

$$\delta : Q \times 2^Q \rightarrow Q$$

*PhD thesis in Theoretical Computer Science*

# Distributed Automata and Logic

Fabian Reiter

Supervised by Olivier Carton.

Defended on 12 December 2017 in front of the following jury:

Olivier Carton (Prof.)	<i>supervisor</i>	IRIF, Université Paris Diderot
Bruno Courcelle (Prof.)	<i>president</i>	LaBRI, Université de Bordeaux
Pierre Fraigniaud (DR CNRS)	<i>examiner</i>	IRIF, Université Paris Diderot
Nicolas Ollinger (Prof.)	<i>examiner</i>	LIFO, Université d'Orléans
Jukka Suomela (Asst. Prof.)	<i>reviewer</i>	Aalto University (Helsinki)
Christine Tasson (MCF)	<i>examiner</i>	IRIF, Université Paris Diderot
Wolfgang Thomas (Prof.)	<i>reviewer</i>	RWTH Aachen University



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Fabian Reiter

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Contact address: [fabian.reiter@gmail.com](mailto:fabian.reiter@gmail.com)



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# Abstract

Distributed automata are finite-state machines that operate on finite directed graphs. Acting as synchronous distributed algorithms, they use their input graph as a network in which identical processors communicate for a possibly infinite number of synchronous rounds. For the *local* variant of those automata, where the number of rounds is bounded by a constant, Hella et al. (2012, 2015) have established a logical characterization in terms of basic modal logic. In this thesis, we provide similar logical characterizations for two more expressive classes of distributed automata.

The first class extends local automata with a global acceptance condition and the ability to alternate between nondeterministic and parallel computations. We show that it is equivalent to monadic second-order logic on graphs. By restricting transitions to be nondeterministic or deterministic, we also obtain two strictly weaker variants for which the emptiness problem is decidable.

Our second class transfers the standard notion of asynchronous algorithm to the setting of nonlocal distributed automata. The resulting machines are shown to be equivalent to a small fragment of least fixpoint logic, and more specifically, to a restricted variant of the modal  $\mu$ -calculus that allows least fixpoints but forbids greatest fixpoints. Exploiting the connection with logic, we additionally prove that the expressive power of those asynchronous automata is independent of whether or not messages can be lost.

We then investigate the decidability of the emptiness problem for several classes of nonlocal automata. We show that the problem is undecidable in general, by simulating a Turing machine with a distributed automaton that exchanges the roles of space and time. On the other hand, the problem is found to be decidable in LOGSPACE for a class of forgetful automata, where the nodes see the messages received from their neighbors but cannot remember their own state.

As a minor contribution, we also give new proofs of the strictness of several set quantifier alternation hierarchies that are based on modal logic.

**Keywords.** Automata, Distributed algorithms, Modal logic, Monadic second-order logic, Graphs.





# Résumé

Les automates distribués sont des machines à états finis qui opèrent sur des graphes orientés finis. Fonctionnant en tant qu'algorithmes distribués synchrones, ils utilisent leur graphe d'entrée comme un réseau dans lequel des processeurs identiques communiquent entre eux pendant un certain nombre (éventuellement infini) de rondes synchrones. Pour la variante *locale* de ces automates, où le nombre de rondes est borné par une constante, Hella et al. (2012, 2015) ont établi une caractérisation logique par des formules de la logique modale de base. Dans le cadre de cette thèse, nous présentons des caractérisations logiques similaires pour deux classes d'automates distribués plus expressives.

La première classe étend les automates locaux avec une condition d'acceptation globale et la capacité d'alterner entre des modes de calcul non-déterministe et parallèle. Nous montrons qu'elle est équivalente à la logique monadique du second ordre sur les graphes. En nous restreignant à des transitions non-déterministes ou déterministes, nous obtenons également deux variantes d'automates strictement plus faibles pour lesquelles le problème du vide est décidable.

Notre seconde classe adapte la notion standard d'algorithme asynchrone au cadre des automates distribués non-locaux. Les machines résultantes sont prouvées équivalentes à un petit fragment de la logique de point fixe, et plus précisément, à une variante restreinte du  $\mu$ -calcul modal qui autorise les plus petits points fixes mais interdit les plus grands points fixes. Profitant du lien avec la logique, nous montrons aussi que la puissance expressive de ces automates asynchrones est indépendante du fait que des messages puissent être perdus ou non.

Nous étudions ensuite la décidabilité du problème du vide pour plusieurs classes d'automates non-locaux. Nous montrons que le problème est indécidable en général, en simulant une machine de Turing par un automate distribué qui échange les rôles de l'espace et du temps. En revanche, le problème s'avère décidable en LOGSPACE pour une classe d'automates oublieux, où les nœuds voient les messages reçus de leurs voisins, mais ne se souviennent pas de leur propre état.

Finalement, à titre de contribution mineure, nous donnons également de nouvelles preuves de séparation pour plusieurs hiérarchies d'alternance de quantificateurs basées sur la logique modale.

**Mots-clés.** Automates, Algorithmes distribués, Logique modale, Logique monadique du second ordre, Graphes.



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Jukka Suomela and Wolfgang Thomas did me the great honor of reviewing a preliminary version of the manuscript. They gave detailed and extremely flattering feedback, and both made an important remark that led to a major improvement of this document: I had failed to include a discussion of perspectives for future research. This shortcoming has now been addressed by the addition of [Chapter 7](#). Furthermore, Wolfgang compiled a very helpful collection of suggestions, which I have tried to incorporate into the present version.

Bruno Courcelle graciously gave his time to read my master's thesis in 2014, although he had never heard of me before. He then informed Géraud Sénizergues, who most kindly invited me to the final conference of the FREC project in Marseille. This opened the door for me into the French community of automata theory, especially since I met my future doctoral advisor at that conference. Thus, it is indirectly through Bruno that I came to Paris.

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We have just crossed the dividing line, where I stop mentioning people by name. This may seem hasty, or even harsh, but there are two good reasons. First, I value my privacy very much and do not want to share personal details in a document that will be publicly available on the Internet. Second, the larger the circle of people I include, the greater the risk of forgetting someone. A simple rule that helps to avoid both of these issues is to mention only people who stand in some direct professional relation to the thesis. Nevertheless, many more have helped me over the years and had a tremendous influence on my life and work. I therefore sincerely hope not to offend anyone by expressing my gratitude in the following simplistic manner:

☞ *Many thanks to my colleagues, friends, and family!* ☞

# 1

## Introduction

The present thesis aims to contribute to the recently initiated development of a *descriptive complexity* theory for *distributed computing*.

**What does this mean?** Descriptive complexity [Imm99] basically compares the expressive powers of certain classes of *algorithms*, or abstract machines, with those of certain classes of logical *formulas*. The Holy Grail, so to speak, is to establish *equivalences* of the form:

*We identify algorithms with abstract machines.*

“Algorithm class  $\mathcal{A}$  has exactly the same power as formula class  $\Phi$ .”

Probably the most famous result in this area is Fagin’s theorem from 1974 [Fag74], which roughly states that a *graph property* can be recognized by a nondeterministic Turing machine in polynomial time if and only if it can be *defined* by a *formula* of existential second-order logic. The theorem thereby provides a logical characterization of the complexity class NPTIME.

Distributed computing [Lyn96, Pel00], on the other hand, studies networks composed of several interconnected processors that share a common goal. The processors communicate with each other by passing messages along the links of the network in order to collectively solve some computational problem. In many cases, this is a *graph* problem, where the considered problem instance is precisely the *graph* defined by the network itself. All processors run the same algorithm concurrently, and often make no prior assumptions about the size and topology of the *graph*. Typical problems that can be solved by such *distributed algorithms* include *graph coloring*, leader election, and the construction of spanning trees and maximal independent sets.

Now, the ultimate objective that motivates this thesis is to develop an extension of descriptive complexity for the classes of algorithms considered in distributed computing. This means that we seek to establish *equivalences* of the form:

“Distributed algorithm class  $\mathcal{A}$  has the same power as formula class  $\Phi$ .”

However, such a statement can only be substantial if we have a precise definition of class  $\mathcal{A}$ . Therefore, we will formally represent distributed algorithms as abstract machines, instead of the more common, but informal, representations in pseudocode.

*Distributed algorithms are abstract machines that communicate.*

**Why is this interesting?** First and foremost, a descriptive complexity theory for distributed computing would offer the same benefits as its classical counterpart does for sequential computing:

a. If distributed algorithm class  $\mathcal{A}$  turns out to be **equivalent** to **formula** class  $\Phi$ , then this provides strong evidence for the naturalness of both classes. Indeed, the definition of any mathematical device may, by itself, seem arbitrary. *Why should distributed machines communicate precisely that way? Why should logical formulas contain precisely those components?* But if two devices, that appear rather different on the surface, turn out to be descriptions of the exact same thing, then this is unlikely to be pure coincidence.

b. Connecting two seemingly unrelated fields – here, distributed computing and logic – can provide new insights into both fields. Some proofs might be easier to perform if one adopts the point of view of one setting rather than the other. Furthermore, some open questions in one field might already have well-known answers in the other. Especially the field of distributed computing could benefit from this, as it is more than a century younger than formal logic, and therefore has had less time to evolve.

*Formal logic dates back to the mid-19th century, while distributed computing started in the 1970s/1980s.*

Second, distributed computing also brings an interesting new perspective to the field of descriptive complexity itself:

c. Distributed algorithms can be evaluated on the same input as logical **formulas**, without any need for encoding that input. More precisely, the network in which a distributed algorithm is executed may be considered identical to the **structure** on which the truth of a corresponding **formula** is evaluated. This stands in sharp contrast to classical descriptive complexity theory. For instance, in the case of Fagin’s theorem, the input of a Turing machine is a binary string that encodes a finite **graph** in form of an adjacency matrix. Hence, the equivalence of nondeterministic polynomial-time Turing machines and existential second-order logic is actually stated with respect to such an encoding.

## 1.1 Background

Let us now take a step back and put the subject into context. We start with a brief summary of some classical results in automata theory, and then turn to more recent developments in distributed computing.

### 1.1.1 Related work in automata theory

Although the field of descriptive complexity theory really started with Fagin’s theorem in the 1970s, the idea of characterizing abstract machines through logical **formulas** had already appeared earlier in automata theory. In the early 1960s, Büchi [Büc60], Elgot [Elg61] and Trakhtenbrot [Tra61] discovered independently of each other that the regular languages, which are recognized by *finite automata* on **words**, are precisely the languages **definable** in *monadic second-order logic*, or **MSOL** (see, e.g., [Tho97b, Thm 3.1]). The latter is an extension of **first-order logic**, which in addition to allowing quantification over elements of a given **domain** (e.g., positions in a **word**), also allows to quantify over sets of such elements. Along with several other equivalent characterizations, in particular through regular expressions [Kle56], regular grammars [Cho56], and finite monoids [Ner58], the **equivalence** between automata and logic helped to legitimize regularity as a highly natural concept in formal

language theory (cf. [Item a](#), above). Furthermore, it proved that the satisfiability and validity problems for [MSOL](#) on [words](#) are decidable, because so are the corresponding problems for finite automata. In this way, the field of logic directly benefited from the connection with automata theory (cf. [Item b](#)). Nowadays, such connections also play a central role in model checking, where one needs to decide whether a system, represented by an automaton, satisfies a given specification, expressed as a logical formula.

About a decade later, the result was generalized from [words](#) to [labeled trees](#) by Thatcher and Wright [[TW68](#)] and Doner [[Don70](#)] (see, e.g., [[Tho97b](#), Thm 3.6]). The corresponding [tree](#) automata can be seen as a canonical extension of finite automata to [trees](#); as far as [MSOL](#) is concerned, the generalization to [trees](#) is even more straightforward, since both [words](#) and [trees](#) are merely special cases of the relational [structures](#) on which logical [formulas](#) are usually evaluated. The other characterizations of regular languages can also be generalized from [words](#) to [trees](#) in a natural manner, and quite remarkably, they all remain equivalent on [trees](#) (see, e.g., [[CDG<sup>+</sup>08](#)]). Hence, the notion of regularity extends directly to [tree languages](#). Moreover, similar [equivalences](#) have been established for several other generalizations of [words](#), such as nested words (see [[AM09](#)]) and Mazurkiewicz traces (see, e.g., [[DM97](#)]).

In contrast, the situation becomes far more complicated if we expand our field of interest to arbitrary finite [graphs](#) (possibly with [node labels](#) and multiple [edge relations](#)). Although some of the characterizations mentioned above can be generalized to [graphs](#) in a meaningful way, they are, in general, no longer equivalent. The logical approach is certainly the easiest to generalize, since [graphs](#) are yet another special case of relational [structures](#). While on [words](#) and [trees](#) the existential fragment of [MSOL](#) ([EMSOL](#)) is already sufficient to characterize regularity, it is strictly less expressive than full [MSOL](#) on [graphs](#), as has been shown by Fagin in [[Fag75](#)]. Similarly, the algebraic approach (based on monoids) has been adapted to [graphs](#) by Courcelle in [[Cou90](#)], and it turns out that [MSOL](#) is strictly less powerful than his notion of recognizability. (The latter is defined in terms of homomorphisms into many-sorted algebras that are finite in each sort.) A common pattern that emerges from such results is that the different characterizations of regularity drift apart as the complexity of the considered [structures](#) increases. In this sense, regularity cannot be considered a natural – or even well-defined – property of [graph languages](#).

To complicate matters even further, the automata-theoretic characterization which is instrumental in the theory of [word](#) and [tree languages](#), does not seem to have a natural counterpart on [graphs](#). A [word](#) or [tree](#) automaton can scan its entire input in a single canonical traversal, which is completely determined by the structure of the input (i.e., left-to-right, for [words](#), and bottom-up, for [trees](#)). On arbitrary [graphs](#), however, there is no sense of a global direction that the automaton could follow, especially since we do not even require [connectivity](#) or acyclicity. This is one of the reasons why much research in the area of [graph languages](#) has focused on [MSOL](#). In the words of Courcelle and Engelfriet [[CE12](#), p. 3]:

*... monadic second-order logic can be viewed as playing the role of “finite automata on graphs” ...*

Another approach, investigated by Thomas in [[Tho91](#)] and [[Tho97a](#)], is to non-deterministically assign a state of the automaton to each [node](#) of the [graph](#), and then check that this assignment satisfies certain local “transition” conditions for

*Fagin’s result was later extended by Matz, Schweikardt and Thomas to yield a complete separation of the [MSOL](#) quantifier alternation hierarchy (see [[MST02](#)]).*

each **node** (specified with respect to **neighboring nodes** within a fixed radius) as well as certain global occurrence conditions at the level of the entire **graph**. The *graph acceptors* devised by Thomas turn out to be equivalent to **EMSOL** on **graphs** of bounded degree. Following up on this idea in [SB99], Schwentick and Barthelmann have also suggested a more general model, which remains very close to a normal form of **EMSOL**, but overcomes the constraint of boundedness on the degree. Both of these **graph** automaton models are legitimate generalizations of classical finite automata, in the sense that they are equivalent to them and can easily simulate them if we restrict the input to (**graphs** representing) **words** or **trees**. However, on arbitrary **graphs**, they are less well-behaved, which is a direct consequence of their **equivalence** with **EMSOL**. In particular, they do not satisfy closure under complementation, and their emptiness problem is undecidable. It is worth noting that both models are somewhat similar to the local distributed algorithms considered in the next section, insofar as they take into account the local view that each **node** has of its fixed-radius neighborhood. This connection has already been recognized and exploited by Göös and Suomela in [GS11, GS16]; we will mention it again below.

### 1.1.2 Related work in distributed computing

Rather surprisingly, the idea of extending descriptive complexity theory to the setting of distributed computing seems to be relatively new. The first research in that direction (of which the author is aware) started in the early 2010s as a collaboration between the Finnish communities of logic and distributed algorithms.

In [HJK<sup>+</sup>12, HJK<sup>+</sup>15], Hella et al. have presented a systematic study of several models of distributed computing that impose restrictions of varying degrees on the communication between the **nodes** of a network. Their most permissive model corresponds to the well-established port-numbering model, where every **node** has a separate communication channel with each of its **neighbors** and is guaranteed that the messages sent and received through that channel relate consistently to the same **neighbor**; the network is anonymous in the sense that **nodes** are not equipped with unique identifiers. In the nomenclature of [HJK<sup>+</sup>12, HJK<sup>+</sup>15], the class of **graph** problems solvable in this model by deterministic *synchronous* algorithms is denoted by  $\mathbf{vv}_c$ . Here, “synchronous” means that all **nodes** of the network share a global clock, thereby allowing the computation to proceed in an infinite sequence of rounds; in each round, all the **nodes** simultaneously exchange messages with their **neighbors**, and then update their local state based on the newly obtained information. Next, by dropping the channel-consistency guarantee, one obtains the class  $\mathbf{vv}$ , where in each round, every **node** sees a vector consisting of all the incoming messages received from its **neighbors**, and generates a vector of outgoing messages that are sent to the **neighbors**; the difference with  $\mathbf{vv}_c$  is that the two vectors are not necessarily sorted in the same order, so the **node** cannot assume that the **neighbor** who sends the  $i$ -th incoming message is the same who receives the  $i$ -th outgoing message. (However, the sorting orders do not change throughout the rounds.) Communication is further restricted in the classes  $\mathbf{mv}$  and  $\mathbf{sv}$ , where the vector of incoming messages is replaced by a multiset and a set, respectively. In the former case, a **node** cannot identify the senders of its incoming messages, whereas in the latter, it cannot even distinguish between several identical messages. Similarly, the classes  $\mathbf{vb}$ ,  $\mathbf{mb}$ , and  $\mathbf{sb}$  are characterized by the fact that the outgoing vector is replaced by a singleton, meaning that a **node** must broadcast the same message to all of its **neighbors**.

The classes of Hella et al.:

	Incoming	Outgoing
$\mathbf{vv}_c$	vector	vector
$\mathbf{vv}$	——	——
$\mathbf{mv}$	multiset	——
$\mathbf{sv}$	set	——
$\mathbf{vb}$	vector	singleton
$\mathbf{mb}$	multiset	——
$\mathbf{sb}$	set	——



The main result of [HJK<sup>+</sup>12, HJK<sup>+</sup>15] is that the preceding classes satisfy the linear order

$$\text{SB} \subsetneq \text{MB} = \text{VB} \subsetneq \text{SV} = \text{MV} = \text{VV} \subsetneq \text{VV}_c.$$

The same order holds for the so-called *local* (or constant-time) versions of these classes, which contain only those **graph** problems that can be solved in a constant number of communication rounds, regardless of the size of the network. (For a relatively recent survey of local algorithms, see [Suo13].)

Most relevant for the present thesis, the same paper also establishes a very natural correspondence between these local classes and several variants of **modal logic**. In particular, a **graph property** lies in  $\text{SB}(1)$ , the local version of **SB**, if and only if it can be defined by a formula of *backward modal logic*. Just like a distributed algorithm, such a formula is evaluated from the local point of view of a particular **node** in the input graph. In order to make a statement about the **incoming neighborhood** of that **node**, **backward modal logic** allows to move the current point of evaluation to one of the **incoming neighbors** by means of a special operator, called *backward modality*. The key insight of Hella et al. is that the nesting depth of these **modalities** corresponds precisely to the running time of the local algorithms that solve problems in  $\text{SB}(1)$ . With this idea in mind, it is possible to derive similar characterizations for the other local classes  $\text{MB}(1), \dots, \text{VV}_c(1)$  in terms of extensions of **backward modal logic** that offer additional types of **modalities** (viz., multimodal and graded modal logic).

*Using the backward version of standard modal logic is merely a presentational choice, motivated by the intuition that the messages of a distributed algorithm should flow in the same direction as the network links on which they travel. The presentation in [HJK<sup>+</sup>12, HJK<sup>+</sup>15] is a bit different.*

Motivated by these results, the connection between distributed algorithms and **modal logic** was further investigated by Kuusisto in [Kuu13a] and [Kuu14]. The first paper lifts the constraint of locality required in [HJK<sup>+</sup>12, HJK<sup>+</sup>15], thereby allowing algorithms with arbitrary running times. Now, for local algorithms, it does not matter whether we impose a restriction on the amount of memory space used by each **node**, because in a constant number of rounds, a **node** can only visit a constant number of different states. Therefore the local algorithms characterized by Hella et al. are implicitly finite-state machines. On the other hand, in the **nonlocal** case considered by Kuusisto, space restrictions have to be made explicit. His papers focus on algorithms for the class **SB**, since results for that class can easily be adapted to the others. In [Kuu13a], particular attention is devoted to a category of such algorithms that act as finite-state semi-deciders; we shall refer to them as *distributed automata*. The main result establishes a logical characterization of **distributed automata** in terms of a new recursive logic dubbed *modal substitution calculus*. In the same vein, it is also shown that the infinite-state generalizations of **distributed automata** recognize precisely those **graph properties** whose complement is definable by the conjunction of a possibly infinite number of **backward modal formulas** (called modal theory). Furthermore, it is proven that on finite **graphs**, **distributed automata** are strictly more expressive than the least-fixpoint fragment of the backward  $\mu$ -calculus. This logic, which we shall refer to simply as the *backward  $\mu$ -fragment*, extends **backward modal logic** with a least fixpoint operator that may not be negated. It thus allows to express statements using least fixpoints, but unlike in the full backward  $\mu$ -calculus, greatest fixpoints are forbidden. Finally, the second paper [Kuu14] makes crucial use of the connection with logic to show that universally halting distributed automata are necessarily local if infinite graphs are allowed into the picture.

Closely related to the work mentioned above, the last decade has also seen active research in *distributed decision* [FF16], a field that aims to develop a counterpart of computational complexity theory for distributed computing. In that context, the

**nodes** of a given network have to collectively decide whether or not their network satisfies some **global property**. Every **node** first computes a local answer, based on the information received from its **neighbors** over several rounds of communication, and then all answers are aggregated to produce a global verdict. Typically, the network is considered to be in a valid state if it has been unanimously accepted by all **nodes**; in other words, the global answer is the logical conjunction of the local answers.

Just as in classical complexity theory, a common approach in distributed decision is to start with some base class of deterministic algorithms, and then extend it with additional features, such as nondeterminism and randomness. However, depending on the underlying model of distributed computing, these additional features can quickly lead to excessive expressive power. For instance, if we add unrestricted nondeterminism to the widely adopted LOCAL model, then the **nodes** can simply guess a representation of the entire network and verify in one round that their guess was correct. Consequently, nondeterministic algorithms in the LOCAL model can already decide every Turing-decidable **graph property** in a single round of communication (see, e.g., [FF16, § 4.1.1]). To make things more interesting, one therefore often imposes a restriction on the number of bits that each **node** can nondeterministically choose; viewing nondeterminism as the ability to “guess and verify”, we refer to the bit strings guessed by the **nodes** as *certificates*. A typically chosen bound on the size of those certificates is logarithmic in the size of the network because this allows each **node** to guess only a constant number of processor identifiers. In stark contrast to the unbounded case, where Turing-decidability is the only limit, there are natural decision problems that cannot be solved by any nondeterministic local algorithm whose certificates are logarithmically bounded. An example of such a problem is to verify whether a given **tree** is a minimum spanning tree, as has been shown by Korman and Kutten in [KK07]. Nevertheless, on **connected graphs**, nondeterminism with logarithmic certificates provides enough power to decide every **property definable** in EMSOL within a constant number of rounds, essentially by using nondeterministic bits to construct a spanning tree and simulate existential **set quantifiers**. This observation has been made by Göös and Suomela in [GS11, GS16], based on the work of Schwentick and Barthelmann mentioned in the previous section.

Once existential quantification has been introduced into the system, a natural follow-up is to complement it with universal quantification; for instance, in classical complexity theory, alternating the two types of quantifiers leads to the polynomial hierarchy, which generalizes the classes NPTIME and co-NPTIME. While not very interesting for the unrestricted LOCAL model with unbounded certificates (where nondeterminism already suffices to decide everything possible), this form of alternation provides a genuine increase of power if we consider distributed algorithms that are oblivious to the **node** identifiers. In [BDFO17], Balliu, D’Angelo, Fraigniaud and Olivetti showed that we require one alternation between universal and existential quantifiers in order to be able to decide every Turing-decidable **property** in the identifier-oblivious variant of the LOCAL model (with unbounded certificates); hence the corresponding alternation hierarchy collapses to its second level. On the other hand, the hierarchy of the standard LOCAL model with certificates of logarithmic size is much less well understood; in particular, it is still open whether or not that hierarchy is infinite. As a first step towards an answer, Feuilloley, Fraigniaud and Hirvonen showed in [FFH16] that if there is equality between the existential and universal versions of a given level in the logarithmic hierarchy, then the entire hierarchy collapses to that level. Furthermore, they could identify a decision problem that lies

*The LOCAL model allows unbounded synchronous communication between Turing-complete processors that are equipped with unique identifiers. Despite the name, algorithms in this model are not necessarily local.*

outside of the hierarchy, which shows that even with the full power of alternation, algorithms whose certificates are logarithmically bounded remain weaker than their unrestricted counterparts.

## 1.2 Contributions

Obviously, developing a descriptive complexity theory for distributed computing is a highly ambitious project, of which the present work can only strive to be a small building block. As its title suggests, this thesis does not deal with the powerful models of computation that are usually considered in distributed computing. Instead, it takes an automata-theoretic approach and focuses on a rather weak model that has already been explored by Hella et al. and Kuusisto, namely *distributed automata*. The main contributions are two new logical characterizations related to that model.

The first covers a variant of *local distributed automata*, extended with a global acceptance condition and the ability to alternate between nondeterministic decisions of the individual processors and the creation of parallel computation branches. This kind of alternation constitutes a canonical generalization of nondeterminism, and is nowadays standard in automata theory. We show that the resulting *alternating local automata with global acceptance* are *equivalent* to *MSOL* on finite *directed graphs*. In spirit, they are similar to the alternation hierarchies considered in the distributed-decision community, even though their expressive power is much more restricted. They also share some similarities with Thomas' *graph* acceptors, as they use a combination of local conditions, checked by the *nodes* based on their *neighborhood*, and global conditions, checked at the level of the entire *graph*. However, both types of conditions are much simpler than in Thomas' model, which allows us to consider *graphs* of unbounded degree. To a certain extent, the *equivalence* with *MSOL* can be considered as a generalization to *graphs* of the classical result of Büchi, Elgot and Trakhtenbrot, although the machines involved are by no means deterministic; whereas on *words* and *trees*, alternation simply provides a more succinct representation of deterministic automata, it turns out to be a crucial ingredient in our case. If we allow only nondeterminism, we get a model that is not closed under complementation, and is even strictly weaker than *EMSOL*, but has a decidable emptiness problem. Interestingly, that model is still powerful enough to characterize precisely the regular languages when restricted to *words* or *trees*. Hence, this work also contributes to the general observation, made in Section 1.1.1, that regularity becomes a moving target when lifted to the setting of *graphs*.

The second main contribution consists in a logical characterization of a fully deterministic class of *nonlocal automata*. As mentioned in Section 1.1.2, Kuusisto has noticed that *distributed automata*, in their unrestricted form, are strictly more powerful than the *backward  $\mu$ -fragment* on finite *graphs*. While it is straightforward to evaluate any *formula* of the *backward  $\mu$ -fragment* via a *distributed automaton*, there also exist *automata* that exploit the fact that a *node* can determine if it receives the same information from all of its *neighbors* at the exact same time. Such a behavior cannot be simulated in the *backward  $\mu$ -fragment*, and actually not even in the much more expressive *MSOL*. However, since the argument relies solely on synchrony, it seems natural to ask whether removing this feature can lead to a distributed automaton model that has the same expressive power as the *backward  $\mu$ -fragment*. To answer this question, we introduce several classes of *asynchronous automata* that

transfer the standard notion of asynchronous algorithm to the setting of finite-state machines. Basically, this means that we eliminate the global clock from the network, thus making it possible for **nodes** to operate at different speeds and for messages to be delayed for arbitrary amounts of time, or even be lost. From the syntactic point of view, an **asynchronous automaton** is the same as a synchronous one, but it has to satisfy an additional semantic condition: its **acceptance behavior** must be independent of any timing-related issues. Taking a closer look at the **automata** obtained by translating **formulas** of the **backward  $\mu$ -fragment**, we can easily see that they are in fact asynchronous. Furthermore, their **state** diagrams are almost acyclic, except that the **states** are allowed to have self-loops; we call this property *quasi-acyclic*. As it turns out, the two properties put together are sufficient to give us the desired characterization: **quasi-acyclic asynchronous automata** are **equivalent** to the **backward  $\mu$ -fragment** on finite **graphs**. Incidentally, this remains true even if we consider a seemingly more powerful variant of **asynchronous automata**, where all messages are guaranteed to be delivered.

Another aspect of **distributed automata** investigated in this thesis are decision problems, and more specifically emptiness problems, where the task is to decide whether a given **automaton** **accepts** on at least one input **graph**. As all the **equivalences** mentioned above are effective, we can immediately settle the decidability of the **emptiness problem** for **local automata**: it is decidable for the basic variant of Hella et al., but undecidable for the alternating extension that we shall consider. This is because the (finite) satisfiability problem is PSPACE-complete for (backward) **modal logic** but undecidable for **MSOL**. The problem is also decidable for our classes of **asynchronous automata**, since (finite) satisfiability for the (backward)  $\mu$ -calculus is EXPTIME-complete. However, the corresponding question for unrestricted, **nonlocal automata** was left open in [Kuu13a]. Here, we answer this question negatively for the general case and also consider it for three special cases. On the positive side, we obtain a LOGSPACE decision procedure for a class of **forgetful automata**, where the **nodes** see the messages received from their **neighbors** but cannot remember their own **state**. When restricted to the appropriate families of **graphs**, these **forgetful automata** are **equivalent** to classical finite **word** automata, but strictly more expressive than finite **tree** automata. On the negative side, we show that the emptiness problem is already undecidable for two heavily restricted classes of **distributed automata**: the *quasi-acyclic* ones, and those that reject immediately if they receive more than one distinct message per round. For the latter class, we present a proof with an unusual twist, where a Turing machine is simulated by a **distributed automaton** in such a way that the roles of space and time are reversed between the two devices.

Finally, as a minor contribution, we investigate the problem of separating **quantifier alternation hierarchies** for several classes of **formulas** that are based on **modal logic**. Essentially, these classes are hybrids, obtained by adding the **set quantifiers** of **MSOL** to some variant of **modal logic**. They are motivated by the above characterizations of **local distributed automata** in terms of (backward) **modal logic** and **MSOL**. The contribution is a toolbox of simple encoding techniques that allow to easily transfer to the **modal** setting the separation results for **MSOL** established by Matz, Schweikardt and Thomas in [MT97, Sch97, MST02]. We thereby provide alternative proofs to similar findings previously reported by Kuusisto in [Kuu08, Kuu15].

## 1.3 Outline

The structure of this thesis is rather straightforward. All the notions that occur in several places are defined in [Chapter 2](#). In particular, there is a simple definition of [distributed automata](#) that subsumes most of the variants we shall consider. The subsequent four chapters (i.e., 3 to 6) are independent of each other and thus can be read in any order. In [Chapter 3](#), we focus on [local distributed automata](#) and present the [alternating variant with global acceptance](#), which is shown to be [equivalent to MSOL](#). [Chapter 4](#) shifts the focus to [nonlocal automata](#); there we introduce the semantic notion of [asynchrony](#) and show that [quasi-acyclic asynchronous automata](#) are captured by the [backward  \$\mu\$ -fragment](#). [Nonlocal automata](#) are also the subject of [Chapter 5](#), where we present both positive and negative decidability results on the [emptiness problem](#) for several restricted classes. Then, in [Chapter 6](#), we switch completely to logic and consider issues related to [quantifier alternation hierarchies](#). Finally, some perspectives for future research are briefly outlined in [Chapter 7](#).

**Note to the reader of the electronic version.** The PDF version of this document makes extensive use of hyperlinks. In addition to the cross-reference links inserted automatically by the standard  $\text{\LaTeX}$  package [hyperref](#), most of the notions defined within the document are linked to their point of definition. This new feature, which concerns both text and mathematical notation, is based on the [knowledge](#) package developed by Thomas Colcombet. Beware that there can be several links within a single symbolic expression; for instance, the expression  $\llbracket \text{BC } \Sigma_{\ell}^{\text{MSO}}(\vec{\text{ML}}) \rrbracket_{\text{DG}}$  contains links to five different concepts:  $\llbracket \dots \rrbracket$ , [BC](#),  [\$\Sigma\_{\ell}^{\text{MSO}}\$](#) ,  [\$\vec{\text{ML}}\$](#) , and [@DG](#).



# 2

## Preliminaries

This chapter introduces essential notation and terminology that will be recurring throughout this thesis. It is meant to be consulted for specific information rather than for consecutive reading. Concepts that are specific to a single chapter, will be introduced later, along with the topic.

### 2.1 Basic notation

We denote the empty set by  $\emptyset$ , the set of Boolean values by  $\mathbb{2} = \{0, 1\}$ , the set of non-negative integers by  $\mathbb{N} = \{0, 1, 2, \dots\}$ , the set of positive integers by  $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ , and the set of integers by  $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ .

Integer intervals of the form  $\{i \in \mathbb{Z} \mid m \leq i \leq n\}$ , where  $m, n \in \mathbb{Z}$  and  $m \leq n$ , will sometimes be denoted by  $[m:n]$ . We may also use the shorthand  $[n] := [1:n]$ , and, by analogy with the Bourbaki notation for real intervals, we indicate that we exclude an endpoint by reversing the square bracket corresponding to that endpoint, e.g.,  $]m:n[ := [m:n] \setminus \{m\}$ .

For any two sets  $S$  and  $T$ , the set of all functions from  $S$  to  $T$  is denoted  $T^S$ . This notation gives rise to two important special cases. First, we write  $\mathbb{2}^S$  for the power set of  $S$ , since we can identify it with the set of all functions from  $S$  to  $\{0, 1\}$ . Second, given  $k \in \mathbb{N}$ , we write  $S^k := S^{[k]}$  for the set of all  $k$ -tuples over  $S$ , since we can identify it with the set of functions from  $[k]$  to  $S$ . All of these notations have another special case in common: the set of binary strings of length  $k$ , denoted  $\mathbb{2}^k$ , can be interpreted as either the function space from  $[k]$  to  $\mathbb{2}$ , or the power set of  $[k]$ , or the set of  $k$ -tuples over  $\mathbb{2}$ . By the first interpretation, the individual letters of a string  $x$  of length  $k$  will be denoted  $x(1), \dots, x(k)$ . Furthermore, we write  $|S|$  for the cardinality of  $S$  and  $|x|$  for the length of  $x$ .

### 2.2 Symbols

Since logic plays an important role in this thesis, it also has an influence on how we present other concepts; in particular, our definition of [directed graphs](#) in [Section 2.4](#)



will refer to the notion of (abstract) **symbol**.

We shall not always make a sharp distinction between **variables** and (non-logical) **constants**. Instead, there is simply a fixed supply of **symbols**, which can serve both as **variables** and as **constants**. Hence, the terms “*variable*” and “*constant*” are just synonyms for “**symbol**”; we will use them whenever we want to clarify the intended role of a **symbol** within a given context.

The set  $\mathbb{S}_0$  contains our **node symbols**, which within **formulas** will represent **nodes** of **structures** such as **graphs**; among them, there is a special **position symbol**  $@$ . Moreover, for every integer  $k \geq 1$ , we let  $\mathbb{S}_k$  denote the set of  **$k$ -ary relation symbols**. All of these sets are infinite and pairwise disjoint. If a **symbol** lies in  $\mathbb{S}_k$ , for  $k \geq 0$ , then we call  $k$  the **arity** of that **symbol**. We also denote the set of all **symbols** by  $\mathbb{S}$ , i.e.,  $\mathbb{S} := \bigcup_{k \geq 0} \mathbb{S}_k$ , and shall often refer to the unary **relation symbols** in  $\mathbb{S}_1$  as **set symbols**.

$\mathbb{S}$  contains both  
variables and  
constants.

**Node symbols** will always be represented by lower-case letters, and **relation symbols** by upper-case ones, often decorated with subscripts. Typically, we use  $x, y, z$  for **node variables** or arbitrary **node symbols**,  $X, Y, Z$  for **set variables** or arbitrary **set symbols**,  $P, Q$  for **set constants**, and  $R, S$  for **relation constants** of higher **arity** or arbitrary **symbols**. (See Section 2.6 for some simple examples.)

## 2.3 Structures

Before we formally introduce **directed graphs** in the next section, we define the more general concept of a relational **structure**. Although the present thesis focuses mainly on variants of **directed graphs**, this top-down approach will allow us to specify the semantics of several types of logical **formulas** in a unified framework, using a consistent notation. In particular, it will be apparent that **modal logic** simply provides an alternative syntax for a certain fragment of **first-order logic** (see Section 2.5).

Let  $\sigma$  be any subset of  $\mathbb{S}$ . A (relational) **structure**  $G$  of **signature**  $\sigma$  consists of a nonempty set of **nodes**  $V^G$  (also called the **domain** of  $G$ ), a **node**  $x^G$  of  $V^G$  for each **node symbol**  $x$  in  $\sigma$ , and a  $k$ -ary relation  $R^G$  on  $V^G$  for each  $k$ -ary **relation symbol**  $R$  in  $\sigma$ . Here,  $x^G$  and  $R^G$  are called  $G$ 's **interpretations** of the **symbols**  $x$  and  $R$ . We may also say that  $G$  is a **structure over**  $\sigma$ , or that  $\sigma$  is the **underlying signature** of  $G$ , and we denote  $\sigma$  by  $\text{sig}(G)$ . In case the **position symbol**  $@$  lies in  $\text{sig}(G)$ , we call  $G$  a **pointed structure** and  $@^G$  the **distinguished node** of  $G$ .

A set of **structures** will be referred to as a **structure language**. As is customary, we are only interested in **structures** up to isomorphism. That is, two **structures** over  $\sigma$  are considered to be equal if there is a bijection between their **domains** that preserves the **interpretations** of all **symbols** in  $\sigma$ . Consequently, our **structure languages** characterize only properties that are invariant under isomorphism.

Let  $G$  be a **structure** and  $\alpha$  be a map of the form  $\{S_1 \mapsto I_1, \dots, S_n \mapsto I_n\}$  that assigns to each **symbol**  $S_i \in \mathbb{S}$ , for  $i \in [n]$ , a suitable **interpretation**  $I_i$  over the **domain** of  $G$ . That is, if  $S_i \in \mathbb{S}_0$ , then  $I_i \in V^G$ , and if  $S_i \in \mathbb{S}_k$ , for  $k \geq 1$ , then  $I_i \subseteq (V^G)^k$ . We use the notation  $G[\alpha]$  to designate the  **$\alpha$ -extended variant** of  $G$ , which is the **structure**  $G'$  obtained from  $G$  by **interpreting** each **symbol**  $S_i$  as  $I_i$ , while maintaining the other **interpretations** provided by  $G$ . More formally, letting  $\sigma = \{S_1, \dots, S_n\}$ , we have  $V^{G'} = V^G$ ,  $\text{sig}(G') = \text{sig}(G) \cup \sigma$ ,  $S_i^{G'} = I_i$  for  $i \in [n]$ , and  $T^{G'} = T^G$  for  $T \in \text{sig}(G) \setminus \sigma$ . Often, we do not want to give an explicit name to the assignment  $\alpha$ , in which case we may denote  $G'$  by  $G[S_1, \dots, S_n \mapsto I_1, \dots, I_n]$ . If the **interpretations** of the **symbols** in  $\sigma$  are clear from context, we may also refer to  $G'$  as the  **$\sigma$ -extended**



*variant* of  $G$ . Furthermore, as we will often consider *pointed variants* of structures, we introduce the shorthand  $G[v] := G[@ \mapsto v]$  for  $v \in V^G$ , and refer to  $G[v]$  as the *v-pointed variant* of  $G$  (i.e., the *variant* of  $G$  with distinguished node  $v$ ).

## 2.4 Different kinds of digraphs

The structures we are actually interested in are several variants of *directed graphs*; these are structures with finite domains and relations of *arity* at most 2. To facilitate lookup and comparison, we present them all in the same section. In the following definitions, let  $s$  and  $r$  be non-negative integers.

An *s-bit labeled, r-relational directed graph*, abbreviated *digraph*, is a finite structure  $G$  of signature  $\{P_1, \dots, P_s, R_1, \dots, R_r\}$ , where  $P_1, \dots, P_s$  are *set symbols*, and  $R_1, \dots, R_r$  are *binary relation symbols*.

The sets  $P_1^G, \dots, P_s^G$ , which we shall call *labeling sets*, determine a (node) *labeling*  $\lambda^G: V^G \rightarrow 2^s$  that assigns a binary string of length  $s$  to each node. More precisely, we define  $\lambda^G$  such that

$$\lambda^G(v)(i) = \begin{cases} 0 & \text{if } v \notin P_i, \\ 1 & \text{otherwise,} \end{cases}$$

for all  $v \in V^G$  and  $i \in [s]$ . Given another mapping  $\zeta: V^G \rightarrow 2^{s'}$  with  $s' \in \mathbb{N}$ , we shall denote by  $G[\zeta]$  the  *$\zeta$ -reabeled variant* of  $G$ , i.e., the  $s'$ -bit labeled digraph  $G'$  that is the same as  $G$ , except that its labeling  $\lambda^{G'}$  is equal to  $\zeta$ .

It is often convenient to regard the labels of an  $s$ -bit labeled digraph as the binary encodings of letters of some finite alphabet  $\Sigma$ . With respect to a given injective map  $f: \Sigma \rightarrow 2^s$ , a  *$\Sigma$ -labeled digraph* is an  $s$ -bit labeled digraph  $G$  such that for every node  $v \in V^G$ , we have  $\lambda^G(v) = f(a)$  for some  $a \in \Sigma$ . Since we do not care about the specific encoding function  $f$ , we will never mention it explicitly, and just call  $G$  a  *$\Sigma$ -labeled, r-relational digraph*.

The binary relations  $R_1^G, \dots, R_r^G$  will be referred to as *edge relations*. If  $uv$  is an *edge* in  $R_i^G$ , then  $u$  is called an *incoming i-neighbor* of  $v$ , or simply an *incoming neighbor*, and  $v$  is called an *outgoing i-neighbor* of  $u$ , or just *outgoing neighbor*. We also say that  $u$  and  $v$  are *adjacent*, and without further qualification, the term *neighbor* refers to both *incoming* and *outgoing neighbors*. The (undirected) *neighborhood* of a node is the set of all of its neighbors, and the *incoming* and *outgoing neighborhoods* are defined analogously. A node without incoming neighbors is called a *source*, whereas a node without outgoing neighbors is called a *sink*.

The class of all  $s$ -bit labeled,  $r$ -relational digraphs is denoted by  $\mathbf{DG}_s^r$ . In case the parameters are  $s = 0$  and  $r = 1$ , we may omit them and use the shorthand  $\mathbf{DG} := \mathbf{DG}_0^1$ . We shall also drop the subscripts on the symbols, and just write  $P$  or  $R$ , if there is only one symbol of a given arity. Furthermore, we denote by  $\mathbf{DG}_\Sigma^r$  the class of all  $\Sigma$ -labeled,  $r$ -relational digraphs.

As can be easily guessed from the previous definitions, a *pointed digraph* is a digraph in which some node has been marked by the position symbol  $@$ , i.e., it is a structure of the form  $G[@ \mapsto v]$ , with  $G \in \mathbf{DG}_s^r$  and  $v \in V^G$ . We write  $@\mathbf{DG}_s^r$  for the set of all  $s$ -bit labeled,  $r$ -relational pointed digraphs, and define  $@\mathbf{DG} := @\mathbf{DG}_0^1$ .

A digraph  $G$  is called an ( $s$ -bit labeled,  $r$ -relational) *undirected graph*, or simply *graph*, if all of its edge relations are irreflexive and symmetric, i.e., if for all  $u, v \in V^G$  and  $i \in [r]$ , it holds that  $uu \notin R_i^G$ , and  $uv \in R_i^G$  if and only if  $vu \in R_i^G$ . The

*Our bracket notation is overloaded, but if one knows the type of  $\zeta$ , the  $\zeta$ -reabeled variant  $G[\zeta]$  of  $G$  should be easy to distinguish from an  $\alpha$ -extended variant  $G[\alpha]$ , as well as from a  $v$ -pointed variant  $G[v]$ .*

corresponding class is denoted by  $\text{GRAPH}_s^r$ , and we may use the shorthand  $\text{GRAPH} := \text{GRAPH}_0^1$ .

A digraph  $G$  is (weakly) *connected* if for every nonempty proper subset  $W$  of  $V^G$ , there exist two nodes  $u \in W$  and  $v \in V^G \setminus W$  that are adjacent.

