On the existence for the Plateau problem in finite dimensional Banach spaces.

Par

Ioann VASILYEV

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Dirigée par:

Professeur Thierry DE PAUW.

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devant le jury composée de :

Thierry DE PAUW  Prof., Univ. Paris Diderot  Directeur de thèse
Hervé PAJOT  Prof., Univ. de Grenoble 1  Rapporteur
Robert HARDT  Prof., Univ. of Rice  Rapporteur
Guy DAVID  Prof., Univ. Paris Saclay  Président du jury
Olivier GUÉDON  Prof., Univ. Paris-Est Mar.-la-Val.  Examineur
Frédéric HÉLEIN  Prof., Univ. Paris Diderot  Examineur
Vincent MILLOT  Mdc., Univ. Paris Diderot  Examineur

1Rapporteur non présent le jour de la soutenance.
À ma famille.
Résumé

Dans cette thèse, nous nous intéressons à l'existence de minimiseurs pour le problème de Plateau. On se place dans le cas des espaces de Banach de dimension finie et dans le contexte des G chaînes rectifiables. Nous commençons par améliorer l'un des théorèmes de Busemann, selon lequel chaque hyperplan dans un espace de Banach de dimension finie admet une projection qu'est une contraction pour la mesure d'Hausdorff des sous-ensembles \( n-1 \) rectifiables (on les appelle ”good projections”). Nous utilisons cette propriété pour montrer que ces projections n’augmentent pas la masse des chaînes rectifiables. On en déduit (en utilisant le théorème de l’approximation forte) la demi-continuité inférieure de la masse sur l’espace des G chaînes rectifiables. Cela nous permet d’établir l’existence mentionnée ci-dessus. En plus, nous avons pu prouver que, dans le cadre du problème énoncé ci-dessus, il y a toujours une solution dont le support soit à l’intérieur de la combinaison convexe du bord. Nous avons également établi une autre démonstration de l’existence des G chaînes minimisantes. Il s’agit des résultats qui disent que la mesure d’Hausdorff est égale à la mesure de Gross et que les mesures de Gross sont semi-continues inférieures. On applique un théorème de Burago et Ivanov pour démontrer l’inégalité triangulaire pour des G cycles polyédriques de dimension deux. Enfin, on l’utilise pour démontrer l’existence des G chaînes rectifiables minimisantes de dimension deux.

Mots clefs.
Problème de Plateau, Théorie géométrique de la mesure, Calcul des variations.
Abstract

In this work we study existence of minimisers for the Plateau problem in case of finite dimensional Banach spaces. We work in the context of polyhedral and rectifiable G chains.

We start with improving one theorem by Busemann, which states that each hyperplane in a finite dimensional Banach space admits area non-increasing projections (which by the way we call good projections). Namely, we prove that these projections do not increase Hausdorff measure of arbitrary (n - 1) rectifiable subsets. We use this property in order to show that those good projections do not increase the mass (the one relative to the Hausdorff measure) of (n - 1) rectifiable G chains. From here we derive the lower semi-continuity of the mass on the space of rectifiable G chains (via using the strong approximation theorem). This in turn gives us the desired existence result. Furthermore, we were able to prove that in the problem stated above there always exists a solution with the support inside of the convex hull of the boundary data.

Along with the “classical” proof discussed above we have also established another proof of the existence of mass-minimizing (n - 1) rectifiable G chains in finite dimensional Banach spaces. Namely, we have proved that for rectifiable subsets of finite dimensional Banach spaces the Hausdorff and the Gross measures coincide. Since those Gross measures are lower semicontinuous, this provides us with an alternative proof of the existence.

We apply one theorem of Burago and Ivanov in our proof of the corresponding triangle inequality for the polyhedral 2-cycles. Finally, we show the existence of mass minimizing 2 rectifiable G chains in finite dimensional Banach spaces.

Key words.
Plateau problem, Geometric measure theory, Calculus of variations.
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Chapter 1

Introduction (français).

1.1 L’histoire.

Résoudre le problème de Plateau c’est une question très célèbre de théorie de la mesure géométrique et du calcul des variations. Ce problème consiste à trouver une surface dont le bord coïncide avec un contour fixé et qui soit d’aire minimale. Nous soulignons que les mots “surface”, “contour” et “aire” utilisés dans la phrase précédente nécessitent une explication. En effet, il y a plusieurs façons possibles d’énoncer le problème de Plateau. Dans le cadre des courants le problème de Plateau est formulé de la manière suivante. Soit $B$ un courant $(m-1)$ dimensionnel intégral et rectifiable dont le support soit compact dans $\mathbb{R}^n$ et tel que $\partial B = 0$. Le but est de trouver un courant intégral et rectifiable de dimension $m$ tel que $\partial S = B$ et tel que $M(S) \leq M(T)$ pour tout les courants intégraux rectifiables et qui satisfont $\partial T = B$.

Il y a une autre formulation possible du problème de Plateau. Pour $m \in [0, \ldots, n]$ soit $\mathcal{R}_m(l_2^n, G)$ le groupe de toutes les chaînes $m$ rectifiables dans l’espace euclidien $\mathbb{R}^n$ avec des coefficients dans un groupe abélien normé et complet $(G, |\cdot|)$. Le groupe $\mathcal{R}_m(l_2^n, G)$ est équipé de la masse $M$, celle associée à la mesure de Hausdorff euclidienne. Le problème de Plateau est le problème suivant

\[
\begin{aligned}
\text{minimiser } & M(T) \\
\text{parmi } & T \in \mathcal{R}_m(l_2^n, G) \text{ tel que } \partial T = B,
\end{aligned}
\]

où $B \in \mathcal{R}_{m-1}(l_2^n, G)$, $\text{spt}(B)$ est compact et $\partial B = 0$.

Grâce aux méthodes de la théorie des courants normaux et intégraux et des chaînes à coefficients dans les groupes, pour le cas des espaces euclidiens, le problème de Plateau admet au moins une solution. Présentons d’abord une description plus détaillée de la situation dans le cadre euclidien. L’une des méthodes possibles pour démontrer l’existence est la méthode directe du calcul des variations. Elle consiste en les deux étapes suivantes: la première est la semi-continuité inférieure de la masse par rapport à une topologie bien choisie et la seconde est la compacité de l’espace des “surfaces” admissibles par rapport à
la même topologie. Les topologies les plus fréquemment utilisées sont la topologie plate et la topologie faible étoile.

Il existe plusieurs preuves différentes de la semi-continuité inférieure de la masse dans les espaces euclidiens. La première méthode ne fonctionne que dans le cadre des courants. Plus précisément, l’étape principale de cette méthode c’est de démontrer la formule suivante:

$$M(T) = \sup\{T(\omega) : \omega \in D^m(\mathbb{R}^n), ||\omega||^* \leq 1\},$$

où $||\ldots||^*$ est la comasse, $||\omega||^* = \sup\{||\langle \xi, \omega \rangle|| : \xi$ est un $m$ vecteur simple unitaire}. On note qu’une fois que cette formule est vérifiée, la semi-continuité inférieure par rapport à la topologie faible étoile suit facilement, car le suprémum de fonctions continues est toujours semi-continue inférieurement.

La seconde méthode est basée sur l’observation suivante. Soit $W \in G(m, \mathbb{R}^n)$ et soit $\pi : \mathbb{R}^n \rightarrow W$ une projection orthogonale sur $W$. Alors Lip($\pi$) $\leq 1$ et donc $\pi$ n’augmente pas les mesures Hausdorff des ensembles rectifiables. On en déduit le fait suivant qui ressemble à l’inégalité triangulaire. Soit $P$ un cycle $m$ polyédrique (i.e. $\partial P = 0$) tel que $P = \sum_{j=1}^N g_j[\sigma_j]$, où $\sigma_j$ sont $m$ sont des simplexes qui ne se chevauchent pas. Alors on a que

$$|g_1|\mathcal{H}^m(\sigma_1) \leq \sum_{j=2}^N |g_j|\mathcal{H}^m(\sigma_j).$$

Cette inégalité découle du théorème de constance. Ensuite, en utilisant les théorèmes d’approximation forte et faible, on peut en déduire la semi-continuité inférieure souhaitée.

La formule de Crofton fournit encore une méthode applicable pour démontrer que la masse est semi-continue inférieurement. Plus précisément, supposons que $A \subset \mathbb{R}^n$ soit un ensemble $m$-rectifiable. Alors, la formule suivante appelée la formule de Crofton (ou la formule intégrale-géométrique) a lieu:

$$\mathcal{H}^m(A) = T_1^m(A) = \int_{G(m,\mathbb{R}^n)} d\gamma_m(W) \int_W \text{card}(\pi_W^{-1}(y) \cap A) dy,$$

où $\pi_W$ est la projection orthogonale sur $W$ et $\gamma_m$ est une mesure $O(m)$ invariante sur la Grassmannienne $G(m, \mathbb{R}^n)$ “bien normalisée”. Grâce à cette formule, on est en mesure de prouver la semi-continuité inférieure de la manière suivante : tout d’abord, on obtient facilement que la masse de Hausdorff $\mathcal{M}$ coïncide avec la masse intégrale-géométrique $\mathcal{M}_\mathcal{I}$ (c’est-à-dire avec celle définie par rapport à la mesure intégrale-géométrique) sur l’espace des chaînes $m$-rectifiable $G$ et que la formule suivante est vraie:

$$\mathcal{M}(T) = \mathcal{M}_\mathcal{I}(T) = \int_{G(m,\mathbb{R}^n)} d\gamma_m(W) \int_W \mathcal{M}(T, \pi_W, y) dy.$$
\[ \mathcal{R}^m(l_2^n, G), \text{ alors les morceaux de dimension zéro satisfont} \]
\[ \int \mathcal{F}(\langle T_i - T, \pi_W, y \rangle) dy \leq \mathcal{F}(T_i - T) \rightarrow 0, \]
où l’inégalité ici découle du fait que \( \pi_W \) à une constante de Lipschitz égale à 1. Donc (peut-être en prenant une sous-suite) on aura que \( \mathcal{F}(\langle T_i - T, \pi_W, y \rangle) \rightarrow 0 \) pour presque tout \( y \). Enfin, le lemme de Fatou et l’observation que la masse est semi-continue inférieurement sur les morceaux de dimension 0 nous permettent de conclure que \( \mathcal{M}_T \) est semi-continue inférieurement.

En ce qui concerne la compacité, l’un des moyens les plus simples de la démontrer consiste en les deux étapes suivantes. La première utilise également que les projections sur les ensembles convexes sont 1-Lipschitziennes, c’est à dire la propriété traitée dans les deux derniers paragraphes. Comme ces projecteurs n’augmentent pas la masse, on peut simplement projeter la suite minimisante sur une boule contenant \( \text{spt}(B) \), voir [L1]. En effet, cette opération fournit une nouvelle suite minimisante qui maintenant reste dans la boule. La deuxième étape concerne le théorème de compacité prouvé dans [Pau14]. Plus en détail, soit \( G \) un groupe localement compact, soit \( K \subseteq \mathbb{R}^n \) un ensemble compact et soit \( C > 0 \) une constante, alors l’ensemble
\[ \mathcal{F}^m(l_2^n, G) \cap \{ T : \mathbb{N}(T) \leq C, \text{supp}(T) \subseteq K \} \]
est \( \mathcal{F} \)-compact. La démonstration de ce théorème est à son tour basée sur le théorème de déformation Brian White. On note que cette méthode garantit l’existence d’une solution dont le support est compact.

1.2 Discription de la thèse.

Malgré cette belle image euclidienne avec plusieurs méthodes applicables décrites dans la section précédente, la situation change lorsque nous prenons un espace de Banach comme espace amiant. Même dans les espaces de Banach de dimension finie, la situation n’est pas assez claire. Il me semble que dans ce dernier cas - qu’on estime de se produire dans les applications - d’autres méthodes devraient être impliquées. En effet, pour les espaces de Banach qui ne possèdent pas de produit scalaire, il n’y a ni dualité, ni formule de Crofton, voir [Sch01]. De plus, il n’est pas difficile de démontrer que dans l’espace \( X = (\mathbb{R}^3, \| \ldots \|_\infty) \) toutes les projections linéaires \( \pi : X \rightarrow W \) sur l’hyperplan \( W = \{ x \in X : x + y + z = 0 \} \) ont une constante de Lipschitz \( C \) strictement supérieure à 1. D’autre part, la compacité n’est pas un vrai problème ici, une fois que l’on accepte des solutions dont le support soit non compact. En effet, un argument diagonal qui ressemble à celui d’Ambrosio et Schmidt [AS13] pourrait être adapté au contexte des \( G \) chaînes dans des espaces de Banach de dimension finie.

Décrits plus en détail la variante du problème de Plateau qui nous intéressera. Supposons que \( (X, \| \ldots \|) \) est un espace de Banach de dimension \( n \). Nous aimerions souligner
ici au tout début que dans chaque espace de Banach de dimension finie \((X, ||\ldots||)\) considéré dans cette thèse, on fixe un système d’Auerbach \(e_1,\ldots,e_n, d_{e_1},\ldots,d_{e_n}\), où \(n\) est la dimension de \(X\). Cela signifie que \(e_1,\ldots,e_n\) sont vecteurs unitaires dans \(X\) et que \(d_{e_1},\ldots,d_{e_n}\) sont vecteurs unitaires dans \(X^*\) tels que pour tout \(i,j\) entre 1 et \(n\), \(d_{e_i}(e_i) = 1\) et \(d_{e_i}(e_j) = 0\), une fois que \(i \neq j\). Nous supposons également qu’une structure euclidienne est fixée sur \(X\) de telle sorte que \(e_1,\ldots,e_n\) soit une base euclidienne orthonormée. La norme euclidienne correspondante est notée \([\ldots]\) et \((\cdot,\cdot)\) représente le produit scalaire. Notons \(\mathcal{R}_m(X,G), m \in [0,\ldots,n]\) le groupe de toutes les chaînes \(m\) - rectifiables avec des coefficients dans un groupe abélien normé complet et localement compact \((G,|\ldots|)\). Equipons le groupe \(\mathcal{R}_m(X,G)\) de la masse \(M\), celle associée à la métrique dans \(X\) donnée par la norme \([\ldots]|\ldots|\). Ensuite, le problème de Plateau correspondant se lit comme suit

\[
\begin{aligned}
\text{minimiser } M(T) \\
\text{parmi } T \in \mathcal{R}_m(X,G) \text{ tels que } \partial T = B,
\end{aligned}
\]

où \(B \in \mathcal{R}_{m-1}(X,G), \text{spt}(B)\) est compact et \(\partial B = 0\). Le but principal de cette thèse est de montrer que ce problème admet une solution dans les cas où \(m = n - 1\) et \(m = 2\).

Décritons maintenant nos résultats principaux. Le théorème classique de Busemann [Bus49] montre que, dans tous les espace de Banach de dimension \(n\), la fonction de Hausdorff - Busemann \((n - 1)\) dimensionnelle est convexe. Nous rappelons ce que cela signifie. Soit \(X\) un espace Banach. Une fonction \(\phi : G\mathcal{C}(m,X) \rightarrow \mathbb{R}\), où \(n > m\) et \(G\mathcal{C}(m,X)\) est le cône de Grassmann constitué de \(m\) - vecteurs simples dans \(X\), est convexe si elle est égale à la restriction d’une norme sur \(\Lambda_m(X)\). La fonction de Hausdorff - Busemann est définie de la façon suivante. Soit \(B\) la boule unitaire de \(X\), \(B = \{x \in X : ||x|| \leq 1\}\) et soit \(G(m,X)\) la \(m\) - Grassmannienne de l’espace \(X\), c’est-à-dire l’ensemble des sous-espaces de dimension \(m\) de \(X\). Pour \(1 \leq m \leq n\) la fonction \(\phi_{BH} : G(m,X) \rightarrow \mathbb{R}_+\) définie comme

\[
\phi_{BH}(W) = \frac{\alpha_m}{\mathcal{H}^m(B \cap W)}
\]

s’appelle la densité Busemann - Hausdorff. Alors le théorème de Busemann dit que la fonction \(\sigma_B : G\mathcal{C}(1,X) \rightarrow \mathbb{R}_+\) définie par \(\sigma_B(v) := ||v|| \cdot \phi_{BH}(v^\perp)\), où \(v^\perp\) est l’hyperplan orthogonal au vecteur \(v\), est convexe.

Puis, Busemann dans son article [Bus50] utilise la convexité révélée afin de montrer que pour tout hyperplan \(W \in G(n - 1,X)\), il existe une projection linéaire \(\pi_W\) avec deux propriétés suivantes. Premièrement, \(\pi_W\) est une projection sur \(W\) et la seconde c’est que

\[
\mathcal{H}^n_\|\ldots\|(\pi_W(A)) \leq \mathcal{H}^n_\|\ldots\|(A)
\]

pour toutes ensembles \(A \subset V\), où \(V\) est un sous-espace linéaire de \(X\) dont la dimension égale \((n - 1)\). Cette deuxième propriété signifie que \(\pi_W\) n’augmente pas la mesure Hausdorff des sous-ensembles d’hyperplans. Nous appelons telles projections “\(W\) - good”. Dans cette thèse, nous avons démontré que ces projections n’augmentaient pas la mesure de Hausdorff de sous-ensembles \((n - 1)\)-rectifiables dans \(X\). Grâce à ces projecteurs, nous
avons pu imiter la deuxième méthode décrite dans la section précédente. En particulier, ces projections ont joué un rôle très important dans notre démonstration de l’inégalité triangulaire pour les cycles. Cette méthode a également utilisé les analogues de Banach des théorèmes d’approximation forte et faible.

