_{2s} (which may sometimes denote simply by $(-\Delta)^{-s}$ when we refer to the associated tempered distribution) is the so-called Riesz kernel of order $2s$, given by

$$I_{2s}(x) := \frac{\gamma_{n,-s}}{|x|^{n-2s}}.$$

In any case, the constant $\gamma_{n,s}$ is given by

$$\gamma_{n,s} := |s| 2^{2s} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}, \quad \forall s \in (-\tfrac{n}{2}, 1).$$

Remark A.1.2. For $s \in (0, \frac{n}{2})$, the Riesz kernel I_{2s} is the fundamental solution of $(-\Delta)^s$, that is, $(-\Delta)^s I_{2s} = \delta_0$ in $\mathscr{S}'$ (we will see later the meaning of $(-\Delta)^s I_{2s}$, since for now $(-\Delta)^s f$ is only defined whenever $f \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and will justify this equality).

Remark A.1.3. If $n = 1$ and $s \in [\frac{1}{2}, 1)$, then $(2\pi|\xi|)^{-2s}$ is not integrable near the origin; in particular, it is not a tempered distribution, so its invert Fourier transform cannot be defined as a tempered distribution, and even less as a function. In other words, we cannot define $(-\Delta)^{-s}$ as a tempered distribution if $n = 1$ and $s \in [\frac{1}{2}, 1)$, and since we rely on the inverse of the s-Laplacian in our proof of regularity, we will often exclude that case. Let us point out that this restriction is only technical: we could define $(-\Delta)^{-s}$ as a distribution by using the so-called Lizorkin space and its dual (see e.g. [95, Chapter 2]), which are better-suited to Riesz kernels, instead of the Schwartz class. Note that if $n \geqslant 2$, $(-\Delta)^s u$ is however defined as a function for all $s \in (-1, 1)$.

We justify in the proof of the next theorem that $(-\Delta)^s u$ is a well-defined function whenever $s \in (-\frac{n}{2}, 1)$, and the choice of the constants $\gamma_{n,s}$.

Theorem A.1.4. *If $s \in (0,1)$ and $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $(-\Delta)^s u$ is well defined and continuous. If $D^2 u$ also belongs to $L^\infty(\mathbb{R}^n)$, then $(-\Delta)^s u \in L^\infty(\mathbb{R}^n)$, and if $D^2 u \in L^1(\mathbb{R}^n)$, then $u \in L^1(\mathbb{R}^n)$, and we have*

$$\mathcal{F}((-\Delta)^s u)(\xi) = (2\pi|\xi|)^{2s}\mathcal{F}(u)(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{A.1.3}$$

If $s \in (-\frac{n}{2}, 0)$ and $u \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $(-\Delta)^s u$ is well defined and belongs to $L^\infty(\mathbb{R}^n)$.

Proof. Let us first tackle the case $s \in (0,1)$. Note that for all $h \in \mathbb{R}^n \setminus \{0\}$,

$$x \mapsto \frac{|2u(x) - u(x+h) - u(x-h)|}{|h|^{n+2s}} \in C^0(\mathbb{R}^n).$$

Since $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, whenever x belongs to some ball B_R we can write

$$\begin{aligned}
&\frac{|2u(x) - u(x+h) - u(x-h)|}{|h|^{n+2s}} \\
&\qquad \leqslant \chi_{\{|h|<1\}} |h|^{-(n+2s-2)} \int_0^1 |D^2 u(x+th)| \, \mathrm{d}t \\
&\qquad\qquad + \chi_{\{|h|\geqslant 1\}} |h|^{-(n+2s)} (2|u(x)| + |u(x+h)| + |u(x-h)|) \\
&\qquad \leqslant C\chi_{\{|h|<1\}} |h|^{-(n+2s-2)} \|D^2 u\|_{L^\infty(B_{R+1})} \\
&\qquad\qquad + C\chi_{\{|h|\geqslant 1\}} |h|^{-(n+2s)} \|u\|_{L^\infty(\mathbb{R}^n)} \in L^1(\mathbb{R}^n)
\end{aligned} \tag{A.1.4}$$

thus the integral in (A.1.1) is defined everywhere and by the theorem of continuity under the integral $(-\Delta)^s u$ is continuous in B_{R+1}. Since R is arbitrary, $(-\Delta)^s u$ is continuous in $\mathbb{R}^n$. In addition, if $D^2 u \in L^\infty(\mathbb{R}^n)$, integrating (A.1.4) we see that

$$|(-\Delta)^s u(x)| \leqslant C\|D^2 u\|_{L^\infty(\mathbb{R}^n)} + C\|u\|_{L^\infty(\mathbb{R}^n)}, \tag{A.1.5}$$

thus $(-\Delta)^s u \in L^\infty(\mathbb{R}^n)$. If $D^2 u \in L^1(\mathbb{R}^n)$, integrating in x and h, we also have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|2u(x) - u(x+h) - u(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \, \mathrm{d}x \\
&\qquad \leqslant \int_{\mathbb{R}^n} \int_{\{|h|<1\}} \frac{|2u(x) - u(x+h) - u(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \, \mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}^n} \int_{\{|h|\geqslant 1\}} \frac{|2u(x) - u(x+h) - u(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \, \mathrm{d}x \\
&\qquad \leqslant \int_{\mathbb{R}^n} \int_{\{|h|<1\}} \int_0^1 |D^2 u(x+th)| \, \mathrm{d}t \frac{1}{|h|^{n+2s-2}} \, \mathrm{d}h \, \mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}^n} \int_{\{|h|\geqslant 1\}} \frac{2|u(x)| + |u(x+h)| + |u(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \, \mathrm{d}x \\
&\qquad \leqslant C\|D^2 u\|_{L^1(\mathbb{R}^n)} + C\|u\|_{L^1(\mathbb{R}^n)} < +\infty.
\end{aligned} \tag{A.1.6}$$

By Fubini's theorem, this shows that $(-\Delta)^s u$ is well defined for almost every $x \in \mathbb{R}^n$ and that $(-\Delta)^s u \in L^1(\mathbb{R}^n)$. In fact, this justifies that we can exchange the Fourier operator and the integrals in (A.1.1), so that

$$\begin{aligned}
\mathcal{F}((-\Delta)^s u)(\xi) &= \frac{\gamma_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (2u(x) - u(x+h) - u(x)) e^{-2i\pi x\cdot\xi} \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^{n+2s}} \\
&= \frac{\gamma_{n,s}}{2} \int_{\mathbb{R}^n} 2\mathcal{F}(u)(\xi) - \mathcal{F}(\tau_h u)(\xi) - \mathcal{F}(\tau_{-h} u)(\xi) \frac{\mathrm{d}h}{|h|^{n+2s}} \\
&= \mathcal{F}(u)(\xi) \frac{\gamma_{n,s}}{2} \int_{\mathbb{R}^n} \frac{2 - e^{2i\pi h\cdot\xi} - e^{-2i\pi h\cdot\xi}}{|h|^{n+2s}} \, \mathrm{d}h \\
&= \mathcal{F}(u)(\xi) \gamma_{n,s} \int_{\mathbb{R}^n} \frac{(1 - \cos(2\pi h\cdot\xi))}{|h|^{n+2s}} \, \mathrm{d}h,
\end{aligned} \tag{A.1.7}$$

where $\tau_h u$ is defined by $\tau_h u(x) := u(x+h)$, $\forall x \in \mathbb{R}^n$. Exploiting the invariance of $h \mapsto \cos(2\pi h \cdot \xi)$ under rotation, after computation we find

$$\int_{\mathbb{R}^n} \frac{2(1-\cos(h\cdot\xi))}{|h|^{n+2s}}\,\mathrm{d}h = \int_{\mathbb{R}^n} \frac{2(1-\cos(|h||\xi|))}{|h|^{n+2s}}\,\mathrm{d}h = \gamma_{n,s}^{-1}\,(2\pi|\xi|)^{2s}$$

so that (A.1.7) yields

$$\mathcal{F}((-\Delta)^s u)(\xi) = (2\pi|\xi|)^{2s}\mathcal{F}(u)(\xi). \tag{A.1.8}$$

The other expression involving the principal value is justified by

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{2u(x)-u(x+h)-u(x-h)}{|h|^{n+2s}}\,\mathrm{d}h \\
&\quad = \lim_{\varepsilon\to 0}\int_{B_\varepsilon^c} \frac{2u(x)-u(x+h)-u(x-h)}{|h|^{n+2s}}\,\mathrm{d}h \\
&\quad = \lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon^c} \frac{u(x)-u(x+h)}{|h|^{n+2s}}\,\mathrm{d}h + \int_{B_\varepsilon^c} \frac{u(x)-u(x-h)}{|h|^{n+2s}}\,\mathrm{d}h\right) \\
&\quad = 2\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(x+h)}{|h|^{n+2s}}\,\mathrm{d}h.
\end{aligned}$$

As for the case $s \in (-\frac{n}{2}, 0)$, the convolution is well defined since there exists some $p > 1$ such that $I_{2|s|} \in L^p(\mathbb{R}^n)$ (e.g. any $p > \frac{n}{n-2s}$), and $u \in L^q(\mathbb{R}^n)$, where $\frac{1}{q} = 1 - \frac{1}{p}$. In addition

$$|I_{2|s|} * u(x)| \leqslant \|I_{2|s|}\|_{L^p(\mathbb{R}^n)}\|u\|_{L^q(\mathbb{R}^n)}$$

so $(-\Delta)^s u \in L^\infty(\mathbb{R}^n)$ in that case as well. To given a meaning to the identity (A.1.3), when s is negative, we need to consider the Fourier transform on tempered distributions rather than on L^1 or L^2 (see Theorem A.1.5). □

Note that for every $s \in (-\frac{n}{2}, 1)$, $(-\Delta)^s u$ is well defined whenever $u \in \mathscr{S}$, but the class of Schwartz functions is unfortunately not stable under the s-Laplacian. By derivation under the integral sign, it is easy to see that $(-\Delta)^s u \in C^\infty(\mathbb{R}^n)$ in that case, but it can have a worse decay at infinity, essentially because $|\xi|^{2s}$ is not smooth at the origin. In particular, with the definition we have for now of the s-Laplacian, we cannot *a priori* apply twice a fractional Laplace operator to a function in the Schwartz class.

The following well-known result justifies the definition of $(-\Delta)^{-s}$.

Theorem A.1.5 ([56, Theorem 2.4.6])**.** *If $s \in (0, \frac{n}{2})$, then $(-\Delta)^{-s} = I_{2s}$ belongs to the class of tempered distributions, and $\mathcal{F}((-\Delta)^{-s}) = (2\pi|\xi|)^{-2s}$ in $\mathscr{S}'$, i.e.,*

$$\langle I_{2s}, \mathcal{F}(\varphi)\rangle_{\mathscr{S}',\mathscr{S}} := \int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi, \quad \forall \varphi \in \mathscr{S}. \tag{A.1.9}$$

We will actually need to apply (A.1.9) to functions φ with less regularity that those in the Schwartz class. First, let us remark that (A.1.9) stands true for any $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by approximation provided that $s < \frac{n}{4}$, as shown in the following corollary.

Corollary A.1.6. *If $s \in (0, \frac{n}{4})$, then we have*

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi, \quad \forall \varphi \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n).$$

Proof. Let $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By density of $\mathscr{S}$ in $L^1(\mathbb{R}^n)$ and in $L^2(\mathbb{R}^n)$, we can find $(\varphi_k)_{k\in\mathbb{N}}$ a sequence of maps in $\mathscr{S}$ converging to φ both in $L^1(\mathbb{R}^n)$ and in $L^2(\mathbb{R}^n)$. By applying a smooth cutoff, we may also assume that $|\varphi_k(x)| \leqslant |\varphi(x)|$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Applying Theorem A.1.5, we have

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi_k)(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi_k(\xi)\,\mathrm{d}\xi, \quad \forall k \in \mathbb{N}. \tag{A.1.10}$$

Note that

$$(2\pi|\xi|)^{-2s}|\varphi_k(\xi)| \leqslant (2\pi|\xi|)^{-2s}|\varphi(\xi)|, \quad \forall \xi \in \mathbb{R}^n,$$

and in addition,

$$\begin{aligned}\int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}|\varphi(\xi)|\,\mathrm{d}\xi &= \int_{B_1} (2\pi|\xi|)^{-2s}|\varphi(\xi)|\,\mathrm{d}\xi + \int_{B_1^{\mathrm{c}}} (2\pi|\xi|)^{-2s}|\varphi(\xi)|\,\mathrm{d}\xi \\ &\leqslant \left(\int_{B_1} (2\pi|\xi|)^{-4s}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \|\varphi\|_{L^2(B_1)} + \|\varphi\|_{L^1(B_1^{\mathrm{c}})} < +\infty\end{aligned}$$

where we used the fact that $\xi \mapsto |\xi|^{-4s} \in L^1(B_1)$ since $n > 4s$. Thus by dominated convergence we obtain

$$\int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi_k(\xi)\,\mathrm{d}\xi \xrightarrow{k\to\infty} \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi. \tag{A.1.11}$$

On the other hand, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} I_{2s}(\xi)|\mathcal{F}(\varphi_k - \varphi)(\xi)|\,\mathrm{d}\xi \\ &\quad= \int_{B_1} I_{2s}(\xi)|\mathcal{F}(\varphi_k - \varphi)(\xi)|\,\mathrm{d}\xi + \int_{B_1^{\mathrm{c}}} I_{2s}(\xi)|\mathcal{F}(\varphi_k - \varphi)(\xi)|\,\mathrm{d}\xi \\ &\quad\leqslant \|I_{2s}\|_{L^1(B_1)}\|\mathcal{F}(\varphi_k - \varphi)\|_{L^\infty(B_1)} + \|I_{2s}\|_{L^2(B_1^{\mathrm{c}})}\|\mathcal{F}(\varphi_k - \varphi)\|_{L^2(B_1^{\mathrm{c}})} \\ &\quad\leqslant \|I_{2s}\|_{L^1(B_1)}\|\varphi_k - \varphi\|_{L^1(\mathbb{R}^n)} + \|I_{2s}\|_{L^2(B_1^{\mathrm{c}})}\|\varphi_k - \varphi\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{A.1.12}$$

where we used the continuity of the Fourier transform from L^1 into L^∞ and from L^2 into itself for the last inequality. The fact that $n > 4s$ ensures that I_{2s} is square integrable at infinity. Passing to the limit in (A.1.12) then yields

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi_k)(\xi)\,\mathrm{d}\xi \xrightarrow{k\to\infty} \int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi,$$

thus combining this with (A.1.10) and (A.1.11) we reach

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi.$$

□

However useful this corollary may be, we do not want to make the assumption that $s \in (0, \frac{n}{4})$, and want to allow s to be in $(0, \frac{n}{2})$. It is not clear that (A.1.9) still stands true for any $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, but we have the following version, which is enough for our needs.