The node labeling  $\lambda^G$  of a  $\Sigma$ -labeled digraph constitutes a valid *coloring* of  $G$  if no two adjacent nodes share the same label, i.e., if  $uv \in R_i^G$  implies  $\lambda^G(u) \neq \lambda^G(v)$ , for all  $u, v \in V^G$  and  $i \in [r]$ . If  $|\Sigma| = k$ , such a coloring is called a *k-coloring* of  $G$ , and any  $r$ -relational digraph for which a *k-coloring* exists is said to be *k-colorable*. Note that, by definition, a digraph that contains self-loops is not *k-colorable* for any  $k$ .

A *directed rooted tree*, or *ditree*, is an ( $s$ -bit labeled)  $r$ -relational digraph  $G$  that has a distinct node  $v_\epsilon$ , called the *root*, such that  $R_i^G \cap R_j^G = \emptyset$  for  $i \neq j$ , and from each node  $v$  in  $V^G$ , there is exactly one way to reach  $v_\epsilon$  by following the directed edges in  $\bigcup_{1 \leq i \leq r} R_i^G$ . A *pointed ditree* is a pointed digraph  $G[v_\epsilon]$ , where  $G$  is a ditree and  $v_\epsilon$  is its root. Moreover, a (pointed)  $r$ -relational ditree is called *ordered* if for  $1 \leq i \leq r$ , every node has at most one incoming  $i$ -neighbor and every node that has an incoming  $(i+1)$ -neighbor also has an incoming  $i$ -neighbor. As a special case, an ordered 1-relational ditree is referred to as a *directed path*, or *dipath*. Accordingly, the distinguished node of a *pointed dipath* is the last node (the one with no outgoing neighbor). The classes of pointed dipaths and pointed ordered ditrees can be identified with the structures on which classical word and tree automata are run. We denote them by  $@\text{DIPATH}_s$  and  $@\text{ODITREE}_s^r$ , respectively.

We shall also consider an important subclass of  $\text{DG}_s^2$  whose members represent rectangular labeled grids (also called pictures). In such a structure  $G$ , each node is identified with a grid cell, and the edge relations  $R_1^G$  and  $R_2^G$  are interpreted as the “vertical” and “horizontal” successor relations, respectively. The unique node that has no predecessor at all is regarded as the “upper-left corner”, and all the usual terminology of matrices applies. Formally,  $G$  is a *s-bit labeled grid* if, for some  $m, n \geq 1$ , it is isomorphic to a structure with domain  $\{1, \dots, m\} \times \{1, \dots, n\}$  and edge relations

$$\begin{aligned} R_1^G &= \{((i, j), (i+1, j)) \mid 1 \leq i < m, 1 \leq j \leq n\}, \\ R_2^G &= \{((i, j), (i, j+1)) \mid 1 \leq i \leq m, 1 \leq j < n\}. \end{aligned}$$

If  $s = 0$ , we refer to  $G$  simply as a *grid*. In alignment with the previous nomenclature, we let  $\text{GRID}$  and  $\text{GRID}_s$  denote the classes of grids and  $s$ -bit labeled grids.

A *digraph language* is a structure language that consist of digraphs with a fixed number of labeling sets and edge relations, i.e., a subset of  $\text{DG}_s^r$ , for some  $s, r \in \mathbb{N}_+$ . The notion is defined analogously for all the other classes of structures introduced above. In particular, a *pointed-digraph language* is a subset of  $@\text{DG}_s^r$ .

## 2.5 The considered logics

As we shall contemplate both classical logic and several variants of modal logic, we introduce them all in a common framework. First we define the syntax and semantics of a generalized language, and then we specify which particular syntactic fragments we are interested in. Some examples will follow in Section 2.6.

Table 2.1 shows how *formulas* are built up, and what they mean. Furthermore, it indicates how to obtain the set  $\text{free}(\varphi)$  of symbols that occur *freely* in a given

Syntax Formula $\psi$	Free symbols Symbol set $\text{free}(\psi)$	Semantics Necessary and sufficient condition for $G \models \psi$
$x$	$\{@, x\}$	$@^G = x^G$
$(x \doteq y)$	$\{x, y\}$	$x^G = y^G$
$X$	$\{@, X\}$	$@^G \in X^G$
$X(x)$	$\{x, X\}$	$x^G \in X^G$
$R(x_0, \dots, x_k)$	$\{x_0, \dots, x_k, R\}$	$(x_0^G, \dots, x_k^G) \in R^G$
$\neg \varphi$	$\text{free}(\varphi)$	not $G \models \varphi$
$(\varphi_1 \vee \varphi_2)$	$\text{free}(\varphi_1) \cup \text{free}(\varphi_2)$	$G \models \varphi_1$ or $G \models \varphi_2$
$\Diamond_R(\varphi_1, \dots, \varphi_k)$	$\{@, R\} \cup \bigcup_{1 \leq i \leq k} \text{free}(\varphi_i)$	For some $v_1, \dots, v_k \in V^G$ such that $(@^G, v_1, \dots, v_k) \in R^G$ , we have $G[@ \mapsto v_i] \models \varphi_i$ for each $i \in \{1, \dots, k\}$ .
$\bar{\Diamond}_R(\varphi_1, \dots, \varphi_k)$	same as above	As above, except for the condition $(v_k, \dots, v_1, @^G) \in R^G$ .
$\Diamond \varphi$	$\text{free}(\varphi) \setminus \{@\}$	$G[@ \mapsto v] \models \varphi$ for some $v \in V^G$
$\exists_x \varphi$	$\text{free}(\varphi) \setminus \{x\}$	$G[x \mapsto v] \models \varphi$ for some $v \in V^G$
$\exists_X \varphi$	$\text{free}(\varphi) \setminus \{X\}$	$G[X \mapsto W] \models \varphi$ for some $W \subseteq V^G$

Here,  $x, x_0, \dots, x_k, y \in \mathbb{S}_0$ ,  $X \in \mathbb{S}_1$ ,  $R \in \mathbb{S}_{k+1}$ , and  $\varphi, \varphi_1, \dots, \varphi_k$  are formulas, for  $k \geq 1$ .

**Table 2.1.** Syntax and semantics of the considered logics.

Class of formulas	Generating grammar
<b>FOL</b> First-order	$\varphi ::= (x \doteq y) \mid X(x) \mid R(x_0, \dots, x_k) \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \exists_x \varphi$
<b>EMSOL</b> Existential <b>MSOL</b>	$\varphi ::= \psi \mid \exists_X \varphi$ , where $\psi \in \text{FOL}$ . Equivalently, $\text{EMSOL} := \Sigma_1^{\text{MSO}}(\text{FOL})$ ; see Section 6.1.
<b>MSOL</b> Monadic second-order	$\varphi ::= (x \doteq y) \mid X(x) \mid R(x_0, \dots, x_k) \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \exists_x \varphi \mid \exists_X \varphi$ Equivalently, $\text{MSOL} := \text{MSO}(\text{FOL})$ .
$\vec{\text{ML}}$ Modal	$\varphi ::= x \mid X \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \Diamond_R(\varphi_1, \dots, \varphi_k)$
$\overleftarrow{\text{ML}}$ Backward modal	$\varphi ::= x \mid X \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \bar{\Diamond}_R(\varphi_1, \dots, \varphi_k)$
$\overleftrightarrow{\text{ML}}$ Bidirectional modal	$\varphi ::= x \mid X \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \Diamond_R(\varphi_1, \dots, \varphi_k) \mid \bar{\Diamond}_R(\varphi_1, \dots, \varphi_k)$
$\vec{\text{ML}}_g$ Modal with global modalities	$\varphi ::= x \mid X \mid \neg \varphi \mid (\varphi_1 \vee \varphi_2) \mid \Diamond_R(\varphi_1, \dots, \varphi_k) \mid \Diamond \varphi$
$\vec{\text{ML}}_g, \overleftrightarrow{\text{ML}}_g$	Analogous to the preceding grammars.
$\text{MSO}(\Phi)$ $\Phi$ extended with set quantifiers	Same grammar as $\Phi$ with the additional choice “ $\mid \exists_X \varphi$ ”.

Here,  $x, x_0, \dots, x_k, y \in \mathbb{S}_0$ ,  $X \in \mathbb{S}_1$ ,  $R \in \mathbb{S}_{k+1}$ , for  $k \geq 1$ , and  $\Phi \in \{\vec{\text{ML}}, \overleftarrow{\text{ML}}, \dots, \vec{\text{ML}}_g, \text{FOL}\}$ .

**Table 2.2.** The considered classes of formulas.

formula  $\varphi$ , i.e., outside the scope of a binding operator. If  $\text{free}(\varphi) \subseteq \sigma$ , we say that  $\varphi$  is a *sentence* over  $\sigma$ . Sometimes, when the notions of “variable” and “constant” are clear from context, we also use the notation  $\varphi(x_1, \dots, x_m, X_1, \dots, X_n)$  to indicate that at most the *variables* given in brackets occur *freely* in  $\varphi$ , i.e., that no other *variables* than  $x_1, \dots, x_m, X_1, \dots, X_n$  lie in  $\text{free}(\varphi)$ . The relation  $\models$  defined in Table 2.1 specifies in which cases a *structure*  $G$  *satisfies*  $\varphi$ , written  $G \models \varphi$ , assuming that  $\varphi$  is a *sentence* over  $\text{sig}(G)$ . Otherwise, we stipulate that  $G \not\models \varphi$ .

Of particular interest for this thesis are those *formulas* in which the *node symbol*  $@$  is considered to be *free*, although it might not occur explicitly. They are evaluated on a *pointed structure*  $G$  from the perspective of the *node*  $@^G$ . Atomic *formulas* of the form  $x$  or  $X$ , with  $x \in \mathbb{S}_0$  and  $X \in \mathbb{S}_1$ , are *satisfied* if  $@^G$  is labeled by the corresponding *symbol*. Using the operator  $\Diamond$ , which is called the *R-diamond*, we can remap the *symbol*  $@$  through existential quantification over the *nodes* in  $G$  that are reachable from  $@^G$  through the relation  $R^G$ . If we want to do the same with respect to the inverse relation of  $R^G$ , we can use the *backward R-diamond*  $\Diamondleftarrow$ . In addition, there is also the *global diamond*  $\Diamond$  (unfortunately often called “universal modality”), which ranges over all *nodes* of  $G$ . It can be considered as the *diamond operator* corresponding to the relation  $V^G \times V^G$ , i.e., the *edge relation* of the complete digraph over  $V^G$ . To facilitate certain descriptions, we shall sometimes treat  $\Diamondleftarrow$  and  $\Diamond$  as special cases of  $\Diamond$ , assuming that they are implicitly associated with the reserved *relation symbols*  $R^{-1}$  and  $\bullet$ , respectively. These *symbols* do not belong to  $\mathbb{S}$ , and therefore cannot be *interpreted* by any *structure*.

Allowing a bit of syntactic sugar, we will make liberal use of the remaining operators of predicate logic, i.e.,  $\wedge$ ,  $\rightarrow$ ,  $\leftrightarrow$ ,  $\forall$ , and we may leave out some parentheses, assuming that  $\vee$  and  $\wedge$  take precedence over  $\rightarrow$  and  $\leftrightarrow$ . Furthermore, we define the abbreviations

$$\begin{aligned} \top &:= @, \quad \perp := \neg @, \quad \text{and} \\ \Box(\varphi_1, \dots, \varphi_k) &:= \neg \Diamond(\neg \varphi_1, \dots, \neg \varphi_k). \end{aligned}$$

Note that it makes sense to define  $\top$  (“true”) as  $@$ , since by definition, the atomic *formula*  $@$  is always *satisfied* at the point of evaluation. Also, the second line remains applicable if one substitutes  $R^{-1}$  or  $\bullet$  for  $R$ . The resulting operators  $\Box$ ,  $\Boxleftarrow$  and  $\Box$  provide universal quantification and are called *boxes* (using the same attributes as for *diamonds*). *Diamonds* and *boxes* are collectively referred to as *modalities* or *modal operators*. In case we restrict ourselves to *structures* that only have a single relation, we may omit the *relation symbol*  $R$ , and just use empty *modalities* such as  $\Diamond$ . Similarly, if the *relation symbols* involved are indexed, like  $R_1, \dots, R_r$ , we associate them with *modalities* of the form  $\Diamond_i$ , for  $1 \leq i \leq r$ .

Let us now turn to the particular classes of *formulas* considered in this thesis, which are specified in Table 2.2. The languages of *first-order logic* (FOL), *existential monadic second-order logic* (EMSOL), and *monadic second-order logic* (MSOL) are defined in the usual way. When evaluated on some *structure*  $G$ , their atomic *formulas* allow to compare *nodes* assigned to *node symbols* in  $\text{sig}(G)$  with respect to the equality relation and any other relation assigned to a *relation symbol* in  $\text{sig}(G)$ . In FOL, we can assign new *interpretations* to *node symbols* by means of existential and universal *quantification over nodes*. In EMSOL, we may additionally *reinterpret* set *symbols* using existential *quantifiers over sets* of *nodes*, and in MSOL, we can also use the corresponding universal *quantifiers*.

The remaining classes of **formulas** can all be qualified as modal languages, insofar as they include **modal operators** instead of the classical **first-order quantifiers**. By performing this change of paradigm, we lose our “bird’s-eye view” of the **structure**  $G$ , and now see it from the local point of view of the **node**  $@^G$ . (For this,  $G$  obviously has to be **pointed**.) In basic **modal logic** ( $\vec{ML}$ ), a **node** “sees” only its **outgoing neighbors**, and thus our domain of quantification is restricted to those **neighbors**. Furthermore, the **position symbol**  $@$  is the only **node symbol** whose **interpretation** can be changed by a **modal operator**. **Backward modal logic** ( $\overleftarrow{ML}$ ) is the variant of  $\vec{ML}$  where a **node** “sees” its **incoming neighbors** instead of its **outgoing neighbors**, whereas **bidirectional modal logic** ( $\overleftrightarrow{ML}$ ) is the combination where a **node** “sees” both **incoming** and **outgoing neighbors**. We will also look at **modal logic with global modalities** ( $\vec{ML}_g$ ), where we regain the possibility to quantify over the entire **domain** of the **structure**, but are still confined to remapping only the **position symbol**  $@$ . The backward and bidirectional variants  $\overleftarrow{ML}_g$  and  $\overleftrightarrow{ML}_g$  are defined analogously. Finally, we also consider crossover versions of **modal logic** that are enriched with the **set quantifiers** of **MSOL**. Given a class of **formulas**  $\Phi$ , we denote by  $\text{MSO}(\Phi)$  the corresponding enriched class. For instance, the **formulas** of  $\text{MSO}(\vec{ML})$  are generated by the grammar

$$\varphi ::= x \mid X \mid \neg\varphi \mid (\varphi_1 \vee \varphi_2) \mid \Diamond(\varphi_1, \dots, \varphi_k) \mid \exists x \varphi,$$

where  $x \in \mathbb{S}_0$ ,  $X \in \mathbb{S}_1$ , and  $R \in \mathbb{S}_{k+1}$ . Note that by this notation,  $\text{MSOL} = \text{MSO}(\text{FOL})$ .

For any class of **formulas**  $\Phi$ , we shall refer to its members as  **$\Phi$ -formulas**. Given a  **$\Phi$ -formula**  $\varphi$ , a class of **structures**  $\mathcal{C}$  (e.g., **DG**), and a **structure**  $G$ , we use the semantic bracket notations  $\llbracket \varphi \rrbracket_{\mathcal{C}}$  and  $\llbracket \varphi \rrbracket_G$  to denote the **structure language defined** by  $\varphi$  over  $\mathcal{C}$ , and the set of **nodes** of  $G$  at which  $\varphi$  holds. More formally,

$$\begin{aligned} \llbracket \varphi \rrbracket_{\mathcal{C}} &:= \{ G \in \mathcal{C} \mid G \models \varphi \}, \quad \text{and} \\ \llbracket \varphi \rrbracket_G &:= \{ v \in V^G \mid G[@ \mapsto v] \models \varphi \}. \end{aligned}$$

Furthermore,  $\llbracket \Phi \rrbracket_{\mathcal{C}}$  denotes the family of **structure languages** that are **definable** in  $\Phi$  (or  **$\Phi$ -definable**) over  $\mathcal{C}$ , i.e.,

$$\llbracket \Phi \rrbracket_{\mathcal{C}} := \{ \llbracket \varphi \rrbracket_{\mathcal{C}} \mid \varphi \in \Phi \}.$$

If  $\mathcal{C}$  is equal to the set of all **structures**, then we do not have to specify it explicitly as a subscript; that is, we may simply write  $\llbracket \varphi \rrbracket$  and  $\llbracket \Phi \rrbracket$  instead of  $\llbracket \varphi \rrbracket_{\mathcal{C}}$  and  $\llbracket \Phi \rrbracket_{\mathcal{C}}$ . Similarly, we use

$$\llbracket \varphi \rrbracket_{\mathcal{C}} := \{ \psi \mid \llbracket \psi \rrbracket_{\mathcal{C}} = \llbracket \varphi \rrbracket_{\mathcal{C}} \}$$

for the **equivalence class** of  $\varphi$  over  $\mathcal{C}$ , and

$$\llbracket \Phi \rrbracket_{\mathcal{C}} := \bigcup_{\varphi \in \Phi} \llbracket \varphi \rrbracket_{\mathcal{C}}$$

for the set of all **formulas** that are **equivalent** over  $\mathcal{C}$  to some **formula** in  $\Phi$ . Again, we may drop the subscript if we do not want to restrict to a particular class of **structures**.

## 2.6 Example formulas

In order to illustrate the syntax introduced in the previous section, we now look at two simple examples.

The first is a great classic that is often used to show how a widely known **graph property** can be expressed in **MSOL** without too much effort.

**Example 2.1** (3-Colorability).

► The following **EMSOL**-formula defines the language of 3-colorable digraphs over **DG**.

$$\begin{aligned} \varphi_3^{\text{color}} := \exists_{X_1, X_2, X_3} \bigg( & \forall_x \big( (X_1(x) \vee X_2(x) \vee X_3(x)) \wedge \\ & \neg(X_1(x) \wedge X_2(x)) \wedge \\ & \neg(X_1(x) \wedge X_3(x)) \wedge \\ & \neg(X_2(x) \wedge X_3(x)) \big) \wedge \\ & \forall_{x,y} \big( R(x,y) \rightarrow \neg(X_1(x) \wedge X_1(y)) \wedge \\ & \neg(X_2(x) \wedge X_2(y)) \wedge \\ & \neg(X_3(x) \wedge X_3(y)) \big) \bigg) \end{aligned}$$

The existentially quantified **set variables**  $X_1, X_2, X_3 \in \mathbb{S}_1$  represent the three possible colors. In the first four lines, we specify that the sets assigned to these **variables** form a partition of the set of **nodes** (possibly with empty components). The remaining three lines constitute the actual definition of a valid **coloring**: no two **adjacent nodes** share the same color, which means that **adjacent nodes** are in different sets. ◀

Our second example is equally simple, but less glamorous because it illustrates a technical issue that will concern us in **Chapter 6**, where we shall work with  $\text{MSO}(\vec{\text{ML}}_g)$  and some variants thereof. As we do not allow **first-order quantification** in **modal logic** with **set quantifiers**, some properties that seem very natural in **FOL** (and thus **MSOL**) become rather cumbersome to express. Nevertheless, translation from **FOL** to  $\text{MSO}(\vec{\text{ML}}_g)$  is always possible because we can simulate **first-order quantifiers** by **set quantifiers** relativized to singletons, which, by extension, also entails the equivalence of **MSOL** and  $\text{MSO}(\vec{\text{ML}}_g)$ . **Example 2.2** presents the basic construction that allows us to do this. We will refer to it several times in **Chapter 6**.

**Example 2.2** (Uniqueness).

► Consider the following **formula** schema, where  $X \in \mathbb{S}_1$ ,  $R \in \mathbb{S}_2$ , and  $\varphi$  can be any  $\vec{\text{ML}}_g$ -formula:

$$\text{see1}_R(\varphi) := \langle R \rangle \varphi \wedge \forall_X \big( \langle R \rangle (\varphi \wedge X) \rightarrow \langle R \rangle (\varphi \rightarrow X) \big).$$

When evaluated on a **pointed structure**  $G$  whose **signature** includes  $\{ @, R \} \cup \text{free}(\varphi)$ , the **formula**  $\text{see1}_R(\varphi)$  states that there is exactly one **node**  $v \in V^G$  reachable from  $@^G$  through an  $R^G$ -edge, such that  $\varphi$  is **satisfied** at  $v$  (i.e., by the **structure**  $G[@ \mapsto v]$ ). In the context of 1-relational **digraphs**, we may use the shorthand  $\text{see1}(\varphi)$  to invoke this schema. Using the same construction with **global modalities**, we also define

$$\text{tot1}(\varphi) := \text{see1}_\bullet(\varphi),$$

which states that there is precisely one **node** in the entire **structure**  $G$  at which  $\varphi$  is **satisfied**. Here,  $G$  does not necessarily have to be **pointed**, and, of course,  $\text{sig}(G)$  does not contain  $\bullet$  (since it is the **symbol** reserved for the total symmetric relation). ◀



Anticipating the notation of Section 6.1, the formulas obtained by the construction in Example 2.2 can be classified as  $[\Pi_1^{\text{MSO}}(\Phi)]$ -formulas, where  $\Phi \in \{\vec{ML}, \vec{ML}, \vec{ML}_g, \vec{ML}_g\}$  depends on the specific modalities that occur in  $\varphi$ .

## 2.7 Distributed automata

We conclude this preliminary chapter by introducing our primary objects of interest. Simply put, a **distributed automaton** is a deterministic finite-state machine  $A$  that reads sets of **states** instead of the usual alphabetic symbols. To **run**  $A$  on a 1-relational **digraph**  $G$ , we place a separate copy of the **machine** on every **node**  $v$  of  $G$ , **initialize** it to a **state** that may depend on  $v$ 's **label**  $\lambda^G(v)$ , and then let all the **nodes** communicate in an infinite sequence of synchronous rounds. In every round, each **node** computes its next **state** as a **function** of its own current **state** and the set of **states** of its **incoming neighbors**. Intuitively, **node**  $v$  broadcasts its current **state**  $q$  to every **outgoing neighbor**, while at the same time collecting the **states** received from its **incoming neighbors** into a set  $S$ ; the successor **state** of  $q$  is then computed as a **function** of  $q$  and  $S$ . Since  $S$  is a set (as opposed to a multiset or a vector),  $v$  cannot distinguish between two **incoming neighbors** that share the same **state**. Now, acting as a semi-decider, the **machine** at **node**  $v$  **accepts** precisely if it visits an **accepting state** at some point in time. Either way, all **machines** of the network keep running and communicating forever. This is because even if a **node** has already **accepted**, it may still obtain new information that affects the **acceptance behavior** of its **outgoing neighbors**.

Let us now define the notion of **distributed automaton** more formally, and generalize it to **digraphs** with an arbitrary number of **edge relations**.

**Definition 2.3** (Distributed automaton).

► A (deterministic, *nonlocal*) **distributed automaton** (**DA**) over  $\Sigma$ -labeled,  $r$ -relational **digraphs** is a tuple  $A = (Q, \delta_0, \delta, F)$ , where  $Q$  is a finite nonempty set of **states**,  $\delta_0: \Sigma \rightarrow Q$  is an **initialization function**,  $\delta: Q \times (\mathcal{P}^Q)^r \rightarrow Q$  is a **transition function**, and  $F \subseteq Q$  is a set of **accepting states**. ◀

Let  $G$  be a  $\Sigma$ -labeled,  $r$ -relational **digraph**. The **run** of  $A$  on  $G$  is an infinite sequence  $\rho = (\rho_0, \rho_1, \rho_2, \dots)$  of maps  $\rho_t: V^G \rightarrow Q$ , called **configurations**, which are defined inductively as follows, for  $t \in \mathbb{N}$  and  $v \in V^G$ :

$$\rho_0(v) = \delta_0(\lambda^G(v)) \quad \text{and} \quad \rho_{t+1}(v) = \delta\left(\rho_t(v), \left(\{\rho_t(u) \mid uv \in R_i^G\}\right)_{1 \leq i \leq r}\right).$$

For  $v \in V^G$ , the automaton  $A$  **accepts** the pointed digraph  $G[v]$  if  $v$  visits an **accepting state** at some point in the **run**  $\rho$  of  $A$  on  $G$ , i.e., if there exists  $t \in \mathbb{N}$  such that  $\rho_t(v) \in F$ . The **pointed-digraph language** of  $A$ , or **pointed-digraph language recognized** by  $A$ , is the set of all **pointed digraphs** that are **accepted** by  $A$ . We denote this **language** by  $\llbracket A \rrbracket_{\text{DG}_\Sigma^r}$ , in analogy to our notation for logical **formulas**. Similarly, given a class of **automata**  $\mathcal{A}$ , we write  $\llbracket \mathcal{A} \rrbracket_{\text{DG}_\Sigma^r}$  for the class of **pointed-digraph languages** over  $\text{DG}_\Sigma^r$  that are **recognized** by some member of  $\mathcal{A}$ ; we call them  **$\mathcal{A}$ -recognizable**.

As usual, two devices (i.e., **automata** or **formulas**) are **equivalent** if they specify (i.e., **recognize** or **define**) the same **language**.

In distributed computing, one often considers algorithms that run in a constant number of synchronous rounds. They are known as **local algorithms** (see, e.g.,

[Suo13]). Here, we use the same terminology for **distributed automata** and give a syntactic definition of **locality** in terms of **state** diagrams. Basically, a **distributed automaton** is **local** if its **state** diagram does not contain any directed cycles, except for self-loops on **sink states**. This is equivalent to requiring that all **nodes** stop changing their **state** after a constant number of rounds.

**Definition 2.4** (Local distributed automaton).

► A **local distributed automaton** (**LDA**) over  $r$ -relational **digraphs** is a **distributed automaton**  $A = (Q, \delta_0, \delta, F)$  whose **state** diagram satisfies the following two conditions:

- a. The only directed cycles are self-loops. That is, for every sequence  $q_1, q_2, \dots, q_n$  of **states** in  $Q$  such that  $q_1 = q_n$  and  $\delta(q_i, \vec{S}_i) = q_{i+1}$  for some  $\vec{S}_i \in (2^Q)^r$ , it must be that all **states** of the sequence are the same.
- b. Self-loops occur only on **sink states**. That is, for every **state**  $q \in Q$ , if  $\delta(q, \vec{S}) = q$  for some  $\vec{S} \in (2^Q)^r$ , then the same must hold for all  $\vec{S} \in (2^Q)^r$ . ◀

Deviating only in nonessential details from the original presentation given by Hella et al. in [HJK<sup>+</sup>12, HJK<sup>+</sup>15], we can now restate their logical characterization of the class **SB(1)** using the terminology introduced above.

**Theorem 2.5** ( $\llbracket \text{LDA} \rrbracket_{\text{DG}_\Sigma^r} = \llbracket \hat{\text{ML}} \rrbracket_{\text{DG}_\Sigma^r}$ ; [HJK<sup>+</sup>12, HJK<sup>+</sup>15]).

► A **pointed-digraph language** is **recognizable** by a **local distributed automaton** if and only if it is **definable** by a **formula of backward modal logic**. There are effective translations in both directions. ◀

The notion of **locality** plays a major role in Chapter 3, where we extend **LDA**'s with the capacity of alternation and a global acceptance condition. Our extension leaves the realm of basic **DA**'s, since we show that it is **equivalent** to **MSOL**, which by [Kuu13a, Prp. 6 & 8] is incomparable with **DA**'s.

On the other hand, in Chapters 4 and 5, we consider a simpler extension of **LDA**'s, which can be seen as a natural intermediate stage between **LDA**'s and **DA**'s. Given the above definition of **local automata**, a rather obvious generalization is to allow self-loops on all **states**, even if they are not **sink states**; we call this property **quasi-acyclic**. More formally, a **quasi-acyclic distributed automaton** (**QDA**) is a **DA** that satisfies Item a of Definition 2.4, but not necessarily Item b. An example of such an **automaton** will be provided in Section 4.1 (Figure 4.1 on page 40).



# 3

## Alternating Local Automata

In this chapter, we transfer the well-established notion of alternating automaton to the setting of **local distributed automata** and combine it with a **global acceptance condition**. This gives rise to a new class of graph automata that **recognize** precisely the **languages** of finite **digraphs definable** in **MSOL**. By restricting transitions to be nondeterministic or deterministic, we also obtain two strictly weaker variants for which the emptiness problem is decidable.

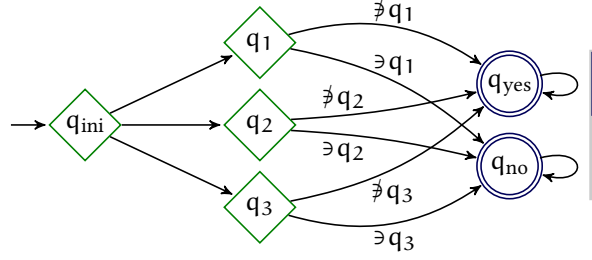
### 3.1 Informal description

We start with an informal description of the adjustments that we make to the basic model of **local automata** (see Section 2.7). Formal definitions will follow in Section 3.2.

The term “*local distributed automaton with global acceptance condition*” ( $\text{LDA}_g$ ) will be used to refer collectively to the deterministic, nondeterministic and alternating versions of our model. Let us first mention the properties they have in common.

**Levels of states.** As for basic **local automata**, the number of communication rounds is limited by a constant. To make this explicit and to simplify the subsequent definition of alternation, we associate a number, called **level**, with every **state**. In most cases, this number indicates the round in which the **state** may occur. We require that potentially **initial states** are at **level 0**, and outgoing transitions from **states** at **level  $i$**  go to **states** at **level  $i + 1$** . There is an exception, however: the **states** at the highest **level**, called the **permanent states**, can also be **initial states** and can have incoming transitions from any **level**. Moreover, all their outgoing transitions are self-loops. The idea is that, once a **node** has reached a **permanent state**, it terminates its local computation, and waits for the other **nodes** in the **digraph** to terminate too.

**Global acceptance.** Unlike for basic **local automata**, the considered input is a **digraph**, not a **pointed digraph**, and consequently the **language** recognized by an  $\text{LDA}_g$  is a **digraph language**. For this reason, once all the **nodes** have reached a **permanent state**, the  $\text{LDA}_g$  ceases to operate as a distributed algorithm, and collects all the reached **permanent states** into a set  $F$ . This set is the sole **acceptance** criterion: if  $F$



**Figure 3.1.**  $A_3^{\text{color}}$ , a nondeterministic  $\text{LDA}_g$  over unlabeled, 1-relational digraphs whose digraph language consists of the 3-colorable digraphs.

is part of the  $\text{LDA}_g$ 's accepting sets, then the input digraph is accepted, otherwise it is rejected. In particular, the automaton cannot detect whether several nodes have reached the same permanent state. This limitation is motivated by the desire to have a simple finite representation of  $\text{LDA}_g$ 's; in other words, the same reason why we do not allow nodes to distinguish between several neighbors that are in the same state.

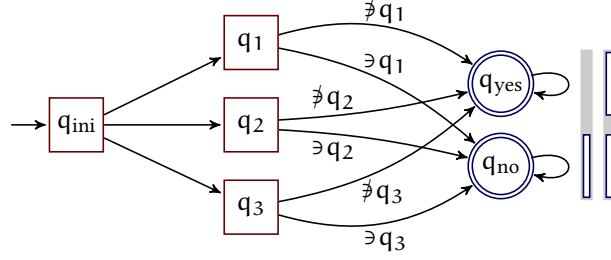
As an introductory example, we translate the MSOL-formula  $\varphi_3^{\text{color}}$  from Example 2.1 in Section 2.6 to the setting of  $\text{LDA}_g$ 's.

**Example 3.1** (3-colorability).

► Figure 3.1 shows the state diagram of a simple nondeterministic  $\text{LDA}_g$   $A_3^{\text{color}}$ . The states are arranged in columns corresponding to their levels, ascending from left to right.  $A_3^{\text{color}}$  expects an unlabeled digraph as input, and accepts it if and only if it is 3-colorable. The automaton proceeds as follows: All nodes of the input digraph are initialized to the state  $q_{\text{ini}}$ . In the first round, each node nondeterministically chooses to go to one of the states  $q_1$ ,  $q_2$  and  $q_3$ , which represent the three possible colors. Then, in the second round, the nodes verify locally that the chosen coloring is valid. If the set received from their incoming neighborhood (only one, since there is only a single edge relation) contains their own state, they go to  $q_{\text{no}}$ , otherwise to  $q_{\text{yes}}$ . The automaton then accepts the input digraph if and only if all the nodes are in  $q_{\text{yes}}$ , i.e.,  $\{q_{\text{yes}}\}$  is its only accepting set. This is indicated by the bar to the right of the state diagram. We shall refer to such a representation of sets using bars as *barcode*. ◀

The last property, which applies only to our most powerful version of  $\text{LDA}_g$ 's, is *alternation*, a generalization of nondeterminism introduced by Chandra, Kozen and Stockmeyer in [CKS81] (there, for Turing machines and other types of word automata).

**Alternation.** In addition to being able to nondeterministically choose between different transitions, nodes can also explore several choices in parallel. To this end, the nonpermanent states of an alternating  $\text{LDA}_g$  ( $\text{ALDA}_g$ ) are partitioned into two types, *existential* and *universal*, such that states on the same level are of the same type. If, in a given round, the nodes are in *existential* states, then they nondeterministically choose a single state to go to in the next round, as described above. In contrast, if they are in *universal* states, then the run of the  $\text{ALDA}_g$  is split into several parallel branches, called *universal branches*, one for each possible combination of choices of the nodes. This procedure of splitting is repeated recursively for each round in which the nodes are in *universal* states. The  $\text{ALDA}_g$  then accepts the input digraph if and only if its acceptance condition is satisfied in every universal branch of the run.



**Figure 3.2.**  $\bar{A}_3^{\text{color}}$ , an alternating  $\text{LDA}_g$  over unlabeled, 1-relational digraphs whose digraph language consists of the digraphs that are *not* 3-colorable.

**Example 3.2** (Non-3-colorability).

► To illustrate the notion of universal branching, consider the  $\text{ALDA}_g \bar{A}_3^{\text{color}}$  shown in Figure 3.2. It is a complement automaton of  $A_3^{\text{color}}$  from Example 3.1, i.e., it accepts precisely those (unlabeled) digraphs that are *not* 3-colorable. States represented as boxes are universal (whereas the diamonds in Figure 3.1 stand for existential states). Given an input digraph with  $n$  nodes,  $\bar{A}_3^{\text{color}}$  proceeds as follows: All nodes are initialized to  $q_{\text{ini}}$ . In the first round, the run is split into  $3^n$  universal branches, each of which corresponds to one possible outcome of the first round of  $A_3^{\text{color}}$  running on the same input digraph. Then, in the second round, in each of the  $3^n$  universal branches, the nodes check whether the coloring chosen in that branch is valid. As indicated by the barcode, the acceptance condition of  $\bar{A}_3^{\text{color}}$  is satisfied if and only if at least one node is in state  $q_{\text{no}}$ , i.e., the accepting sets are  $\{q_{\text{no}}\}$  and  $\{q_{\text{yes}}, q_{\text{no}}\}$ . Hence, the automaton accepts the input digraph if and only if no valid coloring was found in any universal branch. Note that we could also have chosen to make the states  $q_1$ ,  $q_2$  and  $q_3$  existential, since their outgoing transitions are deterministic. Regardless of their type, there is no branching in the second round. ◀

## 3.2 Formal definitions

We now repeat and clarify the notions from Section 3.1 in a more formal setting, beginning with our most general definition of  $\text{LDA}_g$ 's.

**Definition 3.3** (Alternating local distributed automaton).

► An *alternating local distributed automaton with global acceptance condition* ( $\text{ALDA}_g$ ) over  $\Sigma$ -labeled,  $r$ -relational digraphs is a tuple  $A = (\vec{Q}, \delta_0, \delta, \mathcal{F})$ , where

- $\vec{Q} = (Q_{\exists}, Q_{\forall}, Q_P)$  is a collection of states, with  $Q_{\exists}$ ,  $Q_{\forall}$  and  $Q_P \neq \emptyset$  being pairwise disjoint finite sets of *existential*, *universal* and *permanent states*, respectively, also referred to by the notational shorthands
  - $Q := Q_{\exists} \cup Q_{\forall} \cup Q_P$ , for the entire set of *states*,
  - $Q_N := Q_{\exists} \cup Q_{\forall}$ , for the set of *nonpermanent states*,
- $\delta_0: \Sigma \rightarrow Q$  is an *initialization function*,
- $\delta: Q \times (\mathbb{2}^Q)^r \rightarrow \mathbb{2}^Q$  is a (local) *transition function*, and
- $\mathcal{F} \subseteq \mathbb{2}^{Q_P}$  is a set of *accepting sets* of *permanent states*.

The functions  $\delta_0$  and  $\delta$  must be such that one can unambiguously associate with every state  $q \in Q$  a *level*  $l_A(q) \in \mathbb{N}$  satisfying the following conditions:

- **States** on the same *level* are of the same type, i.e., for every  $i \in \mathbb{N}$ ,

$$\{q \in Q \mid l_A(q) = i\} \in (\mathcal{P}^{Q_\exists} \cup \mathcal{P}^{Q_\forall} \cup \mathcal{P}^{Q_P}).$$

- **Initial states** are either on the lowest *level* or *permanent*, i.e., for every  $q \in Q$ ,

$$\exists a \in \Sigma: \delta_0(a) = q \quad \text{implies} \quad l_A(q) = 0 \vee q \in Q_P.$$

- **Nonpermanent states** without incoming transitions are on the lowest *level*, and transitions between *nonpermanent states* go only from one *level* to the next, i.e., for every  $q \in Q_N$ ,

$$l_A(q) = \begin{cases} 0 & \text{if for all } p \in Q \text{ and } \vec{S} \in (\mathcal{P}^Q)^r, \\ & \text{it holds that } q \notin \delta(p, \vec{S}), \\ i + 1 & \text{if there are } p \in Q_N \text{ and } \vec{S} \in (\mathcal{P}^Q)^r \\ & \text{such that } l_A(p) = i \text{ and } q \in \delta(p, \vec{S}). \end{cases}$$

- The **permanent states** are one *level* higher than the highest *nonpermanent ones*, and have only self-loops as outgoing transitions, i.e., for every  $q \in Q_P$ ,

$$l_A(q) = \begin{cases} 0 & \text{if } Q_N = \emptyset, \\ \max\{l_A(q) \mid q \in Q_N\} + 1 & \text{otherwise,} \end{cases}$$

$$\delta(q, \vec{S}) = \{q\} \quad \text{for every } \vec{S} \in (\mathcal{P}^Q)^r. \quad \blacktriangleleft$$

For any  $\text{ALDA}_g A = (\vec{Q}, \delta_0, \delta, \mathcal{F})$ , we define its *length*  $\text{len}(A)$  to be its highest *level*, that is,  $\text{len}(A) := \max\{l_A(q) \mid q \in Q\}$ .

Next, we want to give a formal definition of a *run*. For this, we need the notion of a *configuration*, which can be seen as the global state of an  $\text{ALDA}_g$ .

**Definition 3.4** (Configuration).

► Consider a *digraph*  $G$  and an  $\text{ALDA}_g A = (\vec{Q}, \delta_0, \delta, \mathcal{F})$ . For any map  $\zeta: V^G \rightarrow Q$ , we call the  $Q$ -labeled *variant*  $G[\zeta]$  of  $G$  a *configuration* of  $A$  on  $G$ . If every *node* in  $G[\zeta]$  is labeled by a *permanent state*, we refer to that *configuration* as a *permanent configuration*. Otherwise, if  $G[\zeta]$  is a *nonpermanent configuration* whose *nodes* are labeled exclusively by *existential* and (possibly) *permanent states*, we say that  $G[\zeta]$  is an *existential configuration*. Analogously, the *configuration* is *universal* if it is *nonpermanent* and only labeled by *universal* and (possibly) *permanent states*.

Additionally, we say that a *permanent configuration*  $G[\zeta]$  is *accepting* if the set of *states* occurring in it is *accepting*, i.e., if  $\{\zeta(v) \mid v \in V^G\} \in \mathcal{F}$ . Any other *permanent configuration* is called *rejecting*. *Nonpermanent configurations* are neither *accepting* nor *rejecting*.  $\blacktriangleleft$

The (local) *transition function* of an  $\text{ALDA}_g$  specifies for each *state* a set of potential successors, for a given family of sets of *states*. This can be naturally extended to *configurations*, which leads us to the definition of a *global transition function*.

**Definition 3.5** (Global transition function).

► The *global transition function*  $\delta_g$  of an  $\text{ALDA}_g A = (\bar{Q}, \delta_0, \delta, \mathcal{F})$  over  $\Sigma$ -labeled,  $r$ -relational digraphs assigns to each configuration  $G[\zeta]$  of  $A$  the set of all of its *successor configurations*  $G[\eta]$ , by combining all possible outcomes of local transitions on  $G[\zeta]$ , i.e.,

$$\delta_g: \text{DG}_Q^r \rightarrow 2^{(\text{DG}_Q^r)} \\ G[\zeta] \mapsto \left\{ G[\eta] \mid \bigwedge_{v \in V^G} \eta(v) \in \delta\left(\zeta(v), \left(\{\zeta(u) \mid uv \in R_i^G\}\right)_{i \in [r]}\right) \right\}. \quad \blacktriangleleft$$

We now have everything at hand to formalize the notion of a *run*.

**Definition 3.6** (Run).

► A *run* of an  $\text{ALDA}_g A = (\bar{Q}, \delta_0, \delta, \mathcal{F})$  over  $\Sigma$ -labeled,  $r$ -relational digraphs on a given digraph  $G \in \text{DG}_\Sigma^r$  is an acyclic digraph  $\rho$  whose nodes are configurations of  $A$  on  $G$ , such that

- the *initial configuration*  $G[\delta_0 \circ \lambda^G] \in V^\rho$  is the only *source*,
- every *nonpermanent configuration*  $G[\zeta] \in V^\rho$  with set of *successor configurations*  $\delta_g(G[\zeta]) = \{G[\eta_1], \dots, G[\eta_m]\}$  has
  - exactly one *outgoing neighbor*  $G[\eta_i] \in \delta_g(G[\zeta])$  if  $G[\zeta]$  is *existential*,
  - exactly  $m$  *outgoing neighbors*  $G[\eta_1], \dots, G[\eta_m]$  if  $G[\zeta]$  is *universal*, and
- every *permanent configuration*  $G[\zeta] \in V^\rho$  is a *sink*.

Here, the operator  $\circ$  denotes standard function composition, such that  $(\delta_0 \circ \lambda^G)(v) = \delta_0(\lambda^G(v))$ .

The run  $\rho$  is *accepting* if every permanent configuration  $G[\zeta] \in V^\rho$  is accepting.  $\blacktriangleleft$

An  $\text{ALDA}_g A = (\bar{Q}, \delta_0, \delta, \mathcal{F})$  over  $\Sigma$ -labeled,  $r$ -relational digraphs *accepts* a given digraph  $G \in \text{DG}_\Sigma^r$  if and only if there exists an accepting run  $\rho$  of  $A$  on  $G$ . The digraph language *recognized* by  $A$  is the set

$$\llbracket A \rrbracket_{\text{DG}_\Sigma^r} := \{G \in \text{DG}_\Sigma^r \mid A \text{ accepts } G\}.$$

A digraph language that is recognized by some  $\text{ALDA}_g$  is called  *$\text{ALDA}_g$ -recognizable*. We denote by  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  the class of all such digraph languages.

The  $\text{ALDA}_g A$  is *equivalent* to some MSOL-formula  $\varphi$  if it recognizes precisely the digraph language defined by  $\varphi$  over  $\text{DG}_\Sigma^r$ , i.e., if  $\llbracket A \rrbracket_{\text{DG}_\Sigma^r} = \llbracket \varphi \rrbracket_{\text{DG}_\Sigma^r}$ .

We inductively define that a configuration  $G[\zeta] \in \text{DG}_Q^r$  is *reachable* by  $A$  on  $G$  if either  $G[\zeta] = G[\delta_0 \circ \lambda^G]$ , or  $G[\zeta] \in \delta_g(G[\eta])$  for some configuration  $G[\eta] \in \text{DG}_Q^r$  reachable by  $A$  on  $G$ . In case  $G$  is irrelevant, we simply say that  $G[\zeta]$  is *reachable* by  $A$ .

The automaton  $A$  is called a *nondeterministic  $\text{LDA}_g$  ( $\text{NLDA}_g$ )* if it has no *universal states*, i.e., if  $Q_V = \emptyset$ . If additionally every configuration  $G[\zeta] \in \text{DG}_Q^r$  that is *reachable* by  $A$  has precisely one *successor configuration*, i.e.,  $|\delta_g(G[\zeta])| = 1$ , then we refer to  $A$  as a *deterministic  $\text{LDA}_g$  ( $\text{DLDA}_g$ )*. We denote the classes of  $\text{NLDA}_g$ - and  $\text{DLDA}_g$ -recognizable digraph languages by  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  and  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ .

Let us now illustrate the notion of  $\text{ALDA}_g$  using a slightly more involved example.

**Example 3.7** (Concentric circles).

► Consider the  $\text{ALDA}_g$   $A_{\text{centric}} = (\vec{Q}, \delta_0, \delta, \mathcal{F})$  over  $\{a, b, c\}$ -labeled digraphs represented by the state diagram in Figure 3.3. Again, existential states are represented by diamonds, universal states by boxes, and permanent states by double circles. The short arrows mapping node labels to states indicate the initialization function  $\delta_0$ . For instance,  $\delta_0(a) = q_a$ . The other arrows specify the transition function  $\delta$ . A label on such a transition arrow indicates a requirement on the set of states that a node receives from its incoming neighborhood (only one set, since there is only a single edge relation). For instance,  $\delta(q_b, (\{q_a, q_c\})) = \{q_{b:1}, q_{b:2}\}$ . If there is no label, any set is permitted. Finally, as indicated by the barcode on the far right, the set of accepting sets is  $\mathcal{F} = \{\{q_{a:3}, q_{\text{yes}}\}, \{q_{a:4}, q_{\text{yes}}\}\}$ .

Intuitively,  $A_{\text{centric}}$  proceeds as follows: In the first round, the  $a$ -labeled nodes do nothing but update their state, while the  $b$ - and  $c$ -labeled nodes verify that the labels in their incoming neighborhood satisfy the condition of a valid graph coloring. The  $c$ -labeled nodes additionally check that they do not see any  $a$ 's, and then directly terminate. Meanwhile, the  $b$ -labeled nodes nondeterministically choose one of the markers 1 and 2. In the second round, only the  $a$ -labeled nodes are busy. They verify that their incoming neighborhood consists exclusively of  $b$ -labeled nodes, and that both of the markers 1 and 2 are present, thus ensuring that they have at least two incoming neighbors. Then, they simultaneously pick the markers 3 and 4, thereby creating different universal branches, and the run of the automaton terminates. Finally, the  $\text{ALDA}_g$  checks that all the nodes approve of the digraph (meaning that none of them has reached the state  $q_{\text{no}}$ ), and that in each universal branch, precisely one of the markers 3 and 4 occurs, which implies that there is a unique  $a$ -labeled node.

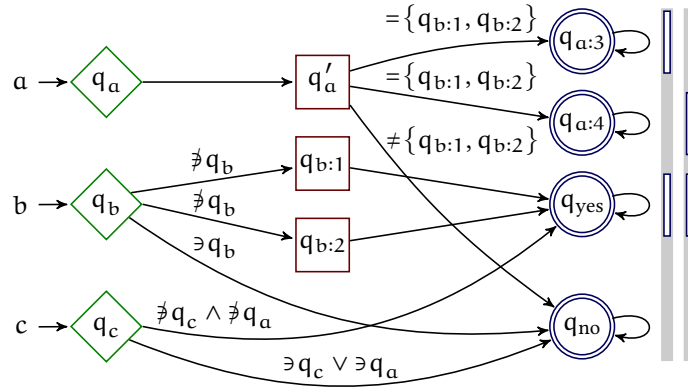
To sum up, the digraph language  $\llbracket A_{\text{centric}} \rrbracket_{\text{DG}_\Sigma^+}$  consists of all the  $\{a, b, c\}$ -labeled, digraphs such that

- the labeling constitutes a valid 3-coloring,
- there is precisely one  $a$ -labeled node  $v_a$ , and
- $v_a$  has only  $b$ -labeled nodes in its undirected neighborhood, and at least two incoming neighbors.

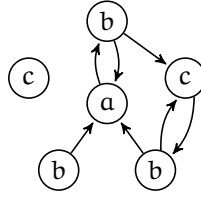
The name “ $A_{\text{centric}}$ ” refers to the fact that, in the (weakly) connected component of  $v_a$ , the  $b$ - and  $c$ -labeled nodes form concentric circles around  $v_a$ , i.e., nodes at distance 1 of  $v_a$  are labeled with  $b$ , nodes at distance 2 (if existent) with  $c$ , nodes at distance 3 (if existent) with  $b$ , and so forth.

Figure 3.4 shows an example of a labeled digraph that lies in  $\llbracket A_{\text{centric}} \rrbracket_{\text{DG}_\Sigma^+}$ . A corresponding accepting run can be seen in Figure 3.5. The leftmost configuration is existential, the next one is universal, and the two double-circled ones are permanent. In the first round, the three nodes that are in state  $q_b$  have a nondeterministic choice between  $q_{b:1}$  and  $q_{b:2}$ . Hence, the second configuration is one of eight possible choices. The branching in the second round is due to the node in state  $q'_a$  which goes simultaneously to  $q_{a:3}$  and  $q_{a:4}$ . In both branches, an accepting configuration is reached, since  $\{q_{a:3}, q_{\text{yes}}\}$  and  $\{q_{a:4}, q_{\text{yes}}\}$  are both accepting sets. Therefore, the entire run is accepting. ◀

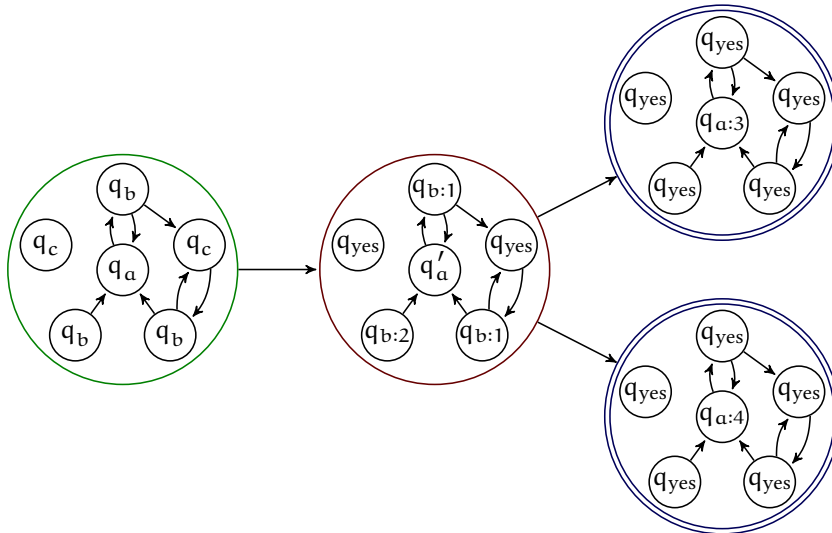
In the following subsections (3.3, 3.4 and 3.5), we derive our results on several properties of  $\text{LDA}_g$ 's.



**Figure 3.3.**  $A_{\text{centric}}$ , an  $\text{ALDA}_g$  over  $\{a, b, c\}$ -labeled digraphs whose digraph language consists of the labeled digraphs that satisfy the following conditions: the labeling constitutes a valid 3-coloring, there is precisely one  $a$ -labeled node  $v_a$ , the undirected neighborhood of  $v_a$  contains only  $b$ -labeled nodes, and  $v_a$  has at least two incoming neighbors.



**Figure 3.4.** An  $\{a, b, c\}$ -labeled, digraph.



**Figure 3.5.** An accepting run of the  $\text{ALDA}_g$  of Figure 3.3 on the labeled digraph shown in Figure 3.4.



### 3.3 Hierarchy and closure properties

By a (node) *projection* we mean a mapping  $h: \Sigma \rightarrow \Sigma'$  between two alphabets  $\Sigma$  and  $\Sigma'$ . With slight abuse of notation, such a mapping is extended to *labeled digraphs* by applying it to each node label, and to *digraph languages* by applying it to each *labeled digraph*. That is, for every  $G \in \text{DG}_\Sigma^r$  and  $L \subseteq \text{DG}_\Sigma^r$ ,

$$h(G) := G[h \circ \lambda^G], \quad \text{and} \quad h(L) := \{h(G) \mid G \in L\},$$

where the operator  $\circ$  denotes function composition, such that  $(h \circ \lambda^G)(v) = h(\lambda^G(v))$ .

**Proposition 3.8** (Closure properties of  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ).

► The class  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  of *ALDA<sub>g</sub>-recognizable digraph languages* is effectively closed under Boolean set operations and under *projection*. ◀

*Proof sketch.* As usual for alternating automata, complementation can be achieved by simply swapping the *existential* and *universal states*, and complementing the *acceptance condition*. That is, for an *ALDA<sub>g</sub>*  $A = ((Q_\exists, Q_\forall, Q_P), \delta_0, \delta, \mathcal{F})$  over  $\Sigma$ -labeled,  $r$ -relational *digraphs*, a complement *automaton* is  $\bar{A} = ((Q_\forall, Q_\exists, Q_P), \delta_0, \delta, \mathbb{2}^{Q_P} \setminus \mathcal{F})$ . This can be easily seen by associating a two-player game with  $A$  and any  $\Sigma$ -labeled,  $r$ -relational *digraph*  $G$ . One player tries to come up with an *accepting run* of  $A$  on  $G$ , whereas the other player seeks to find a (path to a) *rejecting configuration* in any *run* proposed by the adversary. The first player has a winning strategy if and only if  $A$  *accepts*  $G$ . (This game-theoretic characterization will be used and explained more extensively in the proof of [Theorem 3.13](#).) From this perspective, the construction of  $\bar{A}$  corresponds to interchanging the roles and winning conditions of the two players.

For two *ALDA<sub>g</sub>*'s  $A_1$  and  $A_2$ , we can effectively construct an *ALDA<sub>g</sub>*  $A_\cup$  that *recognizes*  $\llbracket A_1 \rrbracket_{\text{DG}_\Sigma^r} \cup \llbracket A_2 \rrbracket_{\text{DG}_\Sigma^r}$  by taking advantage of nondeterminism. The approach is, in principle, very similar to the corresponding construction for nondeterministic finite automata on *words*. In the first round of  $A_\cup$ , each *node* in the input *digraph* nondeterministically and independently decides whether to behave like in  $A_1$  or in  $A_2$ . If there is a consensus, then the *run* continues as it would in the unanimously chosen *automaton*  $A_j$ , and it is *accepting* if and only if it corresponds to an *accepting run* of  $A_j$ . Otherwise, a conflict is detected, either locally by *adjacent nodes* that have chosen different *automata*, or at the latest, when *acceptance* is checked globally (important for *disconnected digraphs*), and in either case the *run* is *rejecting*. (Note that we have omitted some technicalities that ensure that the construction outlined above satisfies all the properties of an *ALDA<sub>g</sub>*.)

Closure under *node projection* is straightforward, again by exploiting nondeterminism. Given an *ALDA<sub>g</sub>*  $A$  with node alphabet  $\Sigma$  and a *projection*  $h: \Sigma \rightarrow \Sigma'$ , we can effectively construct an *ALDA<sub>g</sub>*  $A'$  that *recognizes*  $h(\llbracket A \rrbracket_{\text{DG}_\Sigma^r})$  as follows: For every  $b \in \Sigma'$ , each *node labeled* with  $b$  nondeterministically chooses a new label  $a \in \Sigma$ , such that  $h(a) = b$ . Then, the *automaton*  $A$  is simulated on that new input. ■

**Proposition 3.9** ( $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subset \llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ).

► There are (infinitely many) *ALDA<sub>g</sub>-recognizable digraph languages* that are not *NLDA<sub>g</sub>-recognizable*. ◀

*Proof.* For any constant  $k \geq 1$ , we consider the *language*  $L_{\leq k}^{\text{card}}$  of all *digraphs* that have at most  $k$  *nodes*, i.e.,  $L_{\leq k}^{\text{card}} = \{G \in \text{DG} \mid |V^G| \leq k\}$ . We can easily construct an



$\text{ALDA}_g$  that recognizes this digraph language: In a universal branching, each node goes to  $k + 1$  different states in parallel. The automaton accepts if and only if there is no branch in which the  $k + 1$  states occur all at once. Now, assume for sake of contradiction that  $L_{\leq k}^{\text{card}}$  is also recognized by some  $\text{NLDA}_g$   $A$ , and let  $G$  be a digraph with  $k$  nodes. We construct a variant  $G'$  of  $G$  with  $k + 1$  nodes by duplicating some node  $v$ , together with all of its incoming and outgoing edges. Observe that any accepting run of  $A$  on  $G$  can be extended to an accepting run on  $G'$ , where the copy of  $v$  behaves exactly like  $v$  in every round. ■

**Proposition 3.10** (Closure properties of  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ).

► The class  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  of  $\text{NLDA}_g$ -recognizable digraph languages is effectively closed under union, intersection and projection, but *not* closed under complementation. ◀

*Proof.* For union and projection, we simply use the same constructions as for  $\text{ALDA}_g$ 's (see Proposition 3.8).

Intersection can be handled by a product construction, similar to the one for finite automata on words. Given two  $\text{NLDA}_g$ 's  $A_1$  and  $A_2$ , we construct an  $\text{NLDA}_g$   $A_\otimes$  that operates on the Cartesian product of the state sets of  $A_1$  and  $A_2$ . It simulates the two automata simultaneously and accepts if and only if both of them reach an accepting configuration.

To see that  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  is not closed under complementation, we recall from the proof of Proposition 3.9 that for any  $k \geq 1$ , the language  $L_{\leq k}^{\text{card}}$  of all digraphs that have at most  $k$  nodes is not  $\text{NLDA}_g$ -recognizable. However, complementing the  $\text{ALDA}_g$  given for  $L_{\leq k}^{\text{card}}$  yields an  $\text{NLDA}_g$  that recognizes the complement language  $L_{\geq k+1}^{\text{card}}$ . ■

**Proposition 3.11** ( $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subset \llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ).

► There are (infinitely many)  $\text{NLDA}_g$ -recognizable digraph languages that are not  $\text{DLDA}_g$ -recognizable. ◀

*Proof.* Let  $k \geq 2$ . As mentioned in the proof of Proposition 3.10, the language  $L_{\geq k}^{\text{card}}$  of all digraphs that have at least  $k$  nodes is  $\text{NLDA}_g$ -recognizable. To see that it is not  $\text{DLDA}_g$ -recognizable, consider (similarly to the proof of Proposition 3.9) a digraph  $G$  with  $k - 1$  nodes and a variant  $G'$  with  $k$  nodes obtained from  $G$  by duplicating some node  $v$ , together with all of its incoming and outgoing edges. Given any  $\text{DLDA}_g$   $A$ , the determinism of  $A$  guarantees that  $v$  and its copy  $v'$  behave the same way in the (unique) run of  $A$  on  $G'$ . Hence, if that run is accepting, so is the run on  $G$ . ■

**Proposition 3.12** (Closure properties of  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ).

► The class  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  of  $\text{DLDA}_g$ -recognizable digraph languages is effectively closed under Boolean set operations, but *not* closed under projection. ◀

*Proof.* To complement a  $\text{DLDA}_g$ , we can simply complement its set of accepting sets. The product construction for intersection of  $\text{NLDA}_g$ 's mentioned in Proposition 3.10 remains applicable when restricted to  $\text{DLDA}_g$ 's.

Closure under node projection does not hold because we can, for instance, construct a  $\text{DLDA}_g$  that recognizes the language  $L_{a,b,c}^{\text{occur}}$  of all  $\{a, b, c\}$ -labeled digraphs in which each of the three node labels occurs at least once. However, projection under the mapping  $h: \{a, b, c\} \rightarrow \{\varepsilon\}$ , with  $h(a) = h(b) = h(c) = \varepsilon$  (the empty word), yields the digraph language  $h(L_{a,b,c}^{\text{occur}}) = L_{\geq 3}^{\text{card}}$ , which is not  $\text{DLDA}_g$ -recognizable (see the proof of Proposition 3.11). ■

### 3.4 Equivalence with monadic second-order logic

**Theorem 3.13** ( $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r} = \llbracket \text{MSOL} \rrbracket_{\text{DG}_\Sigma^r}$ ).

► A digraph language is  $\text{ALDA}_g$ -recognizable if and only if it is  $\text{MSOL}$ -definable. There are effective translations in both directions. ◀

*Proof sketch.* ( $\Rightarrow$ ) We start with the direction  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subseteq \llbracket \text{MSOL} \rrbracket_{\text{DG}_\Sigma^r}$ . Let  $A = (\vec{Q}, \delta_0, \delta, \mathcal{F})$  be an  $\text{ALDA}_g$  of length  $n$  over  $\Sigma$ -labeled,  $r$ -relational digraphs. Without loss of generality, we may assume that every configuration reachable by  $A$  has at least one successor configuration and that no permanent configuration is reachable in less than  $n$  rounds. In order to encode the acceptance behavior of  $A$  into an  $\text{MSOL}$ -formula  $\varphi_A$ , we use again the game-theoretic characterization briefly mentioned in the proof sketch of Proposition 3.8. Given  $A$  and some  $G \in \text{DG}_\Sigma^r$ , we consider a game with two players: the *automaton* (player  $\exists$ ) and the *pathfinder* (player  $\forall$ ). This game is represented by an acyclic digraph whose nodes are precisely the configurations reachable by  $A$  on  $G$ . For any two *nonpermanent* configurations  $G[\zeta]$  and  $G[\eta]$ , there is a directed edge from  $G[\zeta]$  to  $G[\eta]$  if and only if  $G[\eta] \in \delta_g(G[\zeta])$ . Starting at the initial configuration  $G[\delta_0 \circ \lambda^G]$ , the two players move through the game together by following directed edges. If the current configuration is *existential*, then the automaton has to choose the next move, if it is *universal*, then the decision belongs to the pathfinder. This continues until some *permanent* configuration is reached. The automaton wins if that *permanent* configuration is *accepting*, whereas the pathfinder wins if it is *rejecting*. A player is said to have a *winning strategy* if it can always win, independently of its opponent's moves. It is straightforward to prove that the automaton has a winning strategy if and only if  $A$  accepts  $G$ . Our  $\text{MSOL}$ -formula  $\varphi_A$  will express the existence of such a winning strategy, and thus be equivalent to  $A$ .

Within  $\text{MSOL}$ , we represent a path  $G[\zeta_0] \cdots G[\zeta_n]$  through the game by a sequence of families of set variables  $\vec{X}_0, \dots, \vec{X}_n$ , where  $\vec{X}_0 = ()$  and  $\vec{X}_i = (X_{i,q})_{q \in Q}$ , for  $1 \leq i \leq n$ . The intention is that each set variable  $X_{i,q}$  is interpreted as the set of nodes  $v \in V^G$  for which  $\zeta_i(v) = q$ . (We do not need set variables to represent  $G[\zeta_0]$ , since the players always start at  $G[\delta_0 \circ \lambda^G]$ .)

Now, for every round  $i$ , we construct a formula  $\varphi_i^{\text{win}}(\vec{X}_i)$  (i.e., with free variables in  $\vec{X}_i$ ), which expresses that the automaton has a winning strategy in the subgame starting at the configuration  $G[\zeta_i]$  represented by  $\vec{X}_i$ . In case  $G[\zeta_i]$  is *existential*, this is true if the automaton has a winning strategy in some *successor configuration* of  $G[\zeta_i]$ , whereas if  $G[\zeta_i]$  is *universal*, the automaton must have a winning strategy in all *successor configurations* of  $G[\zeta_i]$ . This yields the following recursive definition for  $0 \leq i \leq n-1$ :

$$\varphi_i^{\text{win}}(\vec{X}_i) := \exists_{\vec{X}_{i+1}} \left( \varphi_{i+1}^{\text{succ}}(\vec{X}_i, \vec{X}_{i+1}) \wedge \varphi_{i+1}^{\text{win}}(\vec{X}_{i+1}) \right),$$

if level  $i$  of  $A$  is *existential*, and

$$\varphi_i^{\text{win}}(\vec{X}_i) := \forall_{\vec{X}_{i+1}} \left( \varphi_{i+1}^{\text{succ}}(\vec{X}_i, \vec{X}_{i+1}) \rightarrow \varphi_{i+1}^{\text{win}}(\vec{X}_{i+1}) \right),$$

if level  $i$  of  $A$  is *universal*. Here,  $\varphi_{i+1}^{\text{succ}}(\vec{X}_i, \vec{X}_{i+1})$  is an  $\text{FOL}$ -formula expressing that  $\vec{X}_i$  and  $\vec{X}_{i+1}$  represent two configurations  $G[\zeta_i]$  and  $G[\zeta_{i+1}]$  such that  $G[\zeta_{i+1}] \in \delta_g(G[\zeta_i])$ . As our recursion base, we can easily construct a formula  $\varphi_n^{\text{win}}(\vec{X}_n)$  that is *satisfied* if and only if  $\vec{X}_n$  represents an *accepting configuration* of  $A$ .

The desired  $\text{MSOL}$ -formula is  $\varphi_A := \varphi_0^{\text{win}}(\vec{X}_0) = \varphi_0^{\text{win}}()$ .

*This characterization is heavily inspired by the work of Löding and Thomas in [LT00].*

( $\Leftarrow$ ) For the direction  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r} \supseteq \llbracket \text{MSOL} \rrbracket_{\text{DG}_\Sigma^r}$ , we can proceed by induction on the structure of an **MSOL-formula**  $\varphi$ . In order to deal with **free** occurrences of **node symbols**, we encode their **interpretations** into the node labels (which normally encode only the **interpretations** of **set symbols**). Let  $\text{free}_0(\varphi)$  be an abbreviation for  $\text{free}(\varphi) \cap \mathbb{S}_0$ . For  $G \in \text{DG}_\Sigma^r$  and  $\alpha: \text{free}_0(\varphi) \rightarrow V^G$ , we represent  $G[\alpha]$  as the labeled digraph  $G[\lambda^G \times \alpha^{-1}]$  whose labeling  $\lambda^G \times \alpha^{-1}$  assigns to each **node**  $v \in V^G$  the tuple  $(\lambda^G(v), \alpha^{-1}(v))$ , where  $\alpha^{-1}(v)$  is the set of all **node symbols** in  $\text{free}_0(\varphi)$  to which  $\alpha$  assigns  $v$ . We now inductively construct an **ALDA<sub>g</sub>**  $A_\varphi = (\vec{Q}, \delta_0, \delta, \mathcal{F})$  over  $r$ -relational **digraphs** labeled with the alphabet  $\Sigma' = \Sigma \times \mathcal{P}^{\text{free}_0(\varphi)}$ , such that

$$G[\lambda^G \times \alpha^{-1}] \in \llbracket A_\varphi \rrbracket_{\text{DG}_\Sigma^r}, \quad \text{if and only if} \quad G[\alpha] \models \varphi.$$

*Base case.* Let  $x, y \in \mathbb{S}_0$ ,  $X \in \mathbb{S}_1$  and  $i \in [r]$ . If  $\varphi$  is one of the atomic **formulas**  $x \doteq y$  or  $X(x)$ , then, in  $A_\varphi$ , each **node** simply checks that its own label  $(a, M) \in \Sigma \times \mathcal{P}^{\text{free}_0(\varphi)}$  satisfies the condition specified in  $\varphi$  (which, in particular, is the case if  $x, y \notin M$ ). Since this can be directly encoded into the **initialization function**  $\delta_0$ , the **ALDA<sub>g</sub>** has **length** 0. It **accepts** the input **digraph** if and only if every **node** reports that its label satisfies the condition.

The case  $\varphi = R_i(x, y)$  is very similar, but  $A_\varphi$  needs one communication round, after which the **node** assigned to  $y$  can check whether it has received a message through an  $i$ -**edge** from the **node** assigned to  $x$ . Accordingly,  $A_\varphi$  has **length** 1.

*Inductive step.* In case  $\varphi$  is a composed **formula**, we can obtain  $A_\varphi$  by means of the constructions outlined in the proof sketch of **Proposition 3.8** (closure properties of  $\llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}$ ). Let  $\psi$  and  $\psi'$  be **MSOL-formulas** with **equivalent ALDA<sub>g</sub>'s**  $A_\psi$  and  $A_{\psi'}$ , respectively.

If  $\varphi = \neg\psi$ , it suffices to define  $A_\varphi = \bar{A}_\psi$ . Similarly, if  $\varphi = \psi \vee \psi'$ , we get  $A_\varphi$  by applying the union construction on  $A_\psi$  and  $A_{\psi'}$ . (In general, we first have to extend  $A_\psi$  and  $A_{\psi'}$  such that they both operate on the same node alphabet  $\Sigma \times \mathcal{P}^{\text{free}_0(\psi) \cup \text{free}_0(\psi')}$ .)

**Existential quantification** can be handled by **node projection**. If  $\varphi = \exists_X(\psi)$ , with  $X \in \mathbb{S}_1$ , we construct  $A_\varphi$  by applying the **projection** construction on  $A_\psi$ , using a mapping  $h: \Sigma \times \mathcal{P}^{\text{free}_0(\psi)} \rightarrow \tilde{\Sigma} \times \mathcal{P}^{\text{free}_0(\psi)}$  that deletes the **set variable**  $X$  from every label. In other words, the new alphabet  $\tilde{\Sigma}$  encodes the subsets of  $\text{free}(\psi) \cap (\mathbb{S}_1 \setminus \{X\})$ . An analogous approach can be used if  $\varphi = \exists_x(\psi)$ , with  $x \in \mathbb{S}_0$ . The only difference is that, instead of applying the **projection** construction directly on  $A_\psi$ , we apply it on a variant  $A'_\psi$  that operates just like  $A_\psi$ , but additionally checks that precisely one **node** in the input **digraph** is assigned to the **node variable**  $x$ . ■

From **Theorem 3.13** we can immediately infer that it is undecidable whether the **digraph language recognized** by some arbitrary **ALDA<sub>g</sub>** is empty. Otherwise, we could decide the satisfiability problem of **MSOL** on **digraphs**, which is known to be undecidable (a direct consequence of Trakhtenbrot's Theorem, see, e.g., [Lib04, Thm 9.2]).

**Corollary 3.14** (Emptiness problem of **ALDA<sub>g</sub>'s**).

► The emptiness problem for **ALDA<sub>g</sub>'s** is undecidable. ◀

### 3.5 Emptiness problem for nondeterministic automata

At the cost of reduced expressive power, we can also obtain a positive decidability result.

**Proposition 3.15** (Emptiness problem of  $\text{NLDA}_g$ 's).

► The emptiness problem of  $\text{NLDA}_g$ 's is decidable in doubly-exponential time. More precisely, for any  $\text{NLDA}_g A = (\bar{Q}, \delta_0, \delta, \mathcal{F})$  over  $\Sigma$ -labeled,  $r$ -relational digraphs, whether its recognized digraph language  $\llbracket A \rrbracket_{\text{DG}_\Sigma^r}$  is empty or not can be decided in time  $2^k$ , where  $k \in O(r \cdot |Q|^{4 \cdot \text{len}(A)} \cdot \text{len}(A))$ .

Furthermore, whether or not the digraph language  $\llbracket A \rrbracket_{\text{DG}_\Sigma^r}$  contains any *connected, undirected graph* can be decided in time  $2^{2^{k'}}$ , where  $k' \in O(r \cdot |Q| \cdot \text{len}(A))$ . ◀

*Proof sketch.* Let  $G \in \text{DG}_\Sigma^r$ . Since  $\text{NLDA}_g$ 's cannot perform universal branching, we can consider any run of  $A$  on  $G$  as a sequence of configurations  $\rho = G[\zeta_0] \cdots G[\zeta_n]$ , with  $n \leq \text{len}(A)$ . In  $\rho$ , each node of  $G$  traverses one of at most  $|Q|^{\text{len}(A)+1}$  possible sequences of states. Now, assume that  $G$  has more than  $|Q|^{\text{len}(A)+1}$  nodes. Then, by the Pigeonhole Principle, there must be two distinct nodes  $v, v' \in V^G$  that traverse the same sequence of states in  $\rho$ . We construct a smaller digraph  $G'$  by removing  $v'$  from  $G$ , together with its adjacent edges, and adding directed edges from  $v$  to all of the former outgoing neighbors of  $v'$ . If all the nodes in  $G'$  maintain their nondeterministic choices from  $\rho$ , none of them will notice that  $v'$  is missing, and consequently they all behave just as in  $\rho$ . The resulting run  $\rho'$  on  $G'$  is accepting if and only if  $\rho$  is accepting.

Applying this argument recursively, we conclude that if  $\llbracket A \rrbracket_{\text{DG}_\Sigma^r}$  is not empty, then it must contain some labeled digraph that has at most  $|Q|^{\text{len}(A)+1}$  nodes. Hence, the emptiness problem is decidable because the search space is finite. The time complexity indicated above corresponds to the naive approach of checking every digraph with at most  $|Q|^{\text{len}(A)+1}$  nodes.

If we are only interested in *connected, undirected graphs*, the reasoning is very similar, but we have to require a larger minimum number of nodes in order to be able to remove some node without influencing the behavior of the others. In a graph  $G$  with more than  $(|Q| \cdot 2^{r \cdot |Q|})^{\text{len}(A)+1}$  nodes, there must be two distinct nodes  $v, v' \in V^G$  that, in addition to traversing the same sequence of states, also receive the same family of sets of states from their neighborhood in every round. Observe that the automaton will not notice if we merge  $v$  and  $v'$ . The rest of the argument is analogous to the previous scenario. ■

### 3.6 Summary and discussion

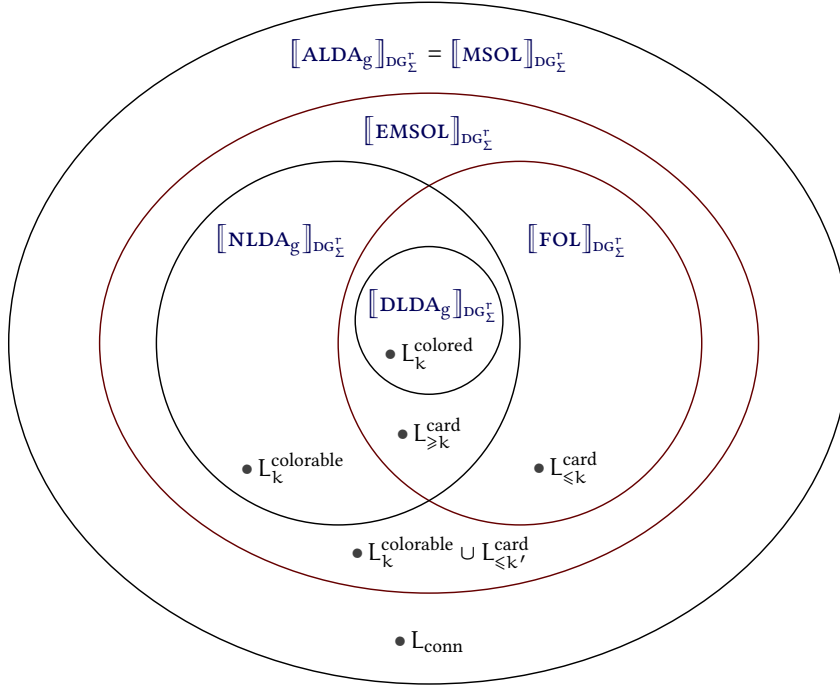
We have introduced  $\text{ALDA}_g$ 's, which are probably the first graph automata in the literature to be equivalent to  $\text{MSOL}$  on digraphs. However, their expressive power results mainly from the use of alternation: we have seen that the deterministic, nondeterministic and alternating variants form a strict hierarchy, i.e.,

$$\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subset \llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subset \llbracket \text{ALDA}_g \rrbracket_{\text{DG}_\Sigma^r}.$$

The corresponding closure and decidability properties are summarized in Table 3.1.

	Closure Properties				Decidability
	Complement	Union	Intersection	Projection	Emptiness
$\text{ALDA}_g$	✓	✓	✓	✓	✗
$\text{NLDA}_g$	✗	✓	✓	✓	✓
$\text{DLDA}_g$	✓	✓	✓	✗	✓

**Table 3.1.** Closure and decidability properties of alternating, nondeterministic, and deterministic  $\text{LDA}_g$ 's.



**Figure 3.6.** Venn diagram relating the classes of digraph languages recognizable by our three flavors of  $\text{LDA}_g$ 's to those definable in  $\text{MSOL}$ ,  $\text{EMSOL}$  and  $\text{FOL}$ .

On an intuitive level, this hierarchy and these closure properties do not seem very surprising. One might even ask: *are  $\text{ALDA}_g$ 's just another syntax for  $\text{MSOL}$ ?* Indeed, universal branchings correspond to universal quantification, and nondeterministic choices to existential quantification. By disallowing universal set quantification in  $\text{MSOL}$  we obtain  $\text{EMSOL}$ , and further disallowing existential set quantification yields  $\text{FOL}$ . Analogously to  $\text{LDA}_g$ 's, the classes of digraph languages definable in these logics form a strict hierarchy, i.e.,

$$[\text{FOL}]_{\text{DG}^r_\Sigma} \subset [\text{EMSOL}]_{\text{DG}^r_\Sigma} \subset [\text{MSOL}]_{\text{DG}^r_\Sigma}.$$

Furthermore, the closure properties of  $[\text{EMSOL}]_{\text{DG}^r_\Sigma}$  and  $[\text{FOL}]_{\text{DG}^r_\Sigma}$  coincide with those of  $[\text{NLDA}_g]_{\text{DG}^r_\Sigma}$  and  $[\text{DLDA}_g]_{\text{DG}^r_\Sigma}$ , respectively. Given that  $[\text{ALDA}_g]_{\text{DG}^r_\Sigma}$  and  $[\text{MSOL}]_{\text{DG}^r_\Sigma}$  are equal, one might therefore expect that the analogous equalities hold for the weaker classes. However, as already hinted by the positive decidability properties in Table 3.1, this is not the case. The actual relationships between the different classes of digraph languages are depicted in Figure 3.6. A glance at this diagram suggests



that  $\text{ALDA}_g$ 's are not simply a one-to-one reproduction of  $\text{MSOL}$ .

*Justification of Figure 3.6.* Fagin has shown in [Fag75] that the language  $L_{\text{conn}}$  of all (weakly) connected digraphs separates  $\llbracket \text{EMSOL} \rrbracket_{\text{DG}_\Sigma^r}$  from  $\llbracket \text{MSOL} \rrbracket_{\text{DG}_\Sigma^r}$ . (Since non-connectivity is  $\text{EMSOL}$ -definable, this also implies that  $\llbracket \text{EMSOL} \rrbracket_{\text{DG}_\Sigma^r}$  is not closed under complementation.) The inclusion  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subseteq \llbracket \text{EMSOL} \rrbracket_{\text{DG}_\Sigma^r}$  holds because we can encode every  $\text{NLDA}_g$  into an  $\text{EMSOL}$ -formula, using the same construction as in the proof sketch of Theorem 3.13. It is also easy to see that we can encode a  $\text{DLDA}_g$  by inductively constructing a family of  $\text{FOL}$ -formulas  $\varphi_{i;q}^{\text{state}}(x)$ , stating that in round  $i$  (of the *unique run* of the automaton), the node assigned to  $x$  is in state  $q$ . Hence,  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \subseteq \llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$ . In the following, let  $k, k' \geq 2$ . The incomparability of  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  and  $\llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$  is witnessed by the language  $L_k^{\text{colorable}}$  of  $k$ -colorable digraphs, which lies within  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  (see Example 3.1) but outside of  $\llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$  (see, e.g., [Lib04]), and the language  $L_{\leq k}^{\text{card}}$  of digraphs with at most  $k$  nodes, which lies outside of  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  (see the proof of Proposition 3.9) but obviously within  $\llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$ . Considering the union language  $L_k^{\text{colorable}} \cup L_{\leq k'}^{\text{card}}$  also tells us that the inclusion of  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \cup \llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$  in  $\llbracket \text{EMSOL} \rrbracket_{\text{DG}_\Sigma^r}$  is strict. Finally, the language  $L_{\geq k}^{\text{card}}$  of digraphs with at least  $k$  nodes separates  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  from  $\llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} \cap \llbracket \text{FOL} \rrbracket_{\text{DG}_\Sigma^r}$  (see the proof of Proposition 3.11). A simple example of a language that lies within  $\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r}$  is the set  $L_k^{\text{colored}}$  of  $\Sigma$ -labeled digraphs whose labelings are valid  $k$ -colorings, with  $|\Sigma| = k$ . ■

Nevertheless, based on the equivalence of  $\text{LDA}$ 's and  $\tilde{\text{ML}}$  established by Hella et al. (Theorem 2.5), we can actually obtain precise logical characterizations of  $\text{DLDA}_g$ 's and  $\text{NLDA}_g$ 's by extending  $\tilde{\text{ML}}$  with global modalities and existential set quantifiers. Adapting the proofs of [HJK<sup>+</sup>12, HJK<sup>+</sup>15] to our setting, it is relatively easy to show that

$$\llbracket \text{DLDA}_g \rrbracket_{\text{DG}_\Sigma^r} = \llbracket \tilde{\text{ML}}_g \rrbracket_{\text{DG}_\Sigma^r} \quad \text{and} \quad \llbracket \text{NLDA}_g \rrbracket_{\text{DG}_\Sigma^r} = \llbracket \Sigma_1^{\text{MSO}}(\tilde{\text{ML}}_g) \rrbracket_{\text{DG}_\Sigma^r}.$$

More generally, one can show a levelwise equivalence with the set quantifier alternation hierarchy of  $\text{MSO}(\tilde{\text{ML}}_g)$ , a rather unconventional logic that is equivalent to  $\text{MSOL}$ . In other words, two corresponding levels of alternation in the frameworks of  $\text{ALDA}_g$ 's and  $\text{MSO}(\tilde{\text{ML}}_g)$  characterize exactly the same digraph languages. Against this backdrop,  $\text{ALDA}_g$ 's may be best described as a machine-oriented syntax for  $\text{MSO}(\tilde{\text{ML}}_g)$ . We shall pick up on this point in the introduction of Chapter 6.

As of the time of writing this thesis, no new results on  $\llbracket \text{MSOL} \rrbracket_{\text{DG}_\Sigma^r}$  have been inferred from the alternative characterization through  $\text{ALDA}_g$ 's. On the other hand, the notion of  $\text{NLDA}_g$  contributes to the general observation, mentioned in Section 1.1.1, that many characterizations of regularity, which are equivalent on words and trees, drift apart on digraphs. To see this, consider  $\text{NLDA}_g$ 's whose input is restricted to those  $\Sigma$ -labeled,  $r$ -relational digraphs that represent words or trees over the alphabet  $\Sigma$ . For words,  $r = 1$  and edges simply go from one position to the next, whereas for ordered trees of arity  $k$ , we set  $r = k$  and require edge relations such that  $uv \in R_i^G$  if and only if  $u$  is the  $i$ -th child of  $v$ . Observe that we can easily simulate any word or tree automaton by an  $\text{NLDA}_g$  of length 2: guess a run of the automaton in the first round (each node nondeterministically chooses some state), then check whether it is a valid accepting run in the second round (transitions are verified locally, and acceptance is determined by the unique sink). This implies that the classes of  $\text{NLDA}_g$ -recognizable and  $\text{MSOL}$ -definable languages collapse on words and trees, and hence

that  $NLDA_g$ 's recognize precisely the regular languages on those restricted structures. (Note that this does not hold for  $DLDA_g$ 's; for instance, it is easy to see that they cannot decide whether a given unary word is of even length.) In this light, the decidability of the emptiness problem for  $NLDA_g$ 's can be seen as an extension to arbitrary digraphs of the corresponding decidability results for finite automata on words and trees.





# 4

## Asynchronous Nonlocal Automata

In this chapter, we introduce a particular class of **nonlocal distributed automata** and show that on finite **digraphs**, they are **equivalent** to the **least-fixpoint fragment** of the **backward  $\mu$ -calculus**, or simply *backward  $\mu$ -fragment*.

For the general case, a logical characterization has been provided by Kuusisto in [Kuu13a]; there he introduced a modal-logic-based variant of Datalog, called *modal substitution calculus*, that captures exactly the class of **nonlocal automata**. Furthermore, [Kuu13a, Prp. 7] shows that these **automata** can easily **recognize** all the **properties definable** in the **backward  $\mu$ -fragment** on finite **digraphs**. On the other hand, the reverse conversion from **nonlocal automata** to the **backward  $\mu$ -fragment** is not possible in general. As explained in [Kuu13a, Prp. 6], it is easy to come up with an **automaton** that makes crucial use of the fact that a **node** can determine whether it receives the same information from all of its **incoming neighbors** at exactly the same time. Such synchronous behavior cannot be simulated in the **backward  $\mu$ -fragment** (and not even in **MSOL**). This leaves open the problem of identifying a subclass of **distributed automata** for which the conversion works in both directions.

Here, we present a very simple solution: it basically suffices to transfer the standard notion of asynchronous algorithm to the setting of **distributed automata**.

The organisation of this chapter is as follows. After giving the necessary formal definitions in Section 4.1, we state and briefly discuss the main result in Section 4.2. The proof is then developed in the last two sections. Section 4.3 presents the rather straightforward translation from logic to automata. The reverse translation is given in Section 4.4, which is a bit more involved and therefore occupies the largest part of the chapter.

### 4.1 Preliminaries

The class of **asynchronous distributed automata** introduced in this chapter, is a special case of the **distributed automata** defined in Section 2.7. We maintain the same syntax as in Definition 2.3, but reintroduce the semantics of (unrestricted) **distributed automata** from a slightly different perspective. In order to keep notation simple, we

do this only for 1-relational **digraphs**, but everything presented here can easily be extended to the multi-relational case.

To **run** a **distributed automaton**  $A$  on a **digraph**  $G$ , we now regard the **edges** of  $G$  as FIFO buffers. Each buffer  $vw$  will always contain a sequence of **states** previously traversed by **node**  $v$ . An adversary chooses when  $v$  evaluates  $\delta$  to push a new **state** to the back of the buffer, and when the current first **state** gets popped from the front. The details are clarified in the following.

A **trace** of an **automaton**  $A = (Q, \delta_0, \delta, F)$  is a finite nonempty sequence  $\sigma = q_1 \dots q_n$  of **states** in  $Q$  such that  $q_i \neq q_{i+1}$  and  $\delta(q_i, S_i) = q_{i+1}$  for some  $S_i \subseteq Q$ . Notice that  $A$  is **quasi-acyclic** if and only if its set of **traces**  $\Omega$  is finite.

For any **states**  $p, q \in Q$  and any (possibly empty) sequence  $\sigma$  of **states** in  $Q$ , we define the unary postfix operators **first**, **last**, **pushlast** and **popfirst** as follows:

$$\begin{aligned} p\sigma.\text{first} &= \sigma p.\text{last} = p, \\ \sigma p.\text{pushlast}(q) &= \begin{cases} \sigma p q & \text{if } p \neq q, \\ \sigma p & \text{if } p = q, \end{cases} \\ p\sigma.\text{popfirst} &= \begin{cases} \sigma & \text{if } \sigma \text{ is nonempty,} \\ p\sigma & \text{if } \sigma \text{ is empty.} \end{cases} \end{aligned}$$

An (asynchronous) **timing** of a **digraph**  $G = (V^G, R^G, \lambda^G)$  is an infinite sequence  $\tau = (\tau_1, \tau_2, \tau_3, \dots)$  of maps  $\tau_t: V^G \cup R^G \rightarrow \mathbb{2}$ , indicating which **nodes** and **edges** are active at time  $t$ , where 1 is assigned infinitely often to every **node** and every **edge**. More formally, for all  $t \in \mathbb{N}_+$ ,  $v \in V^G$  and  $e \in R^G$ , there exist  $i, j > t$  such that  $\tau_i(v) = 1$  and  $\tau_j(e) = 1$ . We refer to this as the **fairness property** of  $\tau$ . As a restriction, we say that  $\tau$  is **lossless-asynchronous** if  $\tau_t(uv) = 1$  implies  $\tau_t(v) = 1$  for all  $t \in \mathbb{N}_+$  and  $uv \in R^G$ . Furthermore,  $\tau$  is called the (unique) **synchronous timing** of  $G$  if  $\tau_t(v) = \tau_t(e) = 1$  for all  $t \in \mathbb{N}_+$ ,  $v \in V^G$  and  $e \in R^G$ .