Comme l’on a déjà mentionné, Busemann, dans son article [Bus49] montre que toutes les densités de Busemann - Hausdorff de dimension \( n - 1 \) sont convexes. 50 ans plus tard un nouveau résultat de ce type est apparu. Burago et Ivanov, dans leur article [BI12], montrent qu’un ensemble plat dans un espace de Banach de dimension finie dont la boule est un polyhèdre convexe possède la plus petite mesure de Hausdorff parmi toutes les surfaces 2-dimensionnelles ayant le même bord. Hélas, en dehors du cas de codimension 1, certaines densités convexes peuvent ne permettre aucune contraction pour la mesure de Hausdorff. Néanmoins, en utilisant les résultats de Burago et Ivanov, nous avons démontré qu’il existe un ”combinaison linéaire” de projections qui est toujours suffisante pour prouver l’inégalité triangulaire pour les 2 cycles polyédriques avec des coefficients dans un groupe abélien et localement compact.

Comme nous l’avons déjà mentionné, il n’existe pas de formule de Crofton dans les espaces de Banach. Malgré ce fait, nous avons pu démontrer que dans les espaces de Banach de dimension finie, en codimension un, il existe une version locale de cette formule, qui introduit une nouvelle version des mesures de Gross. Rappelons la définition de la mesure classique de Hausdorff. Soit \( S \) un sous-ensemble d’un espace de Banach de dimension \( n \) \((X, ||\cdot||)\) et soit \( \delta > 0 \) un nombre réel. Posons pour un entier \( m \) tel que \( 1 \leq m \leq n \)

\[
\mathcal{H}^{m,\delta}_{||\cdot||}(S) := \alpha_m \inf \left\{ \sum_{i=1}^{\infty} \xi^m(A_i) : A_i \subset X \text{ tel que } S \subseteq \bigcup_{i=1}^{\infty} A_i, \text{diam}||\cdot||(A_i) \leq \delta \right\},
\]

où \( \xi^m(A_i) = (\text{diam}||\cdot||(A_i)/2)^m \) et \( \alpha_m := \frac{\pi^m}{\Gamma\left(\frac{m}{2}\right) + 1} \). Ensuite, puisque \( \mathcal{H}^{m,\delta}_{||\cdot||}(S) \) est monotone décroissant en \( \delta \), on peut définir

\[
\mathcal{H}^{m}_{||\cdot||}(S) := \sup_{\delta > 0} \mathcal{H}^{m,\delta}_{||\cdot||}(S) = \lim_{\delta \to 0} \mathcal{H}^{m,\delta}_{||\cdot||}(S).
\]

La mesure \( \mathcal{H}^{m}_{||\cdot||}(S) \) est appelée la mesure de Hausdorff \( m \)-dimensionnelle. Puis, si \( \pi_W \) est une projection \( W \)-good, alors la mesure de Gross c’est celle obtenue par la même construction, mais quand on prend la fonction

\[
\xi^{n-1}_{||\cdot||}(A) = \sup_{W \in G(n-1,X)} \mathcal{H}^{n-1}_{||\cdot||}(\pi_W(A))
\]

au lieu de \( \xi^{n-1} \). Bien sûr, nous trichons ici un peu, car pour certains hyperplans, il peut y en avoir plus qu’une seule \( W \)-good projection. Cela nous oblige à faire un bon choix de tels projecteurs. Il s’est avéré que l’ensemble des projections \( W \)-good est un fermé. On en déduit qu’il existe un choix universellement mesurable \( \text{GP} : G(n-1,X) \to \text{Hom}(X,X) \) de ces projecteurs. Cette condition de mesurabilité universelle s’est avérée suffisante pour prouver que, en codimension 1, la mesure Gross \( \xi^{n-1}_{||\cdot||,\text{GP}} \) coincide avec celle de Hausdorff.
Ensuite, il en découle que les deux masses associées coincident également. Enfin on démontre que la masse de Gross est semi-continue inférieurement.

En ce qui concerne la compacité en codimension un, nous avons pu adapter la méthode qui utilise les projections sur des ensembles convexes, celle décrite dans la section précédente. On a remplacé ces projections par une composition de plusieurs good projections sur des hyperplans. Ensuite on applique des projections euclidiennes sur des points les plus proches à l’intérieur de ces hyperplans. Il est à noter que cette approche permet de conclure qu’il existe toujours une solution du problème de Plateau dont le support soit un sous-ensemble de l’enveloppe convexe du bord. Soulignons que, comme conséquence de ce résultat, on donne une réponse (partielle) à une question posée dans l’article [AS13] d’Ambrosio et Schmidt. D’autre part, le cas de l’espace $l^3$ montre qu’il peut y avoir des chaînes minimisantes dont les supports ne restent pas dans conv(spt($B$)). Soit $u : [0, 1] \times [0, 1] \to \mathbb{R}$ une fonction Lipschitzienne telle que $u = 0$ sur $\partial([0, 1]^2)$ et satisfaisant $\text{Lip}_\infty(u) \leq 1$, où $\text{Lip}_\infty$ c’est la constante de Lipschitz par rapport à la métrique de l’espace $l^2$. On écrit $T_0 = E^2 \cup [0, 1]^2$ et $B = \partial T_0$. Posons $F_u : [0, 1]^2 \to \mathbb{R}^3 : x \to (x, u(x))$ et $T = (F_u)_# T_0$. Alors évidemment $\partial T = \partial T_0$ et il n’est pas très difficile de démontrer que $M_\infty(T) = M_\infty(T_0) = 1$.

Dans le cas de dimension 2, on est obligé d’utiliser un autre argument qui ressemble à celui d’Ambrosio et de Schmidt. Hélas, cette méthode en général ne fournit pas une solution dont le support soit compact.
Chapter 2

Introduction.

2.1 Some history of the subject.

Proving the existence and the regularity for the Plateau problem is a paradigmatic question in the Geometric Measure Theory and the Calculus of Variations. It comprises finding a surface of least area spanned by a given closed contour. We emphasise that the words “surface”, “area” and “spanned” utilised in the previous phrase are in need of an explanation. Indeed, there are several possible ways to state the problem. In the context of integral currents the Plateau problem is formulated in the following way. Let $B$ be a compactly supported $(m – 1)$–dimensional integral rectifiable current in $\mathbb{R}^n$ such that $\partial B = 0$. Then the goal is to find an $m$–dimensional integral rectifiable current $S$ such that $\partial S = B$ and $M(S) \leq M(T)$ for every integral rectifiable current $T$ satisfying $\partial T = B$.

Another possible formulation of the Plateau problem is the following one. For $m \in [0, \ldots, n]$ denote by $\mathcal{R}_m(l^2_n, G)$ the group of all $m$–rectifiable chains in the Euclidean space $\mathbb{R}^n$ with coefficients in a complete locally compact normed Abelian group $(G, |\cdot|)$. The group $\mathcal{R}_m(l^2_n, G)$ is equipped with the mass $M$, the one relative to the Euclidean Hausdorff measure. The corresponding Plateau problem reads as follows

$$\begin{cases}
\text{minimize } M(T) \\
\text{among all } T \in \mathcal{R}_m(l^2_n, G) \text{ such that } \partial T = B,
\end{cases}$$

where $B \in \mathcal{R}_{m-1}(l^2_n, G)$, $\text{spt}(B)$ is compact and $\partial B = 0$.

Thanks to the methods of the theory of normal and integral currents and chains with coefficients in groups, for the case of Euclidean spaces the Plateau problem admits at least one solution. Let us first present a more detailed description of the situation in the Euclidean setting. One of the possible methods of proving the existence is the direct method of calculus of variations. It comprises two different steps: the first one is the lower semicontinuity of the mass with respect to some topology and the second one is the compactness of the space of admissible competitors with respect to the same topology. The most frequently used topologies are the flat topology and the weak star topology.
There are several different proofs of the lower semicontinuity of the mass in Euclidean spaces. The first one works in the context of currents. In more details the principal step of this method is in showing the following formula:

\[ M(T) = \sup\{T(\omega) : \omega \in \mathcal{D}^m(\mathbb{R}^n), ||\omega||^* \leq 1\}, \]

where \( ||\ldots||^* \) stands for the comass, \( ||\omega||^* = \sup\{\langle \xi, \omega \rangle : \xi \text{ is a unit simple } m \text{ vector}\} \).

Note that once such formula holds true the lower semicontinuity with respect to the weak star convergence follows easily, since supremum of continuous functions is always lower semicontinuous.

The second method is based on the following observation. Let \( W \in G(m, \mathbb{R}^n) \) and let \( \pi : \mathbb{R}^n \rightarrow W \) be an orthogonal projection with the range \( W \). Then \( \text{Lip}(\pi) \leq 1 \) and hence \( \pi \) does not increase Hausdorff measures of rectifiable sets. From this observation one can derive the following fact which resembles the triangle inequality. Let \( P \) be a polyhedral \( m \)-cycle (i.e. \( \partial P = 0 \)) such that \( P = \sum_{j=1}^{N} g_j[\sigma_j] \), where \( \sigma_j \) are non-overlapping \( m \)-simplexes. Then

\[ |g_1|\mathcal{H}^m(\sigma_1) \leq \sum_{j=2}^{N} |g_j|\mathcal{H}^m(\sigma_j). \]

It is worth noting that the proof of this inequality requires utilization of the constancy theorem. From here using the strong and the weak approximation theorems one can derive the desired lower semicontinuity.

Crofton formula provides yet another applicable method of showing that the mass is lower semicontinuous. More precisely, let \( A \subset \mathbb{R}^n \) be an \( m \)-rectifiable set. Then the following formula which is called Crofton formula or \( L^1 \)-integral–geometric formula holds true:

\[ \mathcal{H}^m(A) = \mathcal{T}^m(A) = \int_{G(m, \mathbb{R}^n)} d\gamma_m(W) \int_W \text{card}(\pi_W^{-1}(y) \cap A) dy, \]  

(2.2)

where \( \pi_W \) is the orthogonal projection with the range \( W \) and \( \gamma_m \) is the unique “properly normalized” \( O(m) \) invariant measure on the Grassmannian \( G(m, \mathbb{R}^n) \). With the help of this formula one can prove the lower semicontinuity in the following way. First, as an easy consequence, one readily gets that the Hausdorff mass \( \mathcal{M} \) coincides with the integral–geometric mass \( \mathcal{M}_I \) (i.e. the one defined with respect to the integral–geometric measure) on the space of \( m \)-rectifiable \( G \) chains and that the following formula holds true:

\[ \mathcal{M}(T) = \mathcal{M}_I(T) = \int_{G(m, \mathbb{R}^n)} d\gamma_m(W) \int_W \mathcal{M}(T, \pi_W, y) dy. \]

On the other hand, the integral–geometric mass is lower semicontinuous with respect to flat convergence. Indeed, if \( T_i \rightarrow T \) in flat distance with \( T_i, T \in \mathcal{R}^m(l^2, G) \), then the zero dimensional slices satisfy

\[ \int \mathcal{F}(\langle T_i - T, \pi_W, y \rangle) dy \leq \mathcal{F}(T_i - T) \rightarrow 0, \]
where the inequality here follows the fact that $\pi_W$ have Lipschitz constant one. Hence up to a subsequence one has $\mathcal{F}(\langle T_i - T, \pi_W, y \rangle) \to 0$ for almost all $y$. Finally, the Fatou lemma and the observation that the mass is lower semicontinuous on the 0 dimensional slices, allow to conclude that $\mathcal{M}_T$ is lower semicontinuous.

In what concerns the compactness, one of the easiest ways of proving it consists of the following two steps. The first step as well requires 1–Lipschitz property of the nearest point projections on convex sets, discussed in the last two paragraphs. Since those projectors do not increase the mass, one can just project the minimizing sequence onto some ball containing $\text{spt}(B)$, see (2.1). Indeed, this operation provides a new minimizing sequence which stays inside the ball. The second step relates on the compactness theorem proved in [Pau14]. In more details, this result shows that if $G$ is a locally compact group, $K \subseteq \mathbb{R}^n$ is a compact set and $C > 0$ is a constant, then the set

$$\mathcal{F}_m(l_2^n, G) \cap \{ T : \mathbb{N}(T) \leq C, \text{supp}(T) \subseteq K \}$$

is $\mathcal{F}$–compact. The proof of this theorem is in turn based on the deformation theorem for flat $G$ chains by Brian White. It is worth mentioning that this method guarantees the existence of a solution that has compact support.

### 2.2 Description of the thesis.

Despite of a seemingly nice Euclidean picture with several applicable methods described in the previous section, the situation changes dramatically once we let the ambient space become a Banach space. Even in the finite dimensional Banach spaces the situation is not clear enough. It seems that in the latter case—which is likely to arise in applications–other methods should be involved. Indeed, for Banach spaces that possess no Hilbert structure there exists neither duality, nor Crofton formula see [Sch01]. In addition, it is not difficult to show that in the space $X = (\mathbb{R}^3, ||\cdot||_\infty)$ any linear projection $\pi : X \to W$ onto the hyperplane $W = \{ x \in X : x + y + z = 0 \}$ has Lipschitz constant $C$ strictly greater than 1. On the other hand, the compactness is not a real problem here, once one accepts solutions that have non–compact support. Indeed, a diagonal argument that resembles that of Ambrosio and Schmidt [AS13] can be adapted to the context of $G$ chains in finite dimensional Banach spaces.

Let us describe in more details the variant of the Plateau problem we are going to be interested in. Suppose $(X, ||\cdot||)$ is an $n$–dimensional Banach space. We would like to emphasize here in the very beginning that in every finite dimensional Banach space $(X, ||\cdot||)$ considered here, we fix an Auerbach system $e_1, \ldots, e_n, d e_1, \ldots, d e_n$, where $n$ is the dimension of $X$. This means that $e_1, \ldots, e_n$ are unit vectors in $X$ and $d e_1, \ldots, d e_n$ are unit vectors in $X^*$ such that for all $i, j$ between 1 and $n$, $d e_i(e_i) = 1$ and $d e_i(e_j) = 0$, once $i \neq j$. We also suppose that a Euclidean structure is fixed on $X$ in a way that $e_1, \ldots, e_n$ is an orthonormal Euclidean basis. The corresponding Euclidean norm is denoted by $||\cdot||$ and $\langle \cdot, \cdot \rangle$ stands for the corresponding scalar product. Denote by $\mathcal{R}_m(X, G), m \in [0, \ldots, n]$ the group of all $m$–rectifiable chains with coefficients in a complete locally compact normed
Abelian group \((G, |\ldots|)\). Equip the group \(\mathcal{R}_m(X, G)\) with the mass \(\mathbb{M}\), the one relative to the metric in \(X\) induced by the norm \(||\ldots||\). Then the corresponding Plateau problem reads as follows

\[
\begin{aligned}
\text{minimize} & \quad \mathbb{M}(T) \\
\text{among all } & \quad T \in \mathcal{R}_m(X, G) \text{ such that } \partial T = B,
\end{aligned}
\tag{2.3}
\]

where \(B \in \mathcal{R}_{m-1}(X, G), \text{spt}(B)\) is compact and \(\partial B = 0\). The main goal of this thesis is to show that this problem admits a solution in case when \(m = n - 1\) and \(m = 2\).

Let us next describe and motivate our main results. The classical theorem by Busemann \[Bus49\] shows that in any \(n\)-dimensional Banach space \(X\) the \((n - 1)\) dimensional Hausdorff–Busemann area function is extendibly convex. We remind the reader of what this means. Let \(X\) be a Banach space. A function \(\phi : GC(m, X) \to \mathbb{R}\), where \(n > m\) and \(GC(m, X)\) is the Grassmann cone of simple \(m\)–vectors in \(X\), is extendibly convex if it is the restriction of some norm on \(\Lambda_m(X)\). The Hausdorff–Busemann area function in \(X\) is defined as follows. Let \(B\) be the unit ball of \(X\), \(B = \{x \in X : ||x|| \leq 1\}\) and denote by \(G(m, X)\) the \(m\)–Grassmannian of the space \(X\), i.e. the set of all \(m\)–dimensional planes in \(X\). For \(1 \leq m \leq n\) the function \(\sigma_{BH} : G(m, X) \to \mathbb{R}_+\) defined as

\[
\sigma_{BH}(W) = \frac{\alpha_m}{\mathcal{H}^m(B \cap W)}
\]

is called the Busemann–Hausdorff density. Then Busemann’s theorem tells that the function \(\sigma_B : GC(1, X) \to \mathbb{R}_+\) defined as \(\sigma_B(v) := ||v|| \cdot \phi_{BH}(v^\perp)\), where \(v^\perp\) is the hyperplane orthogonal to the vector \(v\), is extendibly convex.

In addition, Busemann in the paper \[Bus50\] used the revealed convexity in order to show that for any hyperplane \(W \in G(n - 1, X)\) there exists a linear projection \(\pi_W\) with the following two properties. First, it has range \(W\) and second is that

\[
\mathcal{H}^{n-1}_{||\ldots||}(\pi_W(A)) \leq \mathcal{H}^{n-1}_{||\ldots||}(A)
\]

for any set \(A \subset V\), where \(V\) is some \((n - 1)\)–dimensional linear subspace of \(X\). The second property means that \(\pi_W\) does not increase the Hausdorff measure of the subsets of hyperplanes. We call such projections \(W\)–good. In this thesis we have shown that these projections do not increase Hausdorff measure of arbitrary \((n - 1)\)–rectifiable subsets of \(X\). With help of these projectors we were able to mimic the second method described in the previous section. In particular those projections played crucial role in our proof of the corresponding version of the triangle inequality for cycles. This method has also required Banach space analogues of the strong and the weak approximation theorems.

As it was already mentioned, Busemann in his paper \[Bus49\] shows that any \((n - 1)\)–dimensional Busemann–Hausdorff density is convex. It was only 50 years later, when a new result of such kind appeared. Namely, in their paper \[BI12\], Burago and Ivanov show that a bounded planar region in a finite dimensional Banach space whose unit ball is a convex polyhedron, has the least two–dimensional Hausdorff measure among all compact smooth two-dimensional surfaces having the same boundary. Alas, outside of the codimension
2.2 Description of the thesis.

one case, some extendibly convex densities can happen to admit no area non-increasing projections. Nevertheless, using the result of Burago and Ivanov we showed that there exists a “linear combination” of projections that is still enough to prove the triangle inequality for polyhedral 2–cycles with coefficients in arbitrary normed locally compact Abelian group.