Corollary A.1.7. *Let $s \in (0, \frac{n}{2})$ and $\varphi \in C^n(\mathbb{R}^n) \cap W^{n,1}(\mathbb{R}^n)$. Then we have*

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi)(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi.$$

Proof. Let us fix $G(x) := e^{-\pi|x|^2}$ the Gaussian in $\mathbb{R}^n$ with unit L^1 norm. Then for every $k \in \mathbb{N}$, we let $\varepsilon_k := 2^{-k}$, $\eta_k(r) := \eta(\varepsilon_k^{-1}r)$ and $G_k(r) := \varepsilon_k^{-n}G(\varepsilon_k^{-1}x)$. We also give ourselves a sequence of cutoff functions $(\eta_l)_{l\in\mathbb{N}}$, such that $\eta_l \in C_c^\infty(0, l+1)$, $\eta_l(r) \in [0,1]$ for all r, $\eta_l(r) = 1$ for $r \in [0,l]$, and $\|\eta_l^{(j)}\|_{L^\infty} \leqslant C$ for all $l \in \mathbb{N}$ and all $j \in \{1,\dots,n\}$, where $C = C(n,j)$. Then we define $\psi_l(x) := \varphi(x)\eta_l(|x|)$ and $\varphi_{k,l}(x) := \psi_l * G_k$. Since ψ_l is compactly supported, and since G_k is smooth and all its derivatives decrease at infinity like a polynomial times $e^{-|x|^2}$, $\psi_l * G_k$ belongs to the class of Schwartz functions, thus by Theorem A.1.5 we have

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\varphi_{k,l})(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi_{k,l}(\xi)\,\mathrm{d}\xi, \tag{A.1.13}$$

for all $k, l \in \mathbb{N}$. Recall that G_k is an approximation of the identity, and observe that

$$(2\pi|\xi|)^{-2s}|\psi_l * G_k(\xi)| \leqslant (2\pi|\xi|)^{-2s}|\psi_l(\xi)|,$$

where the right-hand side belongs to $L^1(\mathbb{R}^n)$ since ψ_l is a continuous, compactly supported function in $\mathbb{R}^n$, and $s < \frac{n}{2}$, which makes $|\xi|^{-2s}$ integrable near the origin. Thus, using that $\psi_l * G_k(x)$ converges everywhere to $\psi_l(x)$ as k goes to infinity, by dominated convergence we find

$$\int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi_{k,l}(\xi)\,\mathrm{d}\xi \xrightarrow{k\to\infty} \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\psi_l(\xi)\,\mathrm{d}\xi. \tag{A.1.14}$$

Applying the dominated convergence theorem once again we easily obtain as well

$$\int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\psi_l(\xi)\,\mathrm{d}\xi \xrightarrow{l\to\infty} \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi(\xi)\,\mathrm{d}\xi. \tag{A.1.15}$$

Let us turn to the left-hand side of (A.1.13). Observe that

$$\mathcal{F}(\varphi_{k,l})(\xi) = \mathcal{F}(\psi_l * G_k)(\xi) = \mathcal{F}(\psi_l)(\xi)\mathcal{F}(G_k)(\xi) = \mathcal{F}(\psi_l)(\xi)G(\varepsilon_k\xi),$$

since $G_k(\xi) = \varepsilon_k^{-n}G(\varepsilon_k^{-1}x)$ and $\mathcal{F}(G) = G$. Thus (A.1.13) rewrites

$$\int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\psi_l)(\xi)G(\varepsilon_k\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\varphi_{k,l}(\xi)\,\mathrm{d}\xi. \tag{A.1.16}$$

We want to use the dominated convergence theorem to pass to the limit in k (and then l) in the left-hand side. First, notice that

$$I_{2s}(\xi)\mathcal{F}(\psi_l)(\xi)G(\varepsilon_k\xi) \xrightarrow{k\to\infty} I_{2s}(\xi)\mathcal{F}(\psi_l)(\xi), \quad \text{for a.e. } \xi.$$

Since $\psi_l(x) = \varphi(x)\eta_l(|x|)$, by the assumptions on φ, we know that $\psi_l \in C^n(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\partial^\alpha\psi_l \in L^1(\mathbb{R}^n)$ for every multi-index α such that $|\alpha| = n$. Let $i \in \{1,\dots,n\}$ and $\alpha \in \mathbb{N}^n$ such that $\alpha_i = n$ and $\alpha_j = 0$ for every $j \neq i$. Then we have

$$|\mathcal{F}(\partial^\alpha\psi_l)(\xi)| = (2\pi|\xi_i|)^n|\mathcal{F}(\psi_l)(\xi)|, \quad \forall\xi \in \mathbb{R}^n,$$

thus

$$\begin{aligned}|\xi_i|^n|\mathcal{F}(\psi_l)(\xi)| \leqslant C|\mathcal{F}(\partial^\alpha\psi_l)(\xi)| &\leqslant C\|\partial^\alpha\psi_l\|_{L^1(\mathbb{R}^n)} \\ &\leqslant C\left(\max_{|\beta|=n}\|\partial^\beta\psi_l\|_{L^1(\mathbb{R}^n)}\right), \quad \forall\xi \in \mathbb{R}^n\setminus\{0\},\end{aligned}$$

for some $C = C(n)$. Since this is true for every $i \in \{1, \dots, n\}$, it follows

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^n |\mathcal{F}(\psi_l)(\xi)| \leqslant C \left(\max_{|\beta|=n} \|\partial^\beta \psi_l\|_{L^1(\mathbb{R}^n)} \right).$$

Now by using the general Leibniz rule and the fact that η_l is constant in B_1 for every $l \in \mathbb{N}$, we easily find that there exists some $C = C(n)$ such that

$$|\partial^\beta \psi_l(x)| \leqslant C \sum_{\beta_1+\beta_2=\beta} |\partial^{\beta_1} \varphi(x)| |\eta_l^{(|\beta_2|)}(x)|,$$

hence

$$|\partial^\beta \psi_l(x)| \leqslant C \left(\max_{|\alpha| \leqslant n} |\partial^\alpha \varphi(x)| \right)$$

for some $C = C(n)$, by our choice of the sequence η_l. In the end, we obtain

$$\sup_{|\xi| \geqslant 1} |\xi|^n |\mathcal{F}(\psi_l)(\xi)| \leqslant C \left(\max_{|\alpha| \leqslant n} \|\partial^\alpha \varphi\|_{L^1(\mathbb{R}^n)} \right) \leqslant C \tag{A.1.17}$$

where C does not depend on k nor l. Now let us write

$$I_{2s}(\xi) |\mathcal{F}(\psi_l)(\xi)| \leqslant I_{2s}(\xi) \|\psi_l\|_{L^1(\mathbb{R}^n)} \leqslant I_{2s}(\xi) \|\varphi\|_{L^1(\mathbb{R}^n)}, \quad \forall \xi \in B_1$$

and, using (A.1.17),

$$I_{2s}(\xi) |\mathcal{F}(\psi_l)(\xi)| \leqslant C |\xi|^{-n} I_{2s}(\xi), \quad \forall \xi \in B_1^{\mathrm{c}},$$

then notice that $I_{2s} \in L^1(B_1)$ and $|\xi|^{-n} I_{2s}(\xi) \leqslant C|\xi|^{2s-2n}$ is integrable in B_1^{c} since $n > 2s$. Thus we can bound

$$I_{2s}(\xi) |\mathcal{F}(\psi_l)(\xi)| G(\varepsilon_k \xi) \leqslant C \left(I_{2s}(\xi) \chi_{B_1}(\xi) + |\xi|^{2s-2n} \chi_{B_1^{\mathrm{c}}}(\xi) \right) \in L^1(\mathbb{R}^n),$$

where C does not depend on k (nor l) and then use dominated converge to deduce, first, that

$$\int_{\mathbb{R}^n} I_{2s}(\xi) \mathcal{F}(\psi_l)(\xi) G(\varepsilon_k \xi) \, \mathrm{d}\xi \xrightarrow{k \to \infty} \int_{\mathbb{R}^n} I_{2s}(\xi) \mathcal{F}(\psi_l)(\xi) \, \mathrm{d}\xi, \tag{A.1.18}$$

and then,

$$\int_{\mathbb{R}^n} I_{2s}(\xi) \mathcal{F}(\psi_l)(\xi) \, \mathrm{d}\xi \xrightarrow{l \to \infty} \int_{\mathbb{R}^n} I_{2s}(\xi) \mathcal{F}(\varphi)(\xi) \, \mathrm{d}\xi, \tag{A.1.19}$$

since $\mathcal{F}(\psi_l)(\xi) = \mathcal{F}(\varphi \eta_l)(\xi)$ converges to $\mathcal{F}(\varphi)(\xi)$ everywhere as well, by continuity of the Fourier transform from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and the fact that $\varphi \eta_l$ obviously converges to φ in $L^1(\mathbb{R}^n)$. Passing to the limit in k then l in (A.1.16) with (A.1.14), (A.1.15), (A.1.18) and (A.1.19) eventually gives the result. □

When $s \in (-\frac{n}{2}, 0)$, even though the fractional Laplacian of a Schwartz function is not rapidly decreasing, it still has some good enough decay at infinity, which is why we introduce the following function spaces.

Definition A.1.8. For every $k \in \mathbb{N}$ and every measurable function u we define the seminorm

$$[u]_{\mathscr{S}_s^k} := \max_{|\alpha|=k} \sup_{x \in \mathbb{R}^n} \left(1 + |x|^{n+2s} \right) |\partial^\alpha u(x)|$$

where the maximum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $\sum_{i=1}^n \alpha_i = k$, and $\partial^\alpha u := \partial^{\alpha_1} \cdots \partial^{\alpha_n} u$. We then define, for any $s \in (-\frac{n}{2}, 1)$, the space

$$\mathscr{S}_s := \left\{ u \in C^\infty(\mathbb{R}^n) \ : \ \forall k \in \mathbb{N}, \ [u]_{\mathscr{S}_s^k} < +\infty \right\}, \tag{A.1.20}$$

endowed with the topology induced by the family of seminorms $[\cdot]_{\mathscr{S}_s^k}$. We denote by $\mathscr{S}_s'$ its topological dual endowed with the weak-$*$ topology.

Note that the topological dual of $\mathscr{S}_s$, $\mathscr{S}_s'$, is a subspace of $\mathscr{S}'$ which is itself a subspace of $\mathscr{D}'(\mathbb{R}^n)$. The following proposition justifies the introduction of those spaces.

Proposition A.1.9. *Let $s \in (0,1)$. Then for any $u \in \mathscr{S}$, $(-\Delta)^s u \in \mathscr{S}_s(\mathbb{R}^n)$, and for every $k \in \mathbb{N}$ and every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = k$, we have*

$$\partial^\alpha (-\Delta)^s u = (-\Delta)^s \partial^\alpha u.$$

More precisely, we have

$$\sup_{x \in \mathbb{R}^n} \left(1 + |x|^{n+2s}\right) |\partial^\alpha (-\Delta)^s u(x)| \leqslant C \max_{|\beta| \leqslant k+2} \left(\sup_{x \in \mathbb{R}^n} \left(1 + |x|^{n+2}\right) |\partial^\beta u(x)| \right), \tag{A.1.21}$$

so that $(-\Delta)^s$ is a continuous linear operator from $\mathscr{S}$ into $\mathscr{S}_s$.

Proof. First let us point out that any $u \in \mathscr{S}$ obviously satisfies the assumptions of Definition A.1.1, thus $(-\Delta)^s u$ is well defined. First let us justify briefly that we can derivate under the integral and obtain $(-\Delta)^s u \in C^\infty(\mathbb{R}^n)$ with $\partial^\alpha (-\Delta)^s u = (-\Delta)^s \partial^\alpha u$ by an easy recursion. For every multi-index α such that $|\alpha| = k$, $\partial^\alpha u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and for every $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} &\frac{|2\partial^\alpha u(x) - \partial^\alpha u(x+h) - \partial^\alpha u(x-h)|}{|h|^{n+2s}} \\ &\leqslant \frac{\chi_{B_1}(h)}{|h|^{n+2s-2}} \Big(\max_{|\beta|=k+2} \|\partial^\beta u\|_{L^\infty(\mathbb{R}^n)} \Big) + \frac{\chi_{B_1^c}(h)}{|h|^{n+2s}} \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} \in L^1(\mathbb{R}^n) \end{aligned} \tag{A.1.22}$$

hence we can apply the theorems of continuity and derivation under the integral to obtain smoothness of $(-\Delta)^s u$ in $\mathbb{R}^n$. In addition, integrating (A.1.22) in h yields

$$\|\partial^\alpha (-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)} \leqslant C \Big[\big(\max_{|\beta|=k+2} \|\partial^\beta u\|_{L^\infty(\mathbb{R}^n)} \big) + \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} \Big], \tag{A.1.23}$$

where C depends only on n and s. To obtain (A.1.21) there remains only to find an upper bound on $\sup\limits_{|x|>1} |x|^{n+2s} |\partial^\alpha (-\Delta)^s u(x)|$. Given x such that $|x| \geqslant 1$, we write

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|2\partial^\alpha u(x) - \partial^\alpha u(x+h) - \partial^\alpha u(x-h)|}{|h|^{n+2s}} \\ &\leqslant \int_{\left\{|h| \leqslant \frac{|x|}{2}\right\}} \left(\max_{|\beta|=k+2} \int_0^1 |\partial^\beta u(x+rh)| \, \mathrm{d}r \right) \frac{\mathrm{d}h}{|h|^{n+2s-2}} \\ &\quad + \int_{\left\{|h| > \frac{|x|}{2}\right\}} 2|\partial^\alpha u(x)| + |\partial^\alpha u(x+h)| + |\partial^\alpha u(x-h)| \frac{\mathrm{d}h}{|h|^{n+2s}}. \end{aligned} \tag{A.1.24}$$

On the one hand, since $u \in \mathscr{S}$, we have

$$\begin{aligned} \max_{|\beta|=k+2} \int_0^1 |\partial^\beta u(x+rh)| \, \mathrm{d}r &\leqslant \left(\int_0^1 \frac{\mathrm{d}r}{1 + |x+rh|^{n+2}} \right) \max_{|\beta|=k+2} \left(\sup_{y \in \mathbb{R}^n} \left(1 + |y|^{n+2}\right) |\partial^\beta u(y)| \right) \\ &\leqslant C |x|^{-(n+2)} \max_{|\beta|=k+2} \left(\sup_{y \in \mathbb{R}^n} \left(1 + |y|^{n+2}\right) |\partial^\beta u(y)| \right) \end{aligned} \tag{A.1.25}$$

for some C depending only on n and s, where we used the fact that $|x+rh| \geqslant \frac{|x|}{2}$ whenever $r \in (0,1)$ and $|h| \leqslant \frac{|x|}{2}$, thus, integrating (A.1.25) yields

$$\begin{aligned}\int_{\{|h|\leqslant\frac{|x|}{2}\}} &\left(\max_{|\beta|=k+2}\int_0^1 |\partial^\beta u(x+rh)|\,\mathrm{d}r\right)\frac{\mathrm{d}h}{|h|^{n+2s-2}} \\ &\leqslant C|x|^{2-2s}|x|^{-(n+2)}\max_{|\beta|=k+2}\Big(\sup_{y\in\mathbb{R}^n}\left(1+|y|^{n+2}\right)|\partial^\beta u(y)|\Big) \\ &\leqslant C|x|^{-(n+2s)}\max_{|\beta|=k+2}\Big(\sup_{y\in\mathbb{R}^n}\left(1+|y|^{n+2}\right)|\partial^\beta u(y)|\Big).\end{aligned} \tag{A.1.26}$$

On the other hand, for $|h| \geqslant \frac{|x|}{2}$,

$$\begin{aligned}\int_{\{|h|>\frac{|x|}{2}\}} &(2|\partial^\alpha u(x)|+|\partial^\alpha u(x+h)|+|\partial^\alpha u(x-h)|)\frac{\mathrm{d}h}{|h|^{n+2s}} \\ &\leqslant 2\Big(\sup_{y\in\mathbb{R}^n}\left(1+|y|^{n+2}\right)|\partial^\alpha u(y)|\Big)\frac{1}{1+|x|^{n+2}}\int_{B^c_{\frac{|x|}{2}}}\frac{1}{|h|^{n+2s}}\,\mathrm{d}t \\ &\qquad + \left|\frac{x}{2}\right|^{-(n+2s)}\|\partial^\alpha u\|_{L^1(\mathbb{R}^n)} \\ &\leqslant C|x|^{-(n+2s)}\Big(\sup_{y\in\mathbb{R}^n}\left(1+|y|^{n+2}\right)|\partial^\alpha u(y)|\Big),\end{aligned} \tag{A.1.27}$$

for some $C = C(n,s)$, where we used

$$\begin{aligned}\|\partial^\alpha u\|_{L^1(\mathbb{R}^n)} &\leqslant \Big(\sup_{x\in\mathbb{R}^n}\left(1+|x|^{n+2}\right)|\partial^\alpha u(x)|\Big)\int_{\mathbb{R}^n}\frac{1}{1+|x|^{n+2}}\,\mathrm{d}x \\ &\leqslant C\Big(\sup_{x\in\mathbb{R}^n}\left(1+|x|^{n+2}\right)|\partial^\alpha u(x)|\Big).\end{aligned}$$

Combining (A.1.24), (A.1.26) and (A.1.27), we find

$$\begin{aligned}\sup_{|x|>1}|x|^{n+2s}|\partial^\alpha(-\Delta)^s u(x)| \leqslant C\Bigg[&\Big(\sup_{x\in\mathbb{R}^n}\left(1+|x|^{n+2}\right)|\partial^\alpha u(x)|\Big) \\ &+\max_{|\beta|=k+2}\Big(\sup_{x\in\mathbb{R}^n}\left(1+|x|^{n+2}\right)|\partial^\beta u(x)|\Big)\Bigg]\end{aligned}$$

hence (A.1.21) with (A.1.23). □

Corollary A.1.10. *Let $s,t \in (0,1)$ such that $s+t<1$. Then for any $\varphi \in \mathscr{S}$, $(-\Delta)^{s+t}\varphi = (-\Delta)^s((-\Delta)^t\varphi) = (-\Delta)^t((-\Delta)^s\varphi) \in \mathscr{S}_{s+t}$.*

Proof. Since s and t play the same role, we need only show $(-\Delta)^{s+t}\varphi = (-\Delta)^s((-\Delta)^t\varphi)$. By Proposition A.1.9, $(-\Delta)^t\varphi \in \mathscr{S}_t$, thus $(-\Delta)^t\varphi$ satisfies all the assumptions of Theorem A.1.4. As a consequence $(-\Delta)^s((-\Delta)^t\varphi)$ is a well-defined function in $L^1(\mathbb{R}^n)$, and we have

$$\mathcal{F}((-\Delta)^s((-\Delta)^t\varphi)) = (2\pi|\xi|)^{2s}\mathcal{F}((-\Delta)^t\varphi) = (2\pi|\xi|)^{2(s+t)}\mathcal{F}(\varphi) = \mathcal{F}((-\Delta)^{s+t}\varphi),$$

hence

$$(-\Delta)^s((-\Delta)^t\varphi) = (-\Delta)^{s+t}\varphi.$$

By Proposition A.1.9, $(-\Delta)^{s+t}\varphi \in \mathscr{S}_{s+t}$, which concludes the proof. □

We can show that $(-\Delta)^s$ is also a continuous linear operator from $\mathscr{S}$ into $\mathscr{S}_s$ when $s \in (-\frac{n}{2}, 0)$, and we can prove this as for the case $s \in (0,1)$, by cutting the integral of (A.1.2) into the two parts $\{|h| \leqslant \frac{|x|}{2}\}$ and $\{|h| > \frac{|x|}{2}\}$ and using the decay rate of $u \in \mathscr{S}$, thus we shall leave the details to the reader.