**Definition 4.1** (Asynchronous Run).

► Let  $A = (Q, \delta_0, \delta, F)$  be a **distributed automaton** over  $s$ -bit **labeled digraphs** and  $\Omega$  be its set of **traces**. Furthermore, let  $G = (V^G, R^G, \lambda^G)$  be an  $s$ -bit **labeled digraph** and  $\tau = (\tau_1, \tau_2, \tau_3, \dots)$  be a **timing** of  $G$ . The (asynchronous) **run** of  $A$  on  $G$  timed by  $\tau$  is the infinite sequence  $\rho = (\rho_0, \rho_1, \rho_2, \dots)$  of **configurations**  $\rho_t: V^G \cup R^G \rightarrow \Omega$ , with  $\rho_t(V^G) \subseteq Q$ , which are defined inductively as follows, for  $t \in \mathbb{N}$ ,  $v \in V^G$  and  $vw \in R^G$ :

$$\begin{aligned} \rho_0(v) &= \rho_0(vw) = \delta_0(\lambda^G(v)), \\ \rho_{t+1}(v) &= \begin{cases} \rho_t(v) & \text{if } \tau_{t+1}(v) = 0, \\ \delta(\rho_t(v), \{\rho_t(uv).\text{first} \mid uv \in R^G\}) & \text{if } \tau_{t+1}(v) = 1, \end{cases} \\ \rho_{t+1}(vw) &= \begin{cases} \rho_t(vw).\text{pushlast}(\rho_{t+1}(v)) & \text{if } \tau_{t+1}(vw) = 0, \\ \rho_t(vw).\text{pushlast}(\rho_{t+1}(v)).\text{popfirst} & \text{if } \tau_{t+1}(vw) = 1. \end{cases} \end{aligned}$$

If  $\tau$  is the **synchronous timing** of  $G$ , we refer to  $\rho$  as the **synchronous run** of  $A$  on  $G$ . ◀

Throughout this chapter, we assume that our **digraphs**, **automata** and logical **formulas** agree on the number  $s$  of **labeling bits**. An **automaton**  $A$  **accepts** a **pointed digraph**  $G[v]$  under **timing**  $\tau$  if  $v$  visits an **accepting state** at some point in the **run**  $\rho$

of  $A$  on  $G$  timed by  $\tau$ , i.e., if there exists  $t \in \mathbb{N}$  such that  $\rho_t(v) \in F$ . If we simply say that  $A$  *accepts*  $G[v]$ , without explicitly specifying a *timing*  $\tau$ , then we stipulate that  $\rho$  is the *synchronous run* of  $A$  on  $G$ . Notice that this is coherent with the definition of *acceptance* presented in Section 2.7.

Given a digraph  $G = (V^G, R^G, \lambda^G)$  and a class  $T$  of *timings* of  $G$ , the automaton  $A$  is called *consistent* for  $G$  and  $T$  if for all  $v \in V^G$ , either  $A$  *accepts*  $G[v]$  under every *timing* in  $T$ , or  $A$  does not *accept*  $G[v]$  under any *timing* in  $T$ . We say that  $A$  is *asynchronous* if it is *consistent* for every possible choice of  $G$  and  $T$ , and *lossless-asynchronous* if it is *consistent* for every choice where  $T$  contains only *lossless-asynchronous timings*. By contrast, we call an automaton *synchronous* if we wish to emphasize that no such consistency requirements are imposed. Intuitively, all automata can operate in the synchronous setting, but only some of them also work reliably in environments that provide fewer guarantees.

We denote by *a-DA*, *la-DA* and *DA* the classes of *asynchronous*, *lossless-asynchronous* and *synchronous automata*, respectively. Similarly, *a-QDA*, *la-QDA* and *QDA* are the corresponding classes of *quasi-acyclic automata*.

Next, we want to introduce the *backward  $\mu$ -fragment*, for which it is convenient to distinguish explicitly between *constants* and *variables*. As our starting point, we consider  $\tilde{ML}$  restricted to  $s$  *set constants* and (arbitrarily many) *unnegated set variables*. Its *formulas* are generated by the grammar

$$\varphi ::= \perp \mid \top \mid P_i \mid \neg P_i \mid X \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond \varphi \mid \Box \varphi,$$

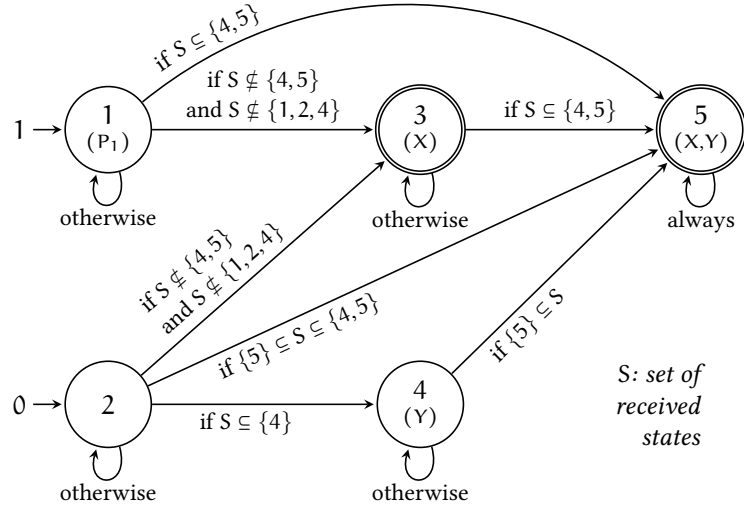
where  $P_i \in \mathbb{S}_1$  is considered to be a *set constant*, for  $1 \leq i \leq s$ , and  $X \in \mathbb{S}_1 \setminus \{P_1, \dots, P_s\}$  is considered to be a *set variable*. Note that this syntax ensures that *set variables* cannot be negated.

Traditionally, the modal  $\mu$ -calculus is defined to comprise individual *fixpoints* which may be nested. However, it is well-known that we can add simultaneous *fixpoints* to the  $\mu$ -calculus without changing its expressive power, and that nested *fixpoints* of the same type (i.e., least or greatest) can be rewritten as non-nested simultaneous ones (see, e.g., [BS07, § 3.7] or [Len05, § 4.3]). The following definition directly takes advantage of this fact. We shall restrict ourselves to the  *$\mu$ -fragment of the backward  $\mu$ -calculus*, abbreviated *backward  $\mu$ -fragment*, where only *least fixpoints* are allowed, and where the usual *modal operators* are replaced by their backward-looking variants. Without loss of generality, we stipulate that each *formula* of the *backward  $\mu$ -fragment* with  $s$  *set constants* is of the form

$$\varphi = \mu \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} . \begin{pmatrix} \varphi_1(P_1, \dots, P_s, X_1, \dots, X_k) \\ \vdots \\ \varphi_k(P_1, \dots, P_s, X_1, \dots, X_k) \end{pmatrix},$$

where  $X_1, \dots, X_k \in \mathbb{S}_1 \setminus \{P_1, \dots, P_s\}$  are considered to be *set variables*, and  $\varphi_1, \dots, \varphi_k$  are *formulas* of  $\tilde{ML}$  with  $s$  *set constants* and *unnegated set variables* that may contain no other *set variables* than  $X_1, \dots, X_k$ . We shall denote the set of *formulas* of the *backward  $\mu$ -fragment* by  $\Sigma_1^\mu(\tilde{ML})$ .

For every digraph  $G = (V^G, R^G, \lambda^G)$ , the tuple  $(\varphi_1, \dots, \varphi_k)$  gives rise to an operator  $f: (\mathcal{2}^{V^G})^k \rightarrow (\mathcal{2}^{V^G})^k$  that takes some valuation of  $\vec{X} = (X_1, \dots, X_k)$  and reassigns to each  $X_i$  the resulting valuation of  $\varphi_i$ . More formally,  $f$  maps  $\vec{W} = (W_1, \dots, W_k)$  to  $(W'_1, \dots, W'_k)$  such that  $W'_i = \llbracket \varphi_i \rrbracket_{G[\vec{X} \mapsto \vec{W}]}$ . Here,  $G[\vec{X} \mapsto \vec{W}]$  is the *extended*



**Figure 4.1.** A quasi-acyclic asynchronous distributed automaton that is equivalent to the formula

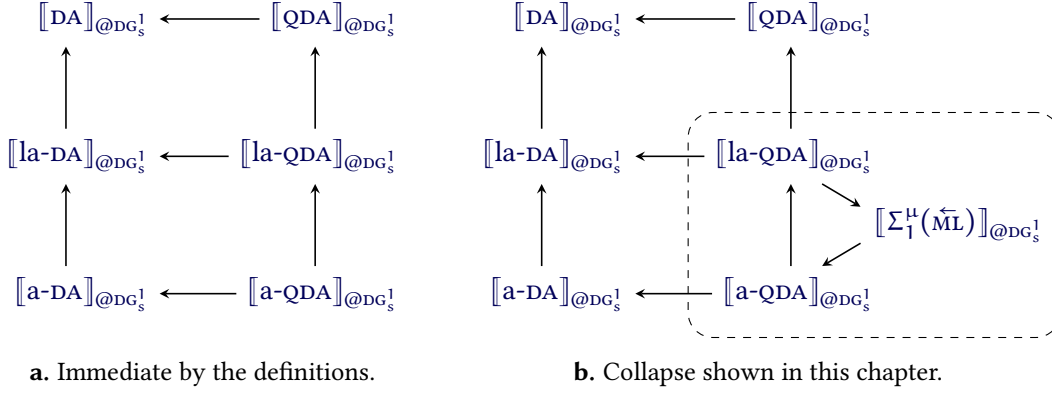
$$\mu \begin{pmatrix} X \\ Y \end{pmatrix} \cdot \left( \begin{pmatrix} (P_1 \wedge Y) \vee \Diamond X \\ \Box Y \end{pmatrix} \right)$$

of the backward  $\mu$ -fragment. A given 1-bit labeled pointed digraph  $G[v]$  is accepted by this automaton if and only if, starting at  $v$  and following  $G$ 's edges in the backward direction, it is possible to reach some node  $u$  labeled with 1 from which it is impossible to reach any directed cycle.

variant of  $G$  that interprets each  $X_i$  as  $W_i$ . A (simultaneous) *fixpoint* of the operator  $f$  is a tuple  $\vec{W} \in (2^{V^G})^k$  such that  $f(\vec{W}) = \vec{W}$ . Since, by definition, set variables occur only positively in formulas, the operator  $f$  is *monotonic*. This means that  $\vec{W} \subseteq \vec{W}'$  implies  $f(\vec{W}) \subseteq f(\vec{W}')$  for all  $\vec{W}, \vec{W}' \in (2^{V^G})^k$ , where set inclusions are to be understood componentwise (i.e.,  $W_i \subseteq W'_i$  for each  $i$ ). Therefore, by virtue of a theorem due to Knaster and Tarski,  $f$  has a *least fixpoint*, which is defined as the unique fixpoint  $\vec{U} = (U_1, \dots, U_k)$  of  $f$  such that  $\vec{U} \subseteq \vec{W}$  for every other fixpoint  $\vec{W}$  of  $f$ . As a matter of fact, the Knaster-Tarski theorem even tells us that  $\vec{U}$  is equal to  $\cap \{\vec{W} \in (2^{V^G})^k \mid f(\vec{W}) \subseteq \vec{W}\}$ , where set operations must also be understood componentwise. Another, perhaps more intuitive, way of characterizing  $\vec{U}$  is to consider the inductively constructed sequence of approximants  $(\vec{U}^0, \vec{U}^1, \vec{U}^2, \dots)$ , where  $\vec{U}^0 = (\emptyset, \dots, \emptyset)$  and  $\vec{U}^{j+1} = f(\vec{U}^j)$ . Since this sequence is monotonically increasing and  $V^G$  is finite, there exists  $n \in \mathbb{N}$  such that  $\vec{U}^n = \vec{U}^{n+1}$ . It is easy to check that  $\vec{U}^n$  coincides with the *least fixpoint*  $\vec{U}$ . For more details and proofs, see, e.g., [GKL<sup>+</sup>07, § 3.3.1].

Having introduced the necessary background, we can finally establish the semantics of  $\varphi$  with respect to  $G$ : the set  $\llbracket \varphi \rrbracket_G = \{v \in V^G \mid G[v] \models \varphi\}$  of nodes at which  $\varphi$  holds is precisely  $U_1$ , the first component of  $\vec{U}$ . Accordingly, the pointed digraph  $G[v]$  lies in the language  $\llbracket \varphi \rrbracket_{\text{DG}_s^1}$  defined by  $\varphi$  if and only if  $v \in U_1$ , and we denote by  $\llbracket \Sigma_1^\mu(\tilde{ML}) \rrbracket_{\text{DG}_s^1}$  the class of all pointed-digraph languages defined by some formula of the backward  $\mu$ -fragment.

Figure 4.1 provides an example of a quasi-acyclic asynchronous distributed automaton and an equivalent formula of the backward  $\mu$ -fragment.



**Figure 4.2.** Hierarchy of the classes of pointed-digraph languages recognizable by distributed automata (DA), depending on whether the automata are synchronous (neither “la” nor “a”), lossless-asynchronous (“la”), asynchronous (“a”), or quasi-acyclic (“Q”). The arrows denote inclusion (e.g.,  $\llbracket \text{la-DA} \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \text{DA} \rrbracket_{\text{DG}_s^1}$ ).

## 4.2 Equivalence with the backward $\mu$ -fragment

Based on the definitions given in Section 4.1, asynchronous automata are a special case of lossless-asynchronous automata, which in turn are a special case of synchronous automata. Furthermore, quasi-acyclicity constitutes an additional (possibly orthogonal) restriction on these models. We thus immediately obtain the hierarchy of classes depicted in Figure 4.2a.

Our main result provides a simplification of this hierarchy: the classes  $\llbracket \text{a-QDA} \rrbracket_{\text{DG}_s^1}$  and  $\llbracket \text{la-QDA} \rrbracket_{\text{DG}_s^1}$  are actually equal to the class of pointed-digraph languages definable in the backward  $\mu$ -fragment. This yields the revised diagram shown in Figure 4.2b.

**Theorem 4.2** ( $\llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1} = \llbracket \text{a-QDA} \rrbracket_{\text{DG}_s^1} = \llbracket \text{la-QDA} \rrbracket_{\text{DG}_s^1}$ ).

► When restricted to finite digraphs, the backward  $\mu$ -fragment is effectively equivalent to the classes of quasi-acyclic asynchronous automata and quasi-acyclic lossless-asynchronous automata. ◀

*Proof.* The forward direction is given by Proposition 4.3 (in Section 4.3), which asserts that  $\llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \text{a-QDA} \rrbracket_{\text{DG}_s^1}$ , and the trivial observation that  $\llbracket \text{a-QDA} \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \text{la-QDA} \rrbracket_{\text{DG}_s^1}$ . For the backward direction, we use Proposition 4.6 (in Section 4.4), which asserts that  $\llbracket \text{la-QDA} \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1}$ . ■

As stated before, synchronous automata are more powerful than the backward  $\mu$ -fragment (and incomparable with MSOL). This holds even if we consider only quasi-acyclic automata, i.e., the inclusion  $\llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \text{QDA} \rrbracket_{\text{DG}_s^1}$  is known to be strict (see [Kuu13a, Prp. 6]). Moreover, an upcoming paper will show that the inclusion  $\llbracket \text{QDA} \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \text{DA} \rrbracket_{\text{DG}_s^1}$  is also strict.

In contrast, it remains open whether quasi-acyclicity is in fact necessary for characterizing  $\llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1}$ . On the one hand, this notion is crucial for our proof (see Proposition 4.6), but on the other hand, no pointed-digraph language separating  $\llbracket \text{a-DA} \rrbracket_{\text{DG}_s^1}$  or  $\llbracket \text{la-DA} \rrbracket_{\text{DG}_s^1}$  from  $\llbracket \Sigma_1^\mu(\tilde{\text{ML}}) \rrbracket_{\text{DG}_s^1}$  has been found so far.

*This may seem counterintuitive at first sight, but it is actually consistent with the standard terminology of distributed computing: an asynchronous algorithm can always serve as a synchronous algorithm (i.e., it can be executed in a synchronous environment), but the converse is not true.*

### 4.3 Computing least fixpoints using asynchronous automata

In this section, we prove the easy direction of the main result. Given a **formula**  $\varphi$  of the **backward  $\mu$ -fragment**, it is straightforward to construct a (synchronous) **distributed automaton**  $A$  that computes on any **digraph** the **least fixpoint**  $\vec{U}$  of the operator associated with  $\varphi$ . As long as it operates in the synchronous setting,  $A$  simply follows the sequence of approximants  $(\vec{U}^0, \vec{U}^1, \dots)$  described in Section 4.1. It is important to stress that the very same observation has previously been made in [Kuu13a, Prp. 7] (formulated from a different point of view). In the following proposition, we refine this observation by giving a more precise characterization of the obtained **automaton**: it is always quasi-acyclic and capable of operating in a (possibly lossy) asynchronous environment.

**Proposition 4.3** ( $\llbracket \Sigma_1^\mu(\vec{ML}) \rrbracket_{@DG_s^1} \subseteq \llbracket a\text{-QDA} \rrbracket_{@DG_s^1}$ ).

► For every **formula** of the **backward  $\mu$ -fragment**, we can effectively construct an **equivalent quasi-acyclic asynchronous automaton**. ◀

*Proof.* Let  $\varphi = \mu(X_1, \dots, X_k).(\varphi_1, \dots, \varphi_k)$  be a **formula** of the **backward  $\mu$ -fragment** with  $s$  **set constants**. Without loss of generality, we may assume that the **subformulas**  $\varphi_1, \dots, \varphi_k$  do not contain any nested **modalities**. To see this, suppose that  $\varphi_i = \Diamond\psi$ . Then  $\varphi$  is **equivalent** to  $\varphi' = \mu(X_1, \dots, X_i, \dots, X_k, Y).(\varphi_1, \dots, \varphi'_i, \dots, \varphi_k, \psi)$ , where  $Y$  is a fresh **set variable** and  $\varphi'_i = \Diamond Y$ . The operator  $\Box$  and Boolean combinations of  $\Diamond$  and  $\Box$  are handled analogously.

We now convert  $\varphi$  into an **equivalent automaton**  $A = (Q, \delta_0, \delta, F)$  with **state set**  $Q = \mathcal{P}\{P_1, \dots, P_s, X_1, \dots, X_k\}$ . The idea is that each **node**  $v$  of the input **digraph** has to remember which of the atomic propositions  $P_1, \dots, P_s, X_1, \dots, X_k$  have, so far, been verified to hold at  $v$ . Therefore, we define the **initialization function** such that  $\delta_0(x) = \{P_i \mid x(i) = 1\}$  for all  $x \in \mathcal{P}^s$ . Let us write  $(q, S) \models \varphi_i$  to indicate that a pair  $(q, S) \in Q \times \mathcal{P}^Q$  satisfies a **subformula**  $\varphi_i$  of  $\varphi$ . This is the case precisely when  $\varphi_i$  holds at any **node**  $v$  that **satisfies** exactly the atomic propositions in  $q$  and whose **incoming neighbors satisfy** exactly the propositions specified by  $S$ . Note that this satisfaction relation is well-defined in our context because the nesting depth of **modal operators** in  $\varphi_i$  is at most 1. With that, the **transition function** of  $A$  can be succinctly described by  $\delta(q, S) = q \cup \{X_i \mid (q, S) \models \varphi_i\}$ . Since  $q \subseteq \delta(q, S)$ , we are guaranteed that the **automaton** is **quasi-acyclic**. Finally, the **accepting set** is given by  $F = \{q \mid X_1 \in q\}$ .

It remains to prove that  $A$  is asynchronous and **equivalent** to  $\varphi$ . For this purpose, let  $G = (V^G, R^G, \lambda^G)$  be an  $s$ -bit **labeled digraph** and  $\vec{U} = (U_1, \dots, U_k) \in (\mathcal{P}^{V^G})^k$  be the **least fixpoint** of the operator  $f$  associated with  $(\varphi_1, \dots, \varphi_k)$ . Due to the asynchrony condition, we must consider an arbitrary **timing**  $\tau = (\tau_1, \tau_2, \dots)$  of  $G$ . The corresponding **run**  $\rho = (\rho_0, \rho_1, \dots)$  of  $A$  on  $G$  timed by  $\tau$  engenders an infinite sequence  $(\vec{W}^0, \vec{W}^1, \dots)$ , where each tuple  $\vec{W}^t = (W_1^t, \dots, W_k^t) \in (\mathcal{P}^{V^G})^k$  specifies the valuation of every **set variable**  $X_i$  at time  $t$ , i.e.,  $W_i^t = \{v \in V^G \mid X_i \in \rho_t(v)\}$ . Since  $A$  is quasi-acyclic and  $V^G$  is finite, this sequence must eventually stabilize at some value  $\vec{W}^\infty$ , and each **node** **accepts** if and only if it belongs to  $W_1^\infty$ . Reformulated this way, our task is to demonstrate that  $\vec{W}^\infty$  equals  $\vec{U}$ , regardless of the **timing**  $\tau$ .

“ $\vec{W}^\infty \subseteq \vec{U}$ ”: We show by induction that  $\vec{W}^t \subseteq \vec{U}$  for all  $t \in \mathbb{N}$ . This obviously holds for  $t = 0$ , since  $\vec{W}^0 = (\emptyset, \dots, \emptyset)$ . Now, consider any node  $v \in V^G$  at an arbitrary



time  $t$ . Let  $q$  be the current state of  $v$  and  $S$  be the set of current **states** of its **incoming neighbors**. Depending on  $\tau$ , it might be the case that  $v$  actually receives some outdated information  $S'$  instead of  $S$ . However, given that the **neighbors'** previous **states** cannot contain more **set variables** than their current ones (by construction), and that **set variables** can only occur positively in each  $\varphi_i$ , we know that  $(q, S') \models \varphi_i$  implies  $(q, S) \models \varphi_i$ . Hence, if  $v$  performs a local transition at time  $t$ , then the only new **set variables** that can be added to its **state** must lie in  $\{X_i \mid (q, S) \models \varphi_i\}$ . On a global scale, this means that  $\vec{W}^{t+1} \setminus \vec{W}^t \subseteq f(\vec{W}^t)$ . Furthermore, by the induction hypothesis, the **monotonicity** of  $f$ , and the fact that  $\vec{U}$  is a **fixpoint**, we have  $f(\vec{W}^t) \subseteq f(\vec{U}) = \vec{U}$ . Putting both together, and again relying on the induction hypothesis, we obtain  $\vec{W}^{t+1} \subseteq \vec{U}$ .

“ $\vec{W}^\infty \supseteq \vec{U}$ ”: For the converse direction, we make use of the Knaster-Tarski theorem, which gives us the equality  $\vec{U} = \bigcap \{\vec{W} \in (2^{V^G})^k \mid f(\vec{W}) \subseteq \vec{W}\}$ . With this, it suffices to show that  $f(\vec{W}^\infty) \subseteq \vec{W}^\infty$ . Consider some time  $t \in \mathbb{N}$  such that  $\vec{W}^{t'} = \vec{W}^\infty$  for all  $t' \geq t$ . Although we know that every **node** has reached its final **state** at time  $t$ , the FIFO buffers of some **edges** might still contain obsolete **states** from previous times. However, the **fairness property** of  $\tau$  guarantees that our customized **popfirst** operation is executed infinitely often at every **edge**, while the **pushlast** operation has no effect because all the **states** remain unchanged. Therefore, there must be a time  $t' \geq t$  from which on each buffer contains only the current **state** of its incoming **node**, i.e.,  $\rho_{t''}(uv) = \rho_t(u)$  for all  $t'' \geq t'$  and  $uv \in R^G$ . Moreover, the **fairness property** of  $\tau$  also ensures that every **node**  $v$  reevaluates the local **transition function**  $\delta$  infinitely often, based on its own current **state**  $q$  and the set  $S$  of **states** in the buffers associated with its **incoming neighbors**. As this has no influence on  $v$ 's **state**, we can deduce that  $\{X_i \mid (q, S) \models \varphi_i\} \subseteq q$ . Consequently, we have  $f(\vec{W}^{t'}) \subseteq \vec{W}^{t'}$ , which is equivalent to  $f(\vec{W}^\infty) \subseteq \vec{W}^\infty$ . ■

## 4.4 Capturing asynchronous runs using least fixpoints

This section is dedicated to proving the converse direction of the main result, which will allow us to translate any **quasi-acyclic lossless-asynchronous automaton** into an **equivalent formula** of the **backward  $\mu$ -fragment** (see Proposition 4.6). Our proof builds on two concepts: the invariance of **distributed automata** under **backward bisimulation** (stated in Proposition 4.4) and an ad-hoc relation “ $\triangleright$ ” that captures the possible behaviors of a fixed **lossless-asynchronous automaton**  $A$  (in a specific sense described in Lemma 4.5).

We start with the notion of **backward bisimulation**, which is defined like the standard notion of bisimulation (see, e.g., [BRV02, Def. 2.16] or [BB07, Def. 5]), except that **edges** are followed in the backward direction. Formally, a **backward bisimulation** between two  $s$ -bit **labeled digraphs**  $G = (V^G, R^G, \lambda^G)$  and  $G' = (V^{G'}, R^{G'}, \lambda^{G'})$  is a binary relation  $B \subseteq V^G \times V^{G'}$  that fulfills the following conditions for all  $vv' \in B$ :

- a.  $\lambda^G(v) = \lambda^{G'}(v')$ ,
- b. if  $uv \in R^G$ , then there exists  $u' \in V^{G'}$  such that  $u'v' \in R^{G'}$  and  $uu' \in B$ , and, conversely,
- c. if  $u'v' \in R^{G'}$ , then there exists  $u \in V^G$  such that  $uv \in R^G$  and  $uu' \in B$ .

We say that the pointed digraphs  $G[v]$  and  $G'[v']$  are *backward bisimilar* if there exists such a backward bisimulation  $B$  relating  $v$  and  $v'$ . It is easy to see that distributed automata cannot distinguish between backward bisimilar structures:

**Proposition 4.4.**

► Distributed automata are invariant under backward bisimulation. That is, for every automaton  $A$ , if two pointed digraphs  $G[v]$  and  $G'[v']$  are backward bisimilar, then  $A$  accepts  $G[v]$  if and only if it accepts  $G'[v']$ . ◀

*Proof.* Let  $B$  be a backward bisimulation between  $G$  and  $G'$  such that  $vv' \in B$ . Since acceptance is defined with respect to the synchronous behavior of the automaton, we need only consider the synchronous runs  $\rho = (\rho_0, \rho_1, \dots)$  and  $\rho' = (\rho'_0, \rho'_1, \dots)$  of  $A$  on  $G$  and  $G'$ , respectively. Now, given that the FIFO buffers on the edges of the digraphs merely contain the current state of their incoming node, it is straightforward to prove by induction on  $t$  that every pair of nodes  $uu' \in B$  satisfies  $\rho_t(u) = \rho'_t(u')$  for all  $t \in \mathbb{N}$ . ■

We now turn to the mentioned relation “ $\triangleright$ ”, which is defined with respect to a fixed automaton. For the remainder of this section, let  $A$  denote an automaton  $(Q, \delta_0, \delta, F)$ , and let  $\Omega$  denote its set of traces. The relation  $\triangleright \subseteq (2^\Omega \times \Omega)$  specifies whether, in a lossless-asynchronous environment, a given trace  $\sigma$  can be traversed by a node whose incoming neighbors traverse the traces of a given set  $\mathfrak{S}$ . Loosely speaking, the intended meaning of  $\mathfrak{S} \triangleright \sigma$  (“ $\mathfrak{S}$  enables  $\sigma$ ”) is the following: Take an appropriately chosen digraph under some lossless-asynchronous timing  $\tau$ , and observe the corresponding run of  $A$  up to a specific time  $t$ ; if node  $v$  was initially in state  $\sigma$ .first and at time  $t$  it has seen its incoming neighbors traversing precisely the traces in  $\mathfrak{S}$ , then it is possible for  $\tau$  to be such that at time  $t$ , node  $v$  has traversed exactly the trace  $\sigma$ . This relation can be defined inductively: As the base case, we specify that for every  $q \in Q$  and  $S \subseteq Q$ , we have  $S \triangleright q.\text{pushlast}(\delta(q, S))$ . For the inductive clause, consider a trace  $\sigma \in \Omega$  and two finite (possibly equal) sets of traces  $\mathfrak{S}, \mathfrak{S}' \subseteq \Omega$  such that the traces in  $\mathfrak{S}'$  can be obtained by appending at most one state to the traces in  $\mathfrak{S}$ . More precisely, if  $\pi \in \mathfrak{S}$ , then  $\pi.\text{pushlast}(p) \in \mathfrak{S}'$  for some  $p \in Q$ , and conversely, if  $\pi' \in \mathfrak{S}'$ , then  $\pi' = \pi.\text{pushlast}(\pi'.\text{last})$  for some  $\pi \in \mathfrak{S}$ . We shall denote this auxiliary relation by  $\mathfrak{S} \Rightarrow \mathfrak{S}'$ . If it holds, then  $\mathfrak{S} \triangleright \sigma$  implies  $\mathfrak{S}' \triangleright \sigma.\text{pushlast}(q)$ , where  $q = \delta(\sigma.\text{last}, \{\pi'.\text{last} \mid \pi' \in \mathfrak{S}'\})$ .

The next step is to show (in Lemma 4.5) that our definition of “ $\triangleright$ ” does indeed capture the intuition given above. To formalize this, we first introduce two further pieces of terminology.

First, the notions of configuration and run can be enriched to facilitate discussions about the past. Let  $\rho = (\rho_0, \rho_1, \dots)$  be a run of  $A$  on a digraph  $G = (V^G, R^G, \lambda^G)$  (timed by some timing  $\tau$ ). The corresponding *enriched run* is the sequence  $\hat{\rho} = (\hat{\rho}_0, \hat{\rho}_1, \dots)$  of *enriched configurations* that we obtain from  $\rho$  by requiring each node to remember the entire trace it has traversed so far. Formally, for  $t \in \mathbb{N}$ ,  $v \in V^G$  and  $e \in R^G$ ,

$$\hat{\rho}_0(v) = \rho_0(v), \quad \hat{\rho}_{t+1}(v) = \hat{\rho}_t(v).\text{pushlast}(\rho_{t+1}(v)) \quad \text{and} \quad \hat{\rho}_t(e) = \rho_t(e).$$

Second, we will need to consider finite segments of timings and enriched runs. A *lossless-asynchronous timing segment* of a digraph  $G$  is a sequence  $\tau = (\tau_1, \dots, \tau_r)$  that could be extended to a whole lossless-asynchronous timing  $(\tau_1, \dots, \tau_r, \tau_{r+1}, \dots)$ .



Likewise, for an initial **enriched configuration**  $\hat{\rho}_0$  of  $G$ , the corresponding **enriched run segment** timed by  $\tau$  is the sequence  $(\hat{\rho}_0, \dots, \hat{\rho}_r)$ , where each  $\hat{\rho}_{t+1}$  is computed from  $\hat{\rho}_t$  and  $\tau_{t+1}$  in the same way as for an entire **enriched run**.

Equipped with the necessary terminology, we can now state and prove a (slightly technical) lemma that will allow us to derive benefit from the relation “ $\triangleright$ ”. This lemma essentially states that if  $\mathfrak{S} \triangleright \sigma$  holds and we are given enough **nodes** that traverse the **traces** in  $\mathfrak{S}$ , then we can take those **nodes** as the **incoming neighbors** of a new **node**  $v$  and delay the messages received by  $v$  in such a way that  $v$  traverses  $\sigma$ , without losing any messages.

**Lemma 4.5.**

► For every **trace**  $\sigma \in \Omega$  and every finite (possibly empty) set of **traces**  $\mathfrak{S} = \{\pi_1, \dots, \pi_\ell\} \subseteq \Omega$  that satisfy the relation  $\mathfrak{S} \triangleright \sigma$ , there are lower bounds  $m_1, \dots, m_\ell \in \mathbb{N}_+$  such that the following statement holds true:

For any  $n_1, \dots, n_\ell \in \mathbb{N}_+$  satisfying  $n_i \geq m_i$ , let  $G$  be a **digraph** consisting of the **nodes**  $(u_i^j)_{i,j}$  and  $v$ , and the **edges**  $(u_i^j v)_{i,j}$ , with index ranges  $1 \leq i \leq \ell$  and  $1 \leq j \leq n_i$ . If we start from the **enriched configuration**  $\hat{\rho}_0$  of  $G$ , where

$$\hat{\rho}_0(u_i^j) = \pi_i, \quad \hat{\rho}_0(u_i^j v) = \pi_i \quad \text{and} \quad \hat{\rho}_0(v) = \sigma.\text{first},$$

then we can construct a (nonempty) **lossless-asynchronous timing segment**  $\tau = (\tau_1, \dots, \tau_r)$  of  $G$ , where  $\tau_t(u_i^j) = 0$  and  $\tau_t(v) = 1$  for  $1 \leq t \leq r$ , such that the corresponding **enriched run segment**  $\hat{\rho} = (\hat{\rho}_0, \dots, \hat{\rho}_r)$  timed by  $\tau$  satisfies

$$\hat{\rho}_{r-1}(u_i^j v) = \pi_i.\text{last} \quad \text{and} \quad \hat{\rho}_r(v) = \sigma. \quad \blacktriangleleft$$

*Proof.* We proceed by induction on the definition of “ $\triangleright$ ”. In the base case, where  $\mathfrak{S} = \{p_1, \dots, p_\ell\} \subseteq Q$  and  $\sigma = q.\text{pushlast}(\delta(q, \mathfrak{S}))$  for some  $q \in Q$ , the statement holds with  $m_1 = \dots = m_\ell = 1$ . This is witnessed by a **timing segment**  $\tau = (\tau_1)$ , where  $\tau_1(u_i^j) = 0$ ,  $\tau_1(v) = 1$ , and  $\tau_1(u_i^j v)$  can be chosen as desired.

For the inductive step, assume that the statement holds for  $\sigma$  and  $\mathfrak{S} = \{\pi_1, \dots, \pi_\ell\}$  with some values  $m_1, \dots, m_\ell$ . Now consider any other set of **traces**  $\mathfrak{S}' = \{\pi'_1, \dots, \pi'_{\ell'}\}$  such that  $\mathfrak{S} \Rightarrow \mathfrak{S}'$ , and let  $\sigma' = \sigma.\text{pushlast}(q)$ , where  $q = \delta(\sigma.\text{last}, \{\pi'_k.\text{last} \mid \pi'_k \in \mathfrak{S}'\})$ . Since  $\mathfrak{S} \triangleright \sigma$ , we have  $\mathfrak{S}' \triangleright \sigma'$ . The remainder of the proof consists in showing that the statement also holds for  $\sigma'$  and  $\mathfrak{S}'$  with some large enough integers  $m'_1, \dots, m'_{\ell'}$ . Let us fix  $m'_k = \sum \{m_i \mid \pi_i.\text{pushlast}(\pi'_k.\text{last}) = \pi'_k\}$ . (As there is no need to find minimal values, we opt for easy expressibility.)

Given any numbers  $n'_1, \dots, n'_{\ell'}$  with  $n'_k \geq m'_k$ , we choose suitable values  $n_1, \dots, n_\ell$  with  $n_i \geq m_i$ , and consider the corresponding **digraph**  $G$  described in the lemma. Because we have  $\mathfrak{S} \Rightarrow \mathfrak{S}'$ , we can assign to each **node**  $u_i^j$  a **state**  $p_i^j$  such that  $\pi_i.\text{pushlast}(p_i^j) \in \mathfrak{S}'$ . Moreover, provided our choice of  $n_1, \dots, n_\ell$  was adequate, we can also ensure that for each  $\pi'_k \in \mathfrak{S}'$ , there are exactly  $n'_k$  **nodes**  $u_i^j$  such that  $\pi_i.\text{pushlast}(p_i^j) = \pi'_k$ . (Note that **nodes** with distinct **traces**  $\pi_i, \pi_{i'} \in \mathfrak{S}$  might be mapped to the same **trace**  $\pi'_k \in \mathfrak{S}'$ , in case  $\pi_{i'} = \pi_i p_i^j$ .) It is straightforward to verify that such a choice of numbers and such an assignment of **states** are always possible, given the lower bounds  $m'_1, \dots, m'_{\ell'}$  specified above.

Let us now consider the **lossless-asynchronous timing segment**  $\tau = (\tau_1, \dots, \tau_r)$  and the corresponding **enriched run segment**  $\hat{\rho} = (\hat{\rho}_0, \dots, \hat{\rho}_r)$  provided by the induction hypothesis. Since the **popfirst** operation has no effect on a **trace** of length 1, we may assume without loss of generality that  $\tau_t(u_i^j v) = 0$  if  $\hat{\rho}_{t-1}(u_i^j v)$  has length 1, for  $t < r$ .

Consequently, if we start from the alternative **enriched configuration**  $\hat{\rho}'_0$ , where

$$\hat{\rho}'_0(u_i^j) = \pi_i.\text{pushlast}(p_i^j), \quad \hat{\rho}'_0(u_i^j v) = \pi_i.\text{pushlast}(p_i^j) \quad \text{and} \quad \hat{\rho}'_0(v) = \sigma.\text{first},$$

then the corresponding **enriched run segment**  $(\hat{\rho}'_0, \dots, \hat{\rho}'_r)$  timed by  $\tau$  can be derived from  $\hat{\rho}$  by simply applying “**pushlast**( $p_i^j$ )” to  $\hat{\rho}_t(u_i^j)$  and  $\hat{\rho}_t(u_i^j v)$ , for  $t < r$ . We thus get

$$\hat{\rho}'_{r-1}(u_i^j v) = \pi_i.\text{last}.\text{pushlast}(p_i^j) \quad \text{and} \quad \hat{\rho}'_r(v) = \sigma.$$

We may also assume without loss of generality that  $\tau_r(u_i^j v) = 1$  if  $\hat{\rho}'_{r-1}(u_i^j v)$  has length 2, since this does not affect  $\hat{\rho}$  and lossless-asynchrony is ensured by  $\tau_r(v) = 1$ . Hence, it suffices to extend  $\tau$  by an additional map  $\tau_{r+1}$ , where  $\tau_{r+1}(u_i^j) = 0$ ,  $\tau_{r+1}(v) = 1$ , and  $\tau_{r+1}(u_i^j v)$  can be chosen as desired. The resulting **enriched run segment**  $(\hat{\rho}'_0, \dots, \hat{\rho}'_{r+1})$  satisfies

$$\begin{aligned} \hat{\rho}'_r(u_i^j v) &= p_i^j = \pi'_k.\text{last} \quad (\text{for some } \pi'_k \in \mathfrak{S}') \quad \text{and} \\ \hat{\rho}'_{r+1}(v) &= \sigma.\text{pushlast}(q) = \sigma'. \end{aligned}$$

■

Finally, we can put all the pieces together and prove the converse direction of Theorem 4.2:

**Proposition 4.6** ( $\llbracket \text{la-QDA} \rrbracket_{\text{DG}_s^1} \subseteq \llbracket \Sigma_1^\mu(\hat{\text{ML}}) \rrbracket_{\text{DG}_s^1}$ ).

► For every **quasi-acyclic lossless-asynchronous automaton**, we can effectively construct an **equivalent formula** of the **backward  $\mu$ -fragment**. ◀

*Proof.* Assume that  $A = (Q, \delta_0, \delta, F)$  is a **quasi-acyclic lossless-asynchronous automaton** over  $s$ -bit **labeled digraphs**. Since it is quasi-acyclic, its set of **traces**  $\Omega$  is finite, and thus we can afford to introduce a separate **set variable**  $X_\sigma$  for each **trace**  $\sigma \in \Omega$ . Making use of the relation “ $\triangleright$ ”, we convert  $A$  into an **equivalent formula**  $\varphi = \mu[X_1, (X_\sigma)_{\sigma \in \Omega}].[\varphi_1, (\varphi_\sigma)_{\sigma \in \Omega}]$  of the **backward  $\mu$ -fragment**, where

$$\varphi_1 = \bigvee_{\substack{\sigma \in \Omega \\ \sigma.\text{last} \in F}} X_\sigma, \tag{a}$$

$$\varphi_q = \bigvee_{\substack{x \in 2^s \\ \delta_0(x) = q}} \left( \bigwedge_{x(i)=1} P_i \wedge \bigwedge_{x(i)=0} \neg P_i \right) \quad \text{for } q \in Q, \quad \text{and} \tag{b}$$

$$\varphi_\sigma = X_{\sigma.\text{first}} \wedge \bigvee_{\substack{\pi \in \Omega \\ \pi \triangleright \sigma}} \left( \left( \bigwedge_{\pi \in \mathfrak{S}} \Diamond X_\pi \right) \wedge \left( \Box \bigvee_{\pi \in \mathfrak{S}} X_\pi \right) \right) \quad \text{for } \sigma \in \Omega \text{ with } |\sigma| \geq 2. \tag{c}$$

Note that this **formula** can be constructed effectively because an inductive computation of “ $\triangleright$ ” must terminate after at most  $|\Omega| \cdot 2^{|\Omega|}$  iterations.

To prove that  $\varphi$  is indeed **equivalent** to  $A$ , let us consider an arbitrary  $s$ -bit **labeled digraph**  $G = (V^G, R^G, \lambda^G)$  and the corresponding **least fixpoint**  $\vec{U} = (U_1, (U_\sigma)_{\sigma \in \Omega}) \in (2^{V^G})^{|\Omega|+1}$  of the operator  $f$  associated with  $(\varphi_1, (\varphi_\sigma)_{\sigma \in \Omega})$ .

The easy direction is to show that for all **nodes**  $v \in V^G$ , if  $A$  **accepts**  $G[v]$ , then  $G[v]$  **satisfies**  $\varphi$ . For that, it suffices to consider the **synchronous enriched run**  $\hat{\rho} = (\hat{\rho}_0, \hat{\rho}_1, \dots)$  of  $A$  on  $G$ . (Any other **run** timed by a **lossless-asynchronous timing** would exhibit the same **acceptance behavior**.) As in the proof of Proposition 4.4, we can simply ignore the **FIFO buffers** on the **edges** of  $G$  because  $\hat{\rho}_t(uv) = \hat{\rho}_t(u).\text{last}$ .

Using this, a straightforward induction on  $t$  shows that every node  $v \in V^G$  satisfies  $\{\hat{\rho}_t(u) \mid uv \in R^G\} \triangleright \hat{\rho}_{t+1}(v)$  for all  $t \in \mathbb{N}$ . (For  $t = 0$ , the claim follows from the base case of the definition of “ $\triangleright$ ”; for the step from  $t$  to  $t + 1$ , we can immediately apply the inductive clause of the definition.) This in turn allows us to prove that each node  $v$  is contained in all the components of  $\vec{U}$  that correspond to a trace traversed by  $v$  in  $\hat{\rho}$ , i.e.,  $v \in U_{\hat{\rho}_t(v)}$  for all  $t \in \mathbb{N}$ . Naturally, we proceed again by induction: For  $t = 0$ , we have  $\hat{\rho}_0(v) = \delta_0(\lambda^G(v)) \in Q$ , hence the subformula  $\varphi_{\hat{\rho}_0(v)}$  defined in equation (b) holds at  $v$ , and thus  $v \in U_{\hat{\rho}_0(v)}$ . For the step from  $t$  to  $t + 1$ , we need to distinguish two cases. If  $\hat{\rho}_{t+1}(v)$  is of length 1, then it is equal to  $\hat{\rho}_t(v)$ , and there is nothing new to prove. Otherwise, we must consider the appropriate subformula  $\varphi_{\hat{\rho}_{t+1}(v)}$  given by equation (c). We already know from the base case that the conjunct  $X_{\hat{\rho}_{t+1}(v).first} = X_{\hat{\rho}_0(v)}$  holds at  $v$ , with respect to any variable assignment that interprets each  $X_\sigma$  as  $U_\sigma$ . Furthermore, by the induction hypothesis,  $X_{\hat{\rho}_t(u)}$  holds at every incoming neighbor  $u$  of  $v$ . Since  $\{\hat{\rho}_t(u) \mid uv \in R^G\} \triangleright \hat{\rho}_{t+1}(v)$ , we conclude that the second conjunct of  $\varphi_{\hat{\rho}_{t+1}(v)}$  must also hold at  $v$ , and thus  $v \in U_{\hat{\rho}_{t+1}(v)}$ . Finally, assuming  $A$  accepts  $G[v]$ , we know by definition that  $\hat{\rho}_t(v).last \in F$  for some  $t \in \mathbb{N}$ . Since  $v \in U_{\hat{\rho}_t(v)}$ , this implies that the subformula  $\varphi_1$  defined in equation (a) holds at  $v$ , and therefore that  $G[v]$  satisfies  $\varphi$ .

For the converse direction of the equivalence, we have to overcome the difficulty that  $\varphi$  is more permissive than  $A$ , in the sense that a node  $v$  might lie in  $U_\sigma$ , and yet not be able to follow the trace  $\sigma$  under any timing of  $G$ . Intuitively, the reason why we still obtain an equivalence is that  $A$  cannot take advantage of all the information provided by any particular run, because it must ensure that for all digraphs, its acceptance behavior is independent of the timing. It turns out that even if  $v$  cannot traverse  $\sigma$ , some other node  $v'$  in an indistinguishable digraph will be able to do so. More precisely, we will show that

$$\begin{aligned} &\text{if } v \in U_\sigma, \text{ then there exists a pointed digraph } G'[v'], \text{ backward} \\ &\text{bisimilar to } G[v], \text{ and a lossless-asynchronous timing } \tau' \text{ of } G', \\ &\text{such that } \hat{\rho}'_t(v') = \sigma \text{ for some } t \in \mathbb{N}, \end{aligned} \quad (*)$$

where  $\hat{\rho}'$  is the enriched run of  $A$  on  $G'$  timed by  $\tau'$ . Now suppose that  $G[v]$  satisfies  $\varphi$ . By equation (a), this means that  $v \in U_\sigma$  for some trace  $\sigma$  such that  $\sigma.last \in F$ . Consequently,  $A$  accepts the pointed digraph  $G'[v']$  postulated in  $(*)$ , based on the claim that  $v'$  traverses  $\sigma$  under timing  $\tau'$  and the fact that  $A$  is lossless-asynchronous. Since  $G[v]$  and  $G'[v']$  are backward bisimilar, it follows from Proposition 4.4 that  $A$  also accepts  $G[v]$ .