As we have already mentioned there exists no Crofton formula in Banach spaces. Despite of this fact, we were able to show that in finite dimensional Banach spaces in the codimension one case, there exists a “local” version of this formula, which brings in a new version of Gross measures. Let us present a more detailed description of those. Recall the definition of the classical Hausdorff measure. Let

\[ S \]

be any subset of an \( n \)-dimensional Banach space \( (X, || \cdot ||) \) and let \( \delta > 0 \) be a real number. For an integer \( m \) such that \( 1 \leq m \leq n \) we define

\[
\mathcal{H}^m(\delta) := \alpha_m \inf \left\{ \sum_{i=1}^{\infty} \xi(A_i) : A_i \subset X \text{ such that } S \subseteq \bigcup_{i=1}^{\infty} A_i, \ diam || \cdot ||(A_i) \leq \delta \right\},
\]

where \( \xi(A_i) = (diam || \cdot ||(A_i)/2)^m \) and \( \alpha_m := \pi^{\frac{m}{2}} / \Gamma(\frac{m}{2} + 1) \). Next, since \( \mathcal{H}^{m,\delta}(S) \) is monotone decreasing in \( \delta \), we are allowed to define

\[
\mathcal{H}^m(S) := \sup_{\delta > 0} \mathcal{H}^{m,\delta}(S) = \lim_{\delta \to 0} \mathcal{H}^{m,\delta}(S).
\]

The measure \( \mathcal{H}^m(S) \) is called the \( m \)-dimensional Hausdorff measure. Now, if \( \pi_W \) is a \( W \)-good projection, then the Gross measure is the one obtained from the very same construction, but with the function

\[
\zeta^{n-1}(A) = \sup_{W \in G(n-1,X)} \mathcal{H}^{n-1}(\pi_W(A))
\]

used as a gauge instead of \( \xi^{n-1} \). Of course, here we cheat a little bit, since for certain hyperplanes there might be more than one \( W \)-good projection. This in turn forces one to make a proper choice of those projectors. It turned out that the set of \( W \)-good projections is closed. This observation further allowed us to prove that there exists a universally measurable choice \( GP : G(n-1,X) \to \text{Hom}(X,X) \) of those projectors. This universal measurability condition turned out to be sufficient to prove that in codimension one the Gross measure \( \mathcal{G}^{n-1} \) coincides with the Hausdorff measure, which in turn implies that the corresponding masses coincide as well. The Gross mass was further shown by us to be lower semicontinuous.

In what concerns the compactness in the codimension one case we were able to adapt the method that requires projections onto convex sets, described in the previous section. We replace these projections with a superposition of a number of good projections onto hyperplanes and further apply Euclidean nearest point projections inside these hyperplanes. It is worth mentioning that this approach allows to conclude that there always exists a solution of the Plateau problem whose support is a subset of the convex hull of the boundary.

We would like to emphasize that as a consequence of this result, we give a (partial) answer to a question raised in the paper [AS13] by Ambrosio and Schmidt. On the other hand, a
simple example of the space $l_\infty^3$ shows that there also exist mass minimizing chains whose supports need not to stay inside of $\text{conv}(\text{spt}(B))$. Indeed let $u : [0,1] \times [0,1] \to \mathbb{R}$ be any Lipschitz function such that $u = 0$ on $\partial([0,1]^2)$ and satisfying $\text{Lip}_\infty(u) \leq 1$, where $\text{Lip}_\infty$ signifies the Lipschitz constant with respect to the $l_\infty^3$ metric. Write $T_0 = E^2 \subset [0,1]^2$ and $B = \partial T_0$. Define $F_u : [0,1]^2 \to \mathbb{R}^3 : x \to (x,u(x))$ and $T = (F_u)_# T_0$. Then obviously $\partial T = \partial T_0$ and it is not difficult to show that $M_\infty(T) = M_\infty(T_0) = 1$.

In the two–dimensional case we have to use another argument which resembles that of Ambrosio and Schmidt. Alas, this method in general does not provide a compact support solution.

2.3 Structure of the thesis.

Here we describe the composition of the chapters of the thesis. In the second chapter we first introduce the main definitions used in the manuscript. Those definitions one can divide into two parts: those related to areas and volumes of sets in Banach spaces and those connected with rectifiable chains with coefficients in groups. We further cite in the second part of the chapter a number of classical results concerning rectifiable sets and chains with coefficients in groups. Among them are coarea formula, B.Kirchheim’s theorem, constancy theorem, Eilenberg’s inequality, Besicovitch covering theorem and compactness theorem for flat chains.

In the third chapter we first give a careful proof (Theorem 4.2.1) of Busemann’s theorem on the convexity of the Hausdorff–Busemann area function. Then we prove Theorem 4.3.2 which shows that there each hyperplane admits area non–increasing projections. Next we derive in Theorem 4.3.4 the fact that these projectors do not increase Hausdorff measure of $(n-1)$–rectifiable subsets. Finally in this chapter we prove Theorem 4.3.5 which says that good projections do not increase the mass of $(n-1)$ polyhedral $G$ chains.

The forth chapter begins with establishing Theorem 5.1.1 where we prove the validity of the “triangle inequality” disclosed before in the codimension one case. Then we prove our Theorem 5.1.2 where we give a necessary and sufficient condition for the lower semicontinuity of the mass on the space of polyhedral chains in finite dimensional Banach spaces. This condition is nothing but the “triangle inequality”. We further derive in Theorem 5.2.1 the lower semicontinuity of the mass on the space of $G$ rectifiable chains via using the strong approximation Theorem 5.2.2. Finally in this chapter we prove our Theorem 5.3.1 where the existence of mass minimising $(n-1)$–rectifiable $G$ chains with compact support is established.

The fifth chapter consists of the following three principal results. The first one is Proposition 6.1.1 where we prove that the set of good projections is closed and with help of which we further derive the fact that there exists a universally measurable selection of good projections. Later in Theorem 6.2.1 we show that $\mathcal{H}_{\ell_{\|\cdot\|}}^{n-1}(A) = \mathcal{G}_{\ell_{\|\cdot\|},G}^{n-1}(A)$ for $(n-1)$ rectifiable subsets of $X$. Theorem 6.2.2, the last result of this chapter, is the one in which we show that the Gross mass $M_G$ is lower semicontinuous with respect to flat norm.

We begin the sixth chapter with a proof of the result of Burago and Ivanov mentioned in
the previous section, our Theorem 7.2.2. We further apply this theorem in our proof of the corresponding triangle inequality for the polyhedral 2–cycles, which is our Theorem 7.3.1. In the last part of this chapter, in Theorem 7.3.2, we prove the corresponding existence result, which in turn uses a compactness argument that resembles that of Ambrosio and Schmidt.
Introduction.
Chapter 3
Rectifiable sets and chains in Banach spaces.

3.1 Some preliminary definitions and remarks.

We begin with a number of useful definitions and remarks.

Definition 3.1.1. (Norm) Let $X$ be a real linear space. We call a function $||\ldots|| : X \to \mathbb{R}$ a norm if

1. for every $x \in X$ one has $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$,
2. for every $x \in X$ and $\alpha \in \mathbb{R}$ one has $||\alpha x|| = |\alpha|||x||$,
3. for all $x, y \in X$ one has $||x + y|| \leq ||x|| + ||y||$.

We shall also call the space $(X,||\ldots||)$ a Banach space.

Definition 3.1.2. (Supporting hyperplane) Let $X$ be a real linear space of dimension $n$, let $S \subset X$ be an arbitrary subset and let $x \in S$. An affine $n-1$ dimensional subspace $W \subset X$ is called a supporting hyperplane of $S$ at $x$ if

1. $x \in W$,
2. $S$ is entirely contained in one of the two closed half-spaces bounded by the $W$.

Remark 3.1.1. Let $X$ be a Banach spaces and let $||\ldots||$ be a norm on $X$. Throughout this thesis $B_{||\ldots||}(x, r)$ with $x \in X$ and $r > 0$ signifies the closed ball or radius $r$ centered at the point $x$ whereas $U_{||\ldots||}(x, r)$ stands for the corresponding open ball.

Definition 3.1.3. (Diameter) For any subset $S \subset X$ of a Banach space $(X,||\ldots||)$ define its diameter as $\text{diam}(S) = \sup\{||x - y|| : x, y \in S\}$. 
Definition 3.1.4. (Hausdorff measure) Let $S$ be any subset of an $n$–dimensional Banach space $(X, ||\ldots||)$ and let $\delta > 0$ be a real number. For an integer $m$ such that $1 \leq m \leq n$ we define

$$\mathcal{H}^{m,\delta}_{||\ldots||}(S) := \alpha_m \inf \left\{ \sum_{i=1}^{\infty} \left( \frac{\text{diam}(A_i)}{2} \right)^m : A_i \subset X \text{ such that } S \subseteq \bigcup_{i=1}^{\infty} A_i, \text{ diam}(A_i) \leq \delta \right\},$$

where $\alpha_m := \frac{\pi^m}{\Gamma(\frac{m}{2} + 1)}$. By definition $\alpha_m = \mathcal{H}^m_2(B)$, where $B = \{ x \in \mathbb{R}^m : ||x||_2 \leq 1 \}$ and $\mathcal{H}^m_2$ is any Euclidean Hausdorff measure.

Next, since $\mathcal{H}^{m,\delta}_{||\ldots||}(S)$ is monotone decreasing in $\delta$, we are allowed to define

$$\mathcal{H}^m_{||\ldots||}(S) := \sup_{\delta > 0} \mathcal{H}^{m,\delta}_{||\ldots||}(S) = \lim_{\delta \to 0} \mathcal{H}^{m,\delta}_{||\ldots||}(S).$$

The measure $\mathcal{H}^m_{||\ldots||}(S)$ is called the $m$–dimensional **Hausdorff measure**.

Remark 3.1.2. In every real linear space $X$ of dimension $n$ considered in this thesis, we fix an Auerbach system $e_1, \ldots, e_n, d_1, \ldots, d_n$, where $n$ is the dimension of $X$. By this we mean that $e_1, \ldots, e_n$ are unit vectors in $X$ and $d_1, \ldots, d_n$ are unit vectors in $X^*$ such that for all $i, j$ between 1 and $n$, $d_i(e_i) = 1$ and $d_i(e_j) = 0$, once $i \neq j$. We also fix a Euclidean structure on $X$ defined in a way that $e_1, \ldots, e_n$ is an orthonormal Euclidean basis. We denote by $|\ldots|$ the corresponding Euclidean norm and by $\langle \cdot, \cdot \rangle$ the corresponding scalar product.

Definition 3.1.5. (Dyadic semi–cubes) Let $X$ as in the previous remark. Let $k \in \mathbb{Z}$, define $Q_k$, a system of **dyadic semi–cubes** of edge $2^{-k}$ as the family of all cubes $Q$ that possess the following form

$$Q = \{ x \in X : \text{ for all } 1 \leq j \leq n, 2^{-k}m_j \leq \langle x, e_j \rangle < 2^{-k}(m_j + 1), \text{ for some } m_1, \ldots, m_n \in \mathbb{Z} \}.$$

Remark 3.1.3. We shall write $\mathcal{H}^m$ for the Hausdorff measure, defined with respect to the Euclidean norm of $X$, described in the previous Remark.

Remark 3.1.4. Note that the measures $\mathcal{H}^{m-1}$ and $\mathcal{H}^{m-1}_{||\ldots||}$ are both Haar invariant measures. Hence, according to [Mat99] Theorem 3.1 they are proportional.

Definition 3.1.6. (Exterior algebra) The $k$–th **exterior algebra** of a vector space $X$, denoted as $\Lambda_k(X)$, is the vector space spanned by elements of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_k$, where $\wedge$ stands for the exterior product (contact [Fed69], 1.3.1) and $x_i \in X, i = 1, 2, \ldots, k$. Let $v_1, \ldots, v_n \in X$ and $w_1, \ldots, w_n \in X^*$, define the parity as

$$\langle v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n \rangle = \det \{ \langle v_i, w_j \rangle \}_{i,j=1}^n.$$

Definition 3.1.7. (Grassmannian) The $k$–th **Grassmannian** of a vector space $X$ denoted $G(k, X)$ is the set of all of $m$–dimensional linear subspaces of $X$.
3.1 Some preliminary definitions and remarks.

Definition 3.1.8. (Grassmann cone) The \( k \)-th Grassmann cone of a vector space \( X \), denoted \( GC_k(X) \), is defined as follows: \( GC_k(X) = \{ \xi \in \Lambda_k(X) : \xi \text{ is a simple vector} \} \). Recall that \( \xi \in \Lambda_k(X) \) is a simple multi-vector if \( \xi = v_1 \wedge \ldots \wedge v_k \) for some \( v_1, \ldots, v_k \in X \).

Definition 3.1.9. (Rectifiable sets) Let \( X \) be a metric space. We call a set \( E \subseteq X \) \( m \)-rectifiable if there exists a countable set of Lipschitz functions \( f_i : A_i \to X \) where \( A_i \subseteq \mathbb{R}^m \) are bounded subsets such that

\[
\mathcal{H}^m \left( E \setminus \bigcup_{i=1}^\infty f_i(A_i) \right) = 0.
\]

Definition 3.1.10. (Approximate tangent spaces) Let \( M \subset X \) be a subset of a finite dimensional Banach space \( X \) that is measurable with respect to the \( m \)-dimensional Hausdorff measure, and such that the restriction measure \( \mathcal{H}^m \upharpoonright M \) is a Radon measure. We say that an \( m \)-dimensional subspace \( P \in G(m, X) \) is an \( m \)-approximate tangent space to \( M \) at a certain point \( x \), denoted \( \text{Tan}_m(M, x) \) (and sometimes \( T_x M \), namely when there is no need to specify \( m \)) if

\[(\mathcal{H}^m \upharpoonright M)_{x, \lambda} \rightharpoonup \mathcal{H}^m \upharpoonright P\]

weakly in the sense of measures. Here for any measure \( \mu \) we denote by \( \mu_{x, \lambda} \) the rescaled and translated measure,

\[\mu_{x, \lambda}(A) := \lambda^{-n} \mu(x + \lambda A)\].

Remark 3.1.5. Let \( M, X \) and \( x \) be as in the previous definition. If an approximate tangent plane \( P \) to \( M \) at \( x \) exists, then it is unique. This follows from [Mat99], Theorem 15.19.

Definition 3.1.11. (Densities of sets) Let \( (X, || \ldots ||) \) be an \( n \)-dimensional Banach space and let \( 0 \leq s \leq n \) be a real number. Suppose that \( E \) is a \( \mathcal{H}^s \)-measurable set in \( X \). The upper and lower Lebesgue \( s \)-density of \( E \) at \( x \in X \) are defined by

\[\Theta^s,*(E, x) = \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha_s r^s},\]

\[\Theta^s_*(E, x) = \liminf_{r \to 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha_s r^s} \]

If they agree the common value

\[\Theta^s(E, x) = \Theta^s_*(E, x) = \Theta^s,*(E, x)\]

is called the \( s \)-density of the set \( E \) at \( x \).

Definition 3.1.12. (Polyhedral chains) Let \( (X, || \ldots ||) \) be an \( n \)-dimensional Banach space and let \( G \) be a complete normed Abelian group. We define \( \mathcal{P}_m(X, G) \), the Abelian group of \( m \) polyhedral chains, as the group of equivalence classes of formal sums of the elements of type

\[P = \sum_{j=1}^N g_j[\sigma_j],\]

where \( g_j \in G \) and \( \sigma_j \) are oriented \( m \)-simplexes.
Definition 3.1.13. (Top-dimensional chains) Let \((X, ||\ldots||)\) be an \(n\)-dimensional Banach space and let \(G\) be a complete normed Abelian group. The group \(P_n(X,G)\) is called the group of top-dimensional polyhedral chains in \(X\).

We adopt the following conventions: if \([\sigma]\) is a simplex endowed with the canonical orientation then we endow \(-[\sigma]\) with the opposite orientation. Moreover \(g(-[\sigma]) = (-g)[\sigma] = -g[\sigma]\); finally any summand whose coefficient is the neutral element of \(G\) may be omitted since it gives null contribution. It follows that \(P = \sum g_i[\sigma_i]\) is the identity element if and only if every \(g_i\) is the neutral element of \(G\), and the inverse element is obtained by \(-P = \sum -g_i[\sigma_i]\), thus the group is well defined.

We say that two simplexes \(\sigma_1\) and \(\sigma_2\) are non-overlapping if \(\text{int}(\sigma_1) \cap \text{int}(\sigma_2) = \emptyset\). We define the support spt\((P)\) of \(P\) as spt\((P) = \bigcup_j \sigma_j\), once \(\sigma_i\) are non-overlapping and the corresponding elements \(g_i \neq 0\).

Definition 3.1.14. (Boundary of polyhedral chains) Define a group homeomorphism \(\partial : P_m(X,G) \to P_{m-1}(X,G)\) called boundary as follows. Let \(P = \sum_j g_j[\sigma_j]\). We first define its value on the basis elements i.e. on simplixes \(\sigma_j\) and then extend it by the formula

\[
\partial P = \sum_j g_j \partial[\sigma_j].
\]

We require \(\partial[\sigma]\) to be the sum of all faces of \(\sigma\) each one endowed with an orientation compatible to the exterior normal vector of \(\sigma\).

Definition 3.1.15. (Mass of polyhedral chains) Let \((X, ||\ldots||)\) be an \(n\)-dimensional Banach space and \(G\) be a complete normed Abelian group. Let \(P \in P_m(X,G)\) be a polyhedral \(m\) chain such that \(P = \sum_{j=1}^N g_j[\sigma_j]\), where \(\sigma_j\) are non-overlapping. We define the mass \(M_{||\ldots||}(P)\) of \(P\), as

\[
M_{||\ldots||}(P) = \sum_{j=1}^N |g_j| H^m_{||\ldots||}(\sigma_j).
\]

Remark 3.1.6. We shall often drop the sign \(||\ldots||\) and simply write \(M\) for the mass instead of \(M_{||\ldots||}\).