Proposition A.1.11. *Let $s \in (-\frac{n}{2}, 0)$. Then $(-\Delta)^s$ is a continuous linear operator from $\mathscr{S}$ into $\mathscr{S}_s$, and*

$$\partial^\alpha((-\Delta)^s u) = (-\Delta)^s(\partial^\alpha u)$$

for every multi-index $\alpha \in \mathbb{N}^n$.

By Propositions A.1.9 and A.1.11, $(-\Delta)^s$ is in fact a continuous linear operator from $\mathscr{S}$ into $\mathscr{S}_s$ for all $s \in (-\frac{n}{2}, 1)$. Now we want to show that the identity $(-\Delta)^{-s} \circ (-\Delta)^s = \mathrm{id}$ holds in $\mathscr{S}$.

Proposition A.1.12. *Let $s \in (0, \min(1, \frac{n}{2}))$. Then we have $(-\Delta)^{-s}((-\Delta)^s u) = u$ for every $u \in \mathscr{S}$.*

Proof. If $u \in \mathscr{S}$, by Proposition A.1.9, $(-\Delta)^s u \in \mathscr{S}_s$, so that $(-\Delta)^s u$ satisfies the assumptions of Definition A.1.1, and $(-\Delta)^{-s}((-\Delta)^s u)$ is a well-defined function in $L^\infty(\mathbb{R}^n)$. Let $\varphi \in \mathscr{S}$. By definition of the convolution, we have

$$\int_{\mathbb{R}^n} (I_{2s} * ((-\Delta)^s u))(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} I_{2s}(x)(\varphi * \widecheck{(-\Delta)^s u})(x)\,\mathrm{d}x \tag{A.1.28}$$

where $\check{f}$ denote the function defined by $\check{f}(x) := f(-x)$. Using the fact that φ and $(-\Delta)^s u \in L^2(\mathbb{R}^n)$, the identities

$$\mathcal{F}^{-1}(f * g) = \mathcal{F}^{-1}(f)\mathcal{F}^{-1}(g) \quad \text{and} \quad \mathcal{F}^{-1}(\check{f}) = \mathcal{F}(f), \quad \forall f, g \in L^2(\mathbb{R}^n),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Notice that $\varphi * \widecheck{(-\Delta)^s u} \in \mathscr{S}_s$, so that $\mathcal{F}^{-1}(\varphi * \widecheck{(-\Delta)^s u}) \in C^n(\mathbb{R}^n) \cap W^{n,1}(\mathbb{R}^n)$. Thus by Corollary A.1.7 we have

$$\begin{aligned}\int_{\mathbb{R}^n} (I_{2s} * ((-\Delta)^s u))(x)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} I_{2s}(\xi)\mathcal{F}(\mathcal{F}^{-1}(\varphi * \widecheck{(-\Delta)^s u}))(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} (2\pi|\xi|)^{-2s}\mathcal{F}^{-1}(\varphi * \widecheck{(-\Delta)^s u})(\xi)\,\mathrm{d}\xi.\end{aligned} \tag{A.1.29}$$

By Theorem A.1.4, using that $(-\Delta)^s u \in \mathscr{S}_s$, we have

$$\mathcal{F}^{-1}(\varphi * \widecheck{(-\Delta)^s u}) = \mathcal{F}^{-1}(\varphi)\mathcal{F}^{-1}(\widecheck{(-\Delta)^s u}) = (2\pi|\xi|)^{2s}\mathcal{F}(\check{\varphi})\mathcal{F}(u) = (2\pi|\xi|)^{2s}\mathcal{F}(\check{\varphi} * u), \tag{A.1.30}$$

and then

$$\int_{\mathbb{R}^n} \mathcal{F}(\check{\varphi} * u)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \mathcal{F}(\check{\varphi})\mathcal{F}(u)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \mathcal{F}(\mathcal{F}(\check{\varphi}))u\,\mathrm{d}\xi = \int_{\mathbb{R}^n} u\varphi\,\mathrm{d}x, \tag{A.1.31}$$

Injecting (A.1.30) and (A.1.31) into (A.1.29), we find

$$\int_{\mathbb{R}^n} (I_{2s} * ((-\Delta)^s u))(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} u\varphi\,\mathrm{d}x,$$

which shows that

$$(-\Delta)^{-s}((-\Delta)^s u)) = u$$

by arbitrariness of $\varphi \in \mathscr{S}_s$. ∎

Remark A.1.13. Again, let us emphasize that $(-\Delta)^{-s}$ is only defined for $s \in (0, \min(1, \frac{n}{2}))$, so that the identity $(-\Delta)^{-s} \circ (-\Delta)^s$ has a meaning only in that case.

By Propositions A.1.9 and A.1.11, we can define the s-Laplacian on the subspace of tempered distributions $\mathscr{S}'_s$ by duality as follows.

Definition A.1.14. If $s \in (-\frac{n}{2}, 0)$ and $T \in \mathscr{S}'_s$, we define $(-\Delta)^s T \in \mathscr{S}'$ by

$$\langle (-\Delta)^s T, \varphi \rangle_{\mathscr{S}', \mathscr{S}} := \langle T, (-\Delta)^s \varphi \rangle_{\mathscr{S}'_s, \mathscr{S}_s}.$$

Among the distributions $T \in \mathscr{S}'_s$, we are especially interested in those given by locally integrable functions.

Definition A.1.15. For $s \in (-\frac{n}{2}, 1)$, we define $L^1_s(\mathbb{R}^n) := L^1_{\text{loc}}(\mathbb{R}^n) \cap (\mathscr{S}_s)'$, and we have the equivalent characterization

$$L^1_s(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, \mathrm{d}x < +\infty \right\}.$$

Since $I_{2s} \in L^1_s \subseteq \mathscr{S}'_s$, $(-\Delta)^s I_{2s}$ defines a tempered distribution, and we can then give a meaning to the identity $(-\Delta)^s I_{2s} = \delta_0$.

Proposition A.1.16. *For $s \in (0, \min(1, \frac{n}{2}))$, we have $(-\Delta)^s I_{2s} = \delta_0$ in $\mathscr{S}'$.*

Proof. The fact that $I_{2s} \in L^1_s(\mathbb{R}^n)$ is clear. Then for $\varphi \in \mathscr{S}$ we have

$$\begin{aligned}
\langle (-\Delta)^s I_{2s}, \varphi \rangle_{\mathscr{S}', \mathscr{S}} &= \langle I_{2s}, (-\Delta)^s \varphi \rangle_{(\mathscr{S}_s)', \mathscr{S}_s} \\
&= \int_{\mathbb{R}^n} I_{2s}(x) \left((-\Delta)^s \varphi(x) \right) \mathrm{d}x \\
&= I_{2s} * \left((-\Delta)^s \check{\varphi} \right)(0) = \left((-\Delta)^{-s} \left((-\Delta)^s \check{\varphi} \right) \right)(0),
\end{aligned}$$

by definition of $(-\Delta)^{-s}$, where $\check{\varphi}(x) := \varphi(-x)$. Since $\varphi \in \mathscr{S}$, by Proposition A.1.12 we have

$$(-\Delta)^{-s} \left((-\Delta)^s \check{\varphi} \right) = \check{\varphi},$$

thus

$$\langle (-\Delta)^s I_{2s}, \varphi \rangle_{\mathscr{S}', \mathscr{S}} = \check{\varphi}(0) = \varphi(0) = \langle \delta_0, \varphi \rangle_{\mathscr{S}', \mathscr{S}}.$$

which gives the result. □

A.2 Elliptic regularity for the distributional s-Laplacian

Now we want to show that if $(-\Delta)^s T = f$ in some open set $\Omega \subseteq \mathbb{R}^n$, where T is a tempered distribution in $\mathscr{S}'_s$, then T morally gains $2s$ fractional derivatives in Ω compared to f. In particular, we obtain that s-harmonic distributions in Ω (i.e., distributions $T \in \mathscr{S}'_s$ satisfying $(-\Delta)^s T = 0$ in Ω) are smooth in Ω. For Sections A.2.2 and A.2.3, we fix an open set $\Omega \subseteq \mathbb{R}^n$ and we always assume $\{n \geqslant 2 \text{ and } s \in (0,1)\}$ or $\{n = 1 \text{ and } s \in (0, \frac{1}{2})\}$. Unfortunately this excludes the case $s \in [\frac{1}{2}, 1)$ in dimension 1.

A.2.1 Regularizing effect of Riesz operators

We need the following well-known regularizing effect of Riesz operators.

Theorem A.2.1. *Let $f \in C^{k,\alpha}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then for any $s \in (0,1)$, defining $g := I_{2s} * f$, $l = \lfloor \alpha + 2s \rfloor \in \{0,1,2\}$ and $\beta = \alpha + 2s - \lfloor \alpha + 2s \rfloor \in [0,1)$, we have*

(i) $g \in C^{k+l,\beta}(\mathbb{R}^n)$ if $\beta > 0$;

(ii) $g \in C^{k+l-1,\delta}(\mathbb{R}^n)$ for every $\delta \in (0,1)$ if $\beta = 0$.

Proof. First, note that by splitting the integral defining $g = I_{2s} * f$ into a part on B_1 and the other on B_1^c, using Hölder inequality we easily see that g is well defined and belongs to $L^\infty(\mathbb{R}^n)$. Then the proof relies on the fact that the Riesz operator[1] $\mathcal{R}_{2s} : f \mapsto (\mathcal{F}^{-1} \circ (2\pi|\xi|)^{-2s} \circ \mathcal{F})(f)$ maps the homogeneous Besov space $\dot{B}^{\alpha,\infty}_{\infty}$ onto $\dot{B}^{\alpha+2s,\infty}_{\infty}$ (see e.g. [112, 5.2.3, Theorem 1]), and the equivalence of the $\dot{B}^{\alpha,\infty}_{\infty}$ seminorm with the seminorm of the Zygmund space $\mathscr{C}^\alpha$ (see [112, 2.2.2/(4)] for the definition of $\mathscr{C}^\alpha$ and [112, 2.5.12, Theorem & Corollary; 5.2.3, Theorem 2 & Remarks 2 to 4] for the equivalence), where the $\mathscr{C}^\alpha$ seminorm is given by

$$[f]_{\mathscr{C}^\alpha} := \sum_{1 \leqslant |\gamma| < k} \|\partial^\gamma f\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma| = k} \sup_{x,h \in \mathbb{R}^n} \frac{|\partial^\gamma f(x+h) + \partial^\gamma f(x-h) - 2\partial^\gamma f(x)|}{|h|^\lambda},$$

where $k := \lfloor \alpha \rfloor$ and $\lambda := \alpha - \lfloor \alpha \rfloor$. It is known that $C^{k,\alpha} = \mathscr{C}^{k+\alpha}$ whenever $k \in \mathbb{N}$ and $\alpha \in (0,1)$ (since $k + \alpha$ is not an integer, see [109, Chapter 5.4, Proposition 8]). Here, since $f \in C^{k,\alpha} = \mathscr{C}^{k+\alpha}$, the fact that $\mathcal{R}_{2s}$ maps $\mathscr{C}^\beta$ onto $\mathscr{C}^{\beta+2s}$ for every $\beta > 0$ shows that $[g]_{\mathscr{C}^{k+\alpha+2s}} < +\infty$, where $g = I_{2s} * f$, meaning that, if $l = \lfloor \alpha + 2s \rfloor$ and $\beta = \alpha + 2s - \lfloor \alpha + 2s \rfloor > 0$,

$$\sum_{1 \leqslant |\gamma| < k+l} \|\partial^\gamma g\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma| = k+l} \sup_{x,h \in \mathbb{R}^n} \frac{|\partial^\gamma g(x+h) + \partial^\gamma g(x-h) - 2\partial^\gamma g(x)|}{|h|^\beta} < +\infty \tag{A.2.1}$$

and if $\beta = 0$,

$$\sum_{1 \leqslant |\gamma| < k+l-1} \|\partial^\gamma g\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma| = k+l-1} \sup_{x,h \in \mathbb{R}^n} \frac{|\partial^\gamma g(x+h) + \partial^\gamma g(x-h) - 2\partial^\gamma g(x)|}{|h|^\delta} < +\infty \tag{A.2.2}$$

for all $\delta \in (0,1)$. Now, we have seen that $g \in L^\infty(\mathbb{R}^n)$ as well, thus we have

$$\sum_{0 \leqslant |\gamma| < k+l} \|\partial^\gamma g\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma| = k+l} \sup_{x,h \in \mathbb{R}^n} \frac{|\partial^\gamma g(x+h) + \partial^\gamma g(x-h) - 2\partial^\gamma g(x)|}{|h|^\beta} < +\infty \tag{A.2.3}$$

and if $\beta = 0$,

$$\sum_{0 \leqslant |\gamma| < k+l-1} \|\partial^\gamma g\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\gamma| = k+l-1} \sup_{x,h \in \mathbb{R}^n} \frac{|\partial^\gamma g(x+h) + \partial^\gamma g(x-h) - 2\partial^\gamma g(x)|}{|h|^\delta} < +\infty \tag{A.2.4}$$

for all $\delta \in (0,1)$. Now we apply the results from [109, Chapter V.4] in order to translate (A.2.3) and (A.2.4) in terms of inclusions into $C^{k,\alpha}$ spaces. By [109, Chapter V.4, (48), Propositions 8 and 9], (A.2.3) implies $v \in C^{k+l,\beta}(\mathbb{R}^n)$, and (A.2.4) implies $v \in C^{k+l-1,\delta}(\mathbb{R}^n)$ for all $\delta \in (0,1)$, which gives the result. ☐

[1] $\mathcal{R}_{2s}$ is defined on the space of tempered distributions modulo polynomials, see [112, p. 5.1.2].

We have the following counterpart on Sobolev spaces.

Theorem A.2.2. *Let $f \in W^{k,p}(\mathbb{R}^n) \cap L^\infty(B_1) \cap L^1(B_1^c)$, where $k \in (0,+\infty)$ and $p \in (1,+\infty)$. Then for any $s \in (0,1)$, letting $g := I_{2s} * f$ we have:*

(i) if k and $k+2s$ are simultaneously integers or nonintegers, then $g \in W^{k+2s,p}(\mathbb{R}^n)$;

(ii) if $k \in \mathbb{N}$, $k+2s \notin \mathbb{N}$, then $g \in W^{k+2s,p}(\mathbb{R}^n)$ when $p \in [2,+\infty)$, and $g \in W^{l,p}(\mathbb{R}^n)$ for all $l \in (0,k+2s)$ when $p \in (1,2)$.

(iii) if $k \notin \mathbb{N}$, $k+2s \in \mathbb{N}$, then $g \in W^{k+2s,p}(\mathbb{R}^n)$ when $p \in [1,2]$, and $g \in W^{l,p}(\mathbb{R}^n)$ for all $l \in (0,k+2s)$ when $p \in (2,+\infty)$.