It remains to verify  $(*)$ . We achieve this by computing the least fixpoint  $\vec{U}$  inductively and proving the statement by induction on the sequence of approximants  $(\vec{U}^0, \vec{U}^1, \dots)$ . Note that we do not need to consider the limit case, since  $\vec{U} = \vec{U}^n$  for some  $n \in \mathbb{N}$ .

The base case is trivially true because all the components of  $\vec{U}^0$  are empty. Furthermore, if  $\sigma$  consists of a single state  $q$ , then we do not even need to argue by induction, as it is evident from equation (b) that for all  $j \geq 1$ , node  $v$  lies in  $U_q^j$  precisely when  $\delta_0(\lambda^G(v)) = q$ . It thus suffices to set  $G'[v'] = G[v]$  and choose the timing  $\tau'$  arbitrarily. Clearly, we have  $\hat{\rho}'_0(v') = \delta_0(\lambda^G(v)) = q$  if  $v \in U_q$ .

On the other hand, if  $\sigma$  is of length at least 2, we must assume that statement  $(*)$  holds for the components of  $\vec{U}^j$  in order to prove it for  $U_\sigma^{j+1}$ . To this end, consider an arbitrary node  $v \in U_\sigma^{j+1}$ . By the first conjunct in (c) and the preceding remarks regarding the trivial cases, we know that  $\delta_0(\lambda^G(v)) = \sigma.first$  (and incidentally that

$j \geq 1$ ). Moreover, the second conjunct ensures the existence of a (possibly empty) set of **traces**  $\mathfrak{S}$  that satisfies  $\mathfrak{S} \triangleright \sigma$  and that represents a “projection” of  $v$ ’s **incoming neighborhood** at stage  $j$ . By the latter we mean that for all  $\pi \in \mathfrak{S}$ , there exists  $u \in V^G$  such that  $uv \in R^G$  and  $u \in U_\pi^j$ , and conversely, for all  $u \in V^G$  with  $uv \in R^G$ , there exists  $\pi \in \mathfrak{S}$  such that  $u \in U_\pi^j$ .

Now, for each **trace**  $\pi \in \mathfrak{S}$  and each **incoming neighbor**  $u$  of  $v$  that is contained in  $U_\pi^j$ , the induction hypothesis provides us with a **pointed digraph**  $G'_{u:\pi}[u'_\pi]$  and a corresponding **timing**  $\tau'_{u:\pi}$ , as described in (\*). We make  $n_{u:\pi} \in \mathbb{N}$  distinct copies of each such **digraph**  $G'_{u:\pi}$ . From this, we construct  $G' = (V^{G'}, R^{G'}, \lambda^{G'})$  by taking the disjoint union of all the  $\sum n_{u:\pi}$  **digraphs**, and adding a single new node  $v'$  with  $\lambda^{G'}(v') = \lambda^G(v)$ , together with all the **edges** of the form  $u'_\pi v'$  (i.e., one such **edge** for each copy of every  $u'_\pi$ ). Given that every  $G'_{u:\pi}[u'_\pi]$  is **backward bisimilar** to  $G[u]$ , we can guarantee that the same holds for  $G'[v']$  and  $G[v]$  by choosing the numbers of **digraph** copies in  $G'$  such that each **incoming neighbor**  $u$  of  $v$  is represented by at least one **incoming neighbor** of  $v'$ . That is, for every  $u$ , we require that  $n_{u:\pi} \geq 1$  for some  $\pi$ .

Finally, we construct a suitable **lossless-asynchronous timing**  $\tau'$  of  $G'$ , which proceeds in two phases to make  $v'$  traverse  $\sigma$  in the corresponding **enriched run**  $\hat{\rho}'$ . In the first phase, where  $0 < t \leq t_1$ , node  $v'$  remains inactive, which means that every  $\tau_t$  assigns 0 to  $v'$  and its incoming **edges**. The **state** of  $v'$  at time  $t_1$  is thus still  $\sigma$ .**first**. Meanwhile, in every copy of each **digraph**  $G'_{u:\pi}$ , the **nodes** and **edges** behave according to **timing**  $\tau'_{u:\pi}$  until the respective copy of  $u'_\pi$  has completely traversed  $\pi$ , whereupon the entire subgraph becomes inactive. By choosing  $t_1$  large enough, we make sure that the FIFO buffer on each **edge** of the form  $u'_\pi v'$  contains precisely  $\pi$  at time  $t_1$ . In the second phase, which lasts from  $t_1 + 1$  to  $t_2$ , the only active parts of  $G'$  are  $v'$  and its incoming **edges**. Since the number  $n_{u:\pi}$  of copies of each **digraph**  $G'_{u:\pi}$  can be chosen as large as required, we stipulate that for every **trace**  $\pi \in \mathfrak{S}$ , the sum of  $n_{u:\pi}$  over all  $u$  exceeds the lower bound  $m_\pi$  that is associated with  $\pi$  when invoking Lemma 4.5 for  $\sigma$  and  $\mathfrak{S}$ . Applying that lemma, we obtain a **lossless-asynchronous timing segment** of the subgraph induced by  $v'$  and its **incoming neighbors**. This segment determines our **timing**  $\tau'$  between  $t_1 + 1$  and  $t_2$  (the other parts of  $G'$  being inactive), and gives us  $\hat{\rho}'_{t_2}(v') = \sigma$ , as desired. Naturally, the remainder of  $\tau'$ , starting at  $t_2 + 1$ , can be chosen arbitrarily, so long as it satisfies the properties of a **lossless-asynchronous timing**.

As a closing remark, note that the **pointed digraph**  $G'[v']$  constructed above is very similar to the standard unraveling of  $G[v]$  into a (possibly infinite) tree. (The set of **nodes** of that tree-unraveling is precisely the set of all directed paths in  $G$  that start at  $v$ ; see, e.g., [BRV02, Def. 4.51] or [BB07, § 3.2]). However, there are a few differences: First, we do the unraveling backwards, because we want to generate a **backward bisimilar structure**, where all the **edges** point toward the **root**. Second, we may duplicate the **incoming neighbors** (i.e., children) of each **node** in the tree, in order to satisfy the lower bounds imposed by Lemma 4.5. Third, we stop the unraveling process at a finite depth (not necessarily the same for each subtree), and place a copy of the original **digraph**  $G$  at every leaf. ■

# 5

## Emptiness Problems

This chapter is concerned with the decidability of the **emptiness problem** for several classes of **nonlocal distributed automata**. Given such an **automaton**, the task is to decide algorithmically whether it **accepts** on at least one input **digraph**. For our main variants of **local automata**, we can easily determine if this is possible, simply on the basis of their logical characterizations: emptiness is decidable for **LDA**'s because they are effectively **equivalent** to  $\tilde{M}L$ , for which the (finite) satisfiability problem is known to be PSPACE-complete; on the other hand, it is undecidable for **ALDA<sub>g</sub>**'s because they are effectively **equivalent** to **MSOL**, for which (finite) satisfiability is undecidable. We have also shown in Section 3.5, that the corresponding problem for **NLDA<sub>g</sub>**'s is decidable, using a simple finite-model argument. Furthermore, by the results on **nonlocal automata** presented in Chapter 4, we know that emptiness is decidable for **a-QDA**'s and **la-QDA**'s, since (finite) satisfiability for the (backward)  $\mu$ -calculus is EXPTIME-complete. However, for **nonlocal automata** in general, the decidability question has been left open by Kuusisto in [Kuu13a]. Indeed, since the logical characterization given there is in terms of the newly introduced modal substitution calculus (for which no decidability results have been previously established), it does not provide us with an immediate answer. Here, we obtain a negative answer for the general case and also consider the question for three subclasses of **nonlocal distributed automata**.

Our first variant, dubbed **forgetful automata**, is characterized by the fact that **nodes** can see their **incoming neighbors' states** but cannot remember their own **state**. Although this restriction might seem very artificial, it bears an intriguing connection to classical automata theory: **forgetful distributed automata** turn out to be **equivalent** to finite **word automata** (and hence **MSOL**) when restricted to **pointed dipaths**, but strictly more expressive than finite **tree automata** (and hence **MSOL**) when restricted to **pointed ordered ditrees**. As shown in [Kuu13a, Prp. 8], the situation is different on arbitrary **digraphs**, where **distributed automata** (and hence **forgetful** ones) are unable to **recognize** non-reachability properties that can be easily expressed in **MSOL**. Hence, none of the two formalisms can simulate the other in general. However, while satisfiability for **MSOL** is undecidable, we obtain a LOGSPACE algorithm that decides the **emptiness problem** for **forgetful distributed automata**.



The preceding decidability result begs the question of what happens if we drop the forgetfulness condition. Motivated by the [equivalence](#) of finite [word automata](#) and [forgetful distributed automata](#), we first investigate this question when restricted to [dipaths](#). In sharp contrast to the forgetful case, we find that for arbitrary [distributed automata](#), it is undecidable whether an [automaton accepts](#) on some [dipath](#). Although our proof follows the standard approach of simulating a Turing machine, it has an unusual twist: we exchange the roles of space and time, in the sense that the space of the simulated Turing machine  $M$  is encoded into the time of the simulating [distributed automaton](#)  $A$ , and conversely, the time of  $M$  is encoded into the space of  $A$ . To lift this result to arbitrary [digraphs](#), we introduce the class of [monovisioned distributed automata](#), where [nodes](#) enter a rejecting [sink state](#) as soon as they see more than one [state](#) in their [incoming neighborhood](#). For every [distributed automaton](#)  $A$ , one can construct a [monovisioned automaton](#)  $A'$  that satisfies the emptiness property if and only if  $A$  does so on [dipaths](#). Hence, the [emptiness problem](#) is undecidable for [monovisioned automata](#), and thus also in general.

Our third and last class consists of the [quasi-acyclic distributed automata](#). The motivation for considering this particular class is threefold. First, quasi-acyclicity may be seen as a natural intermediate stage between [local](#) and [nonlocal distributed automata](#), because [local automata](#) (for which the [emptiness problem](#) is decidable) can be characterized as those [automata](#) whose [state diagram](#) is acyclic as long as we ignore [sink states](#) (see [Section 2.7](#)). Second, the Turing machine simulation mentioned above makes crucial use of directed cycles in the diagram of the simulating [automaton](#), which suggests that cycles might be the source of undecidability. Third, the notion of quasi-acyclic [state diagrams](#) also plays a major role in [Chapter 4](#), where it serves as an ingredient for [a-QDA's](#) and [la-QDA's](#) (for which the [emptiness problem](#) is also decidable). However, contrary to what one might expect from these clues, we show that quasi-acyclicity alone is not sufficient to make the [emptiness problem](#) decidable, thereby giving an alternative proof of undecidability for the general case.

The remainder of this chapter is organized as follows: We first introduce some formal definitions in [Section 5.1](#) and establish the connections between [forgetful distributed automata](#) and classical [word](#) and [tree automata](#) in [Section 5.2](#). Then, in [Section 5.3](#), we show the positive decidability result for [forgetful automata](#). Finally, we establish the negative results for [monovisioned automata](#) in [Section 5.4](#) and for [quasi-acyclic automata](#) in [Section 5.5](#).

## 5.1 Preliminaries

Given a [distributed automaton](#)  $A$ , the (general) [emptiness problem](#) consists in deciding effectively whether the [language](#) of  $A$  is nonempty, i.e., whether there is a [pointed digraph](#)  $G[v]$  that is [accepted](#) by  $A$ . Similarly, the [dipath-emptiness problem](#) is to decide whether  $A$  [accepts](#) some [pointed dipath](#).

We now define [forgetful distributed automata](#), which are characterized by the fact that in each communication round, the [nodes](#) of the input [digraph](#) can see their [neighbors' states](#) but cannot remember their own [state](#). As this entails that they are not able to access their own label by storing it in their [state](#), we instead let them reread that label in each round.

**Definition 5.1** (Forgetful distributed automaton).

- A [forgetful distributed automaton](#) (FDA) over  $\Sigma$ -labeled,  $r$ -relational [digraphs](#) is a

tuple  $A = (Q, q_0, (\delta_a)_{a \in \Sigma}, F)$ , where  $Q$  is a finite nonempty set of **states**,  $q_0 \in Q$  is an **initial state**,  $\delta_a: (\mathbb{2}^Q)^r \rightarrow Q$  is a **transition function associated with label  $a \in \Sigma$** , and  $F \subseteq Q$  is a set of **accepting states**. ◀

The semantics is completely analogous to the one defined in Section 2.7, for the unrestricted automata of Definition 2.3. For a given  $\Sigma$ -labeled,  $r$ -relational digraph  $G$ , the **run**  $\rho$  of  $A$  on  $G$  is the infinite sequence of **configurations**  $(\rho_0, \rho_1, \rho_2, \dots)$ , which are defined inductively as follows, for  $t \in \mathbb{N}$  and  $v \in V^G$ :

$$\rho_0(v) = q_0 \quad \text{and} \quad \rho_{t+1}(v) = \delta_{\lambda^G(v)} \left( \left( \{\rho_t(u) \mid uv \in R_i^G\} \right)_{1 \leq i \leq r} \right).$$

The definition of **acceptance** remains exactly the same as in Section 2.7, i.e., for  $v \in V^G$ , the **pointed digraph**  $G[v]$  is **accepted** by  $A$  if and only if there exists  $t \in \mathbb{N}$  such that  $\rho_t(v) \in F$ .

## 5.2 Comparison with classical automata

The purpose of this section is to motivate our interest in **forgetful distributed automata** by establishing their connection with classical **word** and **tree** automata.

**Proposition 5.2** ( $\llbracket \text{FDA} \rrbracket_{\text{DIPATH}_\Sigma} = \llbracket \text{MSOL} \rrbracket_{\text{DIPATH}_\Sigma}$ ).

► When restricted to the class of **pointed dipaths**, **forgetful distributed automata** are **equivalent** to finite **word** automata, and thus to **MSOL**. ◀

*Proof.* Let us denote a (deterministic) finite **word** automaton over some finite alphabet  $\Sigma$  by a tuple  $B = (P, p_0, \tau, H)$ , where  $P$  is the set of states,  $p_0$  is the initial state,  $\tau: P \times \Sigma \rightarrow P$  is the transition function, and  $H$  is the set of accepting states.

Given such a **word** automaton  $B$ , we construct a **forgetful distributed automaton**  $A = (Q, q_0, (\delta_a)_{a \in \Sigma}, F)$  that simulates  $B$  on  $\Sigma$ -labeled **dipaths**. For this, it suffices to set  $Q = P \cup \{\perp\}$ ,  $q_0 = \perp$ ,  $F = H$ , and

$$\delta_a(S) = \begin{cases} \tau(p_0, a) & \text{if } S = \emptyset, \\ \tau(p, a) & \text{if } S = \{p\} \text{ for some } p \in P, \\ \perp & \text{otherwise.} \end{cases}$$

When  $A$  is **run** on a **dipath**, each **node**  $v$  starts in a waiting phase, represented by  $\perp$ , and remains idle until its predecessor has computed the state  $p$  that  $B$  would have reached just before reading the local letter  $a$  of  $v$ . (If there is no predecessor,  $p$  is set to  $p_0$ .) Then,  $v$  switches to the state  $\tau(p, a)$  and stays there forever. Consequently, the **distinguished last node** of the **dipath** will end up in the state reached by  $B$  at the end of the **word**, and it **accepts** if and only if  $B$  does.

For the converse direction, we convert a given **forgetful distributed automaton**  $A = (Q, q_0, (\delta_a)_{a \in \Sigma}, F)$  into the **word** automaton  $B = (P, p_0, \tau, H)$  with components  $P = \mathbb{2}^Q$ ,  $p_0 = \emptyset$ ,  $H = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$ , and

$$\tau(p, a) = \{q_0\} \cup \begin{cases} \{\delta_a(\emptyset)\} & \text{if } p = p_0, \\ \{\delta_a(\{q\}) \mid q \in p\} & \text{otherwise.} \end{cases}$$

On any  $\Sigma$ -labeled **dipath**  $G$ , our construction guarantees that the set of **states** visited by  $A$  at the  $i$ -th **node** is equal to the state that  $B$  reaches just after processing the

$i$ -th letter of the **word** associated with  $G$ . We can easily verify this by induction on  $i$ : At the first **node**, which is **labeled** with  $a_1$ , **automaton**  $A$  starts in **state**  $q_0$  and then remains forever in **state**  $\delta_{a_1}(\emptyset)$ . **Node** number  $i + 1$  also starts in  $q_0$ , and transitions to  $\delta_{a_{i+1}}(\{q_t^i\})$  at time  $t + 1$ , where  $a_{i+1}$  is the **node**'s own label and  $q_t^i$  is the **state** of its predecessor at time  $t$ . In agreement with this behavior, we know by the induction hypothesis and the definition of  $\tau$  that the state of  $B$  after reading  $a_{i+1}$  is precisely  $\{q_0\} \cup \{\delta_{a_{i+1}}(\{q_t^i\}) \mid t \in \mathbb{N}\}$ . As a result, the final state reached by  $B$  will be accepting if and only if  $A$  visits some **accepting state** at the last **node**. ■

A (deterministic, bottom-up) finite **tree** automaton over  $\Sigma$ -**labeled**,  $r$ -relational **ordered ditrees** can be defined as a tuple  $B = (P, (\tau_k)_{0 \leq k \leq r}, H)$ , where  $P$  is a finite nonempty set of states,  $\tau_k: P^k \times \Sigma \rightarrow P$  is a transition function of arity  $k$ , and  $H \subseteq P$  is a set of accepting states. Such an automaton assigns a state of  $P$  to each **node** of a given **pointed ordered ditree**, starting from the leaves and working its way up to the **root**. If **node**  $v$  is **labeled** with letter  $a$  and its  $k$  children have been assigned the states  $p_1, \dots, p_k$  (following the numbering order of the  $k$  first **edge relations**), then  $v$  is assigned the state  $\tau_k(p_1, \dots, p_k, a)$ . Note that leaves are covered by the special case  $k = 0$ . Based on this, the **pointed ditree** is accepted if and only if the state at the **root** belongs to  $H$ . For a more detailed presentation see, e.g., [Löd12, § 3.3].

**Proposition 5.3** ( $\llbracket \text{FDA} \rrbracket_{\text{@ODITREE}_\Sigma^r} \not\equiv \llbracket \text{MSOL} \rrbracket_{\text{@ODITREE}_\Sigma^r}$ ).

► When restricted to the class of **pointed ordered ditrees**, **forgetful distributed automata** are strictly more expressive than finite **tree** automata, and thus than **MSOL**. ◀

*Proof.* To convert a **tree** automaton  $B = (P, (\tau_k)_{0 \leq k \leq r}, H)$  into a **forgetful distributed automaton**  $A = (Q, q_0, (\delta_a)_{a \in \Sigma}, F)$  that is **equivalent** to  $B$  over  $\Sigma$ -**labeled**,  $r$ -relational **ordered ditrees**, we use a simple generalization of the construction in the proof of Proposition 5.2:  $Q = P \cup \{\perp\}$ ,  $q_0 = \perp$ ,  $F = H$ , and

$$\delta_a(\vec{S}) = \begin{cases} \tau_k(p_1, \dots, p_k, a) & \text{if } \vec{S} = (\{p_1\}, \dots, \{p_k\}, \emptyset, \dots, \emptyset) \text{ with } p_1, \dots, p_k \in P, \\ \perp & \text{otherwise.} \end{cases}$$

In contrast, a conversion in the other direction is not always possible, as can be seen from the following example on binary **ditrees**. Consider the **forgetful distributed automaton**  $A' = (\{\perp, \top, \star\}, \perp, \delta, \{\star\})$ , with

$$\delta(S_1, S_2) = \begin{cases} \perp & \text{if } S_1 = S_2 = \{\perp\} \\ \top & \text{if } S_1, S_2 \in \{\emptyset, \{\top\}\} \\ \star & \text{otherwise.} \end{cases}$$

When run on an **unlabeled**, 2-relational **ordered ditree**,  $A'$  **accepts** at the **root** precisely if the **ditree** is *not* perfectly balanced, i.e., if there exists a **node** whose left and right subtrees have different heights. To achieve this, each **node** starts in the waiting **state**  $\perp$ , where it remains as long as it has two children and those children are also in  $\perp$ . If the **ditree** is perfectly balanced, then all the leaves switch permanently from  $\perp$  to  $\top$  in the first round, their parents do so in the second round, their parents' parents in the third round, and so forth, until the signal reaches the **root**. Therefore, the **root** will transition directly from  $\perp$  to  $\top$ , never visiting **state**  $\star$ , and hence the **pointed ditree** is rejected. On the other hand, if the **ditree** is not perfectly balanced, then



there must be some lowermost internal **node**  $v$  that does not have two subtrees of the same height (in particular, it might have only one child). Since its subtrees are perfectly balanced, they behave as in the preceding case. At some point in time, only one of  $v$ 's children will be in **state**  $\perp$ , at which point  $v$  will switch to **state**  $\star$ . This triggers an upward-propagating chain reaction, eventually causing the **root** to also visit  $\star$ , and thus to **accept**. Note that  $\star$  is just an intermediate **state**; regardless of whether or not the **ditree** is perfectly balanced, every **node** will ultimately end up in  $\top$ .

To prove that  $A'$  is not **equivalent** to any **tree** automaton, one can simply invoke the pumping lemma for regular **tree languages** to show that the complement **language** of  $A'$  is not recognizable by any **tree** automaton. The claim then follows from the fact that regular **tree languages** are closed under complementation. ■

### 5.3 Exploiting forgetfulness

We now give an algorithm deciding the **emptiness problem** for **forgetful distributed automata** (on arbitrary **digraphs**). Its space complexity is linear in the number of **states** of the given automaton. However, as an uncompressed binary encoding of a **distributed automaton** requires space exponential in the number of **states**, this results in LOGSPACE complexity. Obviously, the statement might not hold anymore if the **automaton** were instead represented by a more compact device, such as a logical formula.

#### Theorem 5.4.

► We can decide the **emptiness problem** for **forgetful distributed automata** with LOGSPACE complexity. ◀

*Proof.* Let  $A = (Q, q_0, (\delta_\alpha)_{\alpha \in \Sigma}, F)$  be some **forgetful distributed automaton** over  $\Sigma$ -labeled,  $r$ -relational **digraphs**. Consider the infinite sequence of sets of **states**  $S_0, S_1, S_2, \dots$  such that  $S_t$  contains precisely those **states** that can be visited by  $A$  at some **node** in some **digraph** at time  $t$ . That is,  $q \in S_t$  if and only if there exists a **pointed digraph**  $G[v]$  such that  $\rho_t(v) = q$ , where  $\rho$  is the **run** of  $A$  on  $G$ . From this point of view, the **pointed-digraph language** of  $A$  is nonempty precisely if there is some  $t \in \mathbb{N}$  for which  $S_t \cap F \neq \emptyset$ .

By definition, we have  $S_0 = \{q_0\}$ . Furthermore, exploiting the fact that  $A$  is **forgetful**, we can specify a simple function  $\Delta: \mathcal{P}^Q \rightarrow \mathcal{P}^Q$  such that  $S_{t+1} = \Delta(S_t)$ :

$$\Delta(S) = \{ \delta_\alpha(\vec{T}) \mid \alpha \in \Sigma \text{ and } \vec{T} \in (\mathcal{P}^S)^r \}$$

Obviously,  $S_{t+1} \subseteq \Delta(S_t)$ . To see that  $S_{t+1} \supseteq \Delta(S_t)$ , assume we are given a **pointed digraph**  $G_q[v_q]$  for each state  $q \in S_t$  such that  $v_q$  visits  $q$  at time  $t$  in the **run** of  $A$  on  $G_q$ . (Such a **pointed digraph** must exist by the definition of  $S_t$ .) Now, for any  $\alpha \in \Sigma$  and  $\vec{T} = (T_1, \dots, T_r) \in (\mathcal{P}^{S_t})^r$ , we construct a new **digraph**  $G$  as follows: Starting with a single  $\alpha$ -labeled **node**  $v$ , we add a (disjoint) copy of  $G_q$  for each state  $q$  that occurs in some set  $T_k$ . Then, we add a  $k$ -edge from  $v_q$  to  $v$  if and only if  $q \in T_k$ . Each **node**  $v_q$  behaves the same way in  $G$  as in  $G_q$  because  $v$  has no influence on its **incoming neighbors**. Since  $A$  is **forgetful**, the state of  $v$  at time  $t+1$  depends solely on its own label and its **incoming neighbors' states** at time  $t$ . Consequently,  $v$  visits the state  $\delta_\alpha(\vec{T})$  at time  $t+1$ , and thus  $\delta_\alpha(\vec{T}) \in S_{t+1}$ .

Now, we know that the sequence  $S_0, S_1, S_2 \dots$  must be eventually periodic because its generator function  $\Delta$  maps the finite set  $2^Q$  to itself. Hence, it suffices to consider the prefix of length  $|2^Q|$  in order to determine whether  $S_t \cap F \neq \emptyset$  for some  $t \in \mathbb{N}$ . This leads to the following simple algorithm, which decides the **emptiness problem** for **forgetful automata**.

```

EMPTY(A) :  S  $\leftarrow$  {q0}
             repeat at most  $|2^Q|$  times :
               S  $\leftarrow$   $\Delta(S)$ 
               if  $S \cap F \neq \emptyset$  : return true
             return false

```

It remains to analyze the space complexity of this algorithm. For that, we assume that the binary encoding of  $A$  given to the algorithm contains a lookup table for each **transition function**  $\delta_a$  and a bit array representing  $F$ , which amounts to an asymptotic size of  $\Theta(|\Sigma| \cdot |2^Q|^r \cdot \log |Q|)$  input bits. To implement the procedure **EMPTY**, we need  $|Q|$  bits of working memory to represent the set  $S$  and another  $|Q|$  bits for the loop counter. Furthermore, we can compute  $\Delta(S)$  for any given set  $S \subseteq Q$  by simply iterating over all  $a \in \Sigma$  and  $\vec{T} \in (2^Q)^r$ , and adding  $\delta_a(\vec{T})$  to the returned set if all components of  $\vec{T}$  are subsets of  $S$ . This requires  $\log |\Sigma| + |Q| \cdot r$  additional bits to keep track of the iteration progress,  $\Theta(\log |\Sigma| + |Q| \cdot r + \log \log |Q|)$  bits to store pointers into the lookup tables, and  $|Q|$  bits to store the intermediate result. In total, the algorithm uses  $\Theta(\log |\Sigma| + |Q| \cdot r)$  bits of working memory, which is logarithmic in the size of the input. ■

## 5.4 Exchanging space and time

In this section, we first show the undecidability of the **dipath-emptiness problem** for arbitrary **distributed automata**, and then lift that result to the general **emptiness problem**.

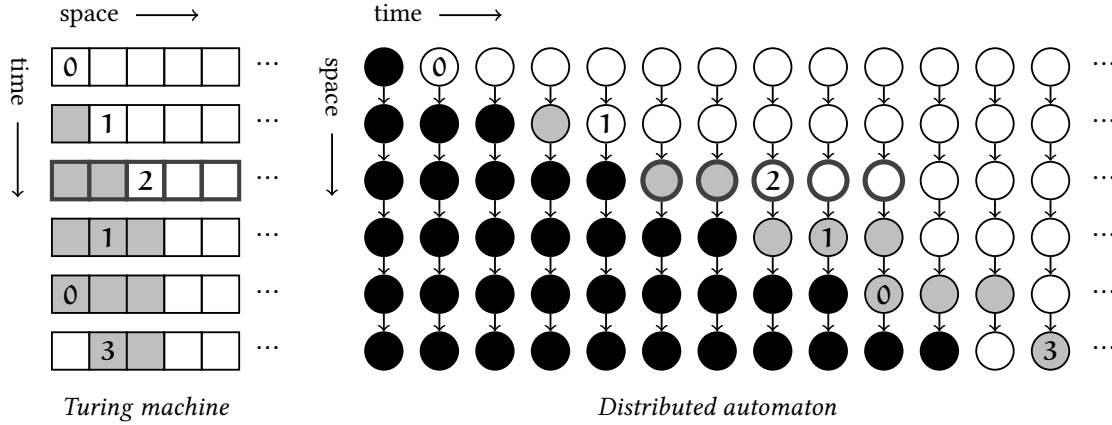
### Theorem 5.5.

► The **dipath-emptiness problem** for **distributed automata** is undecidable. ◀

*Proof sketch.* We proceed by reduction from the halting problem for Turing machines. For our purposes, a Turing machine operates deterministically with one head on a single tape, which is one-way infinite to the right and initially empty. The problem consists of determining whether the machine will eventually reach a designated halting state. We show a way of encoding the computation of a Turing machine  $M$  into the **run** of a **distributed automaton**  $A$  over **unlabeled digraphs**, such that the **language** of  $A$  contains a **pointed dipath** if and only if  $M$  reaches its halting state.

Note that since **dipaths** are oriented, the communication between their **nodes** is only one-way. Hence, we cannot simply represent (a section of) the Turing tape as a **dipath**. Instead, the key idea of our simulation is to exchange the roles of space and time, in the sense that the space of  $M$  is encoded into the time of  $A$ , and the time of  $M$  into the space of  $A$ . Assuming the **language** of  $A$  contains a **dipath**, we will think of that **dipath** as representing the timeline of  $M$ , such that each **node** corresponds to a single point in time in the computation of  $M$ . Roughly speaking, when **running**  $A$ ,

*It turns out that this corresponds to a well-known construction in cellular automata theory; see Section 7.2.2.*



**Figure 5.1.** Exchanging space and time to prove Theorem 5.5. The left-hand side depicts the computation of a Turing machine with state set  $\{0, 1, 2, 3\}$  and tape alphabet  $\{\square, \blacksquare\}$ . On the right-hand side, this machine is simulated by a **distributed automaton** run on a **dipath**. Waiting nodes are represented in black, whereas active nodes display the content of the “currently visited” cell of the Turing machine (i.e., only the third component of the **states** is shown).

the node  $v_t$  corresponding to time  $t$  will “traverse” the configuration  $C_t$  of  $M$  at time  $t$ . Here, “traversing” means that the sequence of **states** of  $A$  visited by  $v_t$  is an encoding of  $C_t$  read from left to right, supplemented with some additional bookkeeping information.

The first element of the **dipath**, node  $v_0$ , starts by visiting a **state** of  $A$  representing an empty cell that is currently read by  $M$  in its initial state. Then it transitions to another **state** that simply represents an empty cell, and remains in such a **state** forever after. Thus  $v_0$  does indeed “traverse”  $C_0$ . We will show that it is also possible for any other node  $v_t$  to “traverse” its corresponding configuration  $C_t$ , based on the information it receives from  $v_{t-1}$ . In order for this to work, we shall give  $v_{t-1}$  a head start of two cells, so that  $v_t$  can compute the content of cell  $i$  in  $C_t$  based on the contents of cells  $i-1$ ,  $i$  and  $i+1$  in  $C_{t-1}$ .

Node  $v_t$  enters an **accepting state** of  $A$  precisely if it “sees” the halting state of  $M$  during its “traversal” of  $C_t$ . Hence,  $A$  **accepts** the **pointed dipath** of length  $t$  if and only if  $M$  reaches its halting state at time  $t$ .

We now describe the inner workings of  $A$  in a semi-formal way. In parallel, the reader might want to have a look at Figure 5.1, which illustrates the construction by means of an example. Let  $M$  be represented by the tuple  $(P, \Gamma, p_0, \square, \tau, p_h)$ , where  $P$  is the set of states,  $\Gamma$  is the tape alphabet,  $p_0$  is the initial state,  $\square$  is the blank symbol,  $\tau: (P \setminus \{p_h\}) \times \Gamma \rightarrow P \times \Gamma \times \{L, R\}$  is the transition function, and  $p_h$  is the halting state. From this, we construct  $A$  as  $(Q, q_0, \delta, F)$ , with the **state** set  $Q = (\{\perp\} \cup (P \times \Gamma) \cup \Gamma)^3$ , the initial **state**  $q_0 = (\perp, \perp, \perp)$ , the **transition function**  $\delta$  specified informally below, and the **accepting** set  $F$  that contains precisely those **states** that have  $p_h$  in their third component. In keeping with the intuition that each **node** of the **dipath** “traverses” a configuration of  $M$ , the third component of its **state** indicates the content of the “currently visited” cell  $i$ . The two preceding components keep track of the recent history, i.e., the second component always holds the content of the previous cell  $i-1$ , and the first component that of  $i-2$ . In the following explanation, we concentrate on updating the third component, tacitly assuming that the other two are kept up to

date. The special symbol  $\perp$  indicates that no cell has been “visited”, and we say that a **node** is in the waiting phase while its third component is  $\perp$ .

In the first round,  $v_0$  sees that it does not have any **incoming neighbor**, and thus exits the waiting phase by setting its third component to  $(p_0, \square)$ , and after that, it sets it to  $\square$  for the remainder of the **run**. Every other **node**  $v_t$  remains in the waiting phase as long as its **incoming neighbor**’s second component is  $\perp$ . This ensures a delay of two cells with respect to  $v_{t-1}$ . Once  $v_t$  becomes active, given the **current state**  $(c_1, c_2, c_3)$  of  $v_{t-1}$ , it computes the third component  $d_3$  of its own **next state**  $(d_1, d_2, d_3)$  as follows: If none of the components  $c_1, c_2, c_3$  “contain the head of  $M$ ”, i.e., if none of them lie in  $P \times \Gamma$ , then it simply sets  $d_3$  to be equal to  $c_2$ . Otherwise, a computation step of  $M$  is simulated in the natural way. For instance, if  $c_3$  is of the form  $(p, \gamma)$ , and  $\tau(p, \gamma) = (p', \gamma', L)$ , then  $d_3$  is set to  $(p', c_2)$ . This corresponds to the case where, at time  $t - 1$ , the head of  $M$  is located to the right of  $v_t$ ’s next “position” and moves to the left. As another example, if  $c_2$  is of the form  $(p, \gamma)$ , and  $\tau(p, \gamma) = (p', \gamma', R)$ , then  $d_3$  is set to  $\gamma'$ . The remaining cases are handled analogously.

Note that, thanks to the two-cell delay between **adjacent nodes**, the head of  $M$  always “moves forward” in the time of  $A$ , although it may move in both directions with respect to the space of  $M$  (see Figure 5.1). ■

To infer from Theorem 5.5 that the general **emptiness problem** for **distributed automata** is also undecidable, we now introduce the notion of **monovisioned automata**, which have the property that **nodes** “expect” to see no more than one **state** in their **incoming neighborhood** at any given time. More precisely, a **distributed automaton**  $A = (Q, \delta_0, \delta, F)$  is **monovisioned** if it has a rejecting **sink state**  $q_{\text{rej}} \in Q \setminus F$ , such that  $\delta(q, S) = q_{\text{rej}}$  whenever  $|S| > 1$  or  $q_{\text{rej}} \in S$  or  $q = q_{\text{rej}}$ , for all  $q \in Q$  and  $S \subseteq Q$ . Obviously, for every **distributed automaton**, we can construct a **monovisioned automaton** that has the same **acceptance behavior** on **dipaths**. Furthermore, as shown by means of the next two lemmas, the **emptiness problem** for **monovisioned automata** is equivalent to its restriction to **dipaths**. All put together, we get the desired reduction from the **dipath-emptiness problem** to the general **emptiness problem**.

**Lemma 5.6.**

► The **language** of a **distributed automaton** is nonempty if and only if it contains a **pointed ditree**. ◀

*Proof sketch.* We slightly adapt the notion of **tree-unraveling**, which is a standard tool in modal logic (see, e.g., [BRV02, Def. 4.51] or [BB07, § 3.2]). Consider any **distributed automaton**  $A$ . Assume that  $A$  **accepts** some **pointed digraph**  $G[v]$ , and let  $t \in \mathbb{N}$  be the first point in time at which  $v$  visits an **accepting state**. Based on that, we can easily construct a **pointed ditree**  $G'[v']$  that is also **accepted** by  $A$ . First of all, the **root**  $v'$  of  $G'$  is chosen to be a copy of  $v$ . On the next level of the **ditree**, the **incoming neighbors** of  $v'$  are chosen to be fresh copies  $u'_1, \dots, u'_n$  of  $v$ ’s **incoming neighbors**  $u_1, \dots, u_n$ . Similarly, the **incoming neighbors** of  $u'_1, \dots, u'_n$  are fresh copies of the **incoming neighbors** of  $u_1, \dots, u_n$ . If  $u_i$  and  $u_j$  have **incoming neighbors** in common, we create distinct copies of those **neighbors** for  $u'_i$  and  $u'_j$ . This process is iterated until we obtain a **ditree** of height  $t$ . It is easy to check that  $v$  and  $v'$  visit the same sequence of **states**  $q_0, q_1, \dots, q_t$  during the first  $t$  communication rounds. ■

**Lemma 5.7.**

► The language of a monovisioned distributed automaton is nonempty if and only if it contains a pointed dipath. ◀

*Proof sketch.* Consider any monovisioned distributed automaton  $A$  whose language is nonempty. By Lemma 5.6,  $A$  accepts some pointed ditree  $G[v]$ . Let  $t \in \mathbb{N}$  be the first point in time at which  $v$  visits an accepting state. Now, it is easy to prove by induction that for all  $i \in \{0, \dots, t\}$ , sibling nodes at depth  $i$  traverse the same sequence of states  $q_0, q_1, \dots, q_{t-i}$  between times 0 and  $t - i$ , and this sequence does not contain the rejecting state  $q_{\text{rej}}$ . Thus,  $A$  also accepts any dipath from some node at depth  $t$  to the root. ■

## 5.5 Timing a firework show

We now show that the emptiness problem is undecidable even for quasi-acyclic automata. This also provides an alternative, but more involved undecidability proof for the general case. Notice that our proof of Theorem 5.5 does not go through if we consider only quasi-acyclic automata.

It is straightforward to see that quasi-acyclicity is preserved under a standard product construction, similar to the one employed for finite automata on words. Hence, we have the following closure property, which will be used in the subsequent undecidability proof.

**Lemma 5.8.**

► The class of languages recognizable by quasi-acyclic distributed automata is closed under union and intersection. ◀

**Theorem 5.9.**

► The emptiness problem for quasi-acyclic distributed automata is undecidable. ◀

*Proof sketch.* We show this by reduction from Post's correspondence problem (PCP). An instance  $P$  of PCP consists of a collection of pairs of nonempty finite words  $(x_i, y_i)_{i \in I}$  over the alphabet  $\{0, 1\}$ , indexed by some finite set of integers  $I$ . It is convenient to view each pair  $(x_i, y_i)$  as a domino tile labeled with  $x_i$  on the upper half and  $y_i$  on the lower half. The problem is to decide if there exists a nonempty sequence  $S = (i_1, \dots, i_n)$  of indices in  $I$ , such that the concatenations  $x_S = x_{i_1} \cdots x_{i_n}$  and  $y_S = y_{i_1} \cdots y_{i_n}$  are equal. We construct a quasi-acyclic automaton  $A$  whose language is nonempty if and only if  $P$  has such a solution  $S$ .

Metaphorically speaking, our construction can be thought of as a perfectly timed “firework show”, whose only “spectator” will see a putative solution  $S = (i_1, \dots, i_n)$ , and be able to check whether it is indeed a valid solution of  $P$ . Our “spectator” is the distinguished node  $v_e$  of the pointed digraph on which  $A$  is run. We assume that  $v_e$  has  $n$  incoming neighbors, one for each element of  $S$ . Let  $v_k$  denote the neighbor corresponding to  $i_k$ , for  $1 \leq k \leq n$ . Similarly to our proof of Theorem 5.5, we use the time of  $A$  to represent the spatial dimension of the words  $x_S$  and  $y_S$ . On an intuitive level,  $v_e$  will “witness” simultaneous left-to-right traversals of  $x_S$  and  $y_S$ , advancing by one bit per time step, and it will check that the two words match. It is the task of each node  $v_k$  to send to  $v_e$  the required bits of the subwords  $x_{i_k}$  and  $y_{i_k}$  at the appropriate times. In keeping with the metaphor of fireworks, the correct timing can be achieved by attaching to  $v_k$  a carefully chosen “fuse”, which is

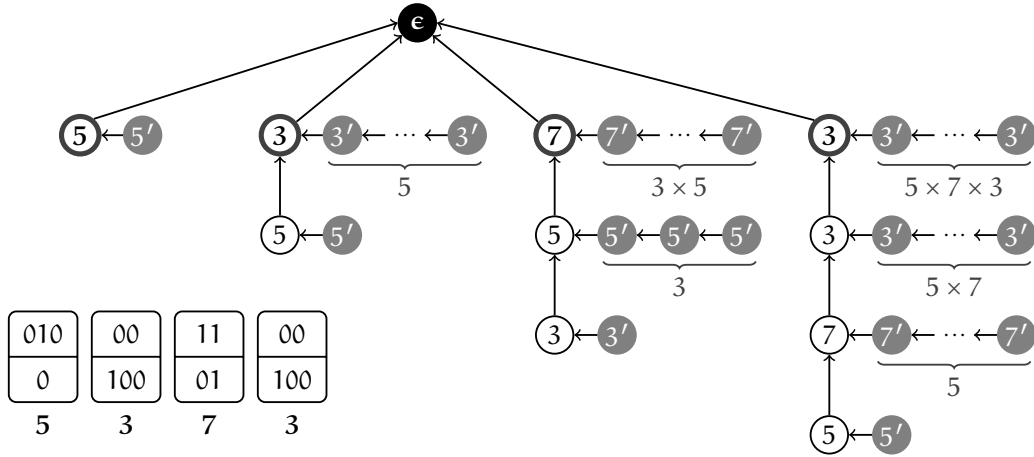


“lit” at time 0. Two separate “fire” signals will travel at different speeds along this (admittedly sophisticated) “fuse”, and once they reach  $v_k$ , they trigger the “firing” of  $x_{i_k}$  and  $y_{i_k}$ , respectively.

We now go into more details. Using the **labeling** of the input **graph**, the **automaton**  $A$  distinguishes between  $2|I| + 1$  different types of **nodes**: two types  $i$  and  $i'$  for each index  $i \in I$ , and one additional type  $\epsilon$  to identify the “spectator”. Motivated by Lemma 5.6, we suppose that the input **graph** is a **pointed ditree**, with a very specific shape that encodes a putative solution  $S = (i_1, \dots, i_n)$ . An example illustrating the following description of such a **ditree**-encoding is given in Figure 5.2. Although  $A$  is not able to enforce all aspects of this particular shape, we will make sure that it **accepts** such a **structure** if its **language** is nonempty. The **root** (and **distinguished node**)  $v_\epsilon$  is the only **node** of type  $\epsilon$ . Its children  $v_1, \dots, v_n$  are of types  $i_1, \dots, i_n$ , respectively. The “fuse” attached to each child  $v_k$  is a chain of  $k - 1$  **nodes** that represents the multiset of indices occurring in the  $(k - 1)$ -prefix of  $S$ . More precisely, there is an induced **dipath**  $v_{k,1} \rightarrow \dots \rightarrow v_{k,k-1} \rightarrow v_k$ , such that the multiset of types of the **nodes**  $v_{k,1}, \dots, v_{k,k-1}$  is equal to the multiset of indices occurring in  $(i_1, \dots, i_{k-1})$ . We do not impose any particular order on those **nodes**. Finally, each **node** of type  $i \in I$  also has an incoming chain of **nodes** of type  $i'$  (depicted in gray in Figure 5.2), whose length corresponds exactly to the product of the types occurring on the part of the “fuse” below that **node**. That is, if we define the alias  $v_{k,k} := v_k$ , then for every **node**  $v_{k,j}$  of type  $i \in I$ , there is an induced **dipath**  $v_{k,j,1} \rightarrow \dots \rightarrow v_{k,j,\ell} \rightarrow v_{k,j}$ , where all the **nodes**  $v_{k,j,1}, \dots, v_{k,j,\ell}$  are of type  $i'$ , and the number  $\ell$  is equal to the product of the types of the **nodes**  $v_{k,1}, \dots, v_{k,j-1}$  (which is 1 if  $j = 1$ ). We shall refer to such a chain  $v_{k,j,1}, \dots, v_{k,j,\ell}$  as a “side fuse”.