Remark 3.1.7. The mass \(M\) is well defined.

Definition 3.1.16. (Flat norm) Define flat norm \(F(P)\) of a polyhedral chain \(P \in P_m(X,G)\) as follows

\[
F(P) = \inf \{M(Q) + M(R) : Q \in P_m(X,G), R \in P_{m+1}(X,G), P = Q + \partial R\}.
\]

Definition 3.1.17. (Flat chains) Let \((X, ||\ldots||)\) be an \(n\)-dimensional Banach space and \(G\) be a complete normed Abelian group. Define the group of the \(m\)-dimensional flat chains denoted by \(F_m(X,G)\), as the \(F\) completion of \(P_m(X,G)\).
3.1 Some preliminary definitions and remarks.

Definition 3.1.18. (Lipschitz chains) Let \((X, \| \cdot \|)\) be an \(n\)-dimensional Banach space and \(G\) be a complete normed Abelian group. We define \(L_m(X,G)\), the Abelian group of \(m\)-Lipschitz chains, as the subgroup of \(F_m(X,G)\) formed by the elements of type

\[ P = \sum_{j=1}^{N} g_j \gamma_j \# [\sigma_j], \]

where \(g_j \in G, \sigma_j \subset \mathbb{R}^m\) are oriented \(m\)-simplexes and \(\gamma_j : \sigma_j \to X\) are Lipschitz mappings.

Definition 3.1.19. (Rectifiable chains) Let \((X, \| \cdot \|)\) be an \(n\)-dimensional Banach space and \(G\) be a complete normed Abelian group. Define the group of the \(m\)-dimensional rectifiable chains denoted by \(R_m(X,G)\), as the \(\mathcal{M}\) completion of \(L_m(X,G)\).

Remark 3.1.8. The support, the restriction and the boundary of rectifiable chains are well defined, recall [PH14]. We shall utilize the symbol \(\ll\) for the restrictions.

Remark 3.1.9. Each \(T \in R_m(X,G)\) is defined by an \(m\)-rectifiable set \(A\), which we will call \(\text{set}(T)\) and a \(\mathcal{H}^m\)-integrable \(G\)-valued multiplicity function which is often denoted by \(\theta_T\). We shall often write

\[ T = \theta_T \mathcal{H}^m \ll \text{set}(T). \]

If \(\text{set}(T) = \gamma (\bigcup_i A_i)\) with \(A_i\) mutually disjoint and where \(\gamma : \bigcup_i A_i \to X\) is Lipschitz, then we shall also write \(T = [\gamma, A_i, g]\). If \(\theta_T\) is a constant, say \(\theta_T \equiv g \in G\), then we simply write \(T = g \text{[set}(T)\text{]}\).

Definition 3.1.20. (Slicing) Let \((X, \| \cdot \|)\) be a Banach space of dimension \(n\). If \(T \in R_m(X,G), T = [\gamma, A_i, g]\) and \(f : X \to \mathbb{R}^k\) is a Lipschitz mapping, then slicing (or slice) \(\langle T, f, y \rangle \in R_{m-k}(X,G)\) is defined for \(\mathcal{H}^k\) almost every \(y\) as

\[ \langle T, f, y \rangle = \pm \theta_T \mathcal{H}^{m-k} \ll (f^{-1}(y) \cap \text{set}(T)), \]

where the sign \(\pm\) depends on the behaviour of \(f\) around the point \(y\), see [PH14]. Note that this is well defined, since the sets \(f^{-1}(y) \cap \text{set}(T)\) are \((m-k)\)-rectifiable for \(\mathcal{H}^k\) almost all \(y\), according to [Fed69], 3.2.22.

Proposition 3.1.1. (see [PH12]) Let \((X, \| \cdot \|)\) and \((Y, || \cdot ||)\) be two finite dimensional Banach spaces of dimensions \(n\) and \(d\) respectively and let \(G\) be a complete normed Abelian group. Suppose that \(S, T \in R_m(X,G)\) and \(f, \phi : X \to Y\) are Lipschitz functions, and \(U \subset X\) is a subset of \(X\). Then

- \(\mathcal{M}(S + T) \leq \mathcal{M}(S) + \mathcal{M}(T)\),
- \(\mathcal{M}(T \ll U) \leq \mathcal{M}(T)\),
- \(\mathcal{M}(\phi \# T) \leq \text{Lip}(\phi)^m \mathcal{M}(T)\),
- \(\int_Y \mathcal{M}(\langle T, f, y \rangle) dy \leq \text{Lip}(f)^d \mathcal{M}(T)\).
3.2 Some useful theorems on rectifiable sets and chains in Banach spaces.

For the reader’s convenience we gather in this section most of theorems on rectifiable sets and chains in Banach spaces cited in this thesis.

First we state the following theorem which is called weak approximation theorem. This result is proved in [Pau14].

Theorem 3.2.1. Assume \((Y, ||\cdot||)\) is a finite dimensional Banach space, \(T \in \mathcal{R}_n(Y, G)\) is so that \(\partial T \in \mathcal{R}_{n-1}(Y, G)\) and \(\text{spt}(T)\) is compact and let \(\varepsilon > 0\). Then there exists \(P \in \mathcal{P}_n(Y, G)\) such that

- \(\mathcal{M}(P) \leq \varepsilon + \mathcal{M}(T)\),
- \(\mathcal{M}(\partial P) \leq \varepsilon + \mathcal{M}(\partial T)\),
- \(\mathcal{F}(T - P) \leq \varepsilon\),
- \(\text{spt}(P) \subseteq B(\text{spt}(T), \varepsilon)\),

where \(B(\text{spt}(T), \varepsilon) = \{y \in Y : ||y - x|| \leq \varepsilon \text{ for some } x \in \text{spt}(T)\}\) is the \(\varepsilon\)-neighborhood of the set \(\text{spt}(T)\).

Second we present a constancy theorem for \(G\) chains in finite dimensional Banach spaces. The proof of this result can be found in [PH14].

Theorem 3.2.2. Let \(Y\) be an \(m\)-dimensional Banach space and let \(G\) be a complete normed Abelian group. If \(T \in \mathcal{F}_m(Y, G), U\) is a connected open subset of \(Y\), and \((\partial T)\cap U = 0\), then \(T\cap U = g[U]\) for some \(g \in G\), see Remark 3.1.9.

The following theorem by B. Kirchheim [Kir94] shows that rectifiable subsets of a metric spaces have density equal to 1 almost everywhere.

Theorem 3.2.3. Let \(X\) be a metric space and let \(A\) be an \(m\)-rectifiable subset of \(X\). Then for \(\mathcal{H}^m\) almost all \(x \in A\) there holds

\[
\lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B(x, r))}{\alpha_m r^m} = 1.
\]

We proceed to the Eilenberg inequality. We give a version borrowed from the book [BZ88].

Theorem 3.2.4. Let \(X\) and \(Y\) be separable metric spaces and let \(f : X \to Y\) be a Lipschitz mapping. Then for any \(A \subseteq X\) and \(1 \leq m \leq n\) there holds

\[
\int_X \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{H}^m(y) \leq \frac{\alpha_m \alpha_{n-m}}{\alpha_n} (\text{Lip}(f))^n \mathcal{H}^n(A),
\]

where \(\int^*\) stands for the upper Lebesgue integral.
3.2 Some useful theorems on rectifiable sets and chains in Banach spaces.

Next we state the area formula for Lipschitz mappings between Euclidean spaces. Our version is the one proved in the book [EG15]. Recall that for a Lipschitz mapping $f : l^n_2 \to l^m_2$, $n \leq m$ for $\mathcal{H}^n$ almost all $x \in l^n_2$ its Jacobian $Jf$ is given by the following formula

$$Jf(x) = \sqrt{\det Df^T(x) \cdot Df(x)},$$

where $Df(x)$ signifies the differential matrix of the mapping $f$ at the point $x$. Indeed, this definition makes sense since by Rademacher’s theorem $f$ is differentiable for $\mathcal{H}^n$ almost all $x \in l^n_2$.

**Theorem 3.2.5.** Let $f : l^n_2 \to l^m_2$ be Lipschitz continuous, $n \leq m$. Then for each Lebesgue measurable subset $A \subset \mathbb{R}^n$,

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} \text{card}(A \cap f^{-1}(y))dy.$$

The following theorem guarantees the almost everywhere existence of approximate tangent spaces of rectifiable sets. One can consult the book [Mat99] for its proof.

**Theorem 3.2.6.** Suppose $E \subset \mathbb{R}^n$ is an $\mathcal{H}^m$ measurable and $m$–rectifiable set with $\mathcal{H}^m(E) < \infty$. Then for $\mathcal{H}^m$ almost every $x \in E$ the approximate tangent space $\text{Tan}_m(E, x)$ exists, is unique and moreover one has $\Theta^m(E, x) = 1$.

Compactness theorem from [Pau14] that we are about to state will play crucial role in our existence theorem in the 4–th chapter.

**Theorem 3.2.7.** If $X$ is a finite dimensional Banach space, $G$ is a locally compact normed Abelian group, $K \subseteq X$ is a compact set and $C > 0$ is a constant, then the set

$$\mathcal{F}_m(X, G) \cap \{ T : M_2(T) + M_2(\partial(T)) \leq C, \text{supp}(T) \subseteq K \},$$

where $M_2$ stands for the Euclidean mass, is $\mathcal{F}$–compact.

The following result by Kirszbraun says that we can extend Lipschitz mappings to the whole space.

**Theorem 3.2.8.** Any Lipschitz mapping from a subset of $l^n_2$ can be extended to a Lipschitz mapping defined on the whole $l^n_2$ with the same Lipschitz constant.

We finally present the Besicovitch covering theorem.

**Theorem 3.2.9.** Let $\mu$ be a Borel regular measure on a finite dimensional Banach space $X$ and let $A \subset X$ be a $\mu$–measurable subset such that $\mu(A) < \infty$. Suppose $F$ is a collection of closed balls such that $\inf\{ r : B(a, r) \in F \} = 0$ for all $a \in A$. Then there exists a countable subcollection disjoint of $F$ that covers $\mu$ almost all of $A$.

The proof of this result can be found in [Fed69] or in [Mat99], Theorem 2.7.
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Chapter 4

Busemann’s theorem and some of its consequences.

Technical by their nature, the results proved in this section will play a crucial role throughout this thesis.

4.1 Some preliminary definitions and remarks.

Let us recall a number of useful definitions and remarks. We first define a density and a volume density.

**Definition 4.1.1.** (Density and volume density) Let \((X, ||\ldots||)\) be an \(n\)--dimensional Banach space and let \(1 \leq m \leq n\). An \(m\)–density function is a continuous function \(\phi : G(m, X) \to \mathbb{R}_+\) that is positively homogenous of degree one. An \(m\)–density \(\phi\) is an \(m\)–volume density function if for all \(a \in G(m, X)\) one has \(\phi(a) \geq 0\) with equality if and only if \(a = 0\).

**Remark 4.1.1.** Let \(X\) and \(m\) be as in the previous definition and let \(\phi\) be a \(m\)–volume density function on \(X\). Then, for a \(m\)–rectifiable set \(S \subset X\) one defines the corresponding volume \(\text{Vol}_\phi(S)\) as

\[
\text{Vol}_\phi(S) := \int_S \phi(T_x S) d\mathcal{H}^m(x),
\]

where \(T_x S\) stands for the corresponding approximate tangent \(m\)–plane, see \(3.1.10\).

We proceed with a very natural example of a volume density.

**Definition 4.1.2.** (Busemann–Hausdorff density) Let \((X, ||\ldots||)\) be an \(n\)--dimensional Banach space with the unit ball \(B = \{x \in X : ||x|| \leq 1\}\). For \(1 \leq m \leq n\) we define the Busemann–Hausdorff density function \(\phi_{BH} : G(m, X) \to \mathbb{R}_+\) as

\[
\phi_{BH}(W) = \frac{\alpha_m}{\mathcal{H}^m(B \cap W)},
\]
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where \( W \in G(m, X) \). We further define one function which is a slight modification of the \( n - 1 \) dimensional Busemann–Hausdorff density. This function will be frequently used in this chapter.

**Definition 4.1.3.** (Function \( \sigma_B \)) Let \((X, \| \ldots \|)\) be an \( n \)-dimensional Banach space with the unit ball \( B = \{ x \in X : \|x\| \leq 1 \} \). Define a function \( \sigma_B : X \to \mathbb{R}_+ \) as follows

\[
\sigma_B(f) := \alpha_{n-1} \frac{|f|}{\mathcal{H}^{n-1}(B \cap f^\perp)} \left( = \phi_{BH}(f^\perp)|f| \right),
\]

where by \( f^\perp \) we denote the hyperplane orthogonal to the vector \( f \).

Let us give two examples of Busemann–Hausdorff functions.

**Remark 4.1.2.** In case when \( X = (\mathbb{R}^2, l_2) \), it is obvious that

\[
\{ x \in X : \sigma_B(x) = 1 \} = \{ x \in X : \|x\|_2 = 1 \}.
\]

**Remark 4.1.3.** In case when \( X = (\mathbb{R}^2, l_\infty) \), direct calculation shows that

\[
\{ x \in X : \sigma_B(x) = 1 \} = \{ x \in X : \|x\|_\infty = 1 \}.
\]

The following property of densities is going to be very important for us.

**Definition 4.1.4.** (Extendibly convex densities) A \( k \)-volume density function \( \phi \) on an \( n \)-dimensional Banach space \( X, n > k \) is said to be **extendibly convex** if it is the restriction of a norm on \( \Lambda^n(X) \) to the cone of simple \( k \)-vectors in \( X \).

**Remark 4.1.4.** Note that in case when \( m = n - 1 \) the notion of extendibly convex density coincides with the usual convexity.

### 4.2 Busemann’s convexity theorem in codimension one.

We recall a classical result concerning sections of convex sets in finite dimensional Banach spaces. This result is a theorem first proved by Herbert Busemann. The proof presented here is taken from the book by Thompson [Tho96].

**Theorem 4.2.1.** (Busemann) Let \((X, \| \ldots \|)\) be an \( n \)-dimensional Banach space with the unit ball \( B = \{ x \in X : \|x\| \leq 1 \} \). Then the function \( \sigma_B \) is convex.

**Proof.** Since \( \sigma_B \) is a positively homogeneous function of degree one, we need only to show that it is subadditive. Take two vectors \( f_1, f_2 \in X \). Define by \( f_3 = f_1 + f_2 \) and by \( L \) the two-dimensional subspace of \( X \) generated by \( f_1 \) and \( f_2 \). Let \( M \) denote the orthogonal complement of \( L \) and let \( f_i, M_i := \text{span} \{ f_i, f \} \). Let \( \phi_1, \phi_2 \geq 0 \) and let

\[
g_i = \phi_i \frac{f_i}{|f_i|}, i = 1, 2.
\]
4.2 Busemann’s convexity theorem in codimension one.

Define vector \( g_3 \) as the intersection of the segment \([g_1, g_2]\) and the ray from 0 through \( f_3 \) (hence \( g_3 := \phi_3 f_3 / |f_3| \) for some real number \( \phi_3 (= \phi_3(\phi_1, \phi_2)) \geq 0 \)). We introduce two more abbreviations concerning areas of slices of \( B \), let

\[
\rho_i(\phi) := \mathcal{H}^{n-2}\left( B \cap \left( M + \phi \frac{f_i}{|f_i|} \right) \right)
\]

for \( \phi \geq 0 \) and \( i = 1, 2, 3 \), and let \( V_i := \int_0^\infty \rho_i(\phi) d\phi \).

We collect most of the introduced notations in the following picture:

Note that the functions \( \rho_i \) are integrable, and that \( V_i = \mathcal{H}^{n-1}(B \cap M_i) \). Both statements follow from the Fubini theorem. We claim that the functions \( \rho_i \) are continuous. Indeed,

\[
\rho_i(\phi) = \mathcal{H}^{n-2}\left( (B \cap M_i) \cap \left( M + \phi \frac{f_i}{|f_i|} \right) \right)
\]

since \((M + \phi f_i / |f_i|) \subset M_i\). Furthermore the set \( B \cap M_i \) is convex and lies in the \((n-1)\)-dimensional space \( M_i \). Hence by Brunn’s lemma [4.2.1] each function \( \rho_i \) is continuous as a composition of two continuous functions, and our claim follows.
Throughout this thesis the symbol \( \triangle \) stands for a triangle.
We will first show that
\[
\frac{|f_1|}{V_1} + \frac{|f_2|}{V_2} \geq \frac{|f_3|}{V_3}. \tag{4.1}
\]
We start with calculating the coefficient \( \phi_3 = \phi_3(\phi_1, \phi_2) \) in two different ways. Comparing the areas of the triangles \( \triangle 0 g_1 g_2, \triangle 0 g_1 g_3 \) and \( \triangle 0 g_2 g_3 \) gives
\[
\phi_1 \phi_2 \sin(g_1 0 g_2) = \phi_1 \phi_3 \sin(g_1 0 g_3) + \phi_2 \phi_3 \sin(g_2 0 g_3),
\]
and hence
\[
\frac{\sin(g_1 0 g_2)}{\phi_3} = \frac{\sin(g_2 0 g_3)}{\phi_1} + \frac{\sin(g_1 0 g_3)}{\phi_2}. \tag{4.2}
\]
Likewise, using the law of sines for the triangles \( \triangle 0 f_1 f_2, \triangle 0 f_1 f_3 \) and \( \triangle 0 f_2 f_3 \) we conclude that
\[
\frac{\sin(f_1 0 f_2)}{|f_3|} = \frac{\sin(f_1 0 f_3)}{|f_2|} = \frac{\sin(f_2 0 f_3)}{|f_1|},
\]
and hence, thanks to the line \( 4.2 \), we infer that
\[
\frac{|f_3|}{\phi_3} = \frac{|f_1|}{\phi_1} + \frac{|f_2|}{\phi_2}. \tag{4.3}
\]
On the other hand, expressing \( g_3 \) as a convex combination of \( g_1 \) and \( g_2 \), we get that \( g_3 = \alpha g_1 + (1 - \alpha) g_2 \) for some \( \alpha \in (0, 1) \). Hence
\[
\frac{\phi_3}{|f_3|} f_3 = \alpha \frac{\phi_1}{|f_1|} f_1 + (1 - \alpha) \frac{\phi_2}{|f_2|} f_2.
\]
Comparing this representation of \( f_3 \) with the definition of \( f_3 \) we conclude that
\[
\alpha = \frac{\phi_3}{|f_3|} \frac{|f_1|}{\phi_1}, (1 - \alpha) = \frac{\phi_3}{|f_3|} \frac{|f_2|}{\phi_2}. \tag{4.4}
\]
As a direct consequence of the definitions of \( g_1, g_2 \) and \( g_3 \) and of the convexity of \( B \), we infer that \( B \cap (M + g_3) \supseteq \alpha[B \cap (M + g_1)] + (1 - \alpha)[B \cap (M + g_2)] \). Hence also
\[
\rho_3(\phi_3) = \mathcal{H}^{n-2}(B \cap (M + g_3)) \geq \mathcal{H}^{n-2}(\alpha[B \cap (M + g_1)] + (1 - \alpha)[B \cap (M + g_2)]). \tag{4.5}
\]
We will need the following lemma, which we present here with a proof. This lemma is sometimes called Brunn’s theorem.