Proof. First, note that by splitting the integral defining $g = I_{2s} * f$ into a part on B_1 and the other on B_1^c, using Hölder inequality and the fact that $f \in L^\infty(B_1) \cap L^1(B_1^c)$, we easily see that g is well defined and belongs to $L^p(\mathbb{R}^n)$. The proof then relies on the boundedness of the Riesz operators $\mathcal{R}_{2s} : f \mapsto (\mathcal{F}^{-1} \circ (2\pi|\xi|)^{-2s} \circ \mathcal{F})(f)$ (as before, defined on tempered distributions modulo polynomials) between homogeneous Triebel-Lizorkin spaces $\dot{F}^k_{p,q}$, embeddings between the corresponding nonhomogeneous spaces and identification with Sobolev spaces. We recall the following well-known facts:

- $\mathcal{R}_{2s}$ maps the homogeneous Triebel-Lizorkin space $\dot{F}^k_{p,q}$ onto $\dot{F}^{k+2s}_{p,q}$ for any $p,q \in (1,+\infty)$ (see [112, Section 5.2.3, Theorem 1]);
- $\dot{F}^k_{p,p} = \dot{W}^{k,p}(\mathbb{R}^n)$ whenever k is not an integer, and $\dot{F}^k_{p,2} = \dot{W}^{k,p}(\mathbb{R}^n)$ when k is an integer, with equivalent seminorms (see [112, 5.2.3, Theorem 1 and Remark 4] for this);
- $F^k_{p,p} = W^{k,p}(\mathbb{R}^n)$ whenever k is not an integer, and $F^k_{p,2} = W^{k,p}(\mathbb{R}^n)$ when k is an integer, with equivalent norms (see [112, 2.2.2, Remark 3/(18,19); 2.3.5/(2); 2.5.6, Theorem; 2.5.7/(9)] for this). In addition $F^k_{p,q} = \dot{F}^k_{p,q} \cap L^p(\mathbb{R}^n)$ and the norm on $F^k_{p,q}$ is equivalent to $[\cdot]_{\dot{F}^k_{p,q}} + \|\cdot\|_{L^p(\mathbb{R}^n)}$.
- we have the continuous embeddings $F^k_{p,q_1} \hookrightarrow F^k_{p,q_2}$ for all $p, q_1, q_2 \in (0,+\infty]$ such that $q_1 \leqslant q_2$, and $F^{k+\varepsilon}_{p,q_1} \hookrightarrow F^k_{p,q_2}$ for all $p, q_1, q_2 \in (0,+\infty]$ and $\varepsilon > 0$ (see [112, 2.3.2, Proposition 2]). These do not have homogeneous counterparts.

There are 4 main different cases:

(1) if k and $k+2s$ are both integers, then we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^k_{p,2} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,2} = \dot{W}^{k+2s,p}(\mathbb{R}^n),$$

thus $g \in \dot{W}^{k+2s,p}(\mathbb{R}^n)$;

(2) if neither k nor $k+2s$ are integers, then we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^k_{p,p} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,p} = \dot{W}^{k+2s,p}(\mathbb{R}^n),$$

thus $g \in \dot{W}^{k+2s,p}(\mathbb{R}^n)$;

(3) if $k \in \mathbb{N}$ and $k+2s \notin \mathbb{N}$ (i.e. $s \neq \frac{1}{2}$), there are three different cases. When $p=2$, then we use

$$\dot{W}^{k,2}(\mathbb{R}^n) = \dot{F}^k_{2,2} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{2,2} = \dot{W}^{k+2s,2}(\mathbb{R}^n) \tag{A.2.5}$$

so $g \in \dot{W}^{k+2s,p}(\mathbb{R}^n)$; when $p \in (2,\infty)$, then we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^k_{p,2} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,2}$$

so $g \in \dot{F}^{k+2s}_{p,2}$, and

$$\dot{F}^{k+2s}_{p,2} \cap L^p(\mathbb{R}^n) = F^{k+2s}_{p,2} \hookrightarrow F^{k+2s}_{p,p} = W^{k+2s,p}(\mathbb{R}^n)$$

implies that $g \in \dot{W}^{k+2s,p}(\mathbb{R}^n)$; when $p \in (1,2)$, for any $l \in (0,k)$, if $l+2s$ is an integer we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^{k}_{p,2} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,2}$$

with

$$\dot{F}^{k+2s}_{p,2} \cap L^p(\mathbb{R}^n) = F^{k+2s}_{p,2} \hookrightarrow F^{l+2s}_{p,2} = W^{l+2s,p}(\mathbb{R}^n),$$

and if $l+2s$ is not an integer we use

$$W^{k,p}(\mathbb{R}^n) = F^{k}_{p,2} \hookrightarrow F^{l}_{p,p} = \dot{F}^{l}_{p,p} \cap L^p(\mathbb{R}^n)$$

with

$$\dot{F}^{l}_{p,p} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{l+2s}_{p,p} = \dot{W}^{l+2s,p}(\mathbb{R}^n)$$

to obtain that $g \in \dot{W}^{l+2s,p}(\mathbb{R}^n)$ for all $l \in (0,k)$;

(4) if $k \notin \mathbb{N}$ and $k+2s \in \mathbb{N}$ (i.e. $s \neq \frac{1}{2}$) there are also three different cases. When $p=2$ we proceed as in (A.2.5); when $p \in (1,2)$, then we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^{k}_{p,p} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,p}$$

and

$$\dot{F}^{k+2s}_{p,p} \cap L^p(\mathbb{R}^n) = F^{k+2s}_{p,p} \hookrightarrow F^{k+2s}_{p,2} = W^{k+2s,p}(\mathbb{R}^n);$$

when $p \in (2,+\infty)$, if $l \in (0,k)$ is such that $l+2s$ is an integer, then we use

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^{k}_{p,p} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,p},$$

with

$$\dot{F}^{k+2s}_{p,p} \cap L^p(\mathbb{R}^n) = F^{k+2s}_{p,p} \hookrightarrow F^{l+2s}_{p,2} = W^{l+2s,p}(\mathbb{R}^n),$$

and if $l+2s$ is not an integer,

$$\dot{W}^{k,p}(\mathbb{R}^n) = \dot{F}^{k}_{p,p} \xrightarrow{\mathcal{R}_{2s}} \dot{F}^{k+2s}_{p,p}$$

with

$$\dot{F}^{k+2s}_{p,p} \cap L^p(\mathbb{R}^n) = F^{k+2s}_{p,p} \hookrightarrow F^{l+2s}_{p,p} = W^{l+2s,p}(\mathbb{R}^n).$$

Reorganizing these cases gives the result, since g also belongs to $L^p(\mathbb{R}^n)$, as we have seen. □

Note that in particular, if $p=2$, then $f * I_{2s} \in W^{k+2s,2}(\mathbb{R}^n)$ whenever $f \in W^{k,2}(\mathbb{R}^n) \cap L^\infty(B_1) \cap L^1(B_1^c)$.

A.2.2 Basic operations and properties of $\mathscr{S}'_s$

Since the distributional setting is not entirely standard, we recall that we can define the convolution of an element of $\mathscr{S}'_s$ with an element of $\mathscr{S}_s$, and show that it produces a smooth function on $\mathbb{R}^n$. For this we need a few technical lemmas.

Lemma A.2.3. *Let $f \in L^1(\mathbb{R}^n)$ with compact support, and $g \in \mathscr{S}_s(\mathbb{R}^n)$. Then $f * g \in \mathscr{S}_s(\mathbb{R}^n)$.*

Proof. Since $f \in L^1(\mathbb{R}^n)$ is compactly supported and $g \in \mathscr{S}_s(\mathbb{R}^n)$ one can easily derivate under the integral recursively to show that $f * g \in C^\infty(\mathbb{R}^n)$. There remains to see that

$$\sup_{x\in\mathbb{R}^n} \left(1 + |x|^{n+2s}\right) |\partial^\alpha (f * g)(x)| < +\infty$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$. Since $\partial^\alpha(f * g) = f * \partial^\alpha g$, we have

$$\|\partial^\alpha (f * g)\|_{L^\infty(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \|\partial^\alpha g\|_{L^\infty(\mathbb{R}^n)} < +\infty. \tag{A.2.6}$$

f being compactly supported, there exists some ball B_R such that $f(x) = 0$ for almost every $x \in B_R^c$. Thus for any x such that $|x| > 2R$, we have

$$\begin{aligned}
|\partial^\alpha (f * g)(x)| &\leqslant \int_{B_R} |f(y)||\partial^\alpha g(x-y)| \,\mathrm{d}y \\
&\leqslant \Big(\sup_{x\in\mathbb{R}^n} \left(1 + |x|^{n+2s}\right) |\partial^\alpha g(x)|\Big) \int_{B_R} \frac{|f(y)|}{(1 + |x-y|^{n+2s})} \,\mathrm{d}y \\
&\leqslant \Big(\sup_{x\in\mathbb{R}^n} \left(1 + |x|^{n+2s}\right) |\partial^\alpha g(x)|\Big) \int_{B_R} \frac{|f(y)|}{\left(1 + |\frac{x}{2}|^{n+2s}\right)} \,\mathrm{d}y \\
&\leqslant \Big(\sup_{x\in\mathbb{R}^n} \left(1 + |x|^{n+2s}\right) |\partial^\alpha g(x)|\Big) \left(\int_{B_R} |f(y)| \,\mathrm{d}y\right) |x|^{-(n+2s)} \,\mathrm{d}y,
\end{aligned} \tag{A.2.7}$$

hence the result, combining (A.2.6) and (A.2.7). ☐

Lemma A.2.4. *Let $F \in \mathscr{S}_s(\mathbb{R} \times \mathbb{R})$, and $T \in \mathscr{S}'_s(\mathbb{R})$. Then $x \mapsto \langle T, F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s} \in C^\infty(\mathbb{R})$ and $\partial_x^k F = \langle T, \partial_x^k F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}$, for all $k \in \mathbb{N}$.*

Proof. The proof follows the standard proof for classical distributions. We show only that $x \mapsto \langle T, F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}$ is everywhere differentiable, and its derivative is given by $x \mapsto \langle T, \partial_x F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}$. The result then follows immediately by induction. By definition of $\mathscr{S}_s(\mathbb{R}\times\mathbb{R})$, for every $x \in \mathbb{R}$, $y \mapsto F(x,y) \in \mathscr{S}_s$ and $y \mapsto \partial_x F(x,y) \in \mathscr{S}_s$ so $\langle T, F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}$ and $\langle T, \partial_x F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}$ are both well defined. Given $x, y \in \mathbb{R}$ and $h \in [-1,1]$. By Taylor's theorem we have

$$F(x+h,y) - F(x,y) - h\partial_x F(x,y) = h^2 \int_0^1 \partial_x^2 F(x+th,y)(1-t)^2 \,\mathrm{d}t.$$

Since $y \mapsto \partial_x F(x,y)$ and $y \mapsto \int_0^1 \partial_x^2 F(x+th,y)(1-t)^2 \,\mathrm{d}t$ belong to $\mathscr{S}_s$, we have

$$\begin{aligned}
&|\langle T, F(x+h,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s} - \langle T, F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s} - h\langle T, \partial_x F(x,\cdot)\rangle_{\mathscr{S}'_s,\mathscr{S}_s}| \\
&\quad\leqslant h^2 |\langle T, \int_0^1 \partial_x^2 F(x+th,\cdot)(1-t)^2 \,\mathrm{d}t\rangle_{\mathscr{S}'_s,\mathscr{S}_s}|.
\end{aligned} \tag{A.2.8}$$

By definition of $\mathscr{S}_s'$, there exists $m > 0$ and C not depending of x and h such that

$$\begin{aligned}
&\left|\langle T, \int_0^1 \partial_x^2 F(x+th,\cdot)(1-t)^2\,\mathrm{d}t\rangle_{\mathscr{S}_s',\mathscr{S}_s}\right| \\
&\leqslant C \sum_{0\leqslant k\leqslant m} \sup_{y\in\mathbb{R}} \left[\left(1+|y|^{n+2s}\right)\left|\partial_y^k \int_0^1 \partial_x^2 F(x+th,y)(1-t)^2\,\mathrm{d}t\right|\right] \\
&\leqslant C \sum_{0\leqslant k\leqslant m} \sup_{(x,y)\in\mathbb{R}^2} \left(1+|(x,y)|^{n+2s}\right)|\partial_x^2\partial_y^k F(x,y)| \leqslant C.
\end{aligned} \tag{A.2.9}$$

By (A.2.8) and (A.2.9) we see that $x \mapsto \langle T, F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s}$ is continuous and differentiable at x, and its derivative is $x \mapsto \langle T, \partial_x F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s}$. □

Lemma A.2.5. *Let $T \in \mathscr{S}_s'$ and $f \in \mathscr{S}_s$, then defining the convolution of T with f by*

$$T * f(x) := \langle T, \tau_{-x}\check{f}\rangle_{\mathscr{S}_s',\mathscr{S}_s},$$

where $\check{f}(x) := f(-x)$ and $\tau_{-x} f := f(\cdot - x)$, we have $T * f \in C^\infty(\mathbb{R}^n)$. In addition, for any $\varphi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} (T*f)(x)\varphi(x)\,\mathrm{d}x = \langle T, f*\varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s}. \tag{A.2.10}$$

Proof. We proceed in two steps.
Step 1. First we show that $T*f$ is a well-defined smooth function. We shall only show that $T*f \in C^1(\mathbb{R}^n)$ and that $\partial_i(T*f) = T*\partial_i f$, as it is then easy to conclude by induction. It is clear that $\tau_{-a} f \in \mathscr{S}_s$ whenever $f \in \mathscr{S}_s$, so that $T*f$ is well defined. Since for any $a, h \in \mathbb{R}^n$, we have

$$\sup_{x\in\mathbb{R}^n} |(\tau_{-(a+h)}\check{f} - \tau_{-a}\check{f})(x)| \leqslant \|\nabla f\|_{L^\infty(\mathbb{R}^n)}|h|,$$

$\tau_{-(a+h)}\check{f}$ converges uniformly to $\tau_{-a}\check{f}$ as h goes to 0. Similarly, one shows that the derivatives $\partial^\alpha(\tau_{-(a+h)}\check{f})$ converge uniformly to $\partial^\alpha(\tau_{-a}\check{f})$ as h goes to 0, for every multi-index $\alpha \in \mathbb{N}^n$. By the topology of $\mathscr{S}_s'$, this implies

$$T*f(a+h) = \langle T, \tau_{-a-h}\check{f}\rangle_{\mathscr{S}_s',\mathscr{S}_s} \xrightarrow{h\to 0} \langle T, \tau_{-a}\check{f}\rangle_{\mathscr{S}_s',\mathscr{S}_s} = T*f(a),$$

hence $T*f$ is continuous. By linearity of T, for any $a \in \mathbb{R}^n$, $h \in \mathbb{R}\setminus\{0\}$ and $i \in \{1,\dots,n\}$ we have

$$T*f(a+he_i) - T*f(a) = (T*\tau_{he_i} f)(a) - (T*f)(a) = \langle T, \tau_{-(a+he_i)}\check{f} - \tau_{-a}\check{f}\rangle_{\mathcal{S}_s',\mathcal{S}_s},$$

thus

$$\frac{T*f(a+he_i) - T*f(a)}{h} = \left\langle T, \frac{\tau_{-(a+he_i)}\check{f} - \tau_{-a}\check{f}}{h}\right\rangle_{\mathcal{S}_s',\mathcal{S}_s}.$$

Now since $f \in \mathscr{S}_s$ and $\check{f} \in \mathscr{S}_s$,

$$\sup_{x\in\mathbb{R}^n}\left|\frac{\tau_{-(a+he_i)}\check{f}(x) - \tau_{-a}\check{f}(x))}{h} - h\partial_i\tau_{-a}\check{f}(x)\right| \leqslant |h|\Big(\max_{|\alpha|=2}\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)}\Big),$$

so that $(\tau_{-(a+he_i)}\check{f} - \tau_{-a}\check{f})/h$ converges uniformly to $\tau_{-a}\widecheck{\partial_i f}$ as h goes to 0. Similarly, we can show that $\partial^\alpha(\tau_{-(a+he_i)}\check{f} - \tau_{-a}\check{f})/h$ converges uniformly to $\partial^\alpha(\tau_{-a}\widecheck{\partial_i f})$ as h goes to 0. Hence by the topology of $\mathscr{S}_s'$, we have

$$\frac{T * f(a+he_i) - T * f(a)}{h} \xrightarrow{h\to 0} \langle T, \tau_{-a}\widecheck{\partial_i f}\rangle = T * \partial_i f(a),$$

i.e. $\partial_i(T*f)(a) = T*\partial_i f(a)$, Proceeding as for the continuity of f, one shows that $T*\partial_i f$ is continuous for all $i\in\{1,\ldots,n\}$, hence $T*f\in C^1(\mathbb{R}^n)$.
Step 2. We show (A.2.10). By Lemma A.2.3, $f*\varphi\in\mathscr{S}_s$ for any $\varphi\in\mathscr{D}(\mathbb{R}^n)$ and $f\in\mathscr{S}_s$, thus $\langle T, f*\varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s}$ is well defined. We show it only for $n=1$, as it is then easy to conclude by induction. Let $\varphi\in\mathscr{D}((a,b),\mathbb{R})$. Then defining

$$F(x,y) := \int_a^x \varphi(t)f(y-t)\,\mathrm{d}t,$$

we check readily that $F\in\mathscr{S}_s(\mathbb{R}^2)$. By Lemma A.2.4, $x\mapsto\langle T, F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s}\in C^\infty(\mathbb{R})$, and we have

$$\partial_x\langle T, F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle T, \partial_x F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s}.$$

Integrating in x between a and b, we find

$$\langle T, F(b,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \int_a^b \langle T, \partial_x, F(x,\cdot)\rangle_{\mathscr{S}_s',\mathscr{S}_s}\,\mathrm{d}x$$

since $F(a,y)=0$ for all $y\in\mathbb{R}$. This reads

$$\langle T, \int_a^b \varphi(t)f(\cdot - t)\,\mathrm{d}t\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \int_a^b \langle T, \varphi(x)f(\cdot - x)\rangle_{\mathscr{S}_s',\mathscr{S}_s}\,\mathrm{d}x.$$

Recalling that φ is compactly supported in (a,b), and noting that

$$\int_a^b \langle T, \varphi(x)f(\cdot - x)\rangle_{\mathscr{S}_s',\mathscr{S}_s}\,\mathrm{d}x = \int_a^b \varphi(x)\langle T, \tau_{-x}\check{f}\rangle_{\mathscr{S}_s',\mathscr{S}_s}\,\mathrm{d}x,$$

it follows

$$\langle T, \varphi * f\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \int_{\mathbb{R}} \varphi(x)(T*f)(x)\,\mathrm{d}x,$$

hence the result. □

A.2.3 Elliptic regularity

We will make use of the following technical lemmas. Although we will only use the next Lemma to estimate the decay at infinity of the derivative of a function f in terms of the decay of f and f'', we find this more general result nice to state and prove.