The **automaton**  $A$  has to perform two tasks simultaneously: First, assuming it is **run** on a **ditree**-encoding of a sequence  $S$ , exactly as specified above, it must verify that  $S$  is a valid solution, i.e., that the words  $x_S$  and  $y_S$  match. Second, it must ensure that the input **graph** is indeed sufficiently similar to such a **ditree**-encoding. In particular, it has to check that the “fuses” used for the first task are consistent with each other. Since, by Lemma 5.8, **quasi-acyclic distributed automata** are closed under intersection, we can consider the two tasks separately, and implement them using two independent **automata**  $A_1$  and  $A_2$ . In the following, we describe both devices in a rather informal manner. The important aspect to note is that they can be easily formalized using **quasi-acyclic state diagrams**.

We start with  $A_1$ , which verifies the solution  $S$ . It takes into account only **nodes** with types in  $I \cup \{\epsilon\}$  (thus ignoring the gray **nodes** in Figure 5.2). At **nodes** of type  $i \in I$ , the **states** of  $A_1$  have two components, associated with the upper and lower halves of the domino  $(x_i, y_i)$ . If a **node** of type  $i$  sees that it does not have any **incoming neighbor**, then the upper and lower components of its **state** immediately start traversing sequences of substates representing the bits of  $x_i$  and  $y_i$ , respectively. Since those substates must keep track of the respective positions within  $x_i$  and  $y_i$ , none of them can be visited twice. After that, both components loop forever on a special substate  $\top$ , which indicates the end of transmission. The other **nodes** of type  $i$  keep each of their two components in a waiting status, indicated by another substate  $\perp$ , until the corresponding component of their **incoming neighbor** reaches its last substate before  $\top$ . This constitutes the aforementioned “fire” signal. Thereupon, they start traversing the same sequences of substates as in the previous case. Note that both components are updated independently of each other, hence there can be an arbitrary time lag between the “traversals” of  $x_i$  and  $y_i$ . Now, assuming the “fuse”



**Figure 5.2.** Timing a “firework show” to prove Theorem 5.9. The domino tiles on the bottom-left visualize the solution  $(5, 3, 7, 3)$  for the instance  $\{3 \mapsto (00, 100), 5 \mapsto (010, 0), 7 \mapsto (11, 01)\}$  of PCP. This solution is encoded into the labeled ditree above, with node types  $\epsilon, 3, 5, 7, 3', 5', 7'$ . Each domino is represented by a bold-highlighted white node of the appropriate type. The “fuse” of such a bold node consists of the chain of white nodes below it, which lists the indices of the preceding dominos in an arbitrary order. Each white node also has a gray “side fuse” whose length is equal to the product of the white types occurring below that node. The “firework show” observed at the root will feature two simultaneous bitstreams, which both represent the sequence 01001100.

of each node  $v_k$  really encodes the multiset of indices occurring in  $(i_1, \dots, i_{k-1})$ , the delay accumulated along that “fuse” will be such that  $v_k$  starts “traversing”  $x_{i_k}$  and  $y_{i_k}$  at the points in time corresponding to their respective starting positions within  $x_S$  and  $y_S$ . That is, for  $x_{i_k}$  it starts at time  $|x_{i_1} \cdots x_{i_{k-1}}| + 1$ , and for  $y_{i_k}$  at time  $|y_{i_1} \cdots y_{i_{k-1}}| + 1$ . Consequently, in each round  $t \leq \min\{|x_S|, |y_S|\}$ , the root  $v_\epsilon$  receives the  $t$ -th bits of  $x_S$  and  $y_S$ . At most two distinct children send bits at the same time, while the others remain in some state  $q \in \{\perp, \top\}^2$ . With this, the behavior of  $A_1$  at  $v_\epsilon$  is straightforward: It enters its only accepting state precisely if all of its children have reached the state  $(\top, \top)$  and it has never seen any mismatch between the upper and lower bits.

We now turn to  $A_2$ , whose job is to verify that the “fuses” used by  $A_1$  are reliable. Just like  $A_1$ , it works under the assumption that the input digraph is a ditree as specified previously, but with significantly reduced guarantees: The root could now have an arbitrary number of children, the “fuses” and “side fuses” could be of arbitrary lengths, and each “fuse” could represent an arbitrary multiset of indices in  $I$ . Again using an approach reminiscent of fireworks, we devise a protocol in which each child  $v$  will send two distinct signals to the root  $v_\epsilon$ . The first signal  $\uparrow_1$  indicates that the current time  $t$  is equal to the product of the types of all the nodes on  $v$ ’s “fuse”. Similarly, the second signal  $\uparrow_2$  indicates that the current time is equal to that same product multiplied by  $v$ ’s own type. To achieve this, we make use of the “side fuses”, along which two additional signals  $\leftarrow_1$  and  $\leftarrow_2$  are propagated. For each node of type  $i \in I$ , the nodes of type  $i'$  on the corresponding “side fuse” operate in a way such that  $\leftarrow_1$  advances by one node per time step, whereas  $\leftarrow_2$  is delayed by  $i$  time units at every node. Hence,  $\leftarrow_1$  travels  $i$  times faster than  $\leftarrow_2$ . Building on that, each node  $v$  of type  $i$  (not necessarily a child of the root) sends  $\uparrow_1$  to its parent, either at time 1,

if it does not have any predecessor on the “fuse”, or one time unit before receiving  $\uparrow_2$  from its predecessor. The latter is possible, because the predecessor also sends a pre-signal  $\uparrow_2^{\text{pre}}$  before sending  $\uparrow_2$ . Then,  $v$  checks that signal  $\leftarrow_1$  from its “side fuse” arrives exactly at the same time as  $\uparrow_2$  from its predecessor, or at time 1 if there is no predecessor. Otherwise, it immediately enters a rejecting *state*. This will guarantee, by induction, that the length of the “side fuse” is equal to the product of the types on the “fuse” below. Finally, two rounds prior to receiving  $\leftarrow_2$ , while that signal is still being delayed by the last *node* on the “side fuse”,  $v$  first sends the pre-signal  $\uparrow_2^{\text{pre}}$ , and then the signal  $\uparrow_2$  in the following round. For this to work, we assume that each *node* on the “side fuse” waits for at least two rounds between receiving  $\leftarrow_2$  from its predecessor and forwarding the signal to its successor, i.e., all indices in  $I$  must be strictly greater than 2. Due to the delay accumulated by  $\leftarrow_2$  along the “side fuse”, the time at which  $\uparrow_2$  is sent corresponds precisely to the length of the “side fuse” multiplied by  $i$ .

Without loss of generality, we require that the set of indices  $I$  contains only prime numbers (as in Figure 5.2). Hence, by the unique-prime-factorization theorem, each multiset of numbers in  $I$  is uniquely determined by the product of its elements. This leads to a simple verification procedure performed by  $A_2$  at the *root*: At time 1, *node*  $v_\epsilon$  checks that it receives  $\uparrow_1$  and not  $\uparrow_2$ . After that, it expects to never again see  $\uparrow_1$  without  $\uparrow_2$ , and remains in a loop as long as it gets either no signal at all or both  $\uparrow_1$  and  $\uparrow_2$ . Upon receiving  $\uparrow_2$  alone, it exits the loop and verifies that all of its children have sent both signals, which is apparent from the *state* of each child. The *root* rejects immediately if any of the expectations above are violated, or if two *nodes* with different types send the same signal at the same time. Otherwise, it enters an *accepting state* after leaving the loop. Now, consider the sequence  $T = (t_1, \dots, t_{n+1})$  of rounds in which  $v_\epsilon$  receives at least one of the signals  $\uparrow_1$  and  $\uparrow_2$ . It is easy to see by induction on  $T$  that successful completion of the procedure above ensures that there is a sequence  $S = (i_1, \dots, i_n)$  of indices in  $I$  with the following properties: For each  $k \in \{1, \dots, n\}$ , the *root* has at least one child  $v_k$  of type  $i_k$  that sends  $\uparrow_1$  at time  $t_k$  and  $\uparrow_2$  at time  $t_{k+1}$ , and the “fuse” of  $v_k$  encodes precisely the multiset of indices occurring in  $(i_1, \dots, i_{k-1})$ . Conversely, each child of  $v_\epsilon$  can be associated in the same manner with a unique element of  $S$ .

To conclude our proof, we have to argue that the *automaton*  $A$ , which simulates  $A_1$  and  $A_2$  in parallel, *accepts* some *labeled pointed digraph* if and only if  $P$  has a solution  $S$ . The “if” part is immediate, since, by construction,  $A$  *accepting a ditree-encoding* of  $S$  is equivalent to  $S$  being a valid solution of  $P$ . To show the “only if” part, we start with a *pointed digraph accepted* by  $A$ , and incrementally transform it into a *ditree-encoding* of a solution  $S$ , while maintaining *acceptance* by  $A$ : First of all, by Lemma 5.6, we may suppose that the *digraph* is a *ditree*. Its *root* must be of type  $\epsilon$ , since  $A$  would not *accept* otherwise. Next, we require that  $A$  raises an alarm at *nodes* that see an unexpected set of *states* in their *incoming neighborhood*, and that this alarm is propagated up to the *root*, which then reacts by entering a rejecting *sink state*. This ensures that the repartition of types is consistent with our specification; for example, that the children of a *node* of type  $i'$  must be of type  $i'$  themselves. We now prune the *ditree* in such a way that *nodes* of type  $i$  keep at most two children and *nodes* of type  $i'$  keep at most one child. (The behavior of the deleted children must be indistinguishable from the behavior of the remaining children, since otherwise an alarm would be raised.) This leaves us with a *ditree* corresponding exactly to the input “expected” by the *automaton*  $A_2$ . Since it is *accepted* by  $A_2$ , this *ditree* must



be very close to an encoding of a solution  $S = (i_1, \dots, i_n)$ , with the only difference that each element  $i_k$  of  $S$  may be represented by several nodes  $v_k^1, \dots, v_k^m$ . However, we know by construction that  $A$  behaves the same on all of these representatives. We can therefore remove the subtrees rooted at  $v_k^2, \dots, v_k^m$ , and thus we obtain a ditree-encoding of  $S$  that is accepted by  $A$ . ■



# 6

## Alternation Hierarchies

In this chapter, we transfer the **set quantifiers** of **MSOL** to the setting of **modal logic** and investigate the resulting alternation hierarchies. More precisely, we establish separation results for the hierarchies that one obtains by alternating existential and universal **set quantifiers** in several logics of the form  $\text{MSO}(\Phi)$ , where  $\Phi$  is some variant of **modal logic**.

Within the context of this thesis, the motivation for such hybrids between modal logic and classical logic stems from their close connection to **local distributed automata**. By [HJK<sup>+</sup>12, HJK<sup>+</sup>15], **LDA**'s are equivalent to  $\tilde{\text{ML}}$  (Theorem 2.5), and by Chapter 3, **ALDA<sub>g</sub>**'s are equivalent to **MSOL** (Theorem 3.13). As mentioned in Section 3.6, the combination of those two results suggests an alternative logical characterization of **ALDA<sub>g</sub>**'s using  $\text{MSO}(\tilde{\text{ML}}_g)$  instead of **MSOL**. The equivalence of  $\text{MSO}(\tilde{\text{ML}}_g)$  and **MSOL** can be easily proven by a standard technique that simulates **node quantifiers** through **set quantifiers** (see, e.g., [Kuu08, Kuu15, § 3]). Yet in some sense,  $\text{MSO}(\tilde{\text{ML}}_g)$  provides a more faithful representation of **ALDA<sub>g</sub>**'s because it preserves the expressive power of each **quantifier** alternation level. For instance, the existential fragment  $\Sigma_1^{\text{MSO}}(\tilde{\text{ML}}_g)$  specifies exactly the same **digraph languages** as **NLDA<sub>g</sub>**'s, whereas **EMSOL** is strictly more powerful (see Section 3.6). Therefore, if we want to precisely examine the power of alternation between nondeterministic decisions and universal branchings in **ALDA<sub>g</sub>**'s, then we can do so from a purely logical perspective using  $\text{MSO}(\tilde{\text{ML}}_g)$ . This has the advantage that, compared to **state diagrams**, **formulas** take up less space and are usually easier to manipulate.

As it turns out, the above considerations are closely related to an old problem in modal logic. Already in 1983, van Benthem asked in [Ben83] whether the syntactic hierarchy obtained by alternating existential and universal **set quantifiers** in  $\text{MSO}(\tilde{\text{ML}})$  induces a corresponding hierarchy on the semantic side. Remaining unanswered, the question was raised again by ten Cate in [Cat06], and finally a positive answer was provided by Kuusisto in [Kuu08, Kuu15]: he showed that  $\text{MSO}(\tilde{\text{ML}})$  induces an *infinite* hierarchy over **pointed digraphs**. This tells us that the hierarchy does not completely collapse at some level, but a priori leaves open whether or not each number of **quantifier** alternations corresponds to a separate semantic level.

Kuusisto's proof builds upon the work of Matz, Schweikardt and Thomas in

*In [Cat06] and [Kuu08, Kuu15],  $\text{MSO}(\tilde{\text{ML}})$  is called SOPML (second-order propositional modal logic).*

[MST02] (elaborating on their previous results in [MT97] and [Sch97]), where they have shown that in the case of  $\text{MSOL}$  on **digraphs**, the alternation hierarchy is *strict*. Thus, each additional alternation between the two types of **set quantifiers** properly extends the family of **definable digraph languages**. Significantly, this separation also holds on **grids**, a more restrictive class of **structures**, where it can be established using techniques from classical automata theory. Furthermore, taken in conjunction with the **equivalence** of  $\text{MSO}(\vec{\text{ML}}_g)$  and  $\text{MSOL}$ , the result on **digraphs** immediately implies that the corresponding hierarchy of  $\text{MSO}(\vec{\text{ML}}_g)$  is *infinite*. But since the alternation levels of that logic are not the same as those of  $\text{MSOL}$ , it does not seem obvious how strictness could be inferred.

The present chapter provides an alternative way of transferring the results of Matz, Schweikardt and Thomas to the **modal** setting. In particular, our method allows to show directly that the **set quantifier** alternation hierarchies of  $\text{MSO}(\vec{\text{ML}})$  and  $\text{MSO}(\vec{\text{ML}}_g)$  are *strict* over (**pointed**) **digraphs**. At first sight, this seems to expand the existing body of knowledge, especially since the strictness question for  $\text{MSO}(\vec{\text{ML}})$  has been mentioned as an open problem in [Kuu08] and [Kuu13b]. However, it turns out that in both cases, strictness is actually a consequence of infiniteness [A. Kuusisto, personal communication, 3 March 2016]. Although this observation has so far not been formally published, it appears to be folklore in the model-theory community. Hence, this chapter contributes new proofs to essentially known results. Just as Kuusisto has done in [Kuu08, Kuu15], we use as a starting point the strictness result of [MST02] for  $\text{MSOL}$  on **grids**. But from there on, the two proof methods diverge considerably.

*To avoid the backward modalities of  $\text{MSO}(\vec{\text{ML}}_g)$ , we work instead with  $\text{MSO}(\vec{\text{ML}}_g)$ , which is called SOPMLE in [Kuu08, Kuu15]. By duality, separating one alternation hierarchy also separates the other.*

The original approach of Kuusisto is mainly based on the fact that one can simulate **first-order quantifiers** by means of **set quantifiers**, combined with a **formula** stating that a set is a singleton. As already mentioned, this can be used to show that  $\text{MSO}(\vec{\text{ML}}_g)$  is **equivalent** to  $\text{MSOL}$ . The spirit of the proof in [Kuu08, Kuu15] is essentially the same for  $\text{MSO}(\vec{\text{ML}})$ , although the details are much more technical, since this logic is less expressive than  $\text{MSOL}$  on arbitrary **pointed structures**. It is precisely the use of additional **second-order quantifiers** that leads to the temporary loss of the specific separation results provided by [MST02].

In contrast, one simple insight will allow us to directly transfer those results: When restricted to the class of **grids**,  $\text{MSO}(\vec{\text{ML}}_g)$  and  $\text{MSOL}$  are more than just **equivalent** – they are *levelwise equivalent*, and consequently all the separation results shown for  $\text{MSOL}$  also hold for  $\text{MSO}(\vec{\text{ML}}_g)$  on **grids**. This approach is based on the observation that the existential fragment of  $\text{MSO}(\vec{\text{ML}}_g)$  can simulate another model, called **tiling systems**, which has been shown to be **equivalent** to the existential fragment of  $\text{MSOL}$  in [GRST96]. On the basis of this new finding, we can then transfer the given separation results from  $\text{MSO}(\vec{\text{ML}}_g)$  on **grids** to other classes of **digraphs** and other extensions of **modal logic**, such as  $\text{MSO}(\vec{\text{ML}})$ . While this works along the same general principle as the *strong first-order reductions* used in [MST02], the additional limitations imposed by **modal logic** force us to introduce custom encoding techniques that cope with the lack of expressive power.

The remainder of this chapter is organized in a top-down manner. After introducing the necessary notation in Section 6.1, we present the main results in Section 6.2, and almost immediately get to the central proof in Section 6.3. The latter relies on several other propositions, but since those are treated as “black boxes”, the main line of reasoning might be comprehensible without reading any further. We then provide all the missing details in the last two sections, which are independent of each

other. Section 6.4 establishes the levelwise **equivalence** of three different alternation hierarchies on **grids**, and may thus be interesting on its own. On the other hand, Section 6.5 is dedicated to encoding functions, which constitute the more technical part of our demonstration.

## 6.1 Preliminaries

Assume we are given some set of **formulas**  $\Phi$ , referred to as **kernel**, which is free of **set quantifiers** and closed under negation (e.g.,  $\vec{M}\vec{L}_g$ ). Then, for  $\ell \geq 0$ , the class  $\Sigma_\ell^{\text{MSO}}(\Phi)$  consists of those **formulas** that one can construct by taking a member of  $\Phi$  and prepending to it at most  $\ell$  consecutive blocks of **set quantifiers**, alternating between existential and universal blocks, such that the first block is existential. Reformulating this solely in terms of existential **quantifiers** and negations, we get

$$\begin{aligned}\Sigma_0^{\text{MSO}}(\Phi) &:= \Phi \quad \text{and} \\ \Sigma_{\ell+1}^{\text{MSO}}(\Phi) &:= \{\exists x \mid x \in \mathbb{S}_1\}^* \cdot \{\neg\varphi \mid \varphi \in \Sigma_\ell^{\text{MSO}}(\Phi)\},\end{aligned}$$

where the second line uses set concatenation and the Kleene star. We define  $\Pi_\ell^{\text{MSO}}(\Phi)$  as the corresponding dual class, i.e., the set of all negations of **formulas** in  $\Sigma_\ell^{\text{MSO}}(\Phi)$ . Generalizing this to arbitrary Boolean combinations, let  $\text{BC } \Sigma_\ell^{\text{MSO}}(\Phi)$  denote the smallest superclass of  $\Sigma_\ell^{\text{MSO}}(\Phi)$  that is closed under negation and disjunction.

The **formulas** in  $\Sigma_\ell^{\text{MSO}}(\Phi)$  and  $\Pi_\ell^{\text{MSO}}(\Phi)$  are said to be in **prenex normal form** with respect to the **kernel**  $\Phi$ . It is well known that every **MSOL-formula** can be transformed into **prenex normal form** with **kernel class** **FOL**. This is based on the observation that **first-order quantifiers** can be replaced by **second-order ones**. Using the construction of Example 2.2 in Section 2.6, it is not difficult to see that the analogue holds for  $\text{MSO}(\vec{M}\vec{L})$ ,  $\text{MSO}(\vec{M}\vec{L})$ ,  $\text{MSO}(\vec{M}\vec{L}_g)$  and  $\text{MSO}(\vec{M}\vec{L}_g)$  with respect to their corresponding **kernel classes**. A more elaborate explanation can be found in [Cat06, Prp. 3].

For the sake of clarity, we break with the tradition of implicit quantification that is customary in modal logic. Instead of evaluating  $\text{MSO}(\vec{M}\vec{L})$ -**formulas** on **non-pointed structures** by means of “hidden” universal quantification, we shall explicitly put a **global box** in front of our **formulas**. This leads to the class

$$\boxed{\Sigma}_\ell^{\text{MSO}}(\vec{M}\vec{L}) := \{\boxed{\cdot}\} \cdot \Sigma_\ell^{\text{MSO}}(\vec{M}\vec{L}).$$

Analogously, we also define  $\boxed{\Pi}_\ell^{\text{MSO}}(\vec{M}\vec{L})$ .

All of our results will be stated in terms of the semantic classes that one obtains by evaluating the preceding **formula classes** on some set of **structures**  $\mathcal{C}$ . On the semantic side, we will additionally consider the class

$$\llbracket \Delta_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} := \llbracket \Sigma_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \cap \llbracket \Pi_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}.$$

Since it is not based on any syntactic counterpart, there is no meaning attributed to the notation  $\Delta_\ell^{\text{MSO}}(\Phi)$  by itself (without the brackets).

## 6.2 Separation results

With the notation in place, we are ready to formally enunciate the main theorem, whose complete proof will be the subject of the remainder of this chapter. It is an

Separation result	Kernel Class $\Phi$	Structures Class $\mathcal{C}$	Levels $\ell \geq \cdot$	Theorem
$\llbracket \Delta_{\ell+1}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\subseteq \llbracket \text{BC } \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$	FOL	GRID, DG, GRAPH	1	6.1 (a) *
	$\vec{\text{ML}}_{\text{g}}, \vec{\text{ML}}_{\text{g}}$	GRID, DG, GRAPH <sup>1</sup>	1	6.2 (a)
$\llbracket \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\subseteq \llbracket \Pi_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$	FOL	GRID, DG, GRAPH	1	6.1 (b) *
	$\vec{\text{ML}}_{\text{g}}, \vec{\text{ML}}_{\text{g}}$	GRID, DG, GRAPH <sup>1</sup>	1	6.2 (b)
	$\vec{\text{ML}}$	@DG	1	6.2 (c)
$\llbracket \Box \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\subseteq \llbracket \Box \Pi_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$	$\vec{\text{ML}}$	DG	2	6.2 (d)

**Table 6.1.** The specific separation results of Theorems 6.1 and 6.2. Theorem 6.1 (marked by asterisks) is due to Matz, Schweikardt and Thomas.

extension to **modal kernel formulas** of the following result of Matz, Schweikardt and Thomas, obtained by combining [MST02, Thm. 1] and [Mat02, Thm. 2.26]<sup>1</sup>:

**Theorem 6.1** (Matz, Schweikardt, Thomas).

► The **set quantifier** alternation hierarchy of **MSOL** is strict over the classes of **grids**, **digraphs** and **undirected graphs**.

A more precise statement of this theorem, referred to as *Theorem 6.1 (a) and (b)*, is given in Table 6.1. ◀

Roughly speaking, the extension provided in the present chapter tells us that the preceding separations are largely maintained if we replace the **first-order kernel** by certain classes of **modal formulas**. To facilitate comparisons, the formal statements of both theorems are presented together in the same table.

**Theorem 6.2** (Main Results).

► The **set quantifier** alternation hierarchies of  $\text{MSO}(\vec{\text{ML}}_{\text{g}})$  and  $\text{MSO}(\vec{\text{ML}}_{\text{g}})$  are strict over the classes of **grids**, **digraphs** and **1-bit labeled undirected graphs**.

Furthermore, the corresponding hierarchies of  $\text{MSO}(\vec{\text{ML}})$  and  $\Box \text{MSO}(\vec{\text{ML}})$  are (mostly) strict over the classes of **pointed digraphs** and **digraphs**, respectively.

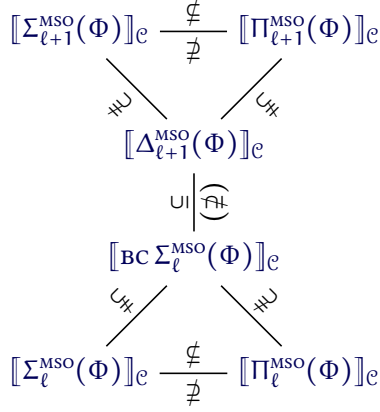
A more precise statement of this theorem, referred to as *Theorem 6.2 (a), (b), (c) and (d)*, is given in Table 6.1. ◀

In particular, the inclusion of  $\llbracket \text{BC } \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$  in  $\llbracket \Delta_{\ell+1}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$  follows from the fact that, when transforming a Boolean combination of  $\Sigma_{\ell}^{\text{MSO}}(\Phi)$ -formulas into *prenex normal form*, one is free to choose whether the resulting formula (with up to  $\ell + 1$  quantifier alternations) should start with an existential or a universal quantifier.

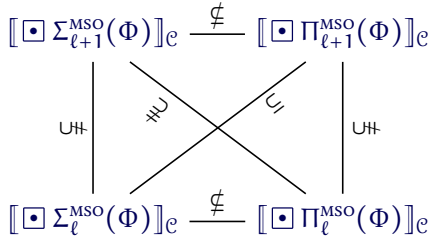
By basic properties of predicate logic and the transitivity of set inclusion, it is easy to infer from Theorem 6.2 the hierarchy diagrams represented in Figures 6.1 and 6.2.

If we take into account all the depicted relations, the diagram in Figure 6.1 is the same as in [MST02] and [Mat02]. Hence, when switching to one of the **modal kernels** that include **global modalities**, i.e.,  $\vec{\text{ML}}_{\text{g}}$  or  $\vec{\text{ML}}_{\text{g}}$ , the separations of Theorem 6.1 are completely preserved on **grids** and **digraphs**. Our proof method also allows us to easily transfer this result to **undirected graphs**, as long as we admit that the vertices may be **labeled** with at least one bit. Additional work would be required to eliminate this condition.

<sup>1</sup> [Mat02, Thm. 2.26] states that  $\llbracket \Sigma_{\ell}^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}} \not\subseteq \llbracket \Pi_{\ell}^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}}$ , which, by duality, also implies  $\llbracket \Sigma_{\ell}^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}} \not\subseteq \llbracket \Pi_{\ell}^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}}$ .



**Figure 6.1.** The set quantifier alternation hierarchies established by Theorem 6.2 (a), (b) and (c). If we include the noninclusion in parentheses, this diagram holds for  $\Phi \in \{\vec{\text{ML}}_g, \vec{\text{ML}}_g\}$  and  $\mathcal{C} \in \{\text{GRID}, \text{DG}, \text{GRAPH}_1^1\}$ . If we ignore that noninclusion, it is also verified for  $\Phi = \vec{\text{ML}}$  and  $\mathcal{C} = @ \text{DG}$ . In both cases, we assume  $\ell \geq 1$ .



**Figure 6.2.** The set quantifier alternation hierarchy implied by Theorem 6.2 (d) for  $\Phi = \vec{\text{ML}}$ ,  $\mathcal{C} = \text{DG}$ , and  $\ell \geq 2$ .

As a spin-off, Theorem 6.2 also provides an extension of some of these separations to  $\vec{\text{ML}}$ , a kernel class without global modalities. Following [Kuu08, Kuu15], we consider the alternation hierarchies of both  $\text{MSO}(\vec{\text{ML}})$  and  $\Box \text{MSO}(\vec{\text{ML}})$ . For the former, which is evaluated on pointed digraphs, Figure 6.1 gives a detailed picture, leaving open only whether the inclusion  $\llbracket \text{BC } \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \subseteq \llbracket \Delta_{\ell+1}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$  is proper. Inferring the strictness of this inclusion from the preceding results does not seem very difficult, but would call for a generalization of our framework. In contrast, the second hierarchy based on  $\vec{\text{ML}}$  is arguably less natural, since every  $\Box \text{MSO}(\vec{\text{ML}})$ -formula is prefixed by a global box, regardless of the occurring set quantifiers. This creates a certain asymmetry between the  $\Sigma_{\ell}^{\text{MSO}}$ - and  $\Pi_{\ell}^{\text{MSO}}$ -levels, which becomes apparent when considering the missing relations in Figure 6.2. Unlike for the other hierarchies, one cannot simply argue by duality to deduce from  $\llbracket \Box \Sigma_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\subseteq \llbracket \Box \Pi_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$  that the converse noninclusion also holds. Nevertheless, the presented result is strong enough to answer the specific strictness question mentioned in [Kuu08]: For arbitrarily high  $\ell$ , we have

$$\llbracket \Box \Sigma_{\ell}^{\text{MSO}}(\vec{\text{ML}}) \rrbracket_{\text{DG}} \not\subseteq \llbracket \Box \Sigma_{\ell+1}^{\text{MSO}}(\vec{\text{ML}}) \rrbracket_{\text{DG}}.$$

## 6.3 Top-level proofs

In accordance with our top-down approach, the present section already provides the proof of our main theorem, where everything comes together. It therefore acts as a gateway to the sections with the technical parts, especially Section 6.5.

### 6.3.1 Figurative inclusions

First of all, we need to introduce the primary tool with which we will transfer separation results from one setting to another. It can be seen as an abstraction of



the *strong first-order reductions* used in [MST02]. Unlike the latter, it is formulated independently of any logical language, which allows us to postpone the technical details to the end of the chapter.

**Definition 6.3** (Figurative Inclusion).

► Consider two sets  $\mathcal{C}$  and  $\mathcal{D}$  and a partial *injective* function  $\mu: \mathcal{C} \rightarrow \mathcal{D}$ . For any two families of subsets  $\mathcal{L} \subseteq 2^{\mathcal{C}}$  and  $\mathcal{M} \subseteq 2^{\mathcal{D}}$ , we say that  $\mathcal{L}$  is *forward included* in  $\mathcal{M}$  *figuro*  $\mu$ , and write  $\mathcal{L} \subseteq_{\mu} \mathcal{M}$ , if for every set  $L \in \mathcal{L}$ , there is a set  $M \in \mathcal{M}$  such that  $\mu(L) = M \cap \mu(\mathcal{C})$ . ◀

Figuratively speaking, the partial bijection  $\mu$  creates a tunnel between  $\mathcal{C}$  and  $\mathcal{D}$ , and all the sets in  $\mathcal{L}$  and  $\mathcal{M}$  are cropped to fit through that tunnel. Two original sets are considered to be equal if their cropped versions are mapped onto each other by  $\mu$ .

We also define the shorthands  $\supseteq_{\mu}$  and  $\equiv_{\mu}$  as natural extensions of the previous notation:  $\mathcal{L} \supseteq_{\mu} \mathcal{M}$ , which is defined as  $\mathcal{M} \subseteq_{\mu^{-1}} \mathcal{L}$ , means that  $\mathcal{M}$  is *backward included* in  $\mathcal{L}$  *figuro*  $\mu$ , and  $\mathcal{L} \equiv_{\mu} \mathcal{M}$ , an abbreviation for the conjunction of  $\mathcal{L} \subseteq_{\mu} \mathcal{M}$  and  $\mathcal{L} \supseteq_{\mu} \mathcal{M}$ , states that  $\mathcal{L}$  is *forward equal* to  $\mathcal{M}$  *figuro*  $\mu$ . All of these relations are referred to as *figurative inclusions*.

Note that ordinary inclusion is a special case of *figurative inclusion*, i.e., for  $\mathcal{C} = \mathcal{D}$ ,

$$\mathcal{L} \subseteq \mathcal{M} \quad \text{if and only if} \quad \mathcal{L} \subseteq_{\text{id}_{\mathcal{C}}} \mathcal{M}.$$

Furthermore, *figurative inclusion* is transitive in the sense that

$$\mathcal{L} \subseteq_{\mu} \mathcal{M} \subseteq_{\nu} \mathcal{N} \quad \text{implies} \quad \mathcal{L} \subseteq_{\nu \circ \mu} \mathcal{N}.$$

(This depends crucially on the fact that  $\nu$  is injective.)

*Proof.* Consider three sets  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$ , two partial injective functions  $\mu: \mathcal{C} \rightarrow \mathcal{D}$  and  $\nu: \mathcal{D} \rightarrow \mathcal{E}$ , and three families of subsets  $\mathcal{L} \subseteq 2^{\mathcal{C}}$ ,  $\mathcal{M} \subseteq 2^{\mathcal{D}}$  and  $\mathcal{N} \subseteq 2^{\mathcal{E}}$ . Assume that we have  $\mathcal{L} \subseteq_{\mu} \mathcal{M} \subseteq_{\nu} \mathcal{N}$ . Choose an arbitrary set  $L \in \mathcal{L}$ . Since  $\mathcal{L} \subseteq_{\mu} \mathcal{M}$ , there must be a set  $M \in \mathcal{M}$  such that  $\mu(L) = M \cap \mu(\mathcal{C})$ . Furthermore, as  $\mathcal{M} \subseteq_{\nu} \mathcal{N}$ , there is also a set  $N \in \mathcal{N}$  such that  $\nu(M) = N \cap \nu(\mathcal{D})$ . Hence,

$$\begin{aligned} (\nu \circ \mu)(L) &= \nu(M \cap \mu(\mathcal{C})) \\ &= \nu(M) \cap (\nu \circ \mu)(\mathcal{C}) \\ &= N \cap \nu(\mathcal{D}) \cap (\nu \circ \mu)(\mathcal{C}) \\ &= N \cap (\nu \circ \mu)(\mathcal{C}). \end{aligned} \tag{*}$$

Equality (\*) holds because  $\nu$  is injective. Since the choice of  $L$  was arbitrary, there is such an  $N \in \mathcal{N}$  for every  $L \in \mathcal{L}$ , and thus  $\mathcal{L} \subseteq_{\nu \circ \mu} \mathcal{N}$ . ■

In our specific context, given a noninclusion  $[\Phi_2]_{\mathcal{C}} \not\subseteq [\Phi_1]_{\mathcal{C}}$ , we shall use the concept of *figurative inclusion* to infer from it another noninclusion  $[\Psi_2]_{\mathcal{D}} \not\subseteq [\Psi_1]_{\mathcal{D}}$ . Here,  $\Phi_1, \Phi_2, \Psi_1, \Psi_2$  and  $\mathcal{C}, \mathcal{D}$  refer to some classes of *formulas* and *structures*, respectively. The key part of the argument will be to construct an appropriate encoding function  $\mu: \mathcal{C} \rightarrow \mathcal{D}$ , in order to apply the following lemma.

**Lemma 6.4.**

► Let  $\mathcal{L}_1, \mathcal{L}_2 \subseteq 2^{\mathcal{C}}$  and  $\mathcal{M}_1, \mathcal{M}_2 \subseteq 2^{\mathcal{D}}$  be families of subsets of some sets  $\mathcal{C}$  and  $\mathcal{D}$ . If there is a *total injective* function  $\mu: \mathcal{C} \rightarrow \mathcal{D}$  such that  $\mathcal{L}_2 \subseteq_{\mu} \mathcal{M}_2$  and  $\mathcal{L}_1 \supseteq_{\mu} \mathcal{M}_1$ , then

$$\mathcal{L}_2 \not\subseteq \mathcal{L}_1 \quad \text{implies} \quad \mathcal{M}_2 \not\subseteq \mathcal{M}_1. \quad \blacktriangleleft$$

We denote the inverse function of  $\mu$  by  $\mu^{-1}$  and the identity function on  $\mathcal{C}$  by  $\text{id}_{\mathcal{C}}$ .

*Proof.* To show the contrapositive, let us suppose that  $\mathcal{M}_2 \subseteq \mathcal{M}_1$ , or, equivalently,  $\mathcal{M}_2 \subseteq_{\text{id}_{\mathcal{D}}} \mathcal{M}_1$ . Then the chain of figurative inclusions

$$\mathcal{L}_2 \subseteq_{\mu} \mathcal{M}_2 \subseteq_{\text{id}_{\mathcal{D}}} \mathcal{M}_1 \subseteq_{\mu^{-1}} \mathcal{L}_1$$

yields  $\mathcal{L}_2 \subseteq_{\text{id}_{\mathcal{C}}} \mathcal{L}_1$ , since  $(\mu^{-1} \circ \text{id}_{\mathcal{D}} \circ \mu) = \text{id}_{\mathcal{C}}$ . (This depends on  $\mu$  being total and injective.) Consequently, we have  $\mathcal{L}_2 \subseteq \mathcal{L}_1$ . ■

In some cases, we can combine two given figurative inclusions in order to obtain a new one that relates the corresponding intersection classes. This property will be very useful for establishing figurative inclusions between classes of the form  $\llbracket \Delta_{\ell}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}}$ .

**Lemma 6.5.**

► Consider two sets  $\mathcal{C}$  and  $\mathcal{D}$ , a partial injective function  $\mu: \mathcal{C} \rightarrow \mathcal{D}$ , and four families of subsets  $\mathcal{L}_1, \mathcal{L}_2 \subseteq 2^{\mathcal{C}}$  and  $\mathcal{M}_1, \mathcal{M}_2 \subseteq 2^{\mathcal{D}}$ . If  $\mu(\mathcal{C})$  is a member of  $\mathcal{M}_1 \cap \mathcal{M}_2$ , and  $\mathcal{M}_1, \mathcal{M}_2$  are both closed under intersection, then

$$\mathcal{L}_1 \subseteq_{\mu} \mathcal{M}_1 \quad \text{and} \quad \mathcal{L}_2 \subseteq_{\mu} \mathcal{M}_2 \quad \text{imply} \quad \mathcal{L}_1 \cap \mathcal{L}_2 \subseteq_{\mu} \mathcal{M}_1 \cap \mathcal{M}_2. \quad \blacktriangleleft$$

*Proof.* Let  $L$  be any set in  $\mathcal{L}_1 \cap \mathcal{L}_2$ . Since  $\mathcal{L}_1 \subseteq_{\mu} \mathcal{M}_1$ , there is, by definition, a set  $M$  in  $\mathcal{M}_1$  such that  $\mu(L) = M \cap \mu(\mathcal{C})$ . Furthermore, we also know that  $\mu(\mathcal{C})$  lies in  $\mathcal{M}_1$ , and that the latter is closed under intersection. Hence,  $\mu(L) \in \mathcal{M}_1$ . Analogously, we also get that  $\mu(L) \in \mathcal{M}_2$ . Finally, knowing that for all  $L$  in  $\mathcal{L}_1 \cap \mathcal{L}_2$ ,  $\mu(L)$  lies in  $\mathcal{M}_1 \cap \mathcal{M}_2$ , we obviously have a sufficient condition for  $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq_{\mu} \mathcal{M}_1 \cap \mathcal{M}_2$ . ■

### 6.3.2 Proving the main theorem

We are now ready to give the central proof of this chapter. Although it makes references to many statements of Sections 6.4 and 6.5, it is formulated in a way that can be understood without having read anything beyond this point.

*Proof of Theorem 6.2.* The basis of our proof shall be laid in Section 6.4, where the case  $s = 0$  of Theorem 6.9 will state the following: When restricted to the class of grids, the set quantifier alternation hierarchies of  $\text{MSOL}$ ,  $\text{MSO}(\vec{\text{ML}}_g)$  and  $\text{MSO}(\vec{\text{ML}}_g)$  are equivalent. More precisely, for every  $\ell \geq 1$  and  $\Xi \in \{\Sigma_{\ell}^{\text{MSO}}, \Pi_{\ell}^{\text{MSO}}, \text{BC} \Sigma_{\ell}^{\text{MSO}}, \Delta_{\ell}^{\text{MSO}}\}$ , it holds that

$$\llbracket \Xi(\text{FOL}) \rrbracket_{\text{GRID}} = \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}} = \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}}.$$

Hence, if we consider only the case  $\mathcal{C} = \text{GRID}$ , the separation results for the kernel class  $\text{FOL}$  stated in Theorem 6.1 (a) and (b) immediately imply those for  $\vec{\text{ML}}_g$  and  $\vec{\text{ML}}_g$  in Theorem 6.2 (a) and (b).

The remainder of the proof now consists of establishing suitable figurative inclusions, in order to transfer these results to other classes of structures and, to some extent, to weaker classes of kernel formulas. For this purpose, we shall introduce in Section 6.5 a notion of translatability between two classes of kernel formulas  $\Phi$  and  $\Psi$ , with respect to a given total injective function  $\mu$  that encodes structures from a class  $\mathcal{C}$  into structures of some class  $\mathcal{D}$ . As will be shown in Lemma 6.14, bidirectional translatability implies

$$\llbracket \Xi(\Phi) \rrbracket_{\mathcal{C}} \equiv_{\mu} \llbracket \Xi(\Psi) \rrbracket_{\mathcal{D}} \quad (*)$$

for all  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}\}$  with  $\ell \geq 0$ . If we can additionally show that  $\mu(\mathcal{C})$  is (at most)  $\Delta_2^{\text{MSO}}(\Psi)$ -definable over  $\mathcal{D}$ , then, by Lemma 6.5, the figurative equality  $(*)$  also holds for  $\Xi = \Delta_{\ell+1}^{\text{MSO}}$  with  $\ell \geq 1$ . Note that the backward part “ $\Xi_\mu$ ” is always true, since  $\mu^{-1}(\mathcal{D})$  is trivially  $\Delta_2^{\text{MSO}}(\Phi)$ -definable over  $\mathcal{C}$ .

The groundwork being in place, we proceed by applying Lemma 6.4 as follows:

- If we have established  $(*)$  for  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}\}$ , then we can transfer the separation

$$\llbracket \Sigma_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\equiv \llbracket \Pi_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \quad (1)$$

to the kernel class  $\Psi$  evaluated on the class of structures  $\mathcal{D}$ .

- Similarly, if  $(*)$  holds for  $\Xi \in \{\text{BC } \Sigma_\ell^{\text{MSO}}, \Delta_{\ell+1}^{\text{MSO}}\}$ , then

$$\llbracket \Delta_{\ell+1}^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \not\equiv \llbracket \text{BC } \Sigma_\ell^{\text{MSO}}(\Phi) \rrbracket_{\mathcal{C}} \quad (2)$$

can also be transferred to  $\Psi$  on  $\mathcal{D}$ .

It remains to provide concrete figurative inclusions to prove the different parts of Theorem 6.2.

(a), (b) The first two parts are treated in parallel. We start by transferring (1) and (2) from grids to digraphs, for the kernel class  $\vec{\text{ML}}_g$ , taking a detour via 2-relational digraphs, and then via 2-bit labeled ones. For all  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}, \Delta_{\ell+1}^{\text{MSO}}\}$  with  $\ell \geq 1$ , we get

$$\begin{aligned} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}} &\equiv_{\text{id}_{\text{GRID}}} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}_0^2} \\ &\equiv_{\mu_1} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}_1^1} \\ &\equiv_{\mu_2} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}}. \end{aligned}$$

The first line is trivial for  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}\}$ , since  $\text{GRID} \subseteq \text{DG}_0^2$ . It also holds for  $\Xi = \Delta_{\ell+1}^{\text{MSO}}$  because  $\text{GRID}$  is  $\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -definable over  $\text{DG}_0^2$ , as shall be demonstrated in Proposition 6.10. The other two lines rely on the existence of adequate injective functions  $\mu_1$  and  $\mu_2$  that allow us to apply Lemmas 6.14 and 6.5 in the way explained above. They will be provided by Propositions 6.15 and 6.16, respectively.

We proceed in a similar way to transfer (1) and (2) from  $\vec{\text{ML}}_g$  to  $\vec{\text{ML}}_g$  on digraphs:

$$\begin{aligned} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}} &\equiv_{\mu_3} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}_2^2} \\ &\equiv_{\mu_1} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}_1^1} \\ &\equiv_{\mu_2} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}}, \end{aligned}$$

for  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}, \Delta_{\ell+1}^{\text{MSO}}\}$  with  $\ell \geq 1$ . The very simple encoding function  $\mu_3$ , which lets us eliminate backward modalities and again use Lemmas 6.14 and 6.5, will be supplied by Proposition 6.17. The encodings  $\mu_1$  and  $\mu_2$  are the same as before, because the properties asserted by Propositions 6.15 and 6.16 hold for both  $\vec{\text{ML}}_g$  and  $\vec{\text{ML}}_g$  as kernel classes. Incidentally, this means we could transfer (1) directly from  $\vec{\text{ML}}_g$  on grids to  $\vec{\text{ML}}_g$  on digraphs, without even mentioning  $\vec{\text{ML}}_g$ .

To show that (1) and (2) are also valid for  $\vec{\text{ML}}_g$  on 1-bit labeled undirected graphs, we establish

$$\llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{DG}} \equiv_{\mu_4} \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRAPH}_1^1},$$

again for all  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}, \Delta_{\ell+1}^{\text{MSO}}\}$  with  $\ell \geq 1$ . The appropriate encoding  $\mu_4$  shall be constructed in Proposition 6.18. Since backward modalities do not offer any additional expressive power on undirected graphs, the separations we obtain also hold for the kernel  $\vec{\text{ML}}_g$ .

(c) Next, to transfer (1) from  $\vec{M}_{L_g}$  on **digraphs** to  $\vec{M}_L$  on **pointed digraphs**, we show that, for  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}\}$  with  $\ell \geq 1$ , we have

$$\llbracket \Xi(\vec{M}_{L_g}) \rrbracket_{\text{DG}} \equiv_{\mu_5} \llbracket \Xi(\vec{M}_L) \rrbracket_{@ \text{DG}}.$$

The injective function  $\mu_5$ , which satisfies the translatability property required to obtain this **figurative equality** via Lemma 6.14, will be provided by Proposition 6.19. Its image  $\mu_5(\text{DG})$  is not  $\text{MSO}(\vec{M}_L)$ -definable, for the simple reason that an  $\text{MSO}(\vec{M}_L)$ -formula is unable to distinguish between two **structures** that are isomorphic when restricted to the **connected** component containing the position marker  $@$ . Hence, we cannot merely apply Lemma 6.5 to show (2). Our approach would have to be refined to take into account equivalence classes of **structures**, which we shall not do in this thesis.