**Lemma 4.2.1.** (Brunn) Let \( (X, \| \ldots \|) \) be an \( n \)-dimensional Banach space and let \( K \subset X \) be a convex body. Define its parallel slice \( K_t \) as an intersection of \( K \) with the hyperplane
\[
W_t = \{ x \in X : x = \sum_{i=1}^n \lambda_i e_i, \lambda_1 = t \},
\]
i.e. \( K_t := K \cap W_t \). Then the function
\[
t \to \left( \mathcal{H}^{n-1}(K_t) \right)^{\frac{1}{n-1}}
\]
is convex.
4.2 Busemann’s convexity theorem in codimension one.

Proof. Let \( s, r, t \in \mathbb{R} \) with \( s = (1 - \lambda)r + \lambda t \) for some \( \lambda \in (0, 1) \), and let \( K_s, K_r, K_t \) be \((n - 1)\)-dimensional orthogonal projections of \( K \), as in the formulation of Lemma 4.2.1. Project by means of the orthgonal projection \( \pi \) the sets \( K_s, K_r, K_t \) onto the hyperplane \( W_t \). We claim that the set \( \pi(K_s) \) contains the set \( \lambda \pi(K_r) + (1 - \lambda)\pi(K_t) \). To show this, take two points \( x \in \pi(K_r) \subset W_t \) and \( y \in \pi(K_t) \subset W_t \), connect the points \( \tilde{x} = (r, x) = \pi^{-1}(x) \cap W_r \) and \( \tilde{y} = (t, y) = \pi^{-1}(y) \cap W_t \) by a straight line. Note that since \( K \) is convex, the segment of this line between the points \( \tilde{x} \) and \( \tilde{y} \) lies entirely inside \( K \). In particular, the point \( \tilde{z} \) lies inside \( K \). Note, that this point belongs also to the hyperplane \( W_s \), whence also to the set \( K_s \). But this means that \( \pi(\tilde{z}) = (1 - \lambda)x + \lambda y \in \pi(K_s) \) and hence the claim follows. Next, since the sets \( \pi(K_s), \pi(K_r) \) and \( \pi(K_t) \) lie in the same hyperplane, we are allowed to use the additive version of the Brunn–Minkowski inequality (see [Fed69], Theorem 3.2.41):

\[
\mathcal{H}^{n-1}(K_t)^{\frac{1}{n-1}} = \mathcal{H}^{n-1}(\pi(K_t))^{\frac{1}{n-1}} \geq \mathcal{H}^{n-1}(\lambda \pi(K_r) + (1 - \lambda)\pi(K_t)))^{\frac{1}{n-1}} \geq \\
\lambda(\mathcal{H}^{n-1}(\pi(K_r)))^{\frac{1}{n-1}} + (1 - \lambda)(\mathcal{H}^{n-1}(\pi(K_t)))^{\frac{1}{n-1}} = \\
\lambda(\mathcal{H}^{n-1}(K_t))^{\frac{1}{n-1}} + (1 - \lambda)(\mathcal{H}^{n-1}(K_t))^{\frac{1}{n-1}},
\]

and the lemma follows.

Note that the sets \( B \cap (M + g_1) \) and \( B \cap (M + g_2) \) lie in the same hyperplane (namely in the hyperplane span\( \{M + g_1, g_1g_2\} \)). We apply Lemma 4.2.1 to this pair of sets and to the space span\( \{M + g_1, g_1g_2\} \). Hence, taking into account estimate (4.5) we obtain the following chain of inequalities:

\[
\rho_3(\phi_3) \geq \mathcal{H}^{n-2}(\alpha[B \cap (M + g_1)] + (1 - \alpha)[B \cap (M + g_2)]) \geq \\
\left(\alpha \rho_1(\phi_1)^{\frac{1}{n-2}} + (1 - \alpha)\rho_2(\phi_2)^{\frac{1}{n-2}}\right)^{n-2}.
\]

We continue our estimates now with the help of the Young inequality:

\[
\rho_3(\phi_3) \geq \left(\rho_1(\phi_1)^{\frac{n}{n-2}} \rho_2(\phi_2)^{\frac{n-2}{n-2}}\right)^{n-2} = \rho_1(\phi_1)^{\alpha} \rho_2(\phi_2)^{1-\alpha}. \tag{4.6}
\]

The inequality that we have just derived is the main part of the proof of the theorem. Note that since the set \( B \) is bounded, in the definition of the integrals \( V_i \) one can consider only bounded domains (intervals) of integration (i.e. the intervals, where \( \rho_i(\phi) \neq 0 \) or in other words we can consider only those \( \phi \) for which \( B \cap (M + \phi f_i | f_i|) \neq \emptyset \)). For \( i = 1, 2 \) we reparameterize the functions \( \rho_i \) in a way that both sets over which we integrate in \( V_i, i = 1, 2 \) become the interval \((0, 1)\). To this end, for each \( t \in (0, 1) \) we define implicit functions \( \phi_1 = \phi_1(t) \) and \( \phi_2 = \phi_2(t) \) by the formulae

\[
t = \frac{1}{V_1} \int_0^{\phi_1(t)} \rho_1(u) du \quad \text{and} \quad t = \frac{1}{V_2} \int_0^{\phi_2(t)} \rho_2(u) du.
\]
Let us now use Young's inequality and the identity (4.3) to estimate the term $V$ further to calculate the derivative of the functions $\phi_1$ and $\phi_2$ for almost all $t \in (0, 1)$:

$$1 = \frac{1}{V_1} \frac{d}{dt} \left( \int_0^{\phi_1(t)} \rho_1(u) du \right) = \frac{1}{V_1} \frac{d}{dt} (R_1(\phi_1(t))) = \frac{1}{V_1} \rho_1(\phi_1(t)) \frac{d\phi_1}{dt}.$$

Since the very same estimate holds for the function $\rho_2$, we can calculate the derivatives of the functions $\phi_1$ and $\phi_2$ for almost all $t \in (0, 1)$:

$$\frac{d\phi_1}{dt} = \frac{V_1}{\rho_1(\phi_1)} \quad \text{and} \quad \frac{d\phi_2}{dt} = \frac{V_2}{\rho_2(\phi_2)}.$$  \hspace{1cm} (4.7)

We define function $\phi_3$ with the help of the identity (4.3):

$$\phi_3(t) = \frac{|f_3|}{|f_1\phi_1(t)| + |f_2\phi_2(t)|}.$$

Next we use (4.7) in order to conclude that $\phi_3$ is differentiable almost everywhere and further to calculate the derivative $\frac{d\phi_3}{dt}$ for almost all $t \in (0, 1)$:

$$\frac{|f_3|}{\phi_3^2} \frac{d\phi_3}{dt} = \frac{|f_1|}{\phi_1^2} \frac{d\phi_1}{dt} + \frac{|f_2|}{\phi_2^2} \frac{d\phi_2}{dt} = \frac{|f_1|V_1}{\phi_1^2 \rho_1(\phi_1)} + \frac{|f_2|V_2}{\phi_2^2 \rho_2(\phi_2)}. \hspace{1cm} (4.8)$$

Having obtained the estimates for the derivatives of the functions $\phi_i$, $i = 1, 2, 3$, we proceed to the estimate of the term $V_3/|f_3|$ from the inequality (4.1). We first use the definition of the term $V_3$ and the inequalities (4.6) and (4.8):

$$\frac{V_3}{|f_3|} = \frac{1}{|f_3|} \int_0^1 \rho_3(t) dt = \frac{1}{|f_3|} \int_0^1 \rho_3(\phi_3(t)) \frac{d\phi_3}{dt} dt \geq \frac{1}{|f_3|} \int_0^1 \rho_1(\phi_1)^\alpha \rho_2(\phi_2)^{1-\alpha} \frac{d\phi_3}{dt} dt \geq$$

$$= \frac{1}{|f_3|} \int_0^1 \rho_1(\phi_1)^\alpha \rho_2(\phi_2)^{1-\alpha} \left( \frac{|f_1|V_1 \phi_3^2}{\phi_1^2 \rho_1(\phi_1)|f_3|} + \frac{|f_2|V_2 \phi_3^2}{\phi_2^2 \rho_2(\phi_2)|f_3|} \right) dt.$$

Let us now use Young’s inequality and the identity (4.3):

$$\frac{V_3}{|f_3|} \geq \int_0^1 \frac{\rho_1(\phi_1)^\alpha \rho_2(\phi_2)^{1-\alpha}}{|f_3|} \left( \frac{V_1 \phi_3^2}{\phi_1 \rho_1(\phi_1)|f_3|} \right)^\alpha \left( \frac{V_2 \phi_3^2}{\phi_2 \rho_2(\phi_2)|f_3|} \right)^{1-\alpha} \frac{|f_3|}{\phi_3} dt =$$

$$= \int_0^1 \left( \frac{V_1}{\phi_1} \right)^\alpha \left( \frac{V_2}{\phi_2} \right)^{1-\alpha} \frac{\phi_3}{|f_3|} dt.$$
4.3 Area non–increasing projections in codimension one.

Thanks to the Young inequality and to the formulae (4.4), we are ready now to finish off the proof of the inequality (4.1):

\[ \frac{V_3}{|f_3|} \geq \int_0^1 \left( \frac{\alpha_1}{V_1} + \frac{(1-\alpha_2)}{V_2} \right)^{-1} \frac{\phi_3}{|f_3|} dt = \left( \frac{|f_1|}{V_1} + \frac{|f_2|}{V_2} \right)^{-1}. \]

Hence we have just proved that \( |f_1|/V_1 + |f_2|/V_2 \geq |f_3|/V_3 \) for all \( f_1, f_2 \) and \( f_3 \) such that \( f_3 = f_1 + f_2 \). Note that this inequality is not exactly what we need in the formulation of the Busemann theorem, since apriori \( V_i \) is not equal to \( H^{n-1}(B \cap f_i^\perp) \). We reduce the desired inequality to the inequality (4.1). To this end, we define lines \( l_i, i = 1, 2 \) as \( l_i := L \cap f_i^\perp \) and denote the corresponding unit vectors by \( h_i, i = 1, 2 \). Note that \( h_i \) is orthogonal to \( f_i \).

Next we take the vectors \( f_i^\ast := |f_i|h_i \) and apply to them the inequality (4.1):

\[ \frac{|f_1|}{V_1} + \frac{|f_2|}{V_2} = \frac{|f_i^\ast|}{V_1} + \frac{|f_i^\ast|}{V_2} = \frac{|f_3|}{V_3}, \]

where \( \tilde{V}_i := H^{n-1}(B \cap f_i^\perp) \). This finishes the proof of Theorem 4.2.1. \( \square \)

4.3 Area non–increasing projections in codimension one.

Next we present another theorem of Busemann which we will frequently use afterwards. This result will provide us with a formula for the Hausdorff measure \( H^{n-1}_{||\cdot||}(U) \) of the Borel subsets of the hyperplanes.

**Theorem 4.3.1.** (Busemann) Let \( (X, ||\cdot||) \) be an \( n \)-dimensional Banach space with the unit ball \( B = \{ x \in X : ||x|| \leq 1 \} \), and let \( W \) be a \((n-1)\)-dimensional linear subspace of \( X \). Then, for every Borel subset \( U \subset W \) the following formula holds true:

\[ H^{n-1}_{||\cdot||}(U) = \phi_{BH}(W) \cdot H^{n-1}(U) \left( = \alpha_{n-1} \frac{H^{n-1}(U)}{H^{n-1}(B \cap W)} \right). \]

**Proof.** Let us write

\[ \mu_{B,W}(U) := \alpha_{n-1} \frac{H^{n-1}(U)}{H^{n-1}(B \cap W)} \]

for brevity. Note that both \( \mu_{B,W} \) and \( H^{n-1}_{||\cdot||} \land W \) are Haar measures on \( W \). According to [Mat99] Theorem 3.1 this means that these measures are proportional. Hence it is sufficient to show that \( \mu_{B,W}(B \cap W) = H^{n-1}_{||\cdot||} \land W(B) \). Since \( \mu_{B,W}(B \cap W) = \alpha_{n-1} \), we are done once we prove that \( H^{n-1}_{||\cdot||} \land W(B) = \alpha_{n-1} \). To this end, we prove one more lemma.

**Lemma 4.3.1.** Let \( (X, ||\cdot||) \) be an \( n \)-dimensional Banach space with the unit ball \( B = \{ x \in X : ||x|| \leq 1 \} \), and let \( \lambda \) be a Haar measure on \( X \). Then for all measurable sets \( U \subset X \) one has

\[ H^n_{||\cdot||}(U) = \alpha_n \frac{\lambda(U)}{\lambda(B)}. \]
Proof. We shall prove two inequalities, namely
\[ \mathcal{H}^n_{\|\cdot\|}(U) \leq \alpha_n \frac{\lambda(U)}{\lambda(B)} \quad \text{and} \quad \mathcal{H}^n_{\|\cdot\|}(U) \geq \alpha_n \frac{\lambda(U)}{\lambda(B)}. \]

We start with the first one. Fix \( \varepsilon > 0 \) and \( \delta > 0 \). Note that since \( U \) is measurable, there exists an open set \( G(= G_\varepsilon) \) such that \( U \subset G \) and \( \lambda(G \setminus U) < \varepsilon / 2 \). Hence, according to the Vitali covering theorem, there also exist countable collections of points \( x'_i \in U \) and of numbers \( \rho'_i < \delta / 2 \) such that the balls \( B(x'_i, \rho'_i) \) are pairwise disjoint and the set \( C := \bigcup_{t=1}^\infty B(x'_i, \rho'_i) \) enjoys \( C \subset G \) and \( \lambda(U \setminus C) = 0 \). Furthermore, since \( \lambda(U \setminus C) = 0 \), there exist another countable collections of points \( x''_i \) and of numbers \( \rho''_i < \delta / 2 \) such that \( U \setminus C \subset \bigcup_{t=1}^\infty B(x''_i, \rho''_i) \) and
\[ \sum_{i=1}^\infty \lambda(B(x''_i, \rho''_i)) < \varepsilon / 2. \]

Combining these collections into one family \( \{B(x_i, \rho_i), i = 1, 2, \ldots\} \) we infer that there holds \( U \subset \bigcup_{i=1}^\infty B(x_i, \rho_i) \) and therefore
\[ \sum_{i=1}^\infty \lambda(B(x_i, \rho_i)) < \varepsilon + \lambda(U). \]

From here we conclude that
\[ \alpha_n \sum_{i=1}^\infty \rho^n_i < \frac{\alpha_n \varepsilon}{\lambda(B)} + \frac{\lambda(U)}{\lambda(B)}. \]

Since \( \varepsilon > 0 \) is arbitrary, the first inequality follows.

Next we proceed to the reciprocal inequality. For its proof we shall need an auxiliary result, which is called the isodiametric inequality.

Lemma 4.3.2. (Isodiametric inequality) Let \( \lambda \) be a Haar measure on a \( d \)-dimensional Banach space \( (X, \|\cdot\|) \) with the unit ball \( B = \{ x \in X : \|x\| \leq 1 \} \), and let \( A \) be a measurable subset of \( X \). Then
\[ 2^d \lambda(A) \leq (\operatorname{diam}(A))^d \lambda(B). \]

Proof. Firstly, the inequality is trivial if either \( A \) is null set, or \( A \) is unbounded, hence we may consider only bounded subsets \( A \subset X \) of positive measure. Secondly, since \( \operatorname{diam}(A) = \operatorname{diam}(\text{cl}(A)) \) (where by \( \text{cl}(A) \) we mean the closure of the set \( A \)), it is sufficient to prove the inequality for closed and bounded (i.e. compact) sets. Thirdly, since the distance function is continuous, for a compact set \( A \) its diameter is attained: there exist some \( a_1 \in A \) and \( a_2 \in A \) such that \( \rho = \operatorname{diam}(A) = \|a_1 - a_2\| \). Hence the ball \( B_{\|\cdot\|}(a_1, \rho) \) contains \( A \). Since the set \( B_{\|\cdot\|}(a_1, \rho) \) is convex, it also contains \( \text{conv}(A) \), the convex hull of the set \( A \). Since \( a_1 \) was an arbitrary point, at which the diameter is attained, we conclude that \( \operatorname{diam}(\text{conv}(A)) = \operatorname{diam}(A) \). Since \( \lambda(\text{conv}(A)) \geq \lambda(A) \), we infer that it is left to prove Lemma 4.3.2 for convex sets.
Let $K$ be a convex body with the diameter $\rho$. Hence $||x - y|| \leq \rho$ for all $x, y \in K$. Therefore, $K + (-K) \subset \rho B$, and the Brunn–Minkowski inequality gives

$$2^d \lambda(K) = \left(\lambda(K)^{\frac{1}{d}} + \lambda(-K)^{\frac{1}{d}}\right)^d \leq \lambda(K + (-K)) \leq \rho^d \lambda(K),$$

which finishes the proof of Lemma 4.3.2.