Lemma A.2.6. *Let $r>0$, $m\in[0,+\infty)$, and $k,l\in\mathbb{N}$ such that $k+1<l$. If $f\in C^l(\mathbb{R}^n)$ satisfies*

$$\sup_{x\in Q_r^c} |x|^m|\partial^\alpha f(x)| < +\infty,$$

for every multi-index $\alpha\in\mathbb{N}^n$ such that $|\alpha|=k$ or $|\alpha|=l$, where $Q_r\subseteq\mathbb{R}^n$ is the open cube $]0,r[^n$, then

$$\sup_{x\in Q_r^c} |x|^m|\partial^\alpha f(x)| < +\infty,$$

for every multi-index α such that $\alpha\in\{k,\ldots,l\}$.

Proof. Up to taking the maximum over all multi-indices α such that $|\alpha| = k$ and $|\alpha| = l$. we may assume that there exists some $M > 0$ such that, for all those multi-indices, we have

$$\sup_{|x|\geqslant r} |x|^m |\partial^\alpha f(x)| \leqslant M.$$

We proceed in two steps.
Step 1. Let us first assume that $n = 1$, and define $p := l - k$, $g := f^{(k)}$, so that $g \in C^p(\mathbb{R})$. Without loss of generality we may assume that x is positive, so that $x \in (r, +\infty)$. For all $i \in \{1, \dots, p-1\}$, we define $\lambda_i := \frac{i}{p-1}$ and $x_i = x + \lambda_i$. Since $g \in C^p(\mathbb{R})$, by Taylor-Lagrange theorem, for each x_i, there exists some $\xi_i \in (x, x_i) \subseteq (x, x+1)$ such that

$$g(x_i) = g(x) + \lambda_i g'(x) + \lambda_i^2 \frac{g^{(2)}(x)}{2} \dots + \lambda_i^{p-1} \frac{g^{(p-1)}(x)}{(p-1)!} + \lambda_i^p \frac{g^{(p)}(\xi_i)}{p!}.$$

Passing all the terms in g and $g^{(p)}$ to the right-hand side, and all the others to the left-hand side, these equalities can be written matricially as

$$AX := \begin{pmatrix} \lambda_1 & \lambda_1^2 & \cdots & \cdots & \lambda_1^{p-1} \\ \lambda_2 & \lambda_2^2 & \cdots & \cdots & \lambda_2^{p-1} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \lambda_{p-1} & \lambda_p^2 & \cdots & \cdots & \lambda_{p-1}^{p-1} \end{pmatrix} \begin{pmatrix} g'(x) \\ \frac{g^{(2)}(x)}{2} \\ \vdots \\ \vdots \\ \frac{g^{(p-1)}(x)}{p!} \end{pmatrix}$$

$$= \begin{pmatrix} g(x_1) - g(x) - \lambda_1^p \frac{g^p(\xi_1)}{p!} \\ g(x_2) - g(x) - \lambda_2^p \frac{g^p(\xi_2)}{p!} \\ \vdots \\ \vdots \\ g(x_{p-1}) - g(x) - \lambda_p^p \frac{g^p(\xi_p)}{p!} \end{pmatrix} =: Y.$$

We remark that A is a Vandermonde matrix, so it is invertible since the λ_i are all distincts. Thus $X = A^{-1}Y$. Note that since x_i and ξ_i belong to $(x, x+1)$ and $x > r$, we have

$$|g(x_i)| \leqslant \Big(\sup_{|x|\geqslant r} |x|^m |g(x)|\Big) |x_i|^{-m} \leqslant M|x|^{-m},$$

and

$$|g^{(p)}(\xi_i)| \leqslant (\sup_{|x|\geqslant r} |x|^m |g^{(p-1)}(x)|) |\xi_i|^{-m} \leqslant M|x|^{-m},$$

as well as (obviously) $|g(x)| \leqslant M|x|^{-m}$, thus $|Y| \leqslant C|x|^{-m}$, where C depends only on M and p. Hence we have

$$|X| \leqslant C\|A^{-1}\| |x|^{-m},$$

which shows that

$$\sup_{|x|\geqslant r} |x|^m |f^{(i)}(x)| \leqslant C, \quad \forall i \in \{k, \dots, l\},$$

where C depends only on M and $p = l - k$, by definition of X and g.
Step 2. Now we turn to the case $n > 1$, and proceed by induction on $p = l - k$. If $p = 1$ then there is nothing to prove. Assume that the lemma is proven for $p \geqslant 1$. Let $f \in C^l(\mathbb{R}^n)$ such that

$$\sup_{x\in Q_r^c} |x|^m |\partial^\alpha f(x)| \leqslant M,$$

for every multi index α such that $|\alpha| = k$ or $|\alpha| = l$, where $l - k = p + 1$. It is enough to show that for any multi-index α such that $|\alpha| = k$, we have

$$\sup_{x \in Q_r^c} |x|^m |\partial^\beta \partial^\alpha f(x)| < +\infty,$$

for every $\beta \in \mathbb{N}^n$ such that $|\beta| \in \{1, \dots, p\}$. Thus we now fix α and consider $g := \partial^\alpha f$. Let $x_0 \in \{0\} \times \mathbb{R}^{n-1}$ be fixed. We define $g_1 \in C^{p+1}(\mathbb{R})$ by $g_1(t) = g(x_0 + te_1)$. Then g_1 satisfies

$$\sup_{|t| \geqslant r} |t|^m |g_1^{(j)}(t)| \leqslant M,$$

for $j = 0$ and $j = p + 1$, thus applying Step 1 we find

$$\sup_{|t| \geqslant r} |t|^m |g_1^{(j)}(t)| \leqslant C,$$

for all $j \in \{0, \dots, p + 1\}$, where C depends only on M and p. By arbitrariness of x_0 and the fact that $g_1^{(j)}(t) = \partial_1^j g(x_0, t)$, this implies

$$\sup_{x \in Q_r^c} |x|^m |\partial_1^j g(x)| \leqslant C,$$

for all $j \in \{0, \dots, p + 1\}$. Proceeding exactly as above, for $i = 2, \dots, n$ instead of $i = 1$, we show that

$$\sup_{x \in Q_r^c} |x|^m |\partial_i^j g(x)| \leqslant C, \quad \forall i \in \{1, \dots, n\}$$

for all $j \in \{0, \dots, p + 1\}$ and some $C = C(M, p)$. In particular, we have

$$\sup_{x \in Q_r^c} |x|^m |\partial_i g(x)| \leqslant C, \quad \forall i \in \{1, \dots, n\},$$

which shows, recalling that $g = \partial^\alpha f$, that $\partial_i \partial^\alpha f$ satisfies the assumptions of the lemma for p (considering 0, $l - k - 1$ and C in place of k, l and M), thus by the induction hypothesis we find

$$\sup_{x \in Q_r^c} |x|^m |\partial^\beta \partial_i \partial^\alpha f(x)| \leqslant C, \quad \forall i \in \{1, \dots, n\},$$

for every multi-index $\beta \in \mathbb{N}^n$ such that $|\beta| \in \{0, \dots, p\}$, where $C = C(M, p)$, hence the result for $p + 1$, by arbitrariness of $i \in \{1, \dots, n\}$ and α such that $|\alpha| = k$. □

Lemma A.2.7. *Let $\psi \in \mathscr{D}(B_{2R}, [0, 1])$ such that $\psi = 1$ in B_R, for some $R \in (0, \frac{1}{2})$, and $F := (1 - \psi) I_{2s}$. Then $(-\Delta)^s F$ is a well defined function and belongs $\mathscr{S}_s$.*

Proof. We proceed in 4 steps.
Step 1. We justify that $(-\Delta)^s F$ is well defined, smooth and that all its derivatives belong to $L^\infty(\mathbb{R}^n)$. Note that $F \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, so $(-\Delta)^s F$ is well defined by Theorem A.1.4. Since $F(x) = I_{2s}$ for every $x \geqslant 2R$, and F is smooth in B_{2R}, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| = k$, by derivation we have

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^{n-2s+k}) |\partial^\alpha F(x)| < +\infty. \tag{A.2.11}$$

In particular for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = 2$ we have

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2-2s}) |\partial^\alpha F(x)| < +\infty,$$

thus $D^2F \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, hence $(-\Delta)^s F \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ by Theorem A.1.4. In fact, using (A.2.11) it is easy to see that we can apply the theorem of derivation under the integral to $(-\Delta)^s F$ and show that $(-\Delta)^s F \in C^\infty(\mathbb{R}^n)$ with

$$\partial^\alpha((-\Delta)^s F) = (-\Delta)^s(\partial^\alpha F). \tag{A.2.12}$$

Proceeding as in (A.1.4) and (A.1.5) for $\partial^\alpha F$ we find

$$\|\partial^\alpha((-\Delta)^s F)\|_{L^\infty(\mathbb{R}^n)} \leqslant C\left(\|\partial^\alpha F\|_{L^\infty(\mathbb{R}^n)} + \max_{|\beta|=|\alpha|+2} \|\partial^\beta F\|_{L^\infty(\mathbb{R}^n)}\right) \tag{A.2.13}$$

for every multi-index $\alpha \in \mathbb{N}^n$. Hence there remains only to prove

$$\sup_{|x|\geqslant r_0} |x|^{-(n+2s)}|\partial^\alpha(-\Delta)^s F(x)| < +\infty$$

for some positive r_0 and every $\alpha \in \mathbb{N}^n$ to conclude that $(-\Delta)^s F \in \mathscr{S}_s$.
Step 2. We show that for every multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k \geqslant 2$, we have

$$\sup_{|x|\geqslant 2} |x|^{n+2s}|\partial^\alpha(-\Delta)^s F(x)| < +\infty. \tag{A.2.14}$$

Let α be a multi-index such that $|\alpha| = k \geqslant 2$. Derivating under the integral, for $x \in B_2^c$, we have

$$\begin{aligned}
|\partial^\alpha(-\Delta)^s F(x)| &\leqslant \int_{\mathbb{R}^n} \frac{|2\partial^\alpha F(x) - \partial^\alpha F(x+h) - \partial^\alpha F(x-h)|}{|h|^{n+2s}}\,\mathrm{d}h \\
&\leqslant \int_{\{|h|\leqslant \frac{|x|}{2}\}} \frac{|2\partial^\alpha F(x) - \partial^\alpha F(x+h) - \partial^\alpha F(x-h)|}{|h|^{n+2s}}\,\mathrm{d}h \\
&\quad + \int_{\{|h|> \frac{|x|}{2}\}} \frac{|2\partial^\alpha F(x) - \partial^\alpha F(x+h) - \partial^\alpha F(x-h)|}{|h|^{n+2s}}\,\mathrm{d}h.
\end{aligned} \tag{A.2.15}$$

Integrating on lines, using (A.2.11) and the fact that $|x+th| \geqslant C|x|$ for some constant C whenever $|x| \geqslant 2$, $|h| \leqslant \frac{|x|}{2}$ and $t \in (0,1)$, it follows

$$\begin{aligned}
&\int_{\{|h|\leqslant \frac{|x|}{2}\}} \frac{|2\partial^\alpha F(x) - \partial^\alpha F(x+h) - \partial^\alpha F(x-h)|}{|h|^{n+2s}}\,\mathrm{d}h \\
&\qquad \leqslant \int_{\{|h|\leqslant \frac{|x|}{2}\}} \left(\max_{|\beta|=k+2} \int_0^1 |\partial^\beta F(x+th)|\,\mathrm{d}t\right) |h|^{-(n+2s-2)}\,\mathrm{d}h \\
&\qquad \leqslant C|x|^{2-2s}\left(\max_{|\beta|=k+2} \sup_{|x|\geqslant 1} |x|^{n-2s+k+2}|\partial^\alpha F(x)|\right)|x|^{-(n-2s+k+2)} \\
&\qquad \leqslant C|x|^{-(n+k)}.
\end{aligned} \tag{A.2.16}$$

Using (A.2.11) again we find as well

$$\begin{aligned}
\int_{\{|h|> \frac{|x|}{2}\}} \frac{|\partial^\alpha F(x)|}{|h|^{n+2s}}\,\mathrm{d}h &\leqslant C|\partial^\alpha F(x)| \int_{B^c_{\frac{|x|}{2}}} \frac{1}{|h|^{n+2s}}\,\mathrm{d}h \\
&\leqslant C|\partial^\alpha F(x)||x|^{-2s} \\
&\leqslant C\Big(\sup_{|x|\geqslant 2}(1+|x|^{n+k-2s})|\partial^\alpha F(x)|\Big)\frac{|x|^{-2s}}{1+|x|^{n+k-2s}} \quad \leqslant C|x|^{-(n+k)}.
\end{aligned} \tag{A.2.17}$$

Since $k \geqslant 2$, by (A.2.11), $\partial^\alpha F \in L^1(\mathbb{R}^n)$, thus

$$\int_{\{|h|>\frac{|x|}{2}\}} \frac{|\partial^\alpha F(x+h)| + |\partial^\alpha F(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \leqslant C|x|^{-n+2s} \|\partial^\alpha F\|_{L^1(\mathbb{R}^n)} \leqslant C|x|^{-n+2s}. \tag{A.2.18}$$

Hence combining (A.2.15) to (A.2.18) and noticing that $n+k > n+2s$ for $k \geqslant 2$, we find

$$|\partial^\alpha (-\Delta)^s F(x)| \leqslant C|x|^{-(n+2s)}, \quad \forall x \in B_2^{\mathrm{c}}, \tag{A.2.19}$$

which gives (A.2.14).