(d) Finally, Proposition 6.19 will also state that  $\mu_5$  can be converted into an encoding  $\mu'_5$ , from  $\text{DG}$  back into  $\text{DG}$ , that satisfies the following **figurative inclusions** for all  $\ell \geq 2$ :

$$\begin{aligned} \llbracket \Sigma_\ell^{\text{MSO}}(\vec{M}_{L_g}) \rrbracket_{\text{DG}} &\subseteq_{\mu'_5} \llbracket \Box \Sigma_\ell^{\text{MSO}}(\vec{M}_L) \rrbracket_{\text{DG}}, \\ \llbracket \Pi_\ell^{\text{MSO}}(\vec{M}_{L_g}) \rrbracket_{\text{DG}} &\subseteq_{\mu'_5} \llbracket \Box \Pi_\ell^{\text{MSO}}(\vec{M}_L) \rrbracket_{\text{DG}}. \end{aligned}$$

Using (1) for  $\vec{M}_{L_g}$  on **digraphs**, and applying Lemma 6.4, we can infer from this that

$$\llbracket \Box \Sigma_\ell^{\text{MSO}}(\vec{M}_L) \rrbracket_{\text{DG}} \not\subseteq \llbracket \Box \Pi_\ell^{\text{MSO}}(\vec{M}_L) \rrbracket_{\text{DG}}. \quad \blacksquare$$

## 6.4 Grids as a starting point

In this section, we establish that the **set quantifier** alternation hierarchies of  $\text{MSOL}$ ,  $\text{MSO}(\vec{M}_{L_g})$  and  $\text{MSO}(\vec{M}_L)$  are **equivalent** on **labeled grids**. In addition, we give a  $[\Pi_1^{\text{MSO}}(\vec{M}_{L_g})]$ -formula that characterizes the class of **grids**.

### 6.4.1 The standard translation

Our first building block is a well-known property of modal logic, which holds even if we do not confine ourselves to the setting of **grids**.

#### Proposition 6.6.

► For every  $\vec{M}_{L_g}$ -formula, there is an **equivalent FOL-formula**, i.e.,

$$\llbracket \vec{M}_{L_g} \rrbracket \subseteq \llbracket \text{FOL} \rrbracket. \quad \blacktriangleleft$$

*Proof.* Given an  $\vec{M}_{L_g}$ -formula  $\varphi$ , we have to construct an **FOL-formula**  $\psi_\varphi$  such that  $G \models \varphi$  if and only if  $G \models \psi_\varphi$ , for every **structure**  $G$ . This is simply a matter of transcribing the semantics of  $\vec{M}_{L_g}$  given in Table 2.1 to the language of **first-order logic**, a method known as the *standard translation* in modal logic (see, e.g., [BRV02, Def. 2.45]). The following table gives a recursive specification of this translation.

$\varphi \in \vec{\text{ML}}_g$	Equivalent formula $\psi_\varphi \in \text{FOL}$
$x$	$@ \doteq x$
$X$	$X(@)$
$\neg\varphi_1$	$\neg\psi_{\varphi_1}$
$\varphi_1 \vee \varphi_2$	$\psi_{\varphi_1} \vee \psi_{\varphi_2}$
$\Diamond(\varphi_1, \dots, \varphi_k)$	$\exists_{x_1, \dots, x_k} (R(@, x_1, \dots, x_k) \wedge \bigwedge_{1 \leq i \leq k} \psi_{\varphi_i}[@ \mapsto x_i])$
$\bar{\Diamond}(\varphi_1, \dots, \varphi_k)$	as above, except $R(x_k, \dots, x_1, @)$
$\Diamond\varphi_1$	$\exists_{@} \psi_{\varphi_1}$

Here,  $x \in \mathbb{S}_0$ ,  $X \in \mathbb{S}_1$ ,  $R \in \mathbb{S}_{k+1}$ ,  $\varphi_1, \dots, \varphi_k \in \vec{\text{ML}}_g$ , for  $k \geq 1$ , and  $x_1, \dots, x_k$  are **node symbols**, chosen such that  $x_i \notin \text{free}(\psi_{\varphi_i})$ . The notation  $\psi_{\varphi_i}[@ \mapsto x_i]$  designates the formula obtained by substituting each **free** occurrence of  $@$  in  $\psi_{\varphi_i}$  by  $x_i$ . ■

### 6.4.2 A detour through tiling systems

By restricting our focus to the class of **labeled grids**, we can take advantage of a well-studied automaton model introduced by Giammarresi and Restivo in [GR92], which is closely related to **MSOL**. A “machine” in this model, called a **tiling system**, is defined as a tuple  $T = (\Sigma, Q, \Theta)$ , where

- $\Sigma = \mathbb{2}^s$  is seen as an alphabet, with  $s \geq 0$ ,
- $Q$  is a finite set of states, and
- $\Theta \subseteq ((\Sigma \times Q) \cup \{\#\})^4$  is a set of  $2 \times 2$ -tiles that may use a fresh letter  $\#$  not contained in  $(\Sigma \times Q)$ .

For a fixed number of bits  $s$ , we denote by  $\text{TS}_s$  the set of all **tiling systems** with alphabet  $\Sigma = \mathbb{2}^s$ .

Given a  $s$ -bit **labeled grid**  $G$ , a **tiling system**  $T \in \text{TS}_s$  operates similarly to a non-deterministic finite automaton generalized to two dimensions. A run of  $T$  on  $G$  is an extended **labeled grid**  $G^\#$ , obtained by nondeterministically labeling each cell of  $G$  with some state  $q \in Q$  and surrounding the entire **grid** with a border consisting of new  $\#$ -**labeled** cells. We consider  $G^\#$  to be a valid run if each of its  $2 \times 2$ -**subgrids** can be identified with some tile in  $\Theta$ . The set recognized by  $T$  consists precisely of those **labeled grids** for which such a run exists. By analogy with our existing notation, we write  $\llbracket \text{TS}_s \rrbracket_{\text{GRID}_s}$  for the class formed by the sets of  $s$ -bit **labeled grids** that are recognized by some **tiling system** in  $\text{TS}_s$ .

Exploiting a locality property of **first-order logic**, Giammarresi, Restivo, Seibert and Thomas have shown in [GRST96] that **tiling systems** capture precisely the existential fragment of **MSOL** on **labeled grids**:

**Theorem 6.7** (Giammarresi, Restivo, Seibert, Thomas).

- For arbitrary  $s \geq 0$ , a set of  $s$ -bit **labeled grids** is **TS**-recognizable if and only if it is  $\Sigma_1^{\text{MSO}}(\text{FOL})$ -definable over  $\text{GRID}_s$ , i.e.,

$$\llbracket \text{TS}_s \rrbracket_{\text{GRID}_s} = \llbracket \Sigma_1^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}_s}.$$

◀

The preceding result is extremely useful for our purposes, because, from the perspective of **modal logic**, it provides a much easier access to **MSOL**. This brings us to the following proposition.

**Proposition 6.8.**

► For arbitrary  $s \geq 0$ , if a set of  $s$ -bit **labeled grids** is **TS**-recognizable, then it is also  $\Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -definable over  $\text{GRID}_s$ , i.e.,

$$\llbracket \text{TS}_s \rrbracket_{\text{GRID}_s} \subseteq \llbracket \Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s}.$$

*Proof.* Let  $T = (\Sigma, Q, \Theta)$  be a **tiling system** with alphabet  $\Sigma = \mathbb{2}^s$ . We have to construct a  $\Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -sentence  $\varphi_T$  over the **signature**  $\{P_1, \dots, P_s, R_1, R_2\}$ , such that each **labeled grid**  $G \in \text{GRID}_s$  satisfies  $\varphi_T$  if and only if it is accepted by  $T$ .

The idea is standard: We represent the states of  $T$  by additional **set symbols**  $(X_q)_{q \in Q}$ , and our **formula** asserts that there exists a corresponding partition of  $V^G$  into  $|Q|$  subsets that represent a run  $G^\#$  of  $T$  on  $G$ . To verify that it is indeed a valid run, we have to check that each  $2 \times 2$ -**subgrid** of  $G^\#$  corresponds to some tile

$$\theta = \begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{bmatrix}$$

in  $\Theta$ . If the entry  $\theta_1$  is different from  $\#$ , we can easily write down an  $\vec{\text{ML}}$ -**formula**  $\varphi_\theta$  that checks at a given position  $v \in V^G$ , whether the  $2 \times 2$ -**subgrid** of  $G^\#$  with upper-left corner  $v$  matches  $\theta$ . Here,  $\theta_1$  is chosen as the representative entry of  $\theta$ , because the upper-left corner of the tile can “see” the other **nodes** by following the directed  $R_1$ - and  $R_2$ -**edges**. Otherwise, if  $\theta_1$  is equal to  $\#$ , there is no such **node**  $v$ , since  $G$  does not contain special border **nodes**. However, we can always choose some other entry  $\theta_i$ , different from  $\#$ , to be the representative of  $\theta$ , and write a **formula**  $\varphi_\theta$  describing the tile from the point of view of a **node** corresponding to  $\theta_i$ . This choice is never arbitrary, because the representative must be able to “see” the other non- $\#$  entries of the tile. Consequently, we divide  $\Theta$  into four disjoint sets  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ , such that  $\Theta_i$  contains those tiles  $\theta$  that are represented by their entry  $\theta_i$ . In order to facilitate the subsequent formalization, we further subdivide each set into partitions according to the  $\#$ -borders that occur within the tiles:  $\Theta_M$  contains the “middle tiles” (all entries different from  $\#$ ),  $\Theta_L$  the “left tiles” (with  $\theta_1$  and  $\theta_3$  equal to  $\#$ ),  $\Theta_{BR}$  the “bottom-right tiles”, and so forth ... Altogether,  $\Theta$  is partitioned into nine subsets, grouped into four types:

$$\begin{aligned} \Theta_1 &= \Theta_M \dot{\cup} \Theta_B \dot{\cup} \Theta_R \dot{\cup} \Theta_{BR} & \Theta_2 &= \Theta_L \dot{\cup} \Theta_{BL} \\ \Theta_3 &= \Theta_T \dot{\cup} \Theta_{TR} & \Theta_4 &= \Theta_{TL} \end{aligned}$$

We now construct the **formula**  $\varphi_T$  in a bottom-up manner, starting with a **subformula**  $\varphi_{\theta_i}$  for each entry  $\theta_i$  other than  $\#$ , for every tile  $\theta \in \Theta$ . Letting  $\theta_i$  be equal to  $(\alpha, q) \in \Sigma \times Q$ , with  $\alpha = \alpha_1 \dots \alpha_s$ , the **formula**  $\varphi_{\theta_i}$  checks at a given position  $v \in V^G$  if the **labeling** of  $v$  matches  $\theta_i$ .

$$\varphi_{\theta_i} = \bigwedge_{\alpha_j=1} P_j \wedge \bigwedge_{\alpha_j=0} \neg P_j \wedge X_q \wedge \bigwedge_{q' \neq q} \neg X_{q'}$$

Building on this, we can define for each tile  $\theta \in \Theta$  the **formula**  $\varphi_\theta$  mentioned above. Since  $\vec{\text{ML}}_g$  does not have **backward modalities**, there is a certain asymmetry

between tiles in  $\Theta_1$ , where the representative can “see” the entire  $2 \times 2$ -subgrid, and the remaining tiles, where the representative must “know” that it lies in the leftmost column or the uppermost row of the grid  $G$ . We shall address this issue shortly, and just assume that information not accessible to the representative is verified by another part of the ultimate formula  $\varphi_T$ . For tiles in  $\Theta_M, \Theta_{BR}, \Theta_L, \Theta_{TL}$ , the definitions of  $\varphi_\theta$  are given in the following table. For tiles in  $\Theta_B, \Theta_R, \Theta_{BL}, \Theta_T, \Theta_{TR}$ , the method is completely analogous.

$\theta$	$\varphi_\theta$
$\Theta_M \ni \begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{bmatrix}$	$\varphi_{\theta_1} \wedge \Diamond \varphi_{\theta_2} \wedge \Diamond \varphi_{\theta_3} \wedge \Diamond \varphi_{\theta_4}$
$\Theta_{BR} \ni \begin{bmatrix} \theta_1 & \# \\ \# & \# \end{bmatrix}$	$\varphi_{\theta_1} \wedge \Box \perp \wedge \Box \perp$
$\Theta_L \ni \begin{bmatrix} \# & \theta_2 \\ \# & \theta_4 \end{bmatrix}$	$\varphi_{\theta_2} \wedge \Diamond \varphi_{\theta_4}$
$\Theta_{TL} \ni \begin{bmatrix} \# & \# \\ \# & \theta_4 \end{bmatrix}$	$\varphi_{\theta_4}$

It remains to mark the top and left borders of  $G$ , using two additional predicates  $Y_T$  and  $Y_L$ , over which we will quantify existentially. To this end, we write an  $\vec{ML}_g$ -formula  $\varphi_{\text{border}}$ , checking that top [resp. left] nodes have no  $R_1$ - [resp.  $R_2$ -] predecessor, that there is a top-left node, and that being top [resp. left] is passed on to the  $R_2$ - [resp.  $R_1$ -] successor, if it exists.

$$\begin{aligned} \varphi_{\text{border}} = & \neg \Diamond (\Diamond Y_T \vee \Diamond Y_L) \wedge \Diamond (Y_T \wedge Y_L) \wedge \\ & \Box ((Y_T \rightarrow \Box Y_T) \wedge (Y_L \rightarrow \Box Y_L)) \end{aligned}$$

Finally, we can put everything together to describe the acceptance condition of  $T$ . Every node  $v \in V^G$  has to ensure that it corresponds to the upper-left corner of some tile in  $\Theta_1$ . Furthermore, nodes in the leftmost column or uppermost row of  $G$  must additionally check that the assignment of states is compatible with the tiles in  $\Theta_2, \Theta_3, \Theta_4$ . This leads to the desired formula  $\varphi_T$ :

$$\begin{aligned} \exists (X_q)_{q \in Q}, Y_T, Y_L \Bigg( & \varphi_{\text{border}} \wedge \\ & \Box \left( \bigvee_{\theta \in \Theta_1} \varphi_\theta \right) \wedge \Box \left( Y_L \rightarrow \bigvee_{\theta \in \Theta_2} \varphi_\theta \right) \wedge \\ & \Box \left( Y_T \rightarrow \bigvee_{\theta \in \Theta_3} \varphi_\theta \right) \wedge \Box \left( Y_T \wedge Y_L \rightarrow \bigvee_{\theta \in \Theta_4} \varphi_\theta \right) \Bigg) \end{aligned}$$

Note that we do not need a separate subformula to check that the interpretations of  $(X_q)_{q \in Q}$  form a partition of  $V^G$ , since this is already done implicitly in the conjunct  $\Box (\bigvee_{\theta \in \Theta_1} \varphi_\theta)$ . ■

### 6.4.3 Equivalent hierarchies on grids

We now have all we need to prove the levelwise equivalence of  $\text{MSOL}$ ,  $\text{MSO}(\vec{ML}_g)$  and  $\text{MSO}(\vec{ML}_g)$  on labeled grids.



**Theorem 6.9.**

► Let  $s \geq 0$ ,  $\ell \geq 1$  and  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}, \Delta_\ell^{\text{MSO}}\}$ . When restricted to the class of  $s$ -bit labeled grids,  $\Xi(\text{FOL})$ ,  $\Xi(\vec{\text{ML}}_g)$  and  $\Xi(\vec{\text{ML}}_g)$  are equivalent, i.e.,

$$\begin{aligned} \llbracket \Xi(\text{FOL}) \rrbracket_{\text{GRID}_s} &= \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s} \\ &= \llbracket \Xi(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s}. \end{aligned} \quad \blacktriangleleft$$

*Proof.* First, we show that the claim holds for the case  $\Xi = \Sigma_1^{\text{MSO}}$  (with arbitrary  $s \geq 0$ ). This can be seen from the following circular chain of inclusions:

$$\begin{aligned} \llbracket \Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s} &\subseteq \llbracket \Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s} & (a) \\ &\subseteq \llbracket \Sigma_1^{\text{MSO}}(\text{FOL}) \rrbracket_{\text{GRID}_s} & (b) \\ &\subseteq \llbracket \text{TS}_s \rrbracket_{\text{GRID}_s} & (c) \\ &\subseteq \llbracket \Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{GRID}_s} & (d) \end{aligned}$$

- (a) The first inclusion follows from the fact that  $\Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g)$  is a syntactic fragment of  $\Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g)$ .
- (b) For the second inclusion, consider any  $\Sigma_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -formula  $\widehat{\varphi} = \exists_{X_1, \dots, X_n}(\varphi)$ , where  $X_1, \dots, X_n$  are set symbols and  $\varphi$  is an  $\vec{\text{ML}}_g$ -formula. By Proposition 6.6, we can replace  $\varphi$  in  $\widehat{\varphi}$  by an equivalent FOL-formula  $\psi_\varphi$ . This results in the  $\Sigma_1^{\text{MSO}}(\text{FOL})$ -formula  $\psi_{\widehat{\varphi}} = \exists_{X_1, \dots, X_n}(\psi_\varphi)$ , which is equivalent to  $\widehat{\varphi}$  on arbitrary structures, and thus, in particular, on  $s$ -bit labeled grids.
- (c) The translation from  $\Sigma_1^{\text{MSO}}(\text{FOL})$  on labeled grids to tiling systems corresponds to the more challenging direction of Theorem 6.7, which is the main result of [GRST96].
- (d) The last inclusion is given by Proposition 6.8.

The general version of the theorem can now be obtained by induction on  $\ell$ . This is straightforward, because the classes  $\Pi_\ell^{\text{MSO}}(\Phi)$ ,  $\text{BC } \Sigma_\ell^{\text{MSO}}(\Phi)$  and  $\Sigma_{\ell+1}^{\text{MSO}}(\Phi)$  are defined syntactically in terms of  $\Sigma_\ell^{\text{MSO}}(\Phi)$ , for any set of kernel formulas  $\Phi$  (see Section 6.1), and if the claim holds for  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}\}$ , then it also holds for the intersection classes of the form  $\llbracket \Delta_\ell^{\text{MSO}}(\Phi) \rrbracket_{\text{GRID}_s}$ . ■

**6.4.4 A logical characterization of grids**

We conclude this section by showing that a single layer of universal set quantifiers is enough to describe grids in  $\text{MSO}(\vec{\text{ML}}_g)$ .

**Proposition 6.10.**

► The set of all grids is  $\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -definable over 2-relational digraphs, i.e.,

$$\text{GRID} \in \llbracket \Pi_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{DG}_0^2}. \quad \blacktriangleleft$$

*Proof.* In the course of this proof, we give a list of properties, items a to f, which are obviously necessary for a 2-relational digraph  $G$  to be a grid, and show how to express them as  $\llbracket \Pi_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket$ -formulas. We argue that the conjunction of all of these properties also constitutes a sufficient condition for being a grid, which immediately provides us with the required formula, since  $\llbracket \Pi_1^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket$  is closed under intersection.

- a. For each relation symbol  $R \in \{R_1, R_2\}$ , every node has at most one  $R$ -predecessor and at most one  $R$ -successor; in other words,  $R_1^G$  and  $R_2^G$  are partial injective functions.

$$\bigwedge_{R \in \{R_1, R_1^{-1}, R_2, R_2^{-1}\}} \forall x \square (\lozenge R x \rightarrow \square R x)$$

- b. Again considering each  $R \in \{R_1, R_2\}$  separately, there is a directed  $R$ -path from every node to an  $R$ -sink, i.e., to some node without  $R$ -successor.

$$\bigwedge_{R \in \{R_1, R_2\}} \forall x (\lozenge x \wedge \square (x \rightarrow \square R x) \rightarrow \lozenge (x \wedge \square R \perp))$$

Taken together, properties a and b state that  $R_1^G$  and  $R_2^G$  each form a collection of directed, acyclic, pairwise vertex-disjoint paths. Let us refer to the first nodes of those paths as  $R_1$ - and  $R_2$ -sources, respectively.

- c. There is precisely one node that is both an  $R_1$ - and an  $R_2$ -source.

$$\text{tot1}(\bar{1} \perp \wedge \bar{2} \perp)$$

(Here,  $\text{tot1}$  is the schema from Example 2.2 in Section 2.6.)

- d. The  $R_1$ -predecessors and  $R_1$ -successors of  $R_2$ -sources must be  $R_2$ -sources themselves.

$$\square (\bar{2} \perp \rightarrow \bar{1} \bar{2} \perp \wedge \bar{1} \bar{2} \perp)$$

By adding c and d to our list of conditions, we ensure that there is an  $R_1$ -path consisting precisely of the  $R_2$ -sources, thereby also forcing the digraph  $G$  to be connected.

- e. If a node has both an  $R_1$ - and an  $R_2$ -successor, then it also has a descendant reachable by first taking an  $R_1$ -edge and then an  $R_2$ -edge.

$$\square (\lozenge \top \wedge \lozenge \top \rightarrow \lozenge \lozenge \top)$$

- f. The relations  $R_1^G$  and  $R_2^G$  commute. This means that following an  $R_1$ -edge and then an  $R_2$ -edge leads to the same node as first taking an  $R_2$ -edge and then an  $R_1$ -edge.

$$\forall x \square (\lozenge \lozenge x \leftrightarrow \lozenge \lozenge x)$$

Considered in conjunction with condition a, there are only two ways to satisfy e and f from the point of view of two nodes  $u, v \in V^G$  that are connected by an  $R_1$ -edge from  $u$  to  $v$ : either both nodes are  $R_2$ -sinks, or they have  $R_2$ -successors  $u'$  and  $v'$ , respectively, with an  $R_1$ -edge from  $u'$  to  $v'$ . Moreover,  $v'$  only possesses an  $R_1$ -successor if  $v$  does. Now, imagine we start from the left border, i.e., from the  $R_1$ -path that consists of all the  $R_2$ -sources, which is provided by properties a to d, and iteratively enforce the requirements just mentioned. Then, in doing so, we propagate the grid topology through the entire digraph. More specifically, the additional requirements of e and f entail that all the  $R_2$ -paths have the same length, and that the nodes lying at a fixed (horizontal) position of those  $R_2$ -paths constitute an independent  $R_1$ -path, ordered in the same way as their respective  $R_2$ -predecessors. ■

## 6.5 A toolbox of encodings

In this section, we provide all the encoding functions used in the proof of [Theorem 6.2](#) (see [Section 6.3.2](#)), and show that they satisfy suitable translatability properties, allowing us to establish the required [figurative inclusions](#). With a view to modularity and reusability, some of our constructions are more general than needed.

Given a set of [symbols](#)  $\sigma$ , the extension  $\sigma \cup \{@\}$  will be abbreviated to  $\sigma_@$ .

### 6.5.1 Encodings that allow for translation

We shall only consider encoding functions that are linear in the following sense:

**Definition 6.11** (Linear Encoding).

► Let  $\mathcal{C}, \mathcal{D}$  be two classes of [structures](#), and  $m, n$  be integers such that  $1 \leq m \leq n$ . A [linear encoding](#) from  $\mathcal{C}$  into  $\mathcal{D}$  with parameters  $m, n$  is a total injective function  $\mu: \mathcal{C} \rightarrow \mathcal{D}$  that assigns to each [structure](#)  $G \in \mathcal{C}$  a [structure](#)  $\mu(G) \in \mathcal{D}$ , whose [domain](#) is composed of  $m$  disjoint copies of the [domain](#) of  $G$  and  $n - m$  additional [nodes](#), i.e.,

$$V^{\mu(G)} = ([1:m] \times V^G) \cup ]m:n]. \quad \blacktriangleleft$$

Given such a [linear encoding](#)  $\mu$  and some  $\vec{M}_g$ -[formula](#)  $\varphi$ , we want to be able to construct a new [formula](#)  $\psi_\varphi$ , such that evaluating  $\varphi$  on  $\mathcal{C}$  is equivalent to evaluating  $\psi_\varphi$  on  $\mu(\mathcal{C})$ . Conversely, we also desire a way of constructing a [formula](#)  $\varphi_\psi$  that is equivalent on  $\mathcal{C}$  to a given [formula](#)  $\psi$  on  $\mu(\mathcal{C})$ . The following two definitions formalize this translatability property for both directions. We then show in [Lemma 6.14](#) that they adequately capture our intended meaning. Although the underlying idea is very simple, the presentation is a bit lengthy because we have to exhaustively cover the structure of  $\vec{M}_g$ -[formulas](#).

**Definition 6.12** (Forward Translation).

► Consider two classes of [structures](#)  $\mathcal{C}$  and  $\mathcal{D}$  over [signatures](#)  $\sigma$  and  $\tau$ , respectively, two classes of [formulas](#)  $\Phi, \Psi \in \{\vec{M}, \vec{M}_g, \vec{M}_g\}$ , and a [linear encoding](#)  $\mu: \mathcal{C} \rightarrow \mathcal{D}$ . We say that  $\mu$  allows for [forward translation](#) from  $\Phi$  to  $\Psi$  if the following properties are satisfied:

- a. For each [node symbol](#) or [set symbol](#)  $P$  in  $\sigma$ , there is a  $\Psi$ -[sentence](#)  $\psi_P$  over  $\tau_@$ , such that

$$G[@ \mapsto u] \models P \quad \text{iff} \quad \mu(G)[@ \mapsto (1, u)] \models \psi_P,$$

for all  $G \in \mathcal{C}$  and  $u \in V^G$ .

- b. For each [relation symbol](#)  $R$  in  $\sigma$  of [arity](#)  $k + 1 \geq 2$ , there is a  $\Psi$ -[sentence](#)  $\psi_R$  over  $\tau_@$  enriched with additional [set symbols](#)  $(Y_i)_{1 \leq i \leq k}$ , such that

$$G[@, (X_i)_{i \leq k} \mapsto u, (U_i)_{i \leq k}] \models \Diamond(X_i)_{i \leq k}$$

if and only if

$$\mu(G)[@, (Y_i)_{i \leq k} \mapsto (1, u), (W_i)_{i \leq k}] \models \psi_R,$$

assuming  $U_i, W_i$  satisfy  $u' \in U_i \Leftrightarrow (1, u') \in W_i$ ,

for all  $G \in \mathcal{C}$ ,  $u \in V^G$ , sets  $(U_i)_{1 \leq i \leq k} \subseteq V^G$  and  $(W_i)_{1 \leq i \leq k} \subseteq V^{\mu(G)}$ , and [set symbols](#)  $(X_i)_{1 \leq i \leq k}$ .

- c. If  $\Phi$  includes **backward modalities**, then for each **relation symbol**  $R$  in  $\sigma$  of **arity** at least 2, there is a  $\Psi$ -formula  $\psi_{R^{-1}}$  that satisfies the property of item  $b$  for  $R^{-1}$  instead of  $R$ .
- d. If  $\Phi$  includes **global modalities**, then there is a  $\Psi$ -formula  $\psi_\bullet$  that satisfies the property of item  $b$  for  $\bullet$  instead of  $R$  and  $k = 1$ .
- e. There is a  $\Psi$ -sentence  $\psi_{\text{ini}}$  over  $\tau$  enriched with an additional **set symbol**  $Y$ , such that

$$G[X \mapsto U] \models \frac{X}{\Diamond X} \quad \text{iff} \quad \mu(G)[Y \mapsto W] \models \psi_{\text{ini}},$$

assuming  $U, W$  satisfy  $u \in U \Leftrightarrow (1, u) \in W$ ,  
 where  $\frac{X}{\Diamond X}$  is  $X$  if  $@ \in \sigma$ , and  $\Diamond X$  otherwise,

for all  $G \in \mathcal{C}$ ,  $U \subseteq V^G$ ,  $W \subseteq V^{\mu(G)}$  and  $X \in \mathbb{S}_1$ . ◀

**Definition 6.13** (Backward Translation).

► Consider two classes of **structures**  $\mathcal{C}$  and  $\mathcal{D}$  over **signatures**  $\sigma$  and  $\tau$ , respectively, two classes of **formulas**  $\Phi, \Psi \in \{\vec{M}\vec{L}, \vec{M}\vec{L}_g, \vec{M}\vec{L}_g\}$ , and a **linear encoding**  $\mu: \mathcal{C} \rightarrow \mathcal{D}$  with parameters  $m, n$ . We say that  $\mu$  allows for **backward translation** from  $\Psi$  to  $\Phi$  if the following properties are satisfied:

- a. For each **node symbol** or **set symbol**  $Q$  in  $\tau$  and all  $h \in [n]$ , there is a  $\Phi$ -sentence  $\varphi_Q^h$  over  $\sigma_@$ , such that

$$G[@ \mapsto u] \models \varphi_Q^h \quad \text{iff} \quad \mu(G)[@ \mapsto v] \models Q,$$

where  $v$  is  $(h, u)$  if  $h \leq m$ , and  $h$  otherwise,

for all  $G \in \mathcal{C}$  and  $u \in V^G$ .

- b. For each **relation symbol**  $S$  in  $\tau$  of **arity**  $k + 1 \geq 2$ , and all  $h \in [n]$ , there is a  $\Phi$ -sentence  $\varphi_S^h$  over  $\sigma_@$  enriched with additional **set symbols**  $(X_i^j)_{1 \leq j \leq n}^{1 \leq i \leq k}$ , such that

$$G[@, (X_i^j)_{1 \leq j \leq n}^{1 \leq i \leq k} \mapsto u, (U_i^j)_{1 \leq j \leq n}^{1 \leq i \leq k}] \models \varphi_S^h$$

if and only if

$$\mu(G)[@, (Y_i)_{1 \leq i \leq k} \mapsto v, (W_i)_{1 \leq i \leq k}] \models \Diamond(Y_i)_{1 \leq i \leq k},$$

where  $v$  is  $(h, u)$  if  $h \leq m$ , otherwise  $h$ , and

$$W_i = \bigcup_{1 \leq j \leq m} (\{j\} \times U_i^j) \cup \bigcup_{m < j \leq n} \{j \mid U_i^j = V^G\},$$

for all  $G \in \mathcal{C}$ , **nodes**  $u \in V^G$ , sets  $(U_i^j)_{1 \leq j \leq m}^{1 \leq i \leq k} \subseteq V^G$  and  $(U_i^j)_{1 \leq j \leq n}^{m < j \leq n} \in \{\emptyset, V^G\}$ , and **set symbols**  $(Y_i)_{1 \leq i \leq k}$ .

- c. If  $\Psi$  includes **backward modalities**, then for each **relation symbol**  $S$  in  $\tau$  of **arity** at least 2, and all  $h \in [n]$ , there is a  $\Phi$ -formula  $\varphi_{S^{-1}}^h$  that satisfies the property of item  $b$  for  $S^{-1}$  instead of  $S$ .
- d. If  $\Psi$  includes **global modalities**, then for all  $h \in [n]$ , there is a  $\Phi$ -formula  $\varphi_\bullet^h$  that satisfies the property of item  $b$  for  $\bullet$  instead of  $S$  and  $k = 1$ .

- e. There is a  $\Phi$ -sentence  $\varphi_{\text{ini}}$  over  $\sigma$  enriched with additional set symbols  $(X^j)^{1 \leq j \leq n}$ , such that

$$\begin{aligned} G[(X^j)^{j \leq n} \mapsto (U^j)^{j \leq n}] &\models \varphi_{\text{ini}} \\ \text{if and only if} \\ \mu(G)[Y \mapsto W] &\models \frac{Y}{\Diamond Y}, \\ \text{where } \frac{Y}{\Diamond Y} &\text{ is } Y \text{ if } @ \in \tau, \text{ otherwise } \Diamond Y, \text{ and} \\ W &= \bigcup_{1 \leq j \leq m} (\{j\} \times U^j) \cup \bigcup_{m < j \leq n} \{j \mid U^j = V^G\}, \end{aligned}$$

for all structures  $G \in \mathcal{C}$ , sets  $(U^j)^{1 \leq j \leq m} \subseteq V^G$  and  $(U^j)^{m < j \leq n} \in \{\emptyset, V^G\}$ , and  $Y \in \mathbb{S}_1$ .  $\blacktriangleleft$

To simplify matters slightly, we shall say that a linear encoding  $\mu$  allows for *bidirectional translation* between  $\Phi$  and  $\Psi$ , if it allows for both forward translation from  $\Phi$  to  $\Psi$  and backward translation from  $\Psi$  to  $\Phi$ . Furthermore, in case  $\Phi = \Psi$ , we may say “within  $\Phi$ ” instead of “between  $\Phi$  and  $\Phi$ ”.

Let us now prove that our notion of translatability is indeed sufficient to imply *figurative inclusion* on the semantic side, even if we bring set quantifiers into play.

**Lemma 6.14.**

► Consider two classes of structures  $\mathcal{C}$  and  $\mathcal{D}$ , a linear encoding  $\mu: \mathcal{C} \rightarrow \mathcal{D}$ , two classes of formulas  $\Phi, \Psi \in \{\vec{M}L, \vec{M}L, \vec{M}L_g, \vec{M}L_g\}$ , and let  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}, \text{BC } \Sigma_\ell^{\text{MSO}}\}$ , for some arbitrary  $\ell \geq 0$ .

- a. If  $\mu$  allows for forward translation from  $\Phi$  to  $\Psi$ , then we have

$$\llbracket \Xi(\Phi) \rrbracket_{\mathcal{C}} \subseteq_{\mu} \llbracket \Xi(\Psi) \rrbracket_{\mathcal{D}}.$$

- b. Similarly, if  $\mu$  allows for backward translation from  $\Psi$  to  $\Phi$ , then we have

$$\llbracket \Xi(\Phi) \rrbracket_{\mathcal{C}} \supseteq_{\mu} \llbracket \Xi(\Psi) \rrbracket_{\mathcal{D}}. \quad \blacktriangleleft$$

*Proof.* Let  $\sigma$  and  $\tau$  be the signatures underlying  $\mathcal{C}$  and  $\mathcal{D}$ , respectively. Parts a and b of the lemma are treated separately in the following proof.

In several places, given some  $\text{MSO}(\vec{M}L_g)$ -formula  $\varphi$ , the need will arise to substitute newly created  $\vec{M}L_g$ -formulas  $\varphi_1, \dots, \varphi_k$  for set symbols  $X_1, \dots, X_k$ . We shall write  $\varphi[(X_i)_{i \leq k} \mapsto (\varphi_i)_{i \leq k}]$  to denote the  $\text{MSO}(\vec{M}L_g)$ -formula that one obtains by simultaneously replacing every free occurrence of each  $X_i$  in  $\varphi$  by the formula  $\varphi_i$ .

- a. For every  $\Xi(\Phi)$ -sentence  $\varphi$  over  $\sigma$ , we must construct a  $\Xi(\Psi)$ -sentence  $\psi_\varphi$  over  $\tau$ , such that  $\psi_\varphi$  says about  $\mu(G)$  the same as  $\varphi$  says about  $G$ , for all structures  $G \in \mathcal{C}$ .

We start by focusing on the kernel classes  $\Phi, \Psi$ , and show the following by induction on the structure of  $\Phi$ -formulas: For every  $\Phi$ -sentence  $\varphi$  over  $\sigma_{@} \cup \mathcal{Z}$ , with  $\mathcal{Z} = \{Z_1, \dots, Z_z\}$  being any collection of set symbols disjoint from  $\sigma$  and  $\tau$  (i.e., free set variables), there is a  $\Psi$ -sentence  $\psi_\varphi^*$  over  $\tau_{@} \cup \mathcal{Z}$  such that

$$\begin{aligned} G[@, (Z_t)_{t \leq z} \mapsto u, (U_t)_{t \leq z}] &\models \varphi \\ \text{if and only if} \\ \mu(G)[@, (Z_t)_{t \leq z} \mapsto (1, u), (W_t)_{t \leq z}] &\models \psi_\varphi^*, \\ \text{assuming } U_t, W_t \text{ satisfy } u' \in U_t &\Leftrightarrow (1, u') \in W_t, \end{aligned}$$

for all **structures**  $G \in \mathcal{C}$ , **nodes**  $u \in V^G$ , and sets  $(U_t)_{1 \leq t \leq z} \subseteq V^G$  and  $(W_t)_{1 \leq t \leq z} \subseteq V^{\mu(G)}$ .

- If  $\varphi = @$  or  $\varphi = Z$ , for some  $Z \in \mathcal{Z}$ , it suffices to set  $\psi_\varphi^* = \varphi$ .
- If  $\varphi = P$ , for some **node symbol** or **set symbol**  $P$  in  $\sigma$ , we exploit that  $\mu$  allows for **forward translation** from  $\Phi$  to  $\Psi$ , and choose  $\psi_\varphi^* = \psi_P$ . Here,  $\psi_P$  is the **formula** postulated by Definition 6.12 a; it fulfills the induction hypothesis, since adding **interpretations** of the **symbols**  $Z_1, \dots, Z_z$  to a **structure** has no influence on whether or not that **structure** satisfies a **sentence** over a **signature** that does not contain these **symbols**.
- If  $\varphi = \neg\varphi_1$  or  $\varphi = \varphi_1 \vee \varphi_2$ , where  $\varphi_1$  and  $\varphi_2$  are **formulas** that satisfy the induction hypothesis, we set  $\psi_\varphi^* = \neg\psi_{\varphi_1}^*$  or  $\psi_\varphi^* = \psi_{\varphi_1}^* \vee \psi_{\varphi_2}^*$ , respectively.
- If  $\varphi = \Diamond(\varphi_i)_{i \leq k}$ , where  $R$  is a **relation symbol** in  $\sigma$  of **arity**  $k+1 \geq 2$ , and  $(\varphi_i)_{i \leq k}$  are  $\Phi$ -**sentences** over  $\sigma_{@} \cup \mathcal{Z}$  satisfying the induction hypothesis, we again use the fact that  $\mu$  allows for **forward translation** from  $\Phi$  to  $\Psi$ . The desired **formula**  $\psi_\varphi^*$  is obtained by substituting  $(\psi_{\varphi_i}^*)_{i \leq k}$  for the **symbols**  $(Y_i)_{i \leq k}$  in the **formula**  $\psi_R$ , whose existence is asserted by Definition 6.12 b, i.e.,

$$\psi_\varphi^* = \psi_R[(Y_i)_{i \leq k} \mapsto (\psi_{\varphi_i}^*)_{i \leq k}].$$

For any integer  $i \in [1:k]$ , let  $U'_i$  be the set of **nodes**  $u' \in V^G$  that satisfy  $\varphi_i$  in  $G[(Z_t)_{t \leq z} \mapsto (U_t)_{t \leq z}]$ , and let  $W'_i$  be the set of **nodes**  $v' \in V^{\mu(G)}$  that satisfy  $\psi_{\varphi_i}$  in  $\mu(G)[(Z_t)_{t \leq z} \mapsto (W_t)_{t \leq z}]$ . By induction hypothesis, we are guaranteed that all the sets  $U'_i, W'_i$  are such that a **node**  $u'$  lies in  $U'_i$  if and only if  $(1, u')$  lies in  $W'_i$ . Thus, we have

$$\begin{aligned} & G[@, (Z_t)_{t \leq z} \mapsto u, (U_t)_{t \leq z}] \models \varphi \\ \text{iff } & G[@, (X_i)_{i \leq k} \mapsto u, (U'_i)_{i \leq k}] \models \Diamond(X_i)_{i \leq k} \\ \text{iff } & \mu(G)[@, (Y_i)_{i \leq k} \mapsto (1, u), (W'_i)_{i \leq k}] \models \psi_R \\ \text{iff } & \mu(G)[@, (Z_t)_{t \leq z} \mapsto (1, u), (W_t)_{t \leq z}] \models \psi_\varphi^*. \end{aligned}$$

- If  $\varphi = \Diamond(\varphi_i)_{i \leq k}$ , assuming  $\Phi$  incorporates **backward modalities**, we obtain  $\psi_\varphi^*$  by applying the same argument as in the previous case, but this time considering  $R^{-1}$  instead of  $R$  and invoking Definition 6.12 c.
- If  $\varphi = \Diamond\varphi_1$ , supposing  $\Phi$  includes **global modalities**, we again follow the same line of reasoning as in the case  $\varphi = \Diamond(\varphi_i)_{i \leq k}$ , referring to Definition 6.12 d and using  $\bullet$  instead of  $R$ , with  $k = 1$ .

Now we can consider the case where the **position symbol**  $@$  is not (re)mapped, and then look beyond the **kernel** classes to finally deal with **set quantifiers**. Arguing once more by structural induction, we extend the preceding claim as follows: For every  $\Xi(\Phi)$ -**sentence**  $\varphi$  over  $\sigma \cup \mathcal{Z}$ , with  $\mathcal{Z} = \{Z_1, \dots, Z_z\}$  as before (possibly empty), there is a  $\Xi(\Psi)$ -**sentence**  $\psi_\varphi$  over  $\tau \cup \mathcal{Z}$  such that

$$\begin{aligned} & G[(Z_t)_{t \leq z} \mapsto (U_t)_{t \leq z}] \models \varphi \\ & \text{if and only if} \\ & \mu(G)[(Z_t)_{t \leq z} \mapsto (W_t)_{t \leq z}] \models \psi_\varphi, \\ & \text{assuming } U_t, W_t \text{ satisfy } u \in U_t \Leftrightarrow (1, u) \in W_t, \end{aligned}$$

for all  $G \in \mathcal{C}$ ,  $(U_t)_{1 \leq t \leq z} \subseteq V^G$  and  $(W_t)_{1 \leq t \leq z} \subseteq V^{\mu(G)}$ .

- If  $\varphi$  lies in the **kernel**  $\Phi$ , we make use of the claim just proven, together with the **formula**  $\psi_{\text{ini}}$  described in **Definition 6.12 e**. We set  $\psi_\varphi = \psi_{\text{ini}}[Y \mapsto \psi_\varphi^*]$ .
  - If  $@$  belongs to  $\sigma$ , the asserted property of  $\psi_{\text{ini}}$  guarantees that  $\varphi$  holds at the initial position  $@^G$  in the  $\mathbb{Z}$ -extended variant of  $G$  if and only if  $\psi_\varphi$  is satisfied by the  $\mathbb{Z}$ -extended variant of  $\mu(G)$ .
  - Otherwise,  $@$  cannot be **free** in  $\varphi$ , since  $\varphi$  is a **sentence** over  $\sigma \cup \mathbb{Z}$ , which also implies that  $\Phi$  incorporates **global modalities**. It follows that  $\varphi$  is **equivalent** to  $\Diamond \varphi$ . Again applying the definition of  $\psi_{\text{ini}}$ , we obtain that the  $\mathbb{Z}$ -extended variant of  $G$  satisfies  $\Diamond \varphi$ , and thus  $\varphi$ , if and only if the  $\mathbb{Z}$ -extended variant of  $\mu(G)$  satisfies  $\psi_\varphi$ .
- If  $\varphi$  is a Boolean combination of **formulas** that satisfy the induction hypothesis, the translation is straightforward, just as in the previous part of the proof.
- If  $\varphi = \exists_{Z_{z+1}} \varphi_1$ , where  $\varphi_1$  is a  $\Xi(\Phi)$ -sentence over  $\sigma \cup \{Z_1, \dots, Z_{z+1}\}$  that satisfies the hypothesis, we choose  $\psi_\varphi = \exists_{Z_{z+1}} \psi_{\varphi_1}$ . To justify this choice, let  $G'$  and  $\mu(G)'$  denote the  $\mathbb{Z}$ -extended variants of  $G$  and  $\mu(G)$ , respectively. We get the following by induction:
  - If choosing  $Z_{z+1} \mapsto U_{z+1}$  leads to **satisfaction** of  $\varphi_1$  in  $G'$ , then choosing  $Z_{z+1} \mapsto \{1\} \times U_{z+1}$  does the same for  $\psi_{\varphi_1}$  in  $\mu(G)'$ .
  - Conversely, if  $Z_{z+1} \mapsto W_{z+1}$  is a **satisfying** choice for  $\psi_{\varphi_1}$  in  $\mu(G)'$ , then so is  $Z_{z+1} \mapsto \{u \mid (1, u) \in W_{z+1}\}$  for  $\varphi_1$  in  $G'$ .

**b.** The proof of the reverse direction of the lemma is very similar to the previous one, but a bit more cumbersome, because each **node** of a **structure**  $G$  has to play the role of several different **nodes** in  $\mu(G)$ . Given any  $\Xi(\Psi)$ -sentence  $\psi$  over  $\tau$ , we need to construct a  $\Xi(\Phi)$ -sentence  $\varphi_\psi$  over  $\sigma$ , such that evaluating  $\varphi_\psi$  on  $G$  is equivalent to evaluating  $\psi$  on  $\mu(G)$ , for all  $G \in \mathcal{C}$ . For the remainder of this proof, let  $m, n$  be the parameters of the **linear encoding**  $\mu$ .