Now let $A_i$ be an arbitrary covering of the set $U$ by sets of diameter less than $\delta$. Then

$$\lambda(U) \leq \lambda\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \lambda(A_i) \leq \sum_{i=1}^{\infty} \left(\frac{\text{diam}(A_i)}{2}\right)^n \lambda(B),$$

where the last inequality follows from Lemma 4.3.2. Hence Lemma 4.3.1 follows.

In order to finish the proof of Theorem 4.3.1 it suffices to apply Lemma 4.3.1 to the space $X$ defined as $X := (W, |||\ldots|||)$, where the norm $|||\ldots|||$ is the one given by the convex set $B \cap W$, the set $U$ defined as the intersection between $B$ and $W$ and any Haar measure $\lambda$ on the space $X$.

The following theorem was once again first discovered by Busemann. This result will play crucial role in the following two chapters, especially in the proofs of the existence for the Plateau problem in codimension one. The proof presented here is the one taken from the article [Bus50].

**Theorem 4.3.2.** (Busemann) Let $(X, |||\ldots|||)$ be an $n$–dimensional Banach space and let $W$ be a $(n - 1)$–dimensional linear subspace of $X$. Then there exists a linear projection $\pi_W : X \to W$ with the range $W$, such that $\mathcal{H}^{n-1}(\pi_W(A)) \leq \mathcal{H}^{n-1}(A)$ for any Borel subset $A \subset V$, where $V$ is some $(n - 1)$–dimensional linear subspace of $X$.

**Proof.** We construct the desired projection by a more or less direct procedure. In more details, the projection will resemble a nearest point projection with respect to some norm given by a very special convex set. We start by constructing this convex set. Note that, thanks to Theorem 4.2.1, we know that the set $K = \{x \in X : \sigma_B(x) \leq 1\}$ is convex. We remind to the reader the definition of the function $\sigma_B : X \to \mathbb{R}_+$, $\sigma_B(x) := a_{n-1}|x|/\mathcal{H}^{n-1}(B \cap x^\perp)$, where by $x^\perp$ we denote the hyperplane orthogonal to the vector $x$. Since the ball $K$ contains the point 0, we can introduce its polar dual set. In more details, the set $K^* := \{x \in X : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$ is called the polar dual of $K$. Note that $K^*$ is a convex symmetric set, containing the point 0. Therefore, it gives rise to the norm $|||\ldots|||$, simply by setting $K^*$ to be the unit ball of this norm. Next, given a hyperplane $W$, let $\omega$ be a vector such that $W$ is a supporting hyperplane (see 3.1.2) of the ball $B_{|||\ldots|||}(\omega, |||\omega|||)$ at the point 0.

We claim that the projection parallel to the vector $\omega$ is area non–increasing. In order to prove this statement, we shall need two more definitions. First we introduce the supporting function $S(K, \cdot) : X \to \mathbb{R}_+$ of the set $K$ as

$$S(K, u) := \sup_{x \in K} \langle u, x \rangle.$$
Notice that the value $S(K, u)$ is equal to the distance from the point 0 to a supporting hyperplane $H$ of the set $K$ at some point $x_0 \in K$ such that the unit outward normal vector to $H$ is equal to $u/|u|$. Let us also define function $\rho(K, \cdot) : X \rightarrow \mathbb{R}_+$, which is called Minkowski functional of $K$, as

$$\rho(K, u) := \sup\{\lambda \in \mathbb{R}_+ : \lambda u \in K\}.$$ 

We infer that since $K$ and $K^*$ are polar dual, one has $S(K^*, u) = 1/\rho(K, u)$. Hence, taking into account the definition of the set $K$, we conclude that

$$\frac{1}{S(K^*, u)} = \rho(K, u) = \mathcal{H}^{n-1}(B \cap u^\perp).$$

(4.9)

Note that it is left to prove the following statement: if $Z$ is a cylinder defined as $Z = \pi^{-1}(A) = \{x \in X : x = a + t\omega, a \in A, t \in \mathbb{R}\}$, then

$$\mathcal{H}^{n-1}_|||\,(Z \cap V) \geq \mathcal{H}^{n-1}_|||\,(A),$$

(4.10)

where $V$ and $A$ are as in the formulation of the theorem. We rewrite both sides of (4.10) using Theorem 4.3.1.

$$\mathcal{H}^{n-1}_|||\,(Z \cap V) = \frac{\mathcal{H}^{n-1}(Z \cap V)}{\mathcal{H}^{n-1}(B \cap V)} = \frac{\mathcal{H}^{n-1}(Z \cap \omega^\perp)}{\mathcal{H}^{n-1}(B \cap V) \cos(\omega, V^\perp)},$$

$$\mathcal{H}^{n-1}_|||\,(A) = \frac{\mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(B \cap W)} = \frac{\mathcal{H}^{n-1}(Z \cap \omega^\perp)}{\mathcal{H}^{n-1}(B \cap W) \cos(\omega, W^\perp)},$$

where by $V^\perp$ we denote the outward unit normal vector of the hyperplane $V$. We collect most of the introduced notations in the picture ?? page 43. Comparing these identities with (4.10) and taking into account formula (4.9) applied to $u = V^\perp$ and $u^\perp = V$, we observe that it is left to prove the following inequality

$$\frac{S(K^*, V^\perp)}{\cos(\omega, V^\perp)} \geq \frac{S(K^*, W^\perp)}{\cos(\omega, W^\perp)}.$$ 

But this is an obvious consequence of the definition of the vector $\omega$. Indeed, the expression $S(K^*, V^\perp)/\cos(\omega, V^\perp)$ is equal to the length of the segment which the hyperplane $V$ intercepts on the ray $r := \{t\omega, t \geq 0\}$, and, analogously, $S(K^*, W^\perp)/\cos(\omega, W^\perp)$ equals the length of the segment which the hyperplane $W$ intercepts on the same ray $r$. The endpoint of the first segment is outside of the set $K^*$, whereas the endpoint of the second one belongs to the boundary of $K^*$. \hfill \Box

**Remark 4.3.1.** Notice that the result of Theorem 4.3.1 is non–trivial. For instance, this can be illustrated by the fact that the Euclidean orthogonal projections machinery seems to be unusefull here.
4.3 Area non-increasing projections in codimension one.
We prove one more theorem, which is a generalization of the formula from the formulation of Theorem [4.3.1] to the case of the \((n - 1)\)-rectifiable subsets of \(X\). We shall need this result while proving Theorem 4.3.4. This theorem was first formulated and proved in the present thesis by I. Vasilyev.

**Theorem 4.3.3.** (I. Vasilyev) Let \((X, ||\ldots||)\) be an \(n\)-dimensional Banach space. If \(A\) is a \((n - 1)\)-rectifiable subset of \(X\), then the following formula holds true:

\[
\mathcal{H}_{||\ldots||}^{n-1}(A) = \int_A \phi_{BH}(T_x A) d\mathcal{H}^{n-1}(x).
\]

**Remark 4.3.2.** Note that the integrand here has sense since \(T_x A \in G(n - 1, X)\) exists almost everywhere, thanks to Theorem 3.2.6.

**Remark 4.3.3.** Theorem 4.3.2 holds in a more general setting. Namely it is also true for \(m\)-rectifiable subsets of \(X\) for all \(1 \leq m \leq n\). Since we are not using this generalization in this thesis, we do not prove it here.

**Proof.** We need one technical lemma before starting the proof of the theorem.

**Lemma 4.3.3.** Let \(M \subset X\) be a \((n - 1)\)-dimensional \(C^1\)-manifold, and let \(x_0 \in M\). Then

\[
\phi_{BH}(T_{x_0} M) = \lim_{r \to 0} \frac{\mathcal{H}_{||\ldots||}^{n-1}(M \cap B(x_0, r))}{\alpha_{n-1} r^{n-1}}.
\]

**Proof.** With no loss of generality we suppose \(x_0 = 0\). For sake of brevity we write \(H := T_0 M\). We also fix an orthonormal basis \(e_1, \ldots, e_n\) of the space \(X\), in a way that \(e_1, \ldots, e_{n-1}\) becomes a basis of \(H\). First of all note that since \(M\) is a \(C^1\)-manifold of dimension \(n - 1\), it is locally a Lipschitz graph. In more details, for every \(\varepsilon > 0\) there exists \(r_0 = r_0(\varepsilon)\) such that for every \(r < r_0\) the set \((M \cap (H \cap B(0, r)) + H^\perp)\) can be written as the image of the map \(F : H \cap B(0, r) \to M \cap ((H \cap B(0, r)) + H^\perp)\) given by the formula \(F(x) = x + e_n f(x)\) for some function \(f : H \cap B(0, r) \to \mathbb{R}\), which is \(\varepsilon\)-Lipschitz. Second of all we claim that for every \(\varepsilon > 0\) and for every \(r < r_0(\varepsilon)\) there holds

\[
(1 + ||e_n||\varepsilon)^{n-1} \mathcal{H}_{||\ldots||}^{n-1}(H \cap B(0, r)) \geq \mathcal{H}_{||\ldots||}^{n-1}(F(H \cap B(0, r))) \geq (1 - ||e_n||\varepsilon)^{n-1} \mathcal{H}_{||\ldots||}^{n-1}(H \cap B(0, r)).
\]

(4.11)

Indeed, note that for all \(x, y \in H \cap B(0, r)\) one has

\[
||x - y|| + |f(x) - f(y)| \cdot ||e_n|| \geq ||F(x) - F(y)|| \geq ||x - y|| - |f(x) - f(y)| \cdot ||e_n||.
\]

But the function \(f(x)\) is \(\varepsilon\)-Lipschitz on \(H \cap B(0, r)\). As a consequence of this fact we get that

\[
(1 + \varepsilon) ||x - y|| \geq ||F(x) - F(y)|| \geq (1 - \varepsilon) ||x - y||,
\]

which in turn yields

\[
(1 + \varepsilon)^{n-1} \mathcal{H}_{||\ldots||}^{n-1}(H \cap B(0, r)) \geq \mathcal{H}_{||\ldots||}^{n-1}(F(H \cap B(0, r))) \geq (1 - \varepsilon)^{n-1} \mathcal{H}_{||\ldots||}^{n-1}(H \cap B(0, r)).
\]
where $\tilde{e} = ||e_n||\varepsilon$ and our claim (4.11) follows. Third of all, we infer that for the fixed $\epsilon$ there exists $r_1 = r_1(\varepsilon)$ such that for all $r < r_1(\varepsilon)$ one has

$$\mathcal{H}^{n-1}(M \cap B(0, r)) \leq \mathcal{H}^{n-1}(F(H \cap B(0, r))) \leq \varepsilon + \mathcal{H}^{n-1}(M \cap B(0, r)).$$

(4.12)

Since the first inequality is trivial, let us concentrate on the second one. Since $M \cap B(0, r) \subseteq F(H \cap B(0, r))$, and since the measures $\mathcal{H}^{n-1}$ and $\mathcal{H}^{n-1}_{||\cdot||}$ are proportional (as both being Haar invariant measures), it is sufficient to show that the measure $\mathcal{H}^{n-1}(E_r)$, where $E_r := F(H \cap B(0, r)) \setminus (M \cap B(0, r))$ is small for $r$ small enough. To this end, we first infer that there exists $(1 - \varepsilon) \leq \gamma \leq 1$ such that for all $r < r_0$ there holds $P_H(E_r) \subseteq B(0, r) \setminus B(0, \gamma r)$, where $P_H$ is the Euclidean projection with the range $H$. For instance one can take $\gamma = 1/(1 + \varepsilon)$ thanks to the fact that $F$ is $(1 + \varepsilon)$–Lipschitz in the ball $B(0, r_0)$. Hence we have that

$$\mathcal{H}^{n-1}(P_H(E_r)) \leq \alpha_{n-1} \left( (\gamma r)^{n-1} - r^{n-1} \right) \leq C r^{n-1} \varepsilon$$

with a constant $C$ depending on $n$ only. Recall that since $F$ is $(1 + \varepsilon)$–Lipschitz, the Jacobian $JF$ (see 3.2.5) satisfies $|JF(x)| \leq (1 + \varepsilon)^n$ for all $x \in B(0, r_0)$. Now we estimate, using the area formula:

$$\mathcal{H}^{n-1}(E_r) = \int_{P_H(E_r)} |JF(x)| d\mathcal{H}^{n-1}(x) \leq C \varepsilon r^{n-1},$$

where the constant $C$ does not depend neither of $\varepsilon$, nor on $r$. Our claim (4.12) now follows. On the other hand,

$$\mathcal{H}^{n-1}_{||\cdot||}(H \cap B(0, r)) = \phi_{BH}(H) \mathcal{H}^{n-1}(H \cap B(0, r)) = \phi_{BH}(H) \alpha_{n-1} r^{n-1}. \quad (4.13)$$

Our conclusion now follows from the estimates (4.11), (4.12) and (4.13).

Now we proceed to the proof of Theorem 4.3.3. We first prove the theorem in case when $A$ is piece of a smooth manifold.

**Lemma 4.3.4.** Let $M$ be a $(n-1)$–dimensional $C^1$–submanifold of $X$, and let $A \subseteq M$ be a Borel set. Then the following formula holds true

$$\mathcal{H}^{n-1}_{||\cdot||}(A) = \int_A \phi_{BH}(T_x M) d\mathcal{H}^{n-1}(x).$$

**Proof.** First, in case when $\mathcal{H}^{n-1}(A) = \infty$ the lemma is trivial. Indeed, this follows from the fact that there exists a constant $C > 0$ such that for all $W \in G(n-1, X)$ there holds $\phi_{BH}(W) \geq C$. Indeed this a consequence of the fact that $\phi_{BH}$ is a continuous function defined on the compact set $G(n-1, X)$. Next we assume $\mathcal{H}^{n-1}(A) < \infty$ and further choose an open set $U$ such that $A \subseteq U$ and $\mathcal{H}^{n-1}(M \cap U) < \infty$. Fix $\varepsilon > 0$. Thanks to Theorem 4.2.1 we know that the function $\phi_{BH} : G(n-1, X) \rightarrow \mathbb{R}$ is convex, hence also continuous. Referring to this fact and also to Lemma 4.3.3 we infer that for every $x \in A$ there exists $\rho_x$ such that
Busemann’s theorem and some of its consequences.

1. $0 < \rho_x \leq \varepsilon$,
2. $M \cap B(x, \rho_x) \subseteq U$,
3. $|\phi_{BH}(T_x M) - \phi_{BH}(T_y M)| \leq \varepsilon$ for all $y \in M \cap B(x, \rho_x)$,
4. for all $0 < r \leq \rho_x$,
   $$\left| \frac{H_{\|\cdot\|}^{n-1}(M \cap B(x, r))}{\alpha_{n-1} r^{n-1}} - \phi_{BH}(T_x M) \right| \leq \varepsilon,$$
5. for all $0 < r \leq \rho_x$,
   $$\left| \frac{H_{\|\cdot\|}^{n-1}(M \cap B(x, r))}{\alpha_{n-1} r^{n-1}} - 1 \right| \leq \varepsilon,$$

where the last condition follows from the fact that $M$ is a smooth manifold. Thus the collection $\mathcal{B} = \{ B(x, r) : x \in A, 0 < r < \rho_x \}$ is a Vitali covering of the set $A$. Hence by the Besicovitch theorem 3.2.9 there exists a countable system of disjoint balls $\{ B(x_i, r_i) \}_{i \in I} \in \mathcal{B}$ such that $H_{\|\cdot\|}^{n-1}(A \setminus \bigcup_{i \in I} B(x_i, r_i)) = 0$. Moreover, since $\phi_{BH}$ is continuous and since the set $G(n - 1, X)$ is compact, we conclude that there exists a constant $C_0$ such that $\phi_{BH}(T_x M) \leq C_0$ for all $x \in M$. Now we estimate

$$\left| H_{\|\cdot\|}^{n-1}(A) - \int_A \phi_{BH}(T_x M) dH^{n-1}(x) \right| =$$

$$\sum_{i \in I} H_{\|\cdot\|}^{n-1}(A \cap B(x_i, r_i)) - \sum_{i \in I} \int_{A \cap B(x_i, r_i)} \phi_{BH}(T_x M) dH^{n-1}(x) \leq$$

$$\sum_{i \in I} \left( \left| \phi_{BH}(T_{x_i} M) \alpha_{n-1} r_{i}^{n-1} - \int_{A \cap B(x_i, r_i)} \phi_{BH}(T_x M) dH^{n-1}(x) \right| + \varepsilon \alpha_{n-1} r_{i}^{n-1} \right) \leq$$

$$\sum_{i \in I} \left( \int_{A \cap B(x_i, r_i)} |\phi_{BH}(T_{x_i} M) - \phi_{BH}(T_x M)| dH^{n-1}(x) + (1 + \phi_{BH}(T_x M)) \varepsilon \alpha_{n-1} r_{i}^{n-1} \right) \leq$$

$$\varepsilon \sum_{i \in I} H^{n-1}(M \cap B(x_i, r_i)) + (1 + C_0) \varepsilon \frac{1}{1 - \varepsilon} \sum_{i \in I} H^{n-1}(M \cap B(x_i, r_i)) \leq$$

$$\varepsilon \left( 1 + \frac{1 + C_0}{1 - \varepsilon} \right) H^{n-1}(M \cap U).$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows. \qed
4.3 Area non–increasing projections in codimension one.