Step 3. We show that

$$\sup_{|x|\geqslant 2} |x|^{n+2s} |(-\Delta)^s F(x)| < +\infty. \tag{A.2.20}$$

Let $\eta \in \mathscr{D}(B_1; [0,+\infty))$ such that

$$\int_{\mathbb{R}^n} \eta(x) \, \mathrm{d}x = 1.$$

For any $\varepsilon > 0$ we define

$$\eta_\varepsilon(x) := \varepsilon^{-n} \eta(\varepsilon^{-1} x),$$

so that $(\eta_\varepsilon)_{\varepsilon>0}$ is an approximation of the dirac δ_0. Since $(-\Delta)^s F$ is continuous, we have

$$(-\Delta)^s F * \eta_\varepsilon(x) \xrightarrow{\varepsilon \to 0} (-\Delta)^s F(x), \quad \forall x \in \mathbb{R}^n. \tag{A.2.21}$$

Since $(-\Delta)^s F$ and η_ε belong to $L^1(\mathbb{R}^n)$, by Theorem A.1.4, we have

$$\mathcal{F}((-\Delta)^s F * \eta_\varepsilon) = \mathcal{F}((-\Delta)^s F)\mathcal{F}(\eta_\varepsilon) = (2\pi|\xi|)^{2s} \mathcal{F}(F)\mathcal{F}(\eta_\varepsilon) = \mathcal{F}(F * (-\Delta)^s(\eta_\varepsilon))$$

thus $(-\Delta)^s F * \eta_\varepsilon = F * ((-\Delta)^s \eta_\varepsilon)$. Let us define $\psi_\varepsilon := (-\Delta)^s \eta_\varepsilon$, and from now on assume $\varepsilon < \frac{1}{2}$. Let us write

$$\begin{aligned} (-\Delta)^s F * \eta_\varepsilon(x) &= F * \psi_\varepsilon(x) \\ &= \int_{B_1} F(y) \psi_\varepsilon(x-y) \, \mathrm{d}y + \int_{B_1^{\mathrm{c}}} F(y) \psi_\varepsilon(x-y) \, \mathrm{d}y. \end{aligned} \tag{A.2.22}$$

Note that $\psi_\varepsilon \in \mathscr{S}_s$ by Proposition A.1.9, since $\eta_\varepsilon \in \mathscr{S}$, and observe that for any $\varepsilon < \frac{1}{2}$ and $x \in B_1^{\mathrm{c}}$, we have

$$\begin{aligned} |\psi_\varepsilon(x)| &\leqslant C \int_{\{|h|\leqslant \frac{|x|}{2}\}} \frac{|2\eta_\varepsilon(x) - \eta_\varepsilon(x+h) - \eta_\varepsilon(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \\ &\quad + C \int_{\{|h|>\frac{|x|}{2}\}} \frac{|2\eta_\varepsilon(x) - \eta_\varepsilon(x+h) - \eta_\varepsilon(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h \\ &= C \int_{\{|h|>\frac{|x|}{2}\}} \frac{|\eta_\varepsilon(x+h) - \eta_\varepsilon(x-h)|}{|h|^{n+2s}} \, \mathrm{d}h, \end{aligned} \tag{A.2.23}$$

since η_ε is compactly supported in $B_{\frac{1}{2}}$ and x, $x+h$ and $x-h$ belong to $B_{\frac{1}{2}}^{\mathrm{c}}$ whenever $x \in B_1^{\mathrm{c}}$ and $|h| \leqslant \frac{|x|}{2}$. Thus from (A.2.23) it follows

$$|\psi_\varepsilon(x)| \leqslant C|x|^{-(n+2s)} \|\eta_\varepsilon\|_{L^1(\mathbb{R}^n)} = C|x|^{-(n+2s)}, \quad \forall x \in B_1^{\mathrm{c}}, \tag{A.2.24}$$

for some C independent of ε and x. For $x \in B_2^c$ and $y \in B_1$, we have $|x-y| \geqslant 1$, thus using (A.2.24) and the fact that $F \in L^\infty(\mathbb{R}^n)$, we infer

$$\begin{aligned}\int_{B_1} |F(y)\psi_\varepsilon(x-y)|\,\mathrm{d}y &\leqslant |x|^{-(n+2s)}\|F\|_{L^\infty(\mathbb{R}^n)}\Big(\sup_{|x|\geqslant 1}|x|^{-(n+2s)}|\psi_\varepsilon(x)|\Big)\\ &\leqslant C|x|^{-(n+2s)},\end{aligned} \tag{A.2.25}$$

where C does not depend on ε nor x. As for the last term on the right-hand side of (A.2.22), recall that $F = I_{2s}$ in $B_{2R}^c \supseteq B_1^c$ since $R \in (0, \frac{1}{2})$, thus

$$\begin{aligned}\int_{B_1^c} F(y)\psi_\varepsilon(x-y)\,\mathrm{d}y &= \int_{B_1^c} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y - \int_{B_1} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y\\ &= (I_{2s} * \psi_\varepsilon)(x) - \int_{B_1} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y.\end{aligned} \tag{A.2.26}$$

Since $\eta_\varepsilon \in \mathscr{S}$, by Proposition A.1.12 we have

$$I_{2s} * \psi_\varepsilon = (-\Delta)^{-s}((-\Delta)^s \eta_\varepsilon) = \eta_\varepsilon,$$

thus for any $x \in B_2^c$, (A.2.26) becomes

$$\begin{aligned}\int_{B_1^c} F(y)\psi_\varepsilon(x-y)\,\mathrm{d}y &= \eta_\varepsilon(x) - \int_{B_1} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y\\ &= -\int_{B_1} I_{2s}(y)\psi_\varepsilon(x-y)\,\mathrm{d}y,\end{aligned} \tag{A.2.27}$$

because η_ε is compactly supported in $B_{\frac{1}{2}}$. Then since $|x-y| \geqslant \frac{|x|}{2} \geqslant 1$ whenever $|x| \geqslant 2$ and $|y| \leqslant 1$, using (A.2.24) and (A.2.27) we estimate

$$\begin{aligned}\int_{B_1^c} |F(y)\psi_\varepsilon(x-y)|\,\mathrm{d}y &\leqslant \int_{B_1} |I_{2s}(y)\psi_\varepsilon(x-y)|\,\mathrm{d}y\\ &\leqslant C\Big(\sup_{|y|\geqslant 1}|y|^{-(n+2s)}|\psi_\varepsilon(y)|\Big)\left(\int_{B_1}|I_{2s}(y)|\,\mathrm{d}y\right)|x|^{-(n+2s)}\\ &\leqslant C|x|^{-(n+2s)}.\end{aligned} \tag{A.2.28}$$

Combining (A.2.22), (A.2.25) and (A.2.28), we find

$$|(-\Delta)^s F * \eta_\varepsilon(x)| \leqslant C|x|^{-(n+2s)}, \quad \forall x \in B_2^c,$$

thus letting ε go to 0, with (A.2.21) we reach

$$|(-\Delta)^s F(x)| \leqslant C|x|^{-(n+2s)}, \quad \forall x \in B_2^c,$$

which gives (A.2.20) and concludes this step.

Step 4. In view of Steps 1 to 3, there remains to show

$$\sup_{|x|\geqslant r_0} |x|^{-(n+2s)}|\partial_i(-\Delta)^s F(x)| < +\infty \tag{A.2.29}$$

for all $i \in \{0, \dots, n\}$, for some positive r_0. We use the interpolation lemma (Lemma A.2.6) with $r = 2$ (since $B_2 \subseteq Q_2$), $k = 0$ and $l = 2$ to conclude directly that (A.2.29) holds in any case for any arbitrary r_0 such that $Q_2 \subseteq B_{r_0}$, which concludes the proof in view of (A.2.13). □

Corollary A.2.8. *Let $\varphi \in \mathscr{S}$, and F be as in Lemma A.2.7. Then the convolution $F * ((-\Delta)^s\varphi)$ is well defined and belongs to $\mathscr{S}_s$.*

Proof. By Lemma A.2.7, $(-\Delta)^s F \in \mathscr{S}_s$, thus $((-\Delta)^s F) * \varphi$ is a well-defined function and belongs to $\mathscr{S}_s(\mathbb{R}^n)$ by Lemma A.2.3. In addition, since $\varphi \in \mathscr{S}$ and $F \in L^1(\mathbb{R}^n)$, using Theorem A.1.4 we have

$$\begin{aligned}\mathcal{F}(F * ((-\Delta)^s\varphi)) = \mathcal{F}(F)\mathcal{F}((-\Delta)^s\varphi) &= (2\pi|\xi|)^{2s}\mathcal{F}(F)\mathcal{F}(\varphi)\\ &= \mathcal{F}((-\Delta)^s F)\mathcal{F}(\varphi) = \mathcal{F}((-\Delta)^s F * \varphi),\end{aligned}$$

hence $F * (-\Delta)^s\varphi = (-\Delta)^s F * \varphi \in \mathscr{S}_s$ by Lemma A.2.7. □

We are now ready to prove that solutions to $(-\Delta)^s T = f$ in $\mathscr{D}(\Omega)$ morally gain $2s$ "fractional derivatives", which immediately gives regularity of s-harmonic distributions.

Theorem A.2.9. *Let $s \in (0, \min(1, \frac{n}{2}))$, $T \in \mathscr{S}_s'$ and $f \in C^{k,\alpha}(\Omega)$. If $(-\Delta)^s T = f$ in $\mathscr{D}'(\Omega)$, i.e.,*

$$\langle(-\Delta)^s T, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x, \quad \forall\varphi \in \mathscr{D}(\Omega),$$

then for every open bounded open set ω such that $\overline{\omega} \subseteq \Omega$, defining $l = \lfloor\alpha + 2s\rfloor \in \{0, 1, 2\}$ and $\beta = \alpha + 2s - \lfloor\alpha + 2s\rfloor \in [0, 1)$, we have

(i) $T \in C^{k+l,\beta}(\omega)$ if $\beta > 0$;

(ii) $T \in C^{k+l-1,\delta}(\omega)$ for every $\delta \in (0, 1)$ if $\beta = 0$.

Proof. First, let us note that $(-\Delta)^s T \in \mathscr{S}_s' \subseteq \mathscr{D}(\Omega)$ by definition of the distributional s-Laplacian. We prove that for any $B_R(x)$ such that $\overline{B_{2R}}(x) \subseteq \Omega$, $R < \frac{1}{2}$, (i) or (ii) holds for $\omega = B_R(x)$, which gives the result by a standard covering argument. Let $B_R(x)$ such that $\overline{B_{2R}}(x) \subseteq \Omega$, and, up to a translation, let us assume from now on that $x = 0$. Let $\varphi \in \mathscr{D}(B_R)$, and note that $(-\Delta)^s\varphi \in \mathscr{S}_s$ by Proposition A.1.9. Since $\varphi \in \mathscr{D}(B_R) \subseteq \mathscr{S}$, by Proposition A.1.12 we have

$$\varphi = (-\Delta)^{-s}((-\Delta)^s\varphi) = I_{2s} * ((-\Delta)^s\varphi), \tag{A.2.30}$$

and we can thus write

$$\langle T, \varphi\rangle_{(\mathscr{S}_s)',\mathscr{S}_s} = \langle T, I_{2s} * ((-\Delta)^s\varphi)\rangle_{(\mathscr{S}_s)',\mathscr{S}_s}.$$

Now consider $\psi \in \mathscr{D}(B_{2R}, [0, 1])$ such that $\psi = 1$ in B_R. Then we write

$$I_{2s} = \psi I_{2s} + (1 - \psi)I_{2s} =: F_1 + F_2,$$

so that

$$\langle T, \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle T, (F_1 + F_2) * ((-\Delta)^s\varphi)\rangle_{(\mathscr{S}_s)',\mathscr{S}_s}.$$

Now $F_1 * ((-\Delta)^s\varphi)$ is a well defined smooth function, since F_1 is a compactly supported distribution, and $(-\Delta)^s\varphi$ is a smooth function. In fact, since $(-\Delta)^s\varphi \in \mathscr{S}_s$ and F_1 is compactly supported, by Lemma A.2.3, $F_1 * ((-\Delta)^s\varphi)$ belongs to $\mathscr{S}_s$. Thus $\langle T, F_1 * ((-\Delta)^s\varphi)\rangle_{(\mathscr{S}_s)',\mathscr{S}_s}$ is well defined. Regarding F_2, by Corollary A.2.8, $F_2 * ((-\Delta)^s\varphi)$ belongs to $\mathscr{S}_s$, thus $\langle T, F_2 * ((-\Delta)^s\varphi)\rangle_{(\mathscr{S}_s)',\mathscr{S}_s}$ is also well defined. Hence by linearity

$$\langle T, \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle T, F_1 * ((-\Delta)^s\varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s} + \langle T, F_2 * ((-\Delta)^s\varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s}. \tag{A.2.31}$$

Note that $F_1 * \varphi \in \mathscr{D}(\Omega) \subseteq \mathscr{S}_s$, thus $(-\Delta)^s(F_1 * \varphi)$ is a well-defined function, and since $F_1 \in L^1(\mathbb{R}^n)$, we have

$$\mathcal{F}(F_1 * ((-\Delta)^s\varphi)) = \mathcal{F}(F_1)\mathcal{F}((-\Delta)^s\varphi) = (2\pi|\xi|)^{2s}\mathcal{F}(F_1)\mathcal{F}(\varphi) = \mathcal{F}((-\Delta)^s(F_1 * \varphi))$$

hence $(-\Delta)^s(F_1 * \varphi) = F_1 * ((-\Delta)^s\varphi)$. As a consequence

$$\langle T, F_1 * ((-\Delta)^s\varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle T, (-\Delta)^s(F_1 * \varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s}. \tag{A.2.32}$$

Now, since φ is a smooth function compactly supported in B_R, and F_1 is compactly supported in B_R, $F_1 * \varphi \in \mathscr{D}(B_{2R}) \subseteq \mathscr{D}(\Omega)$, thus

$$\langle T, (-\Delta)^s(F_1 * \varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \int_{\mathbb{R}^n} f(x)(F_1 * \varphi)(x)\,\mathrm{d}x \tag{A.2.33}$$

because $(-\Delta)^s T = f$ in $\mathscr{D}'(\Omega)$. Since $F_1 * \varphi \in \mathscr{D}(B_{2R})$, there exists $R_0 < 2R$ such that $F_1 * \varphi \in \mathscr{D}(B_{R_0})$. Let us consider another smooth cutoff function $\psi \in \mathscr{D}(B_{2R})$ such that $\psi = 1$ in B_{R_0}, so that $(F_1 * \varphi)f = (F_1 * \varphi)\psi f$, and let us write

$$\int_{\mathbb{R}^n} f(x)(F_1 * \varphi)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\psi(x)(F_1 * \varphi)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} (F_1 * \widetilde{f\psi})(x)\varphi(x)\,\mathrm{d}x. \tag{A.2.34}$$

Recall that $I_{2s} = F_1 + F_2$, thus (A.2.33) and (A.2.34) give

$$\begin{aligned}\langle T, (-\Delta)^s(F_1 * \varphi)\rangle_{\mathscr{S}_s',\mathscr{S}_s} &= \int_{\mathbb{R}^n} (F_1 * \widetilde{f\psi})(x)\varphi(x)\,\mathrm{d}x \\ &= \langle I_{2s} * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} - \langle F_2 * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}}.\end{aligned} \tag{A.2.35}$$

Since $f \in C^{k,\alpha}(\Omega)$ and $\psi \in \mathscr{D}(B_{R_0})$, $\widetilde{f\psi} \in C^{k,\alpha}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and we can apply Theorem A.2.1 to show that $I_{2s} * \widetilde{f\psi}$ morally gains a "fractional" $2s$ derivative. In addition $F_2 * \widetilde{f\psi}$ is smooth, as a convolution of a smooth function, F_2, with a compactly supported distribution, $\widetilde{f\psi}$. Combining (A.2.31) to (A.2.33) and (A.2.35), we find

$$\langle T, \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle I_{2s} * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} - \langle F_2 * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} + \langle T, F_2 * (-\Delta)^s\varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s}. \tag{A.2.36}$$

By Lemma A.2.7, $(-\Delta)^s F_2$ is a well-defined function in $\mathscr{S}_s$, and by Theorem A.1.4

$$\mathcal{F}(((-\Delta)^s F_2) * \varphi) = (2\pi|\xi|)^{2s}\mathcal{F}(F_2)\mathcal{F}(\varphi) = \mathcal{F}(F_2 * ((-\Delta)^s\varphi)),$$

thus

$$F_2 * ((-\Delta)^s\varphi) = ((-\Delta)^s F_2) * \varphi. \tag{A.2.37}$$

Since $T \in \mathscr{S}_s'$ and $(-\Delta)^s F_2 \in \mathscr{S}_s$, by Lemma A.2.5, $T * ((-\Delta)^s F_2)$ defines a smooth function, and we have

$$\langle T, ((-\Delta)^s F_2) * \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle T * ((-\Delta)^s F_2), \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s},$$

hence with (A.2.36) and (A.2.37) it follows

$$\langle T, \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s} = \langle I_{2s} * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} - \langle F_2 * \widetilde{f\psi}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} + \langle T * ((-\Delta)^s F_2), \varphi\rangle_{\mathscr{S}_s',\mathscr{S}_s},$$

i.e. $T = I_{2s} * \widetilde{f\psi} - F_2 * \widetilde{f\psi} + T * ((-\Delta)^s F_2)$ in B_R. Since $T * ((-\Delta)^s F_2)$ and $F_2 * \widetilde{f\psi}$ are smooth functions on $\mathbb{R}^n$, and $\widetilde{f\psi} \in C^{k,\alpha}(\mathbb{R}^n)$, applying Theorem A.2.1 to $I_{2s} * \widetilde{f\psi}$ shows that T satisfies (i) or (ii) for $\omega = B_R$. □

We have the immediate Corollary when the source term f vanishes.

Corollary A.2.10 (Regularity of s-harmonic distributions)**.** *Let $s \in (0, \min(1, \frac{n}{2}))$ and $T \in \mathscr{S}'_s$ such that $(-\Delta)^s T = 0$ in $\mathscr{D}'(\Omega)$, i.e.,*

$$\langle (-\Delta)^s T, \varphi \rangle_{\mathscr{S}', \mathscr{S}} = 0, \quad \forall \varphi \in \mathscr{D}(\Omega),$$

thet T is a smooth function in Ω.