Again, we first deal with the **kernel** classes  $\Phi, \Psi$ , and show the following claim by induction on the structure of  $\Psi$ -formulas: For every  $\Psi$ -sentence  $\psi$  over  $\tau_{@} \cup \mathbb{Z}$  and all  $h \in [n]$ , with  $\mathbb{Z} = \{Z_1, \dots, Z_z\} \subseteq \mathbb{S}_1 \setminus \tau$ , there is a  $\Phi$ -sentence  $\varphi_\psi^h$  over  $\sigma_{@} \cup \tilde{\mathbb{Z}}$ , with  $\tilde{\mathbb{Z}} = \{Z_1^1, \dots, Z_z^n\} \subseteq \mathbb{S}_1 \setminus \sigma$ , such that

$$G[@, (Z_t^j)_{t \leq z}^{j \leq n} \mapsto u, (U_t^j)_{t \leq z}^{j \leq n}] \models \varphi_\psi^h$$

if and only if

$$\mu(G)[@, (Z_t)_{t \leq z} \mapsto v, (W_t)_{t \leq z}] \models \psi,$$

where  $v$  is  $(h, u)$  if  $h \leq m$ , otherwise  $h$ , and

$$W_t = \bigcup_{1 \leq j \leq m} (\{j\} \times U_t^j) \cup \bigcup_{m < j \leq n} \{j \mid U_t^j = V^G\},$$

for all  $G \in \mathcal{C}$ ,  $u \in V^G$ , and sets  $(U_t^j)_{1 \leq j \leq m}^{1 \leq t \leq z} \subseteq V^G$  and  $(U_t^j)_{1 \leq t \leq z}^{m < j \leq n} \in \{\emptyset, V^G\}$ .

- If  $\psi = @$ , it suffices to set  $\varphi_\psi^h = @$ .
- If  $\psi = Z_t$ , for some  $Z_t \in \mathbb{Z}$ , the translation is given by  $\varphi_\psi^h = Z_t^h$ .



- If  $\psi = Q$ , for some **node symbol** or **set symbol**  $Q$  in  $\tau$ , we use the fact that  $\mu$  allows for **backward translation** from  $\Psi$  to  $\Phi$ , and choose  $\varphi_{\psi}^h$  to be the **formula**  $\varphi_Q^h$ , which is provided by **Definition 6.13 a**. The definition asserts that this **formula** fulfills the induction hypothesis for the case where  $G$  and  $\mu(G)$  are not extended using additional **set symbols** from  $\tilde{Z}$  and  $Z$ . But since these **symbols** do not occur **freely** in  $\varphi_Q^h$  and  $Q$ , their **interpretations** do not influence the evaluation of the **formulas**.
- If  $\psi = \neg\psi_1$  or  $\psi = \psi_1 \vee \psi_2$ , where  $\psi_1$  and  $\psi_2$  are **formulas** that satisfy the induction hypothesis, we set  $\varphi_{\psi}^h = \neg\varphi_{\psi_1}^h$  or  $\varphi_{\psi}^h = \varphi_{\psi_1}^h \vee \varphi_{\psi_2}^h$ , respectively.
- If  $\psi = \Diamond(\psi_i)_{i \leq k}$ , where  $S$  is a **relation symbol** in  $\tau$  of **arity**  $k+1 \geq 2$ , and  $(\psi_i)_{i \leq k}$  are  $\Psi$ -**sentences** over  $\tau_{@} \cup Z$  satisfying the hypothesis, we again rely on the premise that  $\mu$  allows for **backward translation** from  $\Psi$  to  $\Phi$ . We construct  $\varphi_{\psi}^h$  by plugging the **formulas**  $(\varphi_{\psi_i}^j)_{i \leq k}^{j \leq n}$  provided by induction into the **formula**  $\varphi_S^h$  of **Definition 6.13 b** as follows:

$$\varphi_{\psi}^h = \varphi_S^h[(X_i^j)_{i \leq k}^{j \leq n} \mapsto (\varphi_{\psi_i}^j)_{i \leq k}^{j \leq n}].$$

For  $1 \leq i \leq k$  and  $1 \leq j \leq n$ , let  $U_i^{j'}$  be the set of **nodes**  $u' \in V^G$  that **satisfy**  $\varphi_{\psi_i}^j$  in  $G[(Z_t^j)_{t \leq z}^{j \leq n} \mapsto (U_t^j)_{t \leq z}^{j \leq n}]$ , and let  $W_i'$  be the set of **nodes**  $v' \in V^{\mu(G)}$  that **satisfy**  $\psi_i$  in  $\mu(G)[(Z_t)_{t \leq z} \mapsto (W_t)_{t \leq z}]$ . The induction hypothesis ensures that

$$W_i' = \bigcup_{1 \leq j \leq m} (\{j\} \times U_i^{j'}) \cup \bigcup_{m < j \leq n} \{j \mid U_i^{j'} = V^G\}.$$

Hence, we obtain the required equivalence as follows:

$$\begin{aligned} & G[@, (Z_t^j)_{t \leq z}^{j \leq n} \mapsto u, (U_t^j)_{t \leq z}^{j \leq n}] \models \varphi_{\psi}^h \\ \text{iff } & G[@, (X_i^j)_{i \leq k}^{j \leq n} \mapsto u, (U_i^{j'})_{i \leq k}^{j \leq n}] \models \varphi_{\psi}^h \\ \text{iff } & \mu(G)[@, (Y_i)_{i \leq k} \mapsto v, (W_i')_{i \leq k}] \models \Diamond(Y_i)_{i \leq k} \\ \text{iff } & \mu(G)[@, (Z_t)_{t \leq z} \mapsto v, (W_t)_{t \leq z}] \models \psi. \end{aligned}$$

- If  $\psi = \bar{\Diamond}(\psi_i)_{i \leq k}$ , supposing  $\Psi$  includes **backward modalities**, we construct  $\varphi_{\psi}^h$  using the same approach as in the previous case, the only difference being that we consider  $S^{-1}$  instead of  $S$  and invoke **Definition 6.13 c** instead of **6.13 b**.
- If  $\psi = \Diamond\psi_1$ , in case  $\Psi$  includes **global modalities**, we again proceed as for the case  $\psi = \Diamond(\psi_i)_{i \leq k}$ , this time using  $\bullet$  instead of  $S$ , with  $k = 1$ , and referring to **Definition 6.13 d**.

Similarly to the proof of part a, we now extend the previous property to cover **formulas** with **set quantifiers**, evaluated on **structures** that may **interpret** the **position symbol**  $@$  arbitrarily. Our induction hypothesis is the following: For every  $\Xi(\Psi)$ -**sentence**  $\varphi_{\psi}$  over  $\tau \cup Z$ , with  $Z = \{Z_1, \dots, Z_z\} \subseteq \mathbb{S}_1 \setminus \tau$  (possibly empty), there is a  $\Xi(\Phi)$ -**sentence**  $\varphi_{\psi}$  over  $\sigma \cup \tilde{Z}$ , with  $\tilde{Z} = \{Z_1^1, \dots, Z_z^n\} \subseteq \mathbb{S}_1 \setminus \sigma$ , such that

$$G[(Z_t^j)_{t \leq z}^{j \leq n} \mapsto (U_t^j)_{t \leq z}^{j \leq n}] \models \varphi_{\psi}$$

if and only if

$$\mu(G)[(Z_t)_{t \leq z} \mapsto (W_t)_{t \leq z}] \models \psi, \quad \text{where}$$

$$W_t = \bigcup_{1 \leq j \leq m} (\{j\} \times U_t^j) \cup \bigcup_{m < j \leq n} \{j \mid U_t^j = V^G\},$$

for all **structures**  $G \in \mathcal{C}$ , and sets  $(\mathcal{U}_t^j)_{1 \leq t \leq z}^{1 \leq j \leq m} \subseteq V^G$  and  $(\mathcal{U}_t^j)_{1 \leq t \leq z}^{m < j \leq n} \in \{\emptyset, V^G\}$ .

- If  $\psi$  belongs to the **kernel** class  $\Psi$ , we apply the claim just proven, and construct  $\varphi_\psi$  by substituting into the **formula**  $\varphi_{\text{ini}}$  provided by **Definition 6.13 e**:  $\varphi_\psi = \varphi_{\text{ini}}[(X^j)^{j \leq n} \mapsto (\varphi_\psi^j)^{j \leq n}]$ . Proceeding analogously to the proof of part a, we have to distinguish whether or not the **position symbol**  $@$  belongs to  $\tau$ . (If it does not,  $\psi$  is necessarily **equivalent** to  $\langle \Diamond \rangle \psi$ .) In both cases, the definition of  $\varphi_{\text{ini}}$  guarantees that the  $\tilde{\mathcal{Z}}$ -**extended variant** of  $G$  **satisfies**  $\varphi_\psi$  if and only if the  $\mathcal{Z}$ -**extended variant** of  $\mu(G)$  **satisfies**  $\psi$ .
- If  $\psi$  is a Boolean combination of **subformulas** that satisfy the induction hypothesis, then  $\varphi_\psi$  is simply the corresponding Boolean combination of the translated **subformulas**.
- If  $\psi = \exists_{Z_{z+1}} \psi_1$ , where  $\psi_1$  is a  $\Xi(\Psi)$ -**sentence** over  $\tau \cup \{Z_1, \dots, Z_{z+1}\}$  that satisfies the induction hypothesis, we choose  $\varphi_\psi$  to be the **formula**

$$\exists_{(Z_{z+1}^j)^{j \leq m}} \left( \bigvee_{N \subseteq ]m:n]} \varphi_{\psi_1} \left[ (Z_{z+1}^j)^{j > m} \mapsto (N(j))^{j > m} \right] \right),$$

with  $N(j) = \top$  if  $j \in N$ , and  $N(j) = \perp$  otherwise. For each set  $N \subseteq ]m:n]$ , let  $\varphi_{\psi_1}^N$  denote the disjunct corresponding to  $N$  in the **formula** above. By induction, we have the following equivalence: the **interpretation map**  $(Z_{z+1}^j)^{j \leq m} \mapsto (\mathcal{U}_{z+1}^j)^{j \leq m}$  leads to **satisfaction** of  $\varphi_{\psi_1}^N$  in the  $\tilde{\mathcal{Z}}$ -**extended variant** of  $G$  if and only if

$$Z_{z+1} \mapsto \bigcup_{1 \leq j \leq m} (\{j\} \times \mathcal{U}_{z+1}^j) \cup N$$

is a **satisfying** choice for  $\psi_1$  in the  $\mathcal{Z}$ -**extended variant** of  $\mu(G)$ . ■

### 6.5.2 Getting rid of multiple edge relations

We now show how to encode a multi-relational **digraph** into a 1-relational one, by inserting additional **labeled nodes** that represent the different **edge relations**.

#### Proposition 6.15.

► For all  $s, r \geq 0$  and  $\Phi \in \{\vec{\text{ML}}_g, \hat{\text{ML}}_g\}$ , there is a **linear encoding**  $\mu$  from  $\text{DG}_s^r$  into  $\text{DG}_{s+r}^1$  that allows for **bidirectional translation** within  $\Phi$ .

Moreover,  $\mu(\text{DG}_s^r)$  is  $\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -**definable** over  $\text{DG}_{s+r}^1$ . ◀

*Proof.* We choose  $\mu$  to be the **linear encoding** that assigns to each  $s$ -bit **labeled**,  $r$ -relational **digraph**  $G$  the  $(s+r)$ -bit **labeled** (1-relational) **digraph**  $\mu(G) = H$  with **domain**  $[r+1] \times V^G$ , **labeling sets**  $P_i^H = \{1\} \times P_i^G$ , for  $1 \leq i \leq s$ , and  $P_{s+i}^H = \{i+1\} \times V^G$ , for  $1 \leq i \leq r$ , and **edge relation**

$$R^H = \bigcup_{1 \leq i \leq r} \left( \left\{ ((1, u), (i+1, u)) \mid u \in V^G \right\} \cup \left\{ ((i+1, u), (1, u)) \mid u \in V^G \right\} \cup \left\{ ((i+1, u), (i+1, u')) \mid (u, u') \in R_i^G \right\} \right).$$

That is, for each **node**  $u \in V^G$  and for  $1 \leq i \leq r$ , we introduce an additional **node** representing the “ $R_i$ -port” of  $u$ , and connect everything accordingly.

Our **forward translation**, from  $\Phi$  on  $\mathbf{DG}_s^r$  to  $\Phi$  on  $\mu(\mathbf{DG}_s^r)$ , is given by

$$\begin{aligned} \psi_{P_i} &= P_i \quad \text{for } 1 \leq i \leq s, \\ \psi_{R_i} &= \Diamond(\psi_{i+1} \wedge \Diamond\Diamond(\psi_1 \wedge Y)) \quad \text{for } 1 \leq i \leq r, \\ \psi_{R_i^{-1}} &= \Diamond(\psi_{i+1} \wedge \bar{\Diamond}\Diamond(\psi_1 \wedge Y)) \quad \text{for } 1 \leq i \leq r, \\ \psi_{\bullet} &= \Diamond(\psi_1 \wedge Y), \\ \psi_{\text{ini}} &= \psi_{\bullet}, \\ \text{where } \psi_1 &= \neg \bigvee_{1 \leq i \leq r} (\psi_{i+1}) \quad \text{("regular")}, \\ \psi_{i+1} &= P_{s+i} \quad \text{for } 1 \leq i \leq r \quad \text{("R}_i\text{-port")}. \end{aligned}$$

Our translation in the other direction, from  $\Phi$  on  $\mu(\mathbf{DG}_s^r)$  to  $\Phi$  on  $\mathbf{DG}_s^r$ , is given by

$$\begin{aligned} \varphi_{P_i}^{h+1} &= \begin{cases} P_i & \text{for } h = 0 \text{ and } 1 \leq i \leq s, \\ \perp & \text{for } h = 0 \text{ and } s+1 \leq i \leq s+r, \\ \top & \text{for } 1 \leq h \leq r \text{ and } i = s+h, \\ \perp & \text{for } 1 \leq h \leq r \text{ and } i \neq s+h, \end{cases} \\ \varphi_R^{h+1} &= \begin{cases} \bigvee_{1 \leq i \leq r} X^{i+1} & \text{for } h = 0, \\ X^1 \vee \Diamond X^{h+1} & \text{for } 1 \leq h \leq r, \end{cases} \\ \varphi_{R^{-1}}^{h+1} &= \begin{cases} \bigvee_{1 \leq i \leq r} X^{i+1} & \text{for } h = 0, \\ X^1 \vee \bar{\Diamond} X^{h+1} & \text{for } 1 \leq h \leq r, \end{cases} \\ \varphi_{\bullet}^{h+1} &= \Diamond(X^1 \vee \dots \vee X^{r+1}) \quad \text{for } 0 \leq h \leq r, \\ \varphi_{\text{ini}} &= \varphi_{\bullet}^1. \end{aligned}$$

We can characterize  $\mu(\mathbf{DG}_s^r)$  over the class  $\mathbf{DG}_{s+r}^1$  by the conjunction of the following  $\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -definable properties, using our helper **formulas**  $(\psi_i)_{1 \leq i \leq r+1}$  from the **forward translation**.

- A “port” that corresponds to a relation symbol  $R_i$  may not be associated with any other relation symbol  $R_j$ , nor be **labeled** with predicates  $(P_j)_{1 \leq j \leq s}$ .

$$\bigwedge_{1 \leq i \leq r} \Box \left( \psi_{i+1} \rightarrow \neg \bigvee_{1 \leq j \leq r, j \neq i} (\psi_{j+1}) \wedge \neg \bigvee_{1 \leq j \leq s} (P_j) \right)$$

- Every “regular **node**” is connected to its  $r$  different “ports”, and nothing else. The uniqueness of each “ $R_i$ -port” can be expressed by the  $[\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)]$ -formula  $\text{see1}(\psi_{i+1})$ , using the construction from **Example 2.2** in **Section 2.6**.

$$\Box \left( \psi_1 \rightarrow \neg \Diamond \psi_1 \wedge \bigwedge_{1 \leq i \leq r} \text{see1}(\psi_{i+1}) \right)$$

- Similarly, each “port” is connected to precisely one “regular **node**” and to an arbitrary number of “ports” of the same **relation symbol**, but not to any other ones.

$$\bigwedge_{1 \leq i \leq r} \Box \left( \psi_{i+1} \rightarrow \text{see1}(\psi_1) \wedge \neg \bigvee_{1 \leq j \leq r, j \neq i} \Diamond \psi_{j+1} \right)$$

- Finally, the links between “regular nodes” and “ports” have to be bidirectional: for each edge from a node of one type to a node of a different type, the corresponding inverse edge must also exist.

$$\bigwedge_{1 \leq i \leq r+1} \forall X \square (\psi_i \wedge X \rightarrow \square (\neg \psi_i \rightarrow \Diamond X))$$

Note that, in combination with the previous properties, this ensures that we have the same total number of nodes for each type  $i \in [1 : r + 1]$ . ■

### 6.5.3 Getting rid of vertex labels

Being able to eliminate multiple edge relations at the cost of additional labeling sets (see Proposition 6.15), our natural next step is to encode labeled digraphs into unlabeled ones.

#### Proposition 6.16.

► For all  $s \geq 1$  and  $\Phi \in \{\vec{ML}_g, \overleftarrow{ML}_g\}$ , there is a linear encoding  $\mu$  from  $DG_s^1$  into  $DG$  that allows for bidirectional translation within  $\Phi$ .

Moreover,  $\mu(DG_s^1)$  is  $\Pi_1^{MSO}(\vec{ML}_g)$ -definable over  $DG$ . ◀

*Proof.* We construct the linear encoding  $\mu$  that assigns to each  $s$ -bit labeled digraph  $G$  the (unlabeled) digraph  $\mu(G) = H$  with domain  $(\{1\} \times V^G) \cup [2 : s + 3]$  and edge relation

$$\begin{aligned} R^H = & \{((1, u), (1, u')) \mid (u, u') \in R^G\} \\ & \cup \{((1, u), 3) \mid u \in V^G\} \\ & \cup \bigcup_{1 \leq i \leq s} \{((1, u), i + 3) \mid u \in P_i^G\} \\ & \cup \{(i + 3, i + 2) \mid 1 \leq i \leq s\} \\ & \cup \{(i + 3, 2) \mid 0 \leq i \leq s\}. \end{aligned}$$

The idea is to introduce a gadget that contains a separate node for each labeling set of the original digraph, and then connect the “regular nodes” to this gadget in a way that corresponds to their respective labeling. The gadget is easily identifiable because it contains the only node in the digraph that has no outgoing edge (namely, node 2). We ensure this by connecting all the “regular nodes” to node 3.

Our forward translation, from  $\Phi$  on  $DG_s^1$  to  $\Phi$  on  $\mu(DG_s^1)$ , is given by

$$\begin{aligned} \psi_{P_i} &= \Diamond \psi_{i+3} \quad \text{for } 1 \leq i \leq s, \\ \psi_R &= \Diamond (\psi_1 \wedge Y), \\ \psi_{R^{-1}} &= \overleftarrow{\Diamond} Y, \\ \psi_\bullet &= \Diamond (\psi_1 \wedge Y), \\ \psi_{ini} &= \psi_\bullet, \end{aligned}$$

where  $\psi_1 = \neg \bigvee_{2 \leq i \leq s+3} (\psi_i)$ ,

$$\psi_2 = \square \perp,$$

$$\psi_3 = \Diamond \psi_2 \wedge \square \psi_2,$$

$$\psi_{i+3} = \Diamond \psi_2 \wedge \Diamond \psi_{i+2} \wedge \square (\psi_2 \vee \psi_{i+2}) \quad \text{for } 1 \leq i \leq s.$$

Our translation in the other direction, from  $\Phi$  on  $\mu(\mathbf{DG}_s^1)$  to  $\Phi$  on  $\mathbf{DG}_s^1$ , is given by

$$\begin{aligned} \varphi_R^h &= \begin{cases} \Diamond X^1 \vee X^3 \vee \bigvee_{1 \leq i \leq s} (P_i \wedge X^{i+3}) & \text{for } h = 1, \\ \perp & \text{for } h = 2, \\ X^2 & \text{for } h = 3, \\ X^2 \vee X^{h-1} & \text{for } 4 \leq h \leq s+3, \end{cases} \\ \varphi_{R^{-1}}^h &= \begin{cases} \overline{\Diamond} X^1 & \text{for } h = 1, \\ \bigvee_{0 \leq i \leq s} X^{i+3} & \text{for } h = 2, \\ \Diamond X^1 \vee X^{h+1} & \text{for } h = 3, \\ \Diamond (P_{h-3} \wedge X^1) \vee X^{h+1} & \text{for } 4 \leq h \leq s+2, \\ \Diamond (P_{h-3} \wedge X^1) & \text{for } h = s+3, \end{cases} \\ \varphi_\bullet^h &= \Diamond (X^1 \vee \dots \vee X^{s+3}) \quad \text{for } 1 \leq h \leq s+3, \\ \varphi_{\text{ini}} &= \varphi_\bullet^1. \end{aligned}$$

Using the helper formulas  $(\psi_i)_{1 \leq i \leq s+3}$  from the forward translation, we can characterize  $\mu(\mathbf{DG}_s^1)$  over  $\mathbf{DG}$  as

$$\Diamond \psi_1 \wedge \bigwedge_{2 \leq i \leq s+3} \text{tot1}(\psi_i) \wedge \Box (\psi_1 \rightarrow \Diamond \psi_3 \wedge \neg \Diamond \psi_2).$$

Here, each  $[\Pi_1^{\text{MSO}}(\vec{\mathbf{ML}}_g)]$ -subformula  $\text{tot1}(\psi_i)$  is obtained through the singleton construction from Example 2.2 in Section 2.6. ■

### 6.5.4 Getting rid of backward modalities

For the sake of completeness, we also describe the encoding that lets us simulate backward modalities by means of an additional edge relation.

**Proposition 6.17.**

- There is a linear encoding  $\mu$  from  $\mathbf{DG}$  into  $\mathbf{DG}_0^2$  that allows for bidirectional translation between  $\vec{\mathbf{ML}}_g$  and  $\vec{\mathbf{ML}}_g$ .  
Moreover,  $\mu(\mathbf{DG})$  is  $\Pi_1^{\text{MSO}}(\vec{\mathbf{ML}}_g)$ -definable over  $\mathbf{DG}_0^2$ . ◀

*Proof.* The encoding is straightforward: to each digraph  $G$ , we assign a copy  $\mu(G) = H$  that is enriched with a second edge relation, which coincides with the inverse of the first. Formally,  $V^H = \{1\} \times V^G$ ,

$$\begin{aligned} R_1^H &= \{((1, u), (1, u')) \mid (u, u') \in R^G\}, \quad \text{and} \\ R_2^H &= \{(v', v) \mid (v, v') \in R_1^H\}. \end{aligned}$$

With this, in order to translate between  $\vec{\mathbf{ML}}_g$  on  $\mathbf{DG}$  and  $\vec{\mathbf{ML}}_g$  on  $\mu(\mathbf{DG})$ , we merely have to replace backward modalities by  $R_2$ -modalities, and vice versa. Hence, when we fix our forward translation, we choose  $\psi_R = \Diamond Y$  and  $\psi_{R^{-1}} = \Diamond Y$ , and for the backward translation we set  $\varphi_{R_1}^1 = \Diamond X^1$  and  $\varphi_{R_2}^1 = \overline{\Diamond} X^1$ .

To define  $\mu(\mathbf{DG})$  over  $\mathbf{DG}_0^2$ , we can use the following  $\Pi_1^{\text{MSO}}(\vec{\mathbf{ML}}_g)$ -formula:

$$\forall X \Box (X \rightarrow \boxed{1} \Diamond X \wedge \boxed{2} \Diamond X)$$

■

### 6.5.5 Getting rid of directed edges

In order to encode a **digraph** into an **undirected graph**, we proceed in a similar manner to the elimination of multiple **edge relations** in Proposition 6.15. Using an ad-hoc trick, we can do this by introducing only one additional **labeling set**.

**Proposition 6.18.**

► There is a **linear encoding**  $\mu$  from **DG** into  $\text{GRAPH}_1^1$  that allows for **bidirectional translation** between  $\vec{\text{ML}}_g$  and  $\vec{\text{ML}}_g$ .

Moreover,  $\mu(\text{DG})$  is  $\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)$ -definable over  $\text{GRAPH}_1^1$ . ◀

*Proof.* A suitable choice for  $\mu$  is to take the function that assigns to every **digraph**  $G$  the 1-bit labeled **undirected graph**  $\mu(G) = H$  with **domain**  $([3] \times V^G) \cup [4:6]$ , **labeling set**  $P^H = [4:6]$ , and **edge relation**  $R^H = \{(v, v') \mid \{v, v'\} \in E^H\}$ , where

$$\begin{aligned} E^H = & \{ \{(1, u), (2, u)\} \mid u \in V^G \} \\ & \cup \{ \{(1, u), (3, u)\} \mid u \in V^G \} \\ & \cup \{ \{(2, u), (3, u')\} \mid (u, u') \in R^G \} \\ & \cup \{ \{(2, u), 4\} \mid u \in V^G \} \\ & \cup \{ \{(3, u), 5\} \mid u \in V^G \} \\ & \cup \{ \{5, 6\} \}. \end{aligned}$$

The idea is that we connect each original **node**  $u \in V^G$  to two new **nodes**, which represent the “outgoing port” and “incoming port” of  $u$ , and use undirected **edges** between “ports” to simulate directed **edges** between “regular **nodes**”. In order to distinguish the different types of **nodes**, we connect them in different ways to the additional  $P$ -labeled **nodes**.

Our **forward translation**, from  $\vec{\text{ML}}_g$  on **DG** to  $\vec{\text{ML}}_g$  on  $\mu(\text{DG})$ , is given by

$$\begin{aligned} \psi_R &= \Diamond(\psi_2 \wedge \Diamond(\psi_1 \wedge Y)), \\ \psi_{R^{-1}} &= \Diamond(\psi_3 \wedge \Diamond(\psi_1 \wedge Y)), \\ \psi_\bullet &= \Diamond(\psi_1 \wedge Y), \\ \psi_{\text{ini}} &= \psi_\bullet, \end{aligned}$$

$$\begin{aligned} \text{where } \psi_1 &= \neg(\psi_2 \vee \dots \vee \psi_6) \quad (\text{“regular”}), \\ \psi_2 &= \Diamond\psi_4 \quad (\text{“outgoing”}), \\ \psi_3 &= \neg P \wedge \Diamond\psi_5 \quad (\text{“incoming”}), \\ \psi_4 &= P \wedge \neg\Diamond P \wedge \Diamond\neg P, \\ \psi_5 &= P \wedge \Diamond P \wedge \Diamond\neg P, \\ \psi_6 &= P \wedge \Diamond P \wedge \neg\Diamond\neg P. \end{aligned}$$

Our **backward translation**, from  $\vec{\text{ML}}_g$  on  $\mu(\text{DG})$  to  $\vec{\text{ML}}_g$  on **DG**, is given by

$$\varphi_P^h = \begin{cases} \perp & \text{for } 1 \leq h \leq 3, \\ \top & \text{for } 4 \leq h \leq 6, \end{cases}$$

$$\varphi_R^h = \begin{cases} X^2 \vee X^3 & \text{for } h = 1, \\ X^1 \vee \Diamond X^3 \vee X^4 & \text{for } h = 2, \\ X^1 \vee \Diamond X^2 \vee X^5 & \text{for } h = 3, \\ \Diamond X^2 & \text{for } h = 4, \\ \Diamond X^3 \vee X^6 & \text{for } h = 5, \\ X^5 & \text{for } h = 6, \\ \varphi_\bullet^h = \Diamond(X^1 \vee \dots \vee X^6) & \text{for } 1 \leq h \leq 6, \\ \varphi_{\text{ini}} = \varphi_\bullet^1. \end{cases}$$

We can define  $\mu(\text{DG})$  over  $\text{GRAPH}_1^1$  with the following  $[\Pi_1^{\text{MSO}}(\vec{\text{ML}}_g)]$ -formula. It makes use of our helper formulas  $(\psi_i)_{1 \leq i \leq 6}$  from the forward translation and the constructions  $\text{see1}(\psi_i)$  and  $\text{tot1}(\psi_i)$  from Example 2.2 in Section 2.6.

$$\begin{aligned} & \bigwedge_{4 \leq i \leq 6} \text{tot1}(\psi_i) \wedge \\ & \Box(\psi_2 \rightarrow \text{see1}(\psi_1) \wedge \Box(\psi_1 \vee \psi_3 \vee \psi_4)) \wedge \\ & \Box(\psi_3 \rightarrow \text{see1}(\psi_1) \wedge \Box(\psi_1 \vee \psi_2 \vee \psi_5)) \wedge \\ & \Box(\psi_1 \rightarrow \text{see1}(\psi_2) \wedge \text{see1}(\psi_3) \wedge \Box(\psi_2 \vee \psi_3)) \end{aligned}$$

The first line states that the three P-labeled nodes are unique, which forces 5 and 6 to be connected. The remaining lines ensure that each “port” is connected to exactly one “regular node”, and, conversely, that every “regular node” is linked to precisely one “outgoing port” and one “incoming port”. As a consequence, the number of “regular nodes” must be the same as the number of “ports” of each type. Furthermore, the formula restricts the types of neighbors each node can have, while the usage of the helper formulas  $\psi_2$  and  $\psi_3$  makes sure that the required connections to the P-labeled nodes are established. Finally, the fact that  $\psi_1$  characterizes the “regular nodes” as the “remaining ones” guarantees that there are no unaccounted-for nodes. ■

### 6.5.6 Getting rid of global modalities

Our last encoding function lets us simulate global modalities by inserting a new node that is bidirectionally connected to all the “regular nodes”.

#### Proposition 6.19.

► There is a linear encoding  $\mu$  from  $\text{DG}$  into  $@\text{DG}$  that allows for bidirectional translation between  $\vec{\text{ML}}_g$  and  $\vec{\text{ML}}$ .

Furthermore,  $\mu$  can be easily adapted into a linear encoding  $\mu'$  from  $\text{DG}$  into  $\text{DG}$  that satisfies the following figurative inclusions, for arbitrary  $\ell \geq 2$ :

$$\begin{aligned} \llbracket \Sigma_\ell^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{DG}} & \subseteq_{\mu'} \llbracket \Box \Sigma_\ell^{\text{MSO}}(\vec{\text{ML}}) \rrbracket_{\text{DG}}, \\ \llbracket \Pi_\ell^{\text{MSO}}(\vec{\text{ML}}_g) \rrbracket_{\text{DG}} & \supseteq_{\mu'} \llbracket \Box \Pi_\ell^{\text{MSO}}(\vec{\text{ML}}) \rrbracket_{\text{DG}}. \end{aligned}$$

*Proof.* We choose  $\mu$  to be the linear encoding that maps each digraph  $G$  to the pointed digraph  $\mu(G) = H$  with domain  $(\{1\} \times V^G) \cup [2:3]$ , position  $@^H = 2$ , and



## edge relation

$$\begin{aligned}
R^H = & \{((1, u), (1, u')) \mid (u, u') \in R^G\} \\
& \cup \{((1, u), 2) \mid u \in V^G\} \\
& \cup \{(2, (1, u)) \mid u \in V^G\} \\
& \cup \{(2, 3)\}.
\end{aligned}$$

One can distinguish **node** 2 from the others because it is connected to 3, which is the only **node** without any outgoing **edge**.

Our **forward translation**, from  $\vec{M}\vec{L}_g$  on  $\mathbf{DG}$  to  $\vec{M}\vec{L}$  on  $\mu(\mathbf{DG})$ , is given by

$$\begin{aligned}
\psi_R &= \Diamond(\psi_1 \wedge Y), \\
\psi_\bullet &= \Diamond(\psi_2 \wedge \Diamond(\psi_1 \wedge Y)), \\
\psi_{\text{ini}} &= \Diamond(\psi_1 \wedge Y), \\
\text{where } \psi_1 &= \Diamond\Diamond\Box\perp \quad \text{and} \quad \psi_2 = \Diamond\Box\perp.
\end{aligned}$$

Our **backward translation**, from  $\vec{M}\vec{L}$  on  $\mu(\mathbf{DG})$  to  $\vec{M}\vec{L}_g$  on  $\mathbf{DG}$ , is given by

$$\begin{aligned}
\varphi_R^h &= \begin{cases} \Diamond X^1 \vee X^2 & \text{for } h = 1, \\ \Diamond X^1 \vee X^3 & \text{for } h = 2, \\ \perp & \text{for } h = 3, \end{cases} \\
\varphi_{\text{ini}} &= \Diamond X^2.
\end{aligned}$$

Turning to the second claim of the proposition, we obtain  $\mu'(G)$  by simply removing the position marker from  $\mu(G)$ , i.e., for every **digraph**  $G$ ,  $\mu'(G)$  is such that  $\mu'(G)[@ \mapsto 2] = \mu(G)$ .

For the **forward figurative inclusions**, let  $\Xi \in \{\Sigma_\ell^{\text{MSO}}, \Pi_\ell^{\text{MSO}}\}$ , for some arbitrary  $\ell \geq 0$ . By applying **Lemma 6.14 a** on  $\mu$ , we get that for every  $\Xi(\vec{M}\vec{L}_g)$ -sentence  $\varphi$  over  $\{R\}$ , there is a  $\Xi(\vec{M}\vec{L})$ -sentence  $\psi_\varphi$  over  $\{ @, R \}$  such that, for all  $G \in \mathbf{DG}$ ,

$$\begin{aligned}
G \models \varphi & \quad \text{iff} \quad \mu'(G)[@ \mapsto 2] \models \psi_\varphi, \\
& \quad \text{iff} \quad \mu'(G) \models \Box(\psi_2 \rightarrow \psi_\varphi).
\end{aligned}$$

Hence,  $\llbracket \Xi(\vec{M}\vec{L}_g) \rrbracket_{\mathbf{DG}} \subseteq_{\mu'} \llbracket \Box \Xi(\vec{M}\vec{L}) \rrbracket_{\mathbf{DG}}$ .

For the **backward figurative inclusion**, we require that  $\ell \geq 2$ . Slightly adapting the proof of **Lemma 6.14 b** to discard the part where we make use of the formula  $\varphi_{\text{ini}}$  from **Definition 6.13 e** (incidentally allowing us to merge the two consecutive induction proofs), it is easy to show the following: Given  $h \in [3]$  and any  $\Pi_\ell^{\text{MSO}}(\vec{M}\vec{L})$ -sentence  $\psi$  over  $\{ @, R \}$ , we can construct a  $\Pi_\ell^{\text{MSO}}(\vec{M}\vec{L}_g)$ -sentence  $\varphi_\psi^h$  over  $\{ @, R \}$  such that, for all  $G \in \mathbf{DG}$  and  $u \in V^G$ ,

$$\begin{aligned}
G[@ \mapsto u] \models \varphi_\psi^h & \quad \text{iff} \quad \mu'(G)[@ \mapsto v] \models \psi, \\
& \quad \text{where } v \text{ is } (h, u) \text{ if } h = 1, \text{ and } h \text{ otherwise.}
\end{aligned}$$

This immediately gives us a way of translating  $\Box\psi$ :

$$G \models \Box(\varphi_\psi^1 \wedge \varphi_\psi^2 \wedge \varphi_\psi^3) \quad \text{iff} \quad \mu'(G) \models \Box\psi.$$

The left-hand side **sentence** can be transformed into **prenex normal form** by simulating the **global box** with a universal **set quantifier**. Checking that a given set is *not* a singleton can be done in  $\Sigma_1^{\text{MSO}}(\vec{M}\vec{L}_g)$ , since the negation is  $\Pi_1^{\text{MSO}}(\vec{M}\vec{L}_g)$ -expressible (see Example 2.2 in Section 2.6). Thus, the given **formula** is equivalent to a  $\Pi_\ell^{\text{MSO}}(\vec{M}\vec{L}_g)$ -formula, and we obtain that  $\llbracket \Pi_\ell^{\text{MSO}}(\vec{M}\vec{L}_g) \rrbracket_{\text{DG}} \preceq_{\mu'} \llbracket \Box \Pi_\ell^{\text{MSO}}(\vec{M}\vec{L}) \rrbracket_{\text{DG}}$ . ■

# 7

## Perspectives

Coming to the end of this thesis, we discuss some ideas for future research. They can be separated into two categories: rather focused questions that directly follow up on the results presented here, and broader questions that aim at the bigger picture.

### 7.1 Focused questions

Let us start with the topics directly related to this work, following roughly the order of discussion in the document.

#### 7.1.1 Is there an alternation level that covers first-order logic?

In Chapter 3, we have related the classes of digraph languages recognizable by our three flavors of  $\text{LDA}_g$ 's to those definable in  $\text{MSOL}$ ,  $\text{EMSOL}$  and  $\text{FOL}$ . As shown in Figure 3.6 on page 33,  $\text{ALDA}_g$ 's cover  $\text{FOL}$  (as a direct consequence of their equivalence to  $\text{MSOL}$ ), whereas  $\text{NLDA}_g$ 's do not. It is also easy to see that every  $\text{NLDA}_g$  is equivalent to an  $\text{NLDA}_g$  of length 1, since each node can simply guess all of its nondeterministic transitions at once, and then verify in one round of communication that its own choices are consistent with those of its neighbors. Furthermore, we know from Chapter 6 (Theorem 6.2 on page 66) that  $\text{ALDA}_g$ 's of length  $\ell + 1$  are strictly more expressive than  $\text{ALDA}_g$ 's of length  $\ell$  (or equivalently, that the set quantifier alternation hierarchy of  $\text{MSO}(\tilde{\text{ML}}_g)$  is strict over digraphs). This means that length-restricted  $\text{ALDA}_g$ 's form an infinite hierarchy of automata classes between  $\text{NLDA}_g$ 's and  $\text{ALDA}_g$ 's.

Against this backdrop, a natural question is whether there exists a bound  $\ell$  such that  $\text{ALDA}_g$ 's of length  $\ell$  can recognize all languages definable in  $\text{FOL}$  on arbitrary digraphs. Note that this would also imply that  $\text{ALDA}_g$ 's of length  $\ell + 1$  fully cover  $\text{EMSOL}$ .

When restricted to digraphs of bounded degree, the answer is positive. This can be seen using Hanf's locality theorem, which basically states that on digraphs of bounded degree, every  $\text{FOL}$ -formula is equivalent to a Boolean combination of conditions of the form “ $r$ -sphere  $H$  occurs at least  $n$  times”, where an  $r$ -sphere is a pointed digraph that represents the  $r$ -neighborhood of its distinguished node (see, e.g., [Tho96, Thm 4.1] or [Lib04, Thm 4.24]). Based on this characterization, it is

relatively easy to show that any **FOL-formula** can be translated to an **ALDA<sub>g</sub>** of length 3. However, for **digraphs** of unbounded degree, the author does not know the answer to the above question.

### 7.1.2 Does asynchrony entail quasi-acyclicity?

As already mentioned in Section 4.2, we make crucial use of **quasi-acyclicity** to prove the **equivalence** of **a-QDA**'s and the **backward  $\mu$ -fragment** (i.e., Theorem 4.2 on page 41). It is however open whether we really need to impose this condition on our **asynchronous automata** in order to be able to convert them into **formulas** of  $\Sigma_1^H(\tilde{ML})$ . **Asynchrony** is a very strong requirement, and it might well be the case that every **asynchronous automaton** is in fact **equivalent** to a **quasi-acyclic** one. Moreover, if this assumption turned out to be true, it would be interesting to know if it extends to **lossless-asynchronous automata**.

### 7.1.3 Is asynchrony decidable?

Another natural question concerning **asynchrony** is whether there exists an algorithm that decides if a given **distributed automaton** is **asynchronous**, or alternatively, if it is **lossless-asynchronous**. Even though we can effectively translate from **quasi-acyclic (lossless-)asynchronous automata** to the **backward  $\mu$ -fragment** (see Proposition 4.6), our translation procedure relies on the guarantee that the given **automaton** is indeed an **la-QDA**. From a practical perspective, it would be advantageous if the procedure could also check that its input is valid. While **quasi-acyclicity** can be easily verified, **(lossless-)asynchrony** seems to present a more challenging problem.

### 7.1.4 Are forgetful automata useful as tree automata?

In Chapter 5, we have seen that **forgetful distributed automata** are strictly more expressive than classical **tree automata** on **ordered ditrees** (Proposition 5.3 on page 52). Moreover, their **emptiness problem** is decidable on arbitrary **digraphs** (Theorem 5.4 on page 53), and since all **distributed automata** satisfy a tree-model property (see Lemma 5.6 on page 56), it is straightforward to adapt our decision procedure to the special case of **ordered ditrees**.

This begs the question whether **forgetful automata** could be of use in typical application areas of **tree automata**, such as program verification and processing of XML-like data. A first step towards an answer would be to investigate their closure properties (which are probably not as nice as those of **tree automata**) and to precisely analyze the complexities of their decision problems. Indeed, the LOGSPACE complexity of the **emptiness problem** (stated in Theorem 5.4) has to be revised for the case of **ordered ditrees** because **distributed automata** that operate exclusively on such **structures** do not have to deal with sets of **states** and thus can be represented more compactly; this leads to higher computational complexity.

### 7.1.5 How powerful are quasi-acyclic automata on dipaths?

We have presented two different constructions to prove the undecidability of the **emptiness problem** for **distributed automata**. The first (Theorem 5.5 on page 54) uses the idea of exchanging space and time to simulate a Turing machine by a **distributed automaton** that runs on a **dipath**. This simulation shows that even the

**dipath-emptiness problem** is undecidable, but it works only if the **state** diagram of the simulating **automaton** may contain cycles. Our second approach (**Theorem 5.9** on page 57) shows that also for **quasi-acyclic automata** the **emptiness problem** is undecidable, but the construction is much more technical and does not work if we restrict ourselves to **dipaths**.

It is part of ongoing work to establish a precise characterization of **quasi-acyclic automata** on **dipaths** in terms of counter machines. As a corollary, this will yield a stronger undecidability result that supersedes the two previous ones.

## 7.2 Broader questions

To conclude, let us expand our focus by suggesting possible extensions and asking how the present work fits into the wider landscape of **graph** automata and distributed computing.

### 7.2.1 What about distributed automata on infinite digraphs?

Although in this thesis we have considered only finite **structures**, this restriction is by no means necessary; **distributed automata** could also run on infinite **digraphs**, and this would not even require changing their definition. It is straightforward to see that the **equivalence** of **ALDA<sub>g</sub>**'s and **MSOL** established in **Chapter 3** immediately extends to the infinite setting (simply by verifying that the given proofs remain applicable). However, this is not the case for all the results presented here. In particular in **Chapter 4**, we have relied on the fact that our **digraphs** are finite to prove the **equivalence** of **quasi-acyclic asynchronous automata** and the **backward  $\mu$ -fragment** (see the proof of **Proposition 4.3** on page 42). It seems that a more powerful **acceptance condition** would be required in order to get a corresponding **equivalence** on infinite **digraphs**.

For future research on **distributed automata**, it would be worthwhile to systematically consider both the finite and the infinite case.

### 7.2.2 What is the overlap with cellular automata?

Obviously, **distributed automata** are closely related to cellular automata. The only noteworthy difference is that **distributed automata** can operate on arbitrary **digraphs**, whereas cellular automata are usually confined to regular **structures**, such as (doubly linked) **grids** or **dipaths**. That is, if we restrict ourselves to the appropriate classes of **digraphs**, then the two models are exactly the same. Furthermore, there is a branch of research concerned with cellular automata as **language** acceptors (see, e.g., [Kut08, § 6.5] or [Ter12]). In order not to “reinvent the wheel”, it is thus important to relate questions arising in the study of **distributed automata** to the existing body of knowledge in cellular automata theory.

An example where this was not done thoroughly enough can be found in **Section 5.4**. As the author has been recently informed by N. Bacquey (on 20 October 2017), the idea of exchanging space and time is well-known within the community of cellular automata. It is for instance documented in [Ter12, Fig. 9], where it is employed to simulate a real-time one-dimensional two-way cellular automaton by a corresponding one-way automaton. Although the presentation and purpose differ considerably from those in the present work, the technical construction is essentially the same.

### 7.2.3 Can we characterize more powerful models?

As mentioned at the beginning of [Chapter 1](#), the original motivation for this thesis was to work toward a descriptive complexity theory for distributed computing. By focusing on [distributed automata](#), we were able to make some progress in that direction, but the main challenge remains to establish logical characterizations of stronger models of computation, powerful enough to cover the kinds of algorithms usually considered in distributed computing. In order to be of practical interest, such a characterization should be in terms of *finite formulas*, just like the one provided by Fagin’s theorem for nondeterministic polynomial-time Turing machines.

There are several ideas “in the air” on how one might characterize distributed finite-state machines equipped with unique identifiers, or even distributed Turing machines subject to certain time and space constraints. As of the time of writing, the author is not aware of any fully developed solution, but new results should be expected in the next few years.

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