Now we derive Theorem 4.3.3 from Lemma 4.3.3. Note that the rectifiability theorem tells us that there exist countably many $C^1$–manifolds $M_i$ such that $\mathcal{H}^{n-1}(A \setminus \bigcup_i M_i) = 0$ and for almost every $x \in A \cap M_i$ one has $T_x A = T_x M_i$, see [Fed69, Theorem 3.2.19]. Next we infer that there exists a disjoint union $A = \bigcup_i A_i \cup (A \setminus \bigcup_i A_i)$ such that $A_i \subset M_i$ are Borel sets and $\mathcal{H}^{n-1}(A \setminus \bigcup_i A_i) = 0$. Indeed, we first define $A_1 = A \cap M_1$, next $A_2 = A \cap M_2 \setminus A_1$, then $A_3 = A \cap M_3 \setminus (A_1 \cup A_2)$ and so on and so far. Now it is obvious that $A_i$ are pairwise disjoint and Borel. Moreover, one has $A \setminus \bigcup_i A_i \subseteq A \setminus \bigcup_i M_i$ hence $\mathcal{H}^{n-1}(A \setminus \bigcup_i A_i) = 0$. Since $A_i \subseteq M_i$ for each $i$ we can now use the previous lemma:

$$\mathcal{H}^{n-1}_{||...||}(A) = \sum_i \int_{A_i} \phi_{BH}(T_x M_i) d\mathcal{H}^{n-1}(x) = \sum_i \int_{A_i} \phi_{BH}(T_x A) d\mathcal{H}^{n-1}(x) = \int_A \phi_{BH}(T_x A) d\mathcal{H}^{n-1}(x),$$

and the proof is complete. \qed

The notion of good projections that we introduce in the following definition is going to be very important for us. These projections will play crucial role in the next chapter.

**Definition 4.3.1.** (Good projections) Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $W \in G(n-1, X)$. We say that a linear projection $\pi_W : X \to W$ is $W$–**good** if

$$\mathcal{H}^{n-1}_{||...||}(\pi_W(A)) \leq \mathcal{H}^{n-1}_{||...||}(A)$$

for any set $A \subset V$, where $V$ is some $(n-1)$–dimensional linear subspace of $X$.

**Remark 4.3.4.** It follows directly from Theorem 4.3.2 that in every finite dimensional Banach space $X$ each $W \in G(n-1, X)$ admits at least one $W$–good projection.

Next we introduce a number of classical definitions regarding measures on finite dimensional Banach spaces.

**Definition 4.3.2.** (Tangent measures) Let $\mu$ be a Radon measure on an $n$–dimensional Banach space $X$ and let $a \in X$. We say that $\nu$ is a $m$–**tangent measure** of the measure $\mu$ at the point $a$ if one has

$$\nu = \lim_{r \to 0} \frac{1}{r^m} T_{a,r,s} \mu,$$

where we take the limit in the sense of weak convergence of measures and the image measure $T_{a,r,s} \mu$ is defined as follows: $T_{a,r,s} \mu(A) = \mu(a + r \cdot A)$. We denote the set of all $m$–tangent measures of $\mu$ at the point $a$ as $\text{Tan}_m(\mu, a)$.

**Definition 4.3.3.** (Densities of measures) Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $0 \leq s \leq n$ be a real number. Suppose that $\mu$ is measure on $X$. The upper and lower $s$–densities of $\mu$ at $x \in X$ are defined by

$$\Theta^s_{||...||}(\mu, x) = \lim_{r \to 0} \frac{\mu(B_{||...||}(x, r))}{\alpha_s r^s},$$

where
Busemann’s theorem and some of its consequences.

\[ \Theta^s_{\|\ldots\|}(\mu, x) = \liminf_{r \to 0} \frac{\mu(B_{\|\ldots\|}(x, r))}{\alpha_s r^s}. \]

If they agree the common value

\[ \Theta^s_{\|\ldots\|}(\mu, x) = \Theta^s_{\|\ldots\|}(\mu, x) = \Theta^s_{\|\ldots\|}(\mu, x) \]

is called the \textbf{s-density} of \( \mu \) at \( x \).

In the following theorem we establish one important property of the good projections, namely that those projectors do not increase the Hausdorff measure of arbitrary \((n-1)\)-rectifiable subsets of \( X \). We will further need this result in our first proof of the existence of mass minimising rectifiable chains in codimension one. This result was first formulated and proved in the present thesis by I.Vasilyev.

\textbf{Theorem 4.3.4.} (I.Vasilyev) Let \((X, \|\ldots\|)\) be an \( n \)-dimensional Banach space. If \( A \) is a \((n-1)\)-rectifiable subset of \( X \), and if \( \pi_W : X \to W \) is a \( W \)-good projection then the following inequality holds true:

\[ \mathcal{H}_{\|\ldots\|}^{n-1}(\pi_W(A)) \leq \mathcal{H}_{\|\ldots\|}^{n-1}(A). \]

\textbf{Proof.} We precede the proof of the theorem with two technical lemmata.

\textbf{Lemma 4.3.5.} In the notation used in the theorem, if \( q \) is a norm on \( X \) and if \( \{\nu_j\}_{j=1}^{\infty} \) is a sequence of Radon measures on \( X \) that converges weakly in the sense of measures to a measure \( \nu \) then for all \( x \) we have

\[ \lim_{j \to +\infty} \nu_j(B_q(x, r)) = \nu(B_q(x, r)), \]

except of a countable number of \( r \) (which may depend on \( x \)).

\textbf{Proof.} Note that since \( \nu_j \to \nu \) weakly one has (see [Mat99], Theorem 1.24)

1. \( \nu(O) \leq \liminf_{j \to +\infty} \nu_j(O) \) for all open subsets \( O \subset X \),

2. \( \nu(K) \geq \limsup_{j \to +\infty} \nu_j(K) \) for all compact subsets \( K \subset X \).

We use these inequalities together with the fact that except of a countable number of \( r \) there holds \( \nu(\partial B_q(x, r)) = 0 \):

\[ \nu(B_q(x, r)) = \nu(\text{int}(B_q(x, r))) \leq \liminf_{j \to +\infty} \nu_j(B_q(x, r)) \leq \limsup_{j \to +\infty} \nu_j(B_q(x, r)) \leq \nu(B_q(x, r)), \]

where the last inequality follows from the fact that \( B_q(x, r) \) is compact since \( X \) is finite dimensional. (Here \( \text{int}(B_q(x, r)) \) denotes the interior of the closed ball \( (B_q(x, r)) \).)

\textbf{Lemma 4.3.6.} In the notation used in the theorem, let \( \mu \) be a Radon measure defined as \( \mu = \mathcal{H}_{\|\ldots\|}^{n-1}\|\ldots\| A \). Then for \( \mu \)-almost all \( x \in A \) one has \( \text{Tan}_{n-1}(\mu, x) = \{\mathcal{H}_{\|\ldots\|}^{n-1}\|\ldots\| W_x \} \) for some hyperplane \( W_x \subset G(n-1, X) \).
4.3 Area non–increasing projections in codimension one.

**Proof.** Denote by $\mu_x$ the unique element of the set $\text{Tan}_{n-1}(\mu, x)$, the uniqueness follows from [Mat99], Theorem 15.19. Since $A$ is rectifiable, according to Theorem 4.3.3 we infer that there exists a function $\theta \in L^1_{\text{loc}}(\mathcal{H}^{n-1} \cap A)$ such that $\mu = \theta \cdot \tilde{\mu}$, where $\tilde{\mu} = \mathcal{H}^{n-1} \cap A$. Note that for $\mu$–almost all $x \in A$ there holds $\text{Tan}_{n-1}(\tilde{\mu}, x) = \{\mathcal{H}^{n-1} \cap W_x\}$, for some $W_x \in G(n-1, X)$. We claim that for $\mu$–almost all $x \in A$ one also has $\mu_x = \theta(x)\mathcal{H}^{n-1} \cap W_x$. Indeed, this follows from [Le03], Proposition 3.12. As a consequence we conclude that for $\mu$–almost all $x, \mu_x = \beta(x)\mathcal{H}^{n-1} \cap W_x$ for some function $\beta \in L^1_{\text{loc}}(\mathcal{H}^{n-1} \cap A)$. On the other hand,

$$\beta(x) = \lim_{r \to 0} \frac{\mu_x(B_{\|\cdot\|}(x, r))}{\alpha_{n-1} r^{n-1}} = \frac{\alpha_{n-1} r^{n-1}}{\mathcal{H}^{n-1}(W_x \cap B_{\|\cdot\|}(x, r))} = \frac{\Theta^{-1}_{\|\cdot\|}(\mu_x, x)}{\Theta^{-1}_{\|\cdot\|}(\mathcal{H}^{n-1} \cap W_x, 0)}.$$

Thanks to Lemma 4.3.5 we know that $\Theta^{-1}_{\|\cdot\|}(\mu_x, x) = 1$ for $\mu$–almost all $x \in A$. Indeed, for $\mu$–almost all $x \in A$

$$\Theta^{-1}_{\|\cdot\|}(\mu_x, x) = \lim_{\rho \to 0} \frac{\mu_x(B_{\|\cdot\|}(x, \rho))}{\alpha_{n-1} \rho^{n-1}} = \lim_{\rho \to 0} \frac{\mu(B_{\|\cdot\|}(x, r\rho))}{\alpha_{n-1} (r\rho)^{n-1}} = \Theta^{-1}_{\|\cdot\|}(\mu, x) = 1,$$

where the last identity follows from B. Kirchheim’s theorem 3.2.3. Referring once again to B. Kirchheim’s theorem we infer that $\Theta^{-1}_{\|\cdot\|}(\mathcal{H}^{n-1} \cap W_x, 0) = 1$ for $\mu$–almost all $x$, and hence $\beta(x) = 1$ for $\mu$–almost all $x$ and the lemma follows. \qed

We proceed to the proof of the theorem keeping the notations $\mu$ and $\mu_x$ introduced in Lemma 4.3.6. First, note that we can replace $A$ by a Borelian subset $\tilde{A} \subseteq A$ such that for all $x \in A$ one has $\text{Tan}_{n-1}(\mu, x) = \{\mathcal{H}^{n-1} \cap W_x\}, W_x \in G(n-1, X)$. Denote $E = \pi_W(A)$. We assume, as we can, that $\mathcal{H}^{n-1}(A) < \infty$. Hence, thanks to the Eilenberg inequality (see 3.2.4 and [Fed69], 2.10.25), for $\mathcal{H}^{n-1}$–almost all $y \in E$ there holds $\text{card}(A \cap \pi_W^{-1}\{y\}) < \infty$. We infer that

1. we can replace $E$ by a set $\tilde{E} \subseteq E$ that satisfies $\mathcal{H}^{n-1}(E \setminus \tilde{E}) = 0$ and such that $\text{card}(A \cap \pi_W^{-1}\{y\}) < \infty$ for all $y \in \tilde{E}$,

2. we can replace $A$ by $A \cap \pi_W^{-1}(\tilde{E})$,

3. we can assume that $\Theta^{-1}_{\|\cdot\|}(\mathcal{H}^{n-1} \cap A, x) = 1$ for all $x \in A$.

The first replacement changes nothing: since $\pi_W$ is a Lipschitz mapping, it maps $\mathcal{H}^{n-1}$–negligible sets to $\mathcal{H}^{n-1}$–negligible sets. The second replacement changes nothing as well, since $\mathcal{H}^{n-1}(A \cap \pi_W^{-1}(\tilde{E})) \leq \mathcal{H}^{n-1}(A \cap \pi_W^{-1}(\tilde{E}))$. We can make the third assumption, because we can replace $A$ by a set $A_1 \subseteq A$, satisfying $\mathcal{H}^{n-1}(A \setminus A_1) = 0$ and $\Theta^{-1}_{\|\cdot\|}(\mathcal{H}^{n-1} \cap A, x) = 1$ for all $x \in A_1$, see Lemma 4.3.6.

Let us define a norm $q$ on the space $X$ as $q(u) = \max\{\|\pi_W(u)\|, \|u - \pi_W(u)\|\}$. Note that for all $y \in \tilde{E}$ there exists $\delta = \delta(y) > 0$ such that for all $x_1, x_2 \in A \cap \pi_W^{-1}(y)$ and for
all \( r \in (0, \delta(y)) \) one has \( B_q(x_1, r) \cap B_q(x_2, r) = \emptyset \) once \( x_1 \) and \( x_2 \) are distinct. Indeed, since \( A \cap \pi^{-1}_W\{y\} \) is a finite set, we can take

\[
\delta(y) = \frac{1}{3} \min_{x \neq x_m \in A \cap \pi^{-1}_W\{y\}} q(x_i - x_m).
\]

Now we claim that for all \( \varepsilon > 0 \) and for all \( y \in \tilde{E} \) there exists \( \tilde{\delta} = \tilde{\delta}(y) > 0 \) such that for all \( x \in A \cap \pi^{-1}_W\{y\} \) and for all but countably many \( r \in (0, \tilde{\delta}) \) one has

\[
|\mu(B_q(x, r)) - \mu_x(B_q(0, r))| \leq \varepsilon \alpha_{n-1} r^{n-1}.
\] (4.14)

Note that, according to Lemma 4.3.5 for every \( \tilde{\varepsilon} > 0 \), there exists \( \tilde{\delta}_1 = \tilde{\delta}_1(x) \) such that for all \( \rho < \tilde{\delta}_1 \) and for all but countably many \( r \in \mathbb{R}_+ \) there holds

\[
\left| \mu_x(B_q(0, r)) - \frac{\mu(B_q(x, r \rho))}{\rho^{n-1}} \right| \leq \tilde{\varepsilon}.
\] (4.15)

On the other hand, since \( A \) is rectifiable and since \( \Theta^{n-1}_{\|\cdot\|}(\mathcal{H}^{n-1}_{\|\cdot\|} \cap A, x) = 1 \), referring to B. Kirchheim’s theorem 3.2.3, we infer that for the fixed \( \varepsilon > 0 \) there exists \( \tilde{\delta}_2 = \tilde{\delta}_2(x) \) such that for all \( \tilde{r} \in (0, \tilde{\delta}_2) \) there holds

\[
\left| \frac{\mu(B_q(x, \tilde{r}))}{\alpha_{n-1} \tilde{r}^{n-1}} - 1 \right| \leq \varepsilon.
\]

Next for \( r < \tilde{\delta}_2 \) we estimate choosing \( \tilde{\varepsilon} = \varepsilon \alpha_{n-1} r^{n-1} \) in (4.15) and further taking \( \rho < \min\{\tilde{\delta}_1, \tilde{\delta}_2/r\} \):

\[
\left| \mu(B_q(x, r)) - \mu_x(B_q(0, r)) \right| \leq \left| \mu_x(B_q(0, r)) - \frac{\mu(B_q(x, r \rho))}{\rho^{n-1}} \right| + \left| \frac{\mu(B_q(x, r \rho))}{\rho^{n-1}} - \alpha_{n-1} r^{n-1} \right| + \left| \alpha_{n-1} r^{n-1} - \mu(B_q(x, r)) \right| \leq 3\varepsilon \alpha_{n-1} r^{n-1},
\]

and it suffices to take \( \tilde{\delta}(y) = \min_{x \in \pi^{-1}(y)} \{\tilde{\delta}_2(x)\} \) in order to finish the proof of the claim (4.14).

Next, referring to the fact that \( A \) is rectifiable and since \( \Theta^{n-1}_{\|\cdot\|}(\mathcal{H}^{n-1}_{\|\cdot\|} \cap A, x) = 1 \), we infer that for every \( y \in \tilde{E} \) and for every \( x \in \pi^{-1}_W(y) \) for every \( \varepsilon > 0 \) there exists \( \delta' \) such that for all \( r < \delta' \) there holds

\[
\left| \frac{\mathcal{H}^{n-1}_{\|\cdot\|}(A \cap B_q(x, r))}{\alpha_{n-1} r^{n-1}} - \phi_B(W_x) \right| \leq \varepsilon.
\]

Hence if \( \varepsilon \) is small enough, for all \( r \leq \delta' \) we can guarantee

\[
C_0 r^{n-1} \leq \mathcal{H}^{n-1}_{\|\cdot\|}(A \cap B_q(x, r)),
\] (4.16)
where the constant $C_0$ depends only on the dimension $n$.