If instead of using Theorem A.2.1 to end the proof of Theorem A.2.9, we use its counterpart on Sobolev spaces, i.e., Theorem A.2.2, we get the following theorem.

Theorem A.2.11. *Let $s \in (0, \min(1, \frac{n}{2}))$, $T \in \mathscr{S}'_s$ and $f \in W^{k,p}(\Omega)$. If $(-\Delta)^s T = f$ in $\mathscr{D}'(\Omega)$, then we have*

(i) if k and $k + 2s$ are simultaneously integers or nonintegers, then $T \in W^{k+2s,p}_{\text{loc}}(\Omega)$;

(ii) if $k \in \mathbb{N}$, $k + 2s \notin \mathbb{N}$, then $g \in W^{k+2s,p}_{\text{loc}}(\Omega)$ when $p \in [2, +\infty)$, and $T \in W^{l,p}_{\text{loc}}(\Omega)$ for all $l \in (0, k + 2s)$ when $p \in (1, 2)$;

(iii) if $k \notin \mathbb{N}$, $k + 2s \in \mathbb{N}$, then $T \in W^{k+2s,p}_{\text{loc}}(\Omega)$ when $p \in [1, 2]$, and $T \in W^{l,p}_{\text{loc}}(\Omega)$ for all $l \in (0, k + 2s)$ when $p \in (2, +\infty)$.

A.3 Regularity of weakly s-harmonic functions

In this section we give another proof of smoothness for functions u satisfying $(-\Delta)^s u = 0$ in a weak sense, but in a more specific functional setting (the one from Chapters 1 and 2), and we obtain bounds on the L^∞ norm of all the derivatives of u. We recall briefly this functional setting and how it relates to the distributional setting of .

Definition A.3.1. For $s \in (0, 1)$, we define the fractional s-energy in a bounded open subset $\Omega \subseteq \mathbb{R}^n$

$$\mathcal{E}_s(u, \Omega) := \frac{\gamma_{n,s}}{4} \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y + \frac{\gamma_{n,s}}{2} \iint_{\Omega \times (\mathbb{R}^n \setminus \Omega)} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y,$$

and the Hilbert space

$$\widehat{H}^s(\Omega) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^n) \ : \ \mathcal{E}_s(u, \Omega) < +\infty \right\}$$

endowed with the norm

$$\|u\|^2_{\widehat{H}^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + \mathcal{E}_s(u, \Omega).$$

Any map $u \in \widehat{H}^s(\Omega)$ defines a tempered distribution in $\mathscr{S}'_s$, as is shown in the following proposition.

Proposition A.3.2. *For $s \in (0, 1)$, Ω an open subset of $\mathbb{R}^n$, and $u \in \widehat{H}^s(\Omega)$, we have*

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, \mathrm{d}x \leqslant C \|u\|_{\widehat{H}^s(\Omega)},$$

for some C depending only on n, s and Ω. In particular $\widehat{H}^s(\Omega) \subseteq L^1_s(\mathbb{R}^n) \subseteq \mathscr{S}'_s$.

Proof. Let $x_0 \in \Omega$ and $\rho > 0$ such that $B_\rho(x_0) \subseteq \Omega$. Then by [81, Lemma 2.1], we have

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(1+|x-x_0|)^{n+2s}}\,\mathrm{d}x \leqslant C\left(\mathcal{E}_s(u, B_\rho(x_0)) + \|u\|^2_{L^2(B_\rho(x_0))}\right) \leqslant C\|u\|^2_{\widehat{H}^s(\Omega)}.$$

Now for $|x| \geqslant |x_0|$, we have $1+|x-x_0| \leqslant 2(1+|x|)$, thus

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{1+|x|^{n+2s}}\,\mathrm{d}x \leqslant C\|u\|^2_{\widehat{H}^s(\Omega)}.$$

Then by Jensen inequality, since $(1+|x|^{n+2s})^{-1}\,\mathrm{d}x$ is a finite measure, we get

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,\mathrm{d}x \leqslant C\left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{1+|x|^{n+2s}}\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C\|u\|_{\widehat{H}^s(\Omega)},$$

hence the result. □

Remark A.3.3. In Chapter 2, for any $u \in \widehat{H}^s(\Omega)$, we defined $(-\Delta)^s u$ as a linear form on $\widehat{H}^s(\Omega)$ by

$$\begin{aligned}\langle(-\Delta)^s u, \varphi\rangle_{(\widehat{H}^s)', \widehat{H}^s} := \frac{\gamma_{n,s}}{2} \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ + \gamma_{n,s} \iint_{\Omega\times(\mathbb{R}^n\setminus\Omega)} \frac{(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for all $\varphi \in \widehat{H}^s(\Omega)$. It is easy to check that $\mathscr{S} \subseteq \widehat{H}^s(\Omega)$, and that, with our distributional definition of $(-\Delta)^s u \in \mathscr{S}'$ for $u \in \mathscr{S}'_s$, we have

$$\langle(-\Delta)^s u, \varphi\rangle_{(\widehat{H}^s)', \widehat{H}^s} = \langle(-\Delta)^s u, \varphi\rangle_{\mathscr{S}'_s, \mathscr{S}_s},$$

i.e. both definitions coincide on $\mathscr{S}$.

Now we want to give another proof of the fact that any $u \in \widehat{H}^s(D_1)$ which satisfies $(-\Delta)^s u = 0$ weakly in $\left(\widehat{H}^s(\Omega)\right)'$ is smooth, and obtain Caccioppoli type estimates for the fractional Laplacian. There are two possibilites for this: we can either prove both the smoothness and the estimates using the fractional harmonic extension, already-known Hölder regularity results for degenerate elliptics equations and a bootstrap argument, *or* we can take a shortcut (except when $s \in [\frac{1}{2}, 1)$ in dimension 1) and use the smoothness of s-harmonic distributions, which we already know from Section A.2, to apply a shorter bootstrap argument to get only the estimates. We implement the former strategy, which does not rely on the regularity results from the previous section, and includes the case $n = 1$ and $s \in [\frac{1}{2}, 1)$, which were missing from Theorems A.2.9 and A.2.11.

Proposition A.3.4 (Hölder-regularity of s-harmonic functions)**.** *Let $s \in (0,1)$ and $u \in \widehat{H}^s(D_{2R})$ with $R < 1$. If $(-\Delta)^s u = 0$ in $\left(\widehat{H}^s(D_R)\right)'$, then $u \in C^{0,\alpha}(D_{\frac{R}{2}})$ for some $\alpha = \alpha(n,s) \in (0,1)$, and we have*

$$R^{2\alpha}[u]^2_{C^{0,\alpha}(D_{\frac{R}{2}})} \leqslant C_1 R^{2s-n-2}\|u^{\mathrm{e}}\|^2_{L^2(B^+_R, |z|^a\mathrm{d}\mathbf{x})} \tag{A.3.1}$$

for some $C_1 = C_1(n,s)$, where u^{e} is the fractional harmonic extension of u.

Remark A.3.5. If we want to write estimate (A.3.1) without the fractional harmonic extension, proceeding as in the proof of [80, Lemma 2.7], we can show that

$$R^{2s-n-2}\|u^{\mathrm{e}}\|^2_{L^2(B_R,|z|^a\mathrm{d}\mathbf{x})} \leqslant CR^{2s}\int_{\mathbb{R}^n}\frac{|u(x)|^2}{(R^2+|x|^2)^{\frac{n+2s}{2}}}\,\mathrm{d}x$$
$$\leqslant C\left(R^{2s-n}\mathcal{E}_s(u,D_R)+R^{-n}\|u\|^2_{L^2(D_R)}\right),$$

for some $C=C(n,s)$.

Proof. Since $u\in\widehat{H}^s(D_R)$, the fractional harmonic extension u^{e} of u is well defined and belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\cup\partial^0B^+_R,|z|^a\,\mathrm{d}\mathbf{x})\cap L^2_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+},|z|^a\,\mathrm{d}\mathbf{x})$ (see Chapter 2). In the weak sense, it satisfies

$$\begin{cases}\operatorname{div}(z^a\nabla u^{\mathrm{e}})=0 & \text{in } B^+_R\\ u^{\mathrm{e}}=u & \text{on } \partial^0(B^+_R)\simeq D_R\end{cases}.$$

We also know that

$$\int_{\mathbb{R}^{n+1}_+}\nabla u^{\mathrm{e}}\cdot\nabla\Phi\,|z|^a\mathrm{d}\mathbf{x}=\langle(-\Delta)^su,\Phi_{|\mathbb{R}^n}\rangle_{\mathscr{S}',\mathscr{S}},$$

for all Φ smooth and compactly supported in $B^+_R\cup\partial^0B^+_R$. Since $(-\Delta)^su=0$ in $\big(\widehat{H}^s(D_R)\big)'$, we find

$$\int_{\mathbb{R}^{n+1}_+}\nabla u^{\mathrm{e}}\cdot\nabla\Phi\;z^a\,\mathrm{d}\mathbf{x}=0,\quad\forall\Phi\in\mathscr{D}(B^+_R\cup\partial^0B^+_R).$$

Thus defining u^{e} on the ball $B_R\subseteq\mathbb{R}^{n+1}$ by even reflection, we get that $u^{\mathrm{e}}\in H^1(B_R,|z|^a\mathrm{d}\mathbf{x})$ satisfies

$$\operatorname{div}(|z|^a\nabla u^{\mathrm{e}})=0\quad\text{weakly in } H^1(B_R,|z|^a\mathrm{d}\mathbf{x}),$$

and using [40, Theorem 2.3.12] we get that $u^{\mathrm{e}}\in C^{0,\alpha}(B_{\frac{R}{2}})$ for some $\alpha\in(0,1)$ depending only on n and s, with

$$[u^{\mathrm{e}}]_{C^{0,\alpha}(B_{\frac{R}{2}})}\leqslant\frac{C}{R^\alpha}\left(R^{2s-n-2}\int_{B_R}|u^{\mathrm{e}}(\mathbf{x})|^2\,|z|^a\mathrm{d}\mathbf{x}\right)^{\frac12},$$

hence (A.3.1) □

We could probably use a similar strategy to show that any solution to $(-\Delta)^su=f$ morally gains $2s$ fractional derivatives which would include the case $n=1$ and $s\in[\frac12,1)$ missing from Theorems A.2.9 and A.2.11, but we only focus on the simplest case $f=0$.

Theorem A.3.6 (Full regularity for weakly s-harmonic functions). *Let $s\in(0,1)$ and $u\in\widehat{H}^s(D_{2R})$ with $R<1$. If $(-\Delta)^su=0$ in $\big(\widehat{H}^s(D_R)\big)'$, then $u\in C^\infty(D_{\frac{R}{8}})$, and for all $l\in\mathbb{N}$ we have*

$$|R|^{2l}\sum_{|\beta|=l}\|\partial^\beta u\|^2_{L^\infty(D_{\frac{R}{8}})}\leqslant C_lR^{2s-n-2}\|u^{\mathrm{e}}\|^2_{L^2(B^+_R,|z|^a\mathrm{d}\mathbf{x})},$$

where C_l depends only on l, n and s and u^{e} is the fractional harmonic extension of u.

Proof. This is actually a standard bootstrap argument, albeit a bit technical. By Proposition A.3.4, we know that $u^{\mathrm{e}} \in C^{0,\alpha}(B_{\frac{R}{2}})$ for some $\alpha = \alpha(n,s) \in (0,1)$, with the estimate

$$R^{2\alpha}[u^{\mathrm{e}}]^2_{C^{0,\alpha}(B_{\frac{R}{2}})} \leqslant C_1 R^{2s-n-2} \|u^{\mathrm{e}}\|_{L^2(B_r^+,|z|^a\mathrm{d}\mathbf{x})}, \tag{A.3.2}$$

where $C = C(n,s)$.
Step 1. Let us show that $u \in C^{0,\beta}(D_{\frac{R}{4}})$, for some $\beta \in (\frac{1}{2},1)$, with the estimate

$$R^{2\beta}[u^{\mathrm{e}}]^2_{C^{0,\beta}(B_{\frac{R}{4}})} \leqslant CR^{2s-n-2}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}.$$

In fact, we show that there exists some integer k larger than 2 such that, $u^{\mathrm{e}} \in C^{0,\beta}(D_{r_k})$ for some $\beta \in [1/2,1)$, where $r_k := (1/4 + 4^{-k})R$, with the estimate

$$R^{2\beta}[u^{\mathrm{e}}]^2_{C^{0,\beta}(B_{r_k})} \leqslant CR^{2s-n-2}\|u^{\mathrm{e}}\|^2_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})},$$

for some C depending only on n, s. Let us take $k = 2$ to begin with. If $\alpha > 1/2$ then there is nothing to do. If $\alpha = 1/2$, (A.3.2) is obviously also true for e.g. $\alpha = 2/3$, so we continue with this value instead. Thus we assume that $\alpha < 1/2$. For any nonvanishing $h \in B_{\frac{R}{2}}$, consider the map

$$w_h(\mathbf{x}) := \frac{u^{\mathrm{e}}(\mathbf{x}+h) - u^{\mathrm{e}}(\mathbf{x})}{|h|^\alpha},$$

defined on $B_{r_{k-1}}$. Note that $r_k + 8^{-k}R < r_{k-1}$ and that w_h also satisfies

$$\operatorname{div}(|z|^a\nabla w_h) = 0$$

weakly in $H^1(B_{r_k},|z|^a\mathrm{d}\mathbf{x})$, thus by [40, Theorem 2.3.12] we have

$$R^{2\alpha}[w_h]^2_{C^{0,\alpha}(B_{r_k+8^{-k}R})} \leqslant CR^{2s-n-2}\|w_h\|^2_{L^2(B_{r_{k-1}},|z|^a\mathrm{d}\mathbf{x})}, \tag{A.3.3}$$

where C depends only n and s. Observe that

$$\begin{aligned}\|w_h\|^2_{L^2(B_{r_{k-1}},|z|^a\mathrm{d}\mathbf{x})} &= \int_{B_{r_{k-1}}} \frac{|u^{\mathrm{e}}(\mathbf{x}+he_i) - u^{\mathrm{e}}(\mathbf{x})|^2}{|h|^{2\alpha}}|z|^a\,\mathrm{d}\mathbf{x}\\ &\leqslant C[u^{\mathrm{e}}]^2_{C^{0,\alpha}(B_{r_{k-1}})}\int_{B_R}|z|^a\,\mathrm{d}\mathbf{x}\\ &\leqslant CR^{n-2s+2}[u^{\mathrm{e}}]^2_{C^{0,\alpha}(B_{r_{k-1}})}\\ &\leqslant CR^{-2\alpha}\|u^{\mathrm{e}}\|^2_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})},\end{aligned} \tag{A.3.4}$$

for some C depending only on n and s, where we used (A.3.2). Using (A.3.3) with (A.3.4), we find

$$\frac{|w_h(\mathbf{x}) - w_h(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} \leqslant CR^{-2\alpha}R^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}, \tag{A.3.5}$$

for all $\mathbf{x},\mathbf{y} \in B_{r_k+8^{-k}R}$ and any $h \in B_{\frac{R}{2}}$. In particular, for any $\mathbf{x} \in B_{r_k+16^{-k}R}$ and $h \in B_{16^{-k}R}\setminus\{0\}$, we have

$$\frac{|w_h(\mathbf{x}) - w_h(\mathbf{x}-h)|}{|h|^\alpha} \leqslant CR^{-2\alpha}R^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})},$$

thus for all $h \in B_{16^{-k}R} \setminus \{0\}$, by definition of w_h,

$$\sup_{\mathbf{x} \in B_{r_k + 16^{-k}R}} |u^{\mathrm{e}}(\mathbf{x}+h) - 2u^{\mathrm{e}}(\mathbf{x}) + u^{\mathrm{e}}(\mathbf{x}-h)| \leqslant CR^{-2\alpha} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})} |h|^{2\alpha}, \quad \text{(A.3.6)}$$

where $C = C(n,s)$. Let $\varphi \in \mathscr{D}(B_{r_k + 32^{-k}R}, [0,1])$ be a cutoff function such that $\varphi = 1$ in B_{r_k}, $\|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \leqslant CR^{-1}$ and $\|D^2 \varphi\|_{L^\infty(\mathbb{R}^n)} \leqslant CR^{-2}$, where C depends only on n and k. Then defining $v := \varphi u^{\mathrm{e}}$ and

$$D_h^2 v(\mathbf{x}) := v(\mathbf{x}+h) - 2v(\mathbf{x}) + v(\mathbf{x}-h),$$

we claim that

$$\sup_{|h|>0} \frac{\|D_h^2 v\|_{L^\infty(\mathbb{R}^n)}}{|h|^{2\alpha}} \leqslant CR^{-2\alpha} R^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})}, \quad \text{(A.3.7)}$$

where C depends only on k, n and s. To prove (A.3.7), let us first show that

$$\|u^{\mathrm{e}}\|_{L^\infty(B_R)} \leqslant CR^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})}. \quad \text{(A.3.8)}$$