For every $y \in \tilde{E}$ denote $\Delta(y) = \min\{\delta(y), \delta'(y)\}$. According to the Vitali covering theorem there exists a sequence of balls $B_j = B_q(y_j, r_j) \cap W$ in $W$ such that for all $j$ the radii $r_j$ satisfy $0 < r_j < \Delta(y_j)$ and are outside of the bad set of radii from the line \[\ref{eq:bdy}], the balls $B_j$ are mutually disjoint and $\mathcal{H}^{n-1}_{|||\cdot|||}(\tilde{E} \setminus \bigcup_{j=1}^\infty B_j) = 0$. Hence

$$\mathcal{H}^{n-1}_{|||\cdot|||}(\pi_W(A)) = \mathcal{H}^{n-1}_{|||\cdot|||}(\tilde{E}) \leq \sum_{j=1}^\infty \mathcal{H}^{n-1}_{|||\cdot|||}(B_j). \quad \tag{4.17}$$

For each $j$ denote $k_j = \text{card}(A \cap \pi_W^{-1}\{y_j\})$. Let $x_{j,1}, \ldots, x_{j,k_j}$ be the elements of the set $A \cap \pi_W^{-1}\{y_j\}$. We write $W_j = \bigcup_{k=1}^{k_j} (x_{j,k} + W_{x_{j,k}})$ for sake of brevity. We claim that

$$\left| \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)) - \mathcal{H}^{n-1}_{|||\cdot|||}((x_{j,k} + W_{x_{j,k}}) \cap B_q(x_{j,k}, r_j)) \right| \leq C_1 \varepsilon \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)), \quad \tag{4.18}$$

for some universal constant $C_1$. Indeed, we first note that the inequality \[\ref{eq:bdy} \text{ gives}

$$\left| \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)) - \mathcal{H}^{n-1}_{|||\cdot|||}((x_{j,k} + W_{x_{j,k}}) \cap B_q(x_{j,k}, r_j)) \right| \leq \varepsilon \alpha_{n-1} r_j^{n-1}.$$ 

On the other hand, since $r_j < \delta'(y)$, we conclude from \[\ref{eq:bdy} \text{ that for } \varepsilon \text{ small enough there holds } \varepsilon \alpha_{n-1} r_j^{n-1} \leq C_1 \varepsilon \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)) \text{ and our claim follows.}$$

The line \[\ref{eq:bdy} \text{ now gives:}

$$\sum_{k=1}^{k_j} \mathcal{H}^{n-1}_{|||\cdot|||}(W_j \cap B_q(x_{j,k}, r_j)) = \mathcal{H}^{n-1}_{|||\cdot|||}(W_j \cap \bigcup_{k=1}^{k_j} B_q(x_{j,k}, r_j)) \leq (1 + C_1 \varepsilon) \sum_{k=1}^{k_j} \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)), \quad \tag{4.19}$$

thanks to the fact that the balls $B_q(y_j, r_j)$ are mutually disjoint.

Let us use the definition of the norm $q$. Note that $W_j \cap B_q(x_{j,k}, r_j) = W_j \cap \pi_W^{-1}(B_q(y_j, r_j))$ and hence $\pi_W(W_j \cap B_q(x_{j,k}, r_j)) = W \cap B_q(y_j, r_j)$. Since the projection $\pi_W$ is $W$–good we have that

$$\mathcal{H}^{n-1}_{|||\cdot|||}(B_j) = \mathcal{H}^{n-1}_{|||\cdot|||}(W \cap B_q(y_j, r_j)) \leq \mathcal{H}^{n-1}_{|||\cdot|||}(W_j \cap B_q(x_{j,k}, r_j)).$$

This inequality together with the estimate \[\ref{eq:bdy} \text{ allows us to write}

$$\mathcal{H}^{n-1}_{|||\cdot|||}(B_j) \leq \sum_{k=1}^{k_j} \mathcal{H}^{n-1}_{|||\cdot|||}(B_j) \leq \sum_{k=1}^{k_j} \mathcal{H}^{n-1}_{|||\cdot|||}(W_j \cap B_q(x_{j,k}, r_j)) \leq (1 + C_1 \varepsilon) \sum_{k=1}^{k_j} \mathcal{H}^{n-1}_{|||\cdot|||}(A \cap B_q(x_{j,k}, r_j)).$$
We finish our estimates recalling the line (4.17):

\[ H^{n-1}_{||...||}(\pi_W(A)) \leq (1 + C_1 \varepsilon) \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} H^{n-1}_{||...||}(A \cap B_q(x_{j,k}, r_j)) \leq (1 + C_1 \varepsilon) H^{n-1}_{||...||}(A), \]

where the last inequality follows from the fact that the balls \( \{B_q(x_{j,k}, r_j) : j = 1, 2, \ldots; k = 1, \ldots k_j \} \) are mutually disjoint. The theorem will follow after taking a limit when \( \varepsilon \) tends to 0.

The following theorem, the last one in this chapter, is the one where we derive the fact that good projectors do not increase the Hausdorff mass of \((n-1)\) polyhedral \(G\) chains in \(X\). We will further rely on this result in our proof of the “triangle inequality” for \((n-1)\) cycles.

**Theorem 4.3.5.** (I.Vasilyev) Let \((X, ||...||)\) be an \(n\)-dimensional Banach space. If \(P \in \mathcal{P}_{n-1}(X, G)\) (i.e. \(P\) is a \((n-1)\) polyhedral \(G\) chain) and if \(\pi_W : X \to W\) is a \(W\)-good projection then the following inequality holds true:

\[ \mathcal{M}(\pi_W \#(P)) \leq \mathcal{M}(P). \]

**Remark 4.3.5.** This result was first formulated and proved in the present thesis by I.Vasilyev.

**Proof.** We write \(\pi\) instead of \(\pi_W\) here for sake of brevity. Suppose that

\[ P = \sum_{i=1}^{N} g_i[\sigma_i] \]

with \(\sigma_i\) non-overlapping simplexes. Consider a family of simplexes \(\pi(\sigma_1), \ldots, \pi(\sigma_N)\) and choose a non-overlapping representation of \(\pi \# P\), say

\[ \pi \# P = \sum_{k=1}^{M} h_k[\tau_k]. \]

Define for each \(k\) from 1 to \(M\) a number \(a_k\) as \(a_k := \text{card}(i : \pi^{-1}(\tau_k) \cap \text{int}(\sigma_i) \neq \emptyset)\) and an increasing function \(\phi_k : [1, \ldots, a_k] \to [1, \ldots, N]\) defined inductively as follows, \(\phi_k(1) = \min\{i : \pi^{-1}(\tau_k) \cap \text{int}(\sigma_i) \neq \emptyset\}\) and for all \(j\) from 1 to \(a_k\)

\[ \phi_k(j) = \min\{i : \pi^{-1}(\tau_k) \cap \text{int}(\sigma_i) \neq \emptyset, i > \phi_k(j-1)\}. \]

Hence \(\tau_k = \pi(\sigma_{\phi_k(j)} \cap \pi^{-1}(\tau_k))\) for each \(j \in [1, \ldots, a_k]\). Define further \(m(k) \in \{\phi_k(j)\}_{j=1}^{a_k}\) such that

\[ H^{n-1}_{||...||}(\sigma_{m(k)} \cap \pi^{-1}(\tau_k)) \leq H^{n-1}_{||...||}(\sigma_{\phi_k(j)} \cap \pi^{-1}(\tau_k)) \]
for all \( l \in \{ \phi_k(j) \}_{j=1}^{a_k} \). Next we estimate using the fact that \( h_k = \sum_j g_{\phi_k(j)} \), the triangle inequality and the fact that \( \pi \) is \( W \)-good

\[
\mathbb{M}(\pi \# P) = \sum_{k=1}^{M} |h_k| \mathcal{H}_{\|\cdot\|\|}^{n-1}(\tau_k) \leq \sum_{k=1}^{M} \left| \sum_{j=1}^{a_k} g_{\phi_k(j)} \right| \mathcal{H}_{\|\cdot\|\|}^{n-1}(\pi(\sigma_{m(k)} \cap \pi^{-1}(\tau_k))) \leq \sum_{k=1}^{M} \sum_{j=1}^{a_k} |g_{\phi_k(j)}| \mathcal{H}_{\|\cdot\|\|}^{n-1}(\pi(\sigma_{\phi_k(j)} \cap \pi^{-1}(\tau_k))) \leq \sum_{i=1}^{N} |g_i| \sum_{k=1}^{M} \mathcal{H}_{\|\cdot\|\|}^{n-1}(\sigma_i \cap \pi^{-1}(\tau_k)) \leq \sum_{i=1}^{N} |g_i| \mathcal{H}_{\|\cdot\|\|}^{n-1}(\sigma_i) = \mathbb{M}(P).
\]

\( \square \)
Busemann’s theorem and some of its consequences.
Chapter 5

The “classical” proof of the existence of mass minimizing \((n - 1)\)-rectifiable \(G\) chains.

This chapter is devoted to the first proof of the existence of mass minimizing \((n - 1)\)-rectifiable \(G\) chains in \(n\)-dimensional Banach spaces.

Remark 5.0.6. The theorems proved in this chapter were first formulated and proved here by I. Vasilyev.

5.1 The lower semicontinuity on the space of \((n - 1)\) polyhedral \(G\) chains.

Next we establish a “triangle inequality” for the \((n - 1)\) polyhedral cycles.

Theorem 5.1.1. Let \((X, ||...||)\) be an \(n\)-dimensional Banach space with the unit ball \(B = \{x \in X : ||x|| \leq 1\}\) and let \(G\) be a complete normed Abelian group. Let \(P \in \mathcal{P}_{n-1}(X,G)\) be a \((n - 1)\)-cycle (i.e. \(\partial P = 0\)) such that

\[
P = \sum_{j=1}^{N} g_{j} [\sigma_{j}],
\]

where \(\sigma_{j}\) are non–overlapping \((n - 1)\)-simplexes. Then

\[
|g_{1}| \mathcal{H}_{||...||}^{n-1}(\sigma_{1}) \leq \sum_{j=2}^{N} |g_{j}| \mathcal{H}_{||...||}^{n-1}(\sigma_{j}).
\]

Proof. For \(j\) between 1 and \(N\) let \(W_{j}\) be the hyperplane such that \(\sigma_{j} \subseteq W_{j}\) and let \(\pi : X \to W_{1}\) be any \(W_{1}\)-good projection. Denote

\[
Q = \sum_{j \geq 2} g_{j} [\sigma_{j}].
\]
The “classical” proof of the existence of mass minimizing $(n-1)$–rectifiable $G$ chains.

Hence $P = g_1[\sigma_1] + Q$ and further $\pi#P = g_1[\sigma_1] + \pi#Q$. Note that $\pi#P$ is a top–dimensional $G$ chain (see Definition 3.1.13) in the space $(W_1, \ldots)$, where the norm $\ldots$ is defined by the convex set $B \cap W_1$. But $\partial \pi#P = 0$, and by the constancy theorem (see Theorem 3.2.2) we conclude that $g_1[\sigma_1] = -\pi#Q$. Now we estimate

$$\mathcal{M}(g_1)[\sigma_1] = |g_1|\mathcal{H}^n_{\|\ldots\|}(\sigma_1) = \mathcal{M}(\pi#Q) \leq \mathcal{M}(Q) = \sum_{j \geq 2} |g_j|\mathcal{H}^n_{\|\ldots\|}(\sigma_j), \quad (5.1)$$

where the inequality in (5.1) follows from the fact that $\pi$ is $W_1$–good, see Theorem 4.3.5. Hence the theorem follows.

Next we are going to prove that the “triangle inequality”, obtained in the previous theorem is a necessary and sufficient condition for the lower semicontinuity of the mass on the space of polyhedral chains.

**Theorem 5.1.2.** Let $(X, \|\ldots\|)$ be an $n$–dimensional Banach space, let $G$ be a complete normed Abelian group and let $m$ be an integer such that $1 \leq m \leq n-1$. Then the following conditions are equivalent:

1. $\mathcal{M}$ is lower semicontinuous with respect to flat convergence on $\mathcal{P}_m(X, G)$.
2. If $P = \sum_{j=1}^{N} g_j[\sigma_j] \in \mathcal{P}_m(X, G)$, where $\sigma_j$ are non–overlapping, is a cycle (which means that $\partial P = 0$) then

$$|g_1|\mathcal{H}^m_{\|\ldots\|}(\sigma_1) \leq \sum_{j=2}^{N} |g_i|\mathcal{H}^m_{\|\ldots\|}(\sigma_j).$$

**Proof.** $2 \Rightarrow 1$

We start with a very special case when $P = g[\sigma]$, i.e. when the chain $P$ consists of only one simplex. Suppose that $\mathcal{F}(P - P_i) \to 0$ as $i$ tends to $\infty$ for some $P_i \in \mathcal{P}_m(X, G)$. By the definition of flat convergence there exists a sequence of $m$–chains $\{Q_i\}_{i=1}^\infty$ and a sequence of $(m+1)$–chains $\{R_i\}_{i=1}^\infty$ such that $P - P_i = Q_i + \partial R_i$ and $\mathcal{M}(Q_i) + \mathcal{M}(R_i) \to 0$ when $i$ tends to $\infty$. Note that $\partial(P - P_i - Q_i) = \partial(\partial R_i) = 0$, so we can apply 2 to the chain $(P - P_i - Q_i)$ and obtain

$$\mathcal{M}(P) \leq \lim inf_i \mathcal{M}(P_i + Q_i) \leq \lim inf_i \mathcal{M}(P_i) + \mathcal{M}(Q_i) \leq \lim inf_i \mathcal{M}(P_i),$$

and hence 1 follows.

We pass to the general case. Suppose that $P_i \to P$ in flat norm for a sequence $P_i \in \mathcal{P}_m(X, G)$. For every $j$ between 1 and $N$ denote by $W_j$ the $m$–plane containing the simplex $\sigma_j$. Let $F_{j,\eta}$ be a $(m+1)$–convex polyhedron defined in the following way. If $\sigma_j = \text{conv}\{a_{1,j}, \ldots, a_{m+1,j}\}$, then we define $F_{j,\eta}$ as $F_{j,\eta} = \text{conv}\{a_{1,j}, \ldots, a_{m+1,j}, v_{n,j}\}$ where

$$v_{n,j} = \frac{1}{n} \sum_{k=1}^{m+1} a_{k,i} + \eta \omega_j,$$
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for some (fixed) nonzero vector \(\omega_j\) orthogonal to the \(m\)-plane \(W_j\) and a small number \(\eta\) to be defined in a moment. In more details, we claim that there exists \(\eta\) depending on \(\sigma_1, \ldots, \sigma_n\) only such that \(F_{i,\eta}\) and \(F_{j,\eta}\) do not overlap once \(i \neq j\). First note that it is sufficient to find for each pair \((i, j)\) a number \(\eta_{i,j}\) such that \(F_{i,\eta_{i,j}}\) and \(F_{j,\eta_{i,j}}\) are non-overlapping. Second, consider two simplexes, say \(\sigma_1\) and \(\sigma_2\) (which are non-overlapping by our assumption). Assume, as we can, that \(\sigma_1 \cap \sigma_2 \neq \emptyset\), hence, possibly after renumbering, there exists \(0 < l < m + 1\) such that \(a_{1,1} = a_{1,2} = b_1, a_{2,1} = a_{2,2} = b_2, \ldots, a_{l,1} = a_{l,2} = b_l\). Translate both simplexes to zero so that \(b_1 = 0\). Next, we infer that there exists a hyperplane \(W\) with the unit normal \(\omega\) such that \(\text{conv}\{0, b_1, \ldots, b_l\} \subset W\) and \(\langle x, \omega \rangle > 0\) for all \(x \in \sigma_1 \setminus \text{conv}\{0, b_1, \ldots, b_l\}\) and \(\langle x, \omega \rangle < 0\) for all \(x \in \sigma_2 \setminus \text{conv}\{0, b_1, \ldots, b_l\}\). Note that we are done once we show that there exists \(\eta > 0\) such that for all \(x \in \text{int}(F_{1,\eta})\) there holds \(\langle x, \omega \rangle > 0\) and for all \(x \in \text{int}(F_{2,\eta})\) one has \(\langle x, \omega \rangle < 0\). Fix \(\eta > 0\) and suppose that there exists \(x \in W \cap \text{int}(F_{1,\eta})\). Then

\[
x = \sum_{k=1}^{m+1} a_{k} a_{1,k} + \alpha_{0} \left( n \sum_{k=1}^{m+1} a_{1,k} + n \omega_1 \right)
\]

with \(\alpha_{i_0} > 0\) for some \(i_0 \in \{l + 1, \ldots, m + 1\}\), \(\alpha_0 > 0\) and \(\langle x, \omega \rangle = 0\). Hence

\[
0 = \langle x, \omega \rangle = \sum_{k=1}^{m+1} \left( \alpha_k + \frac{\alpha_0}{n} \right) \langle a_{1,k}, \omega \rangle + \eta \langle \omega_1, \omega \rangle = \frac{\alpha_0}{n} \langle a_{1,i_0}, \omega \rangle + \alpha_0 \eta \langle \omega_1, \omega \rangle,
\]

since \(\alpha_0 > 0\) and since \(\langle a_{1,i_0}, \omega \rangle > 0\), this gives a contradiction for \(\eta\) small enough, and our claim follows. We further infer that, using a construction which is very similar to the one above, one can suppose that the polyhedrons \(F_{i,\eta}\) are “two–sided”. In more details, let us define polyhedrons \(\tilde{F}_{j,\eta}\) and \(\tilde{F}_{i,\eta}\). The same argumentation as above shows us that one can find \(\eta\) small enough to guarantee that \(\tilde{F}_{j,\eta}\) and \(\tilde{F}_{i,\eta}\) have no overlapping once \(i \neq j\). We shall further write \(F_j\) instead of \(\tilde{F}_{j,\eta}\).

Next, let \(u_j\) be a function defined as

\[
u_j(x) = \min\{\text{dist}_\infty(P_{W_j}(x), \sigma_j^c), \text{dist}_\infty(x, F_j^c)\},
\]

where \(P_{W_j}\) is the orthogonal projection onto \(W_j\), \(S^c\) stands for the complement of a set \(S\) and \(\text{dist}_\infty\) is defined in the following way:

\[
\text{dist}_\infty(x, S) = \inf_{z \in S} \|x - z\|_\infty,
\]

where \(\|\ldots\|_\infty\) is the supremum norm with respect to the fixed Euclidean structure. Note that it follows from the construction of the polyhedrons \(F_j\) that the sets \(\{x : u_j(x) > \kappa\}\) are pairwise disjoint whenever \(\kappa > 0\). We infer that for every \(j \in \{1, \ldots, N\}\) and for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for all \(r \in (0, \delta)\) one has

\[
\mathbb{M}(g_j[\sigma_j] - g_j[\sigma_j \cap \{x : u_j(x) > r\}]) \leq \varepsilon. \tag{5.2}
\]
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We claim that for every $j \in [1, \ldots, N]$ there exists $\tilde{\delta} \in [\delta/2, \delta]$ and a sequence $\delta_i \in (\delta/2, \delta)$ such that $\delta_i \to \tilde{\delta}$

$$\mathcal{F}(P \bigl\{ x : u_j(x) > \tilde{\delta} \bigr\} - P_