For $\mathbf{x} \in B_R$, integrating in $\mathbf{y}$ over B_R the inequality

$$|u^{\mathrm{e}}(\mathbf{x})| \leqslant |u^{\mathrm{e}}(\mathbf{x}) - u^{\mathrm{e}}(\mathbf{y})| + |u^{\mathrm{e}}(\mathbf{y})|$$

we find

$$|u^{\mathrm{e}}(\mathbf{x})| \leqslant CR^{-n} \int_{B_R} |u^{\mathrm{e}}(\mathbf{x}) - u^{\mathrm{e}}(\mathbf{y})| \, \mathrm{d} + R^{-n} \int_{B_R} |u^{\mathrm{e}}(\mathbf{y})| \, \mathrm{d}$$

thus, using the α-Hölder continuity of u^{e}, (A.3.2) and Cauchy-Schwarz inequality, it follows

$$\begin{aligned}
|u^{\mathrm{e}}(\mathbf{x})| &\leqslant CR^{-n} \int_{B_R} |u^{\mathrm{e}}(\mathbf{x}) - u^{\mathrm{e}}(\mathbf{y})| \, \mathrm{d}\mathbf{y} + R^{-n} \int_{B_R} |u^{\mathrm{e}}(\mathbf{y})| \, \mathrm{d}\mathbf{y} \\
&\leqslant CR^{-n} R^{\frac{2s-n-2-2\alpha}{2}} \int_{B_R} |\mathbf{x}-\mathbf{y}|^\alpha \, \mathrm{d}\mathbf{y} + R^{-n} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})} \left(\int_{B_R} |z|^{-a} \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \\
&\leqslant CR^{-n} R^{\frac{2s-n-2-2\alpha}{2}} R^{n+\alpha+1} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})} + R^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})} \\
&\leqslant CR^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})},
\end{aligned} \quad \text{(A.3.9)}$$

hence (A.3.8). Now since φ is compactly supported in B_R, with (A.3.9) and by definition of v, we infer

$$\|D_h^2 v\|_{L^\infty(\mathbb{R}^n)} \leqslant 4 \|u^{\mathrm{e}}\|_{L^\infty(B_R)} \leqslant CR^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})},$$

so that, for any h such that $|h| \geqslant \frac{R}{32^k}$, we have

$$\frac{\|D_h^2 v\|_{L^\infty(\mathbb{R}^n)}}{|h|^{2\alpha}} \leqslant CR^{\frac{2s-n-4\alpha}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})} \leqslant CR^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R, |z|^a \mathrm{d}\mathbf{x})}, \quad \text{(A.3.10)}$$

where $C = C(n,s,k)$, since we assumed $\alpha \in (0, \frac{1}{2})$. Hence to establish (A.3.7) there remains only look at h such that $|h| < \frac{R}{32^k}$. Now for such an h, $D_h^2 v$ is compactly supported in $B_{r_k + 16^{-k}R}$, so we need only look at $\mathbf{x} \in B_{r_k + 16^{-k}R}$. By Taylor's theorem and our choice of φ, we have

$$|\varphi(\mathbf{x}+h) - \varphi(\mathbf{x}) - \nabla \varphi(\mathbf{x}) \cdot h| \leqslant C|h|^2 R^{-2}, \quad \forall \mathbf{x} \in \mathbb{R}^{n+1},$$

and the same holds for $-h$, thus for any $\mathbf{x} \in B_{r_k+16^{-k}R}$ and $|h| \geqslant \frac{R}{32^k}$, we have

$$|D_h^2\, v(\mathbf{x})| \leqslant |D_h^2\, u^{\mathrm{e}}(\mathbf{x})| + |h||\nabla\varphi(\mathbf{x})||u^{\mathrm{e}}(\mathbf{x}+h) - u^{\mathrm{e}}(\mathbf{x}-h)| \\ + C|h|^2R^{-2}\left(|u^{\mathrm{e}}(\mathbf{x}+h)| + |u^{\mathrm{e}}(\mathbf{x}-h)|\right). \tag{A.3.11}$$

Using the fact that $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} \leqslant CR^{-1}$, (A.3.8) and (A.3.6), from (A.3.11) it follows

$$|D_h^2\, v(\mathbf{x})| \leqslant C\left(R^{-2\alpha}R^{\frac{2s-n-2}{2}}|h|^{2\alpha} + R^{\frac{2s-n-2}{2}}|h|\right)\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})} \\ \leqslant CR^{-2\alpha}R^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}|h|^{2\alpha} \tag{A.3.12}$$

for all $\mathbf{x} \in B_{r_k+16^{-k}R}$, $|h| < \frac{R}{32^k}$, where we also used the facts that $|h|^2R^{-2} \leqslant C|h|R^{-1}$ and $\alpha \in (0,\frac{1}{2})$, for some C depending only on n, s and k. Claim (A.3.7) follows by combining the cases (A.3.10) and (A.3.11). From (A.3.7), since $\beta := 2\alpha < 1$, by [109, Chapter V.4, (48), Propositions 8 and 9] we get that $v \in C^{0,\beta}(\mathbb{R}^n)$ for $\beta := 2\alpha$, with the estimate

$$[v]_{C^{0,\beta}(\mathbb{R}^n)} \leqslant CR^{-2\alpha}R^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})},$$

where C depends only on n, s and k. In particular, since $v = u^{\mathrm{e}}$ in B_{r_k}, $u^{\mathrm{e}} \in C^{0,\beta}(B_{r_k})$ with

$$R^{\beta}[u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_k})} = R^{2\alpha}[u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_k})} \leqslant CR^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}.$$

There are three possible cases:

- $1/2 < \beta < 1$. We are done.
- $0 < \beta < 1/2$. Now u^{e} is $C^{0,\beta}$ in B_{r_k}, with the estimate

$$R^{\beta}[u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_k})} \leqslant CR^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}.$$

  We then reapply Step 1 to get that u^{e} is $C^{0,2\beta}$ in $B_{r_{k+1}}$, and so on. Since the Hölder exponent is doubled each time, after a finite number of steps we will find that u is $C^{0,\beta}$ for some $\beta > 1/2$ with the estimate

$$R^{\beta}[u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_k})} \leqslant CR^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})},$$

  where C depends only on n, s and the final k. Note that the final k depends only on n, s and the number of times Step 1 is repeated, which depends on how small α is, which itself depends only on n and s, so C depends only on n and s.
- $\beta = 1/2$. In particular, u^{e} is Hölder continuous with exponent $\beta' = \frac{3}{8} < \frac{1}{2}$ and the right estimate. We treat it like the second case, taking $\beta = \frac{3}{8}$, only this time reapplying Step 1 yields $u^{\mathrm{e}} \in C^{0,3/4}(B_{r_{k+1}})$, and we are also done.

Step 2. Take $\beta \in (1/2,1)$ given by Step 1. We show that for all multi-indices $\gamma = (\gamma_1,\ldots,\gamma_n,0)$ (note that there is no derivative in z), $\partial^\gamma u^{\mathrm{e}}$ is well defined and belongs to $C^{0,\beta}(B_{1/8})$, with the estimate

$$R^{|\gamma|+\beta}[\partial^\gamma u^{\mathrm{e}}]_{C^{0,\beta}(B_{1/8})} \leqslant CR^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})} \tag{A.3.13}$$

where C depends only on N, s and $|\gamma|$. Let us proceed by induction. For $l \in \mathbb{N}$, let $r_k := R/8+8^{-(k+1)}R \leqslant \frac{R}{2}$, and assume that $\partial^\gamma u^{\mathrm{e}}$ is well defined and belongs to $C^{0,\beta}(B_{r_k})$, with

$$R^{|\gamma|+\beta}[\partial^\gamma u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_k})} \leqslant C_lR^{\frac{2s-n-2}{2}}\|u^{\mathrm{e}}\|_{L^2(B_R^+,|z|^a\mathrm{d}\mathbf{x})}, \tag{A.3.14}$$

for every multi-index $\gamma = (\gamma_1, \dots, \gamma_n, 0)$ such that $|\gamma| \leqslant l$, which has already been proven for $l = 0$ in Step 1. Let $\gamma = (\gamma_1, \dots, \gamma_n, 0)$ such that $|\gamma| = l$, and $i \in \{1, \dots, n\}$. For any nonvanishing $h \in B_{\frac{R}{2}} \setminus \{0\}$, consider the map

$$w_h(\mathbf{x}) := \frac{\partial^\gamma u^{\mathrm{e}}(\mathbf{x} + he_i) - \partial^\gamma u^{\mathrm{e}}(\mathbf{x})}{|h|^\beta},$$

defined on B_{r_k}. Since there is no derivative in z and $\partial^\gamma u^{\mathrm{e}} \in C^0(B_{r_k})$, w_h satisfies

$$\operatorname{div}(|z|^a \nabla w_h) = 0$$

weakly in $H^1(B_{r_k}, |z|^a \mathrm{d}\mathbf{x})$, thus proceeding as in Step 1 and using the induction assumption (A.3.14), we find

$$\frac{|w_h(\mathbf{x}) - w_h(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\beta} \leqslant C R^{-l-2\beta} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})}, \tag{A.3.15}$$

for all $x, y \in B_{r_{k+1} + 8^{-(k+2)} R}$ and any $h \in B_{\frac{R}{2}}$, where C depends only on n, s and k. Still proceeding as in Step 1, using a cutoff function $\varphi \in \mathscr{D}(B_{r_{k+1} + 16^{-(k+2)}})$ such that $\varphi = 1$ in $B_{r_{k+1} + 32^{-(k+2)}}$, and letting $v := \varphi u^{\mathrm{e}}$, we get

$$\sup_{|h|>0} \frac{\|D_h^2 v\|_{L^\infty(\mathbb{R}^n)}}{|h|^{2\beta}} \leqslant C R^{-l-2\beta} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})}.$$

Now since $\frac{1}{2} < \beta < 1$, by [109, Chapter V.4, (48), Propositions 8 and 9] this implies that v is actually differentiable everywhere in the direction e_i, and so is u^{e} in the ball $B_{r_{k+1}}$. We can then let h go to zero in (A.3.15) to get

$$\frac{|\partial_i \partial^\gamma u^{\mathrm{e}}(\mathbf{x}) - \partial_i \partial^\gamma u^{\mathrm{e}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\beta} \leqslant C R^{-l-2\beta} R^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})} \qquad \forall \mathbf{x}, \mathbf{y} \in B_{r_{k+1}}, \mathbf{x} \neq \mathbf{y},$$

meaning that $\partial_i \partial^\gamma u^{\mathrm{e}} \in C^{0,\beta}(B_{r_{k+1}})$ with the estimate

$$\begin{aligned} R^{1+l+\beta} \|\partial_i \partial^\gamma u^{\mathrm{e}}\|_{C^{0,\beta}(B_{r_{k+1}})} &\leqslant D_{l+1} R^{1-\beta} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})} \\ &\leqslant D_{l+1} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})}, \end{aligned}$$

where D_{l+1} depends only on k, n and s. In view of the arbitrariness of i, taking C_{l+1} the maximum of the D_{l+1} over all possible multi-indices $\gamma = (\gamma_1, \dots, \gamma_n, 0)$ such that $|\gamma| = l$ and all $i \in \{1, \dots, n\}$, we get that $\partial^\gamma u^{\mathrm{e}} \in C^{0,\beta}(B_{r_{k+1}})$ for all multi-indices γ such that $|\gamma| = l + 1$, with the estimate

$$R^{l+1+\beta} [\partial^\gamma u^{\mathrm{e}}]_{C^{0,\beta}(B_{r_{k+1}})} \leqslant C_{l+1} R^{\frac{2s-n-2}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})},$$

We have thus proven (A.3.14) for all $n \in \mathbb{N}$, and in particular (A.3.13).
Now it is easy to conclude. Proceeding as (A.3.9) to prove (A.3.8), with $\partial^\gamma u^{\mathrm{e}}$ instead of u^{e}, and using the β-Hölder continuity of $\partial^\gamma u^{\mathrm{e}}$ and estimate (A.3.13), we find as well

$$R^{|\gamma|} \|\partial^\gamma u\|_{L^\infty(D_{\frac{R}{8}})} \leqslant C R^{\frac{2s-n}{2}} \|u^{\mathrm{e}}\|_{L^2(B_R^+, |z|^a \mathrm{d}\mathbf{x})}, \tag{A.3.16}$$

where C depends only on $|\gamma|$, n and s. □

Using the regularity results already obtained in Section A.2 in the distributional setting, we can in fact drop the assumption that $u \in \widehat{H}^s(\Omega)$ and assume only that u is square integrable w.r.t. to the measure $(1+|x|^{n+2s})^{-1}\,\mathrm{d}x$, and still obtain L^∞ estimates of the derivatives.

Corollary A.3.7. *Let $s \in (0, \min(1, \frac{n}{2}))$, and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(1+|x|^2)^{\frac{n+2s}{2}}}\,\mathrm{d}x < +\infty.$$

If $(-\Delta)^s u = 0$ in $\mathscr{D}'(D_R)$ (where $(-\Delta)^s u$ is defined as a tempered distribution in Section A.2), then $u \in C^\infty(D_{\frac{R}{16}})$ and we have

$$|R|^{2l} \sum_{|\gamma|=l} \|\partial^\gamma u\|^2_{L^\infty(D_{\frac{R}{16}})} \leqslant C_l R^{2s} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{(R^2+|x|^2)^{\frac{n+2s}{2}}}\,\mathrm{d}x$$

where C_l depends only on l, n and s.

Proof. Note that u belongs to $L^1_s(\mathbb{R}^n)$ by Cauchy-Schwarz inequality, thus its distributional s-laplacian is well defined. Then by the regularity results of the previous section, we automatically get that u is smooth in D_R. With this we can now show that u is in $\widehat{H}^s(D_{\frac{R}{2}})$. Indeed, splitting $\mathcal{E}_s(u, D_{\frac{R}{2}})$ into two parts and using the smoothness of u in D_R, we have

$$\begin{aligned}
\mathcal{E}_s(u, D_{\frac{R}{2}}) &= \frac{\gamma_{n,s}}{4} \iint_{D_{\frac{R}{2}} \times D_{\frac{R}{2}}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{\gamma_{n,s}}{2} \iint_{D_{\frac{R}{2}} \times D^c_{\frac{R}{2}}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{\gamma_{n,s}}{2} \iint_{D_{\frac{3R}{4}} \times D_{\frac{3R}{4}}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{\gamma_{n,s}}{2} \iint_{D_{\frac{R}{2}} \times D^c_{\frac{3R}{4}}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C\|\nabla u\|^2_{L^\infty(D_{\frac{3R}{4}})} \iint_{D_{\frac{3R}{4}} \times D_{\frac{3R}{4}}} \frac{1}{|x-y|^{n+2s-2}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \gamma_{n,s} \iint_{D_{\frac{R}{2}} \times D^c_{\frac{3R}{4}}} \frac{|u(x)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y + \gamma_{n,s} \iint_{D_{\frac{R}{2}} \times D^c_{\frac{3R}{4}}} \frac{|u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{A.3.17}$$

Since $2s-2 < n$,

$$\iint_{D_{\frac{R}{2}} \times D_{\frac{R}{2}}} \frac{1}{|x-y|^{n+2s-2}}\,\mathrm{d}x\,\mathrm{d}y < +\infty. \tag{A.3.18}$$

and by Fubini's theorem and a change of variables, we have,

$$\iint_{D_{\frac{R}{2}} \times D^c_{\frac{3R}{4}}} \frac{|u(x)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y = \|u\|^2_{L^2(D_{\frac{R}{2}})} \int_{D^c_{\frac{R}{4}}} \frac{1}{|y|^{n+2s}}\,\mathrm{d}y < +\infty. \tag{A.3.19}$$

As for the last term on the right-hand side of (A.3.17), using that $|x-y| \geqslant C_R(1+|y|)$, for every $x \in D_{\frac{3R}{4}}$ and $y \in D^c_{\frac{3R}{4}}$, for some C_R depending only on R, we have

$$\iint_{D_{\frac{R}{2}} \times D^c_{\frac{3R}{4}}} \frac{|u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \leqslant |D_{\frac{R}{2}}| C_R \int_{D^c_{\frac{3R}{4}}} \frac{|u(y)|^2}{(1+|y|)^{n+2s}}\,\mathrm{d}y < +\infty, \tag{A.3.20}$$

thus combining (A.3.17) to (A.3.20) we see that $\mathcal{E}_s(u, D_{\frac{R}{2}})$ is finite, hence we can use Theorem A.3.6 and Remark A.3.5 to conclude. ☐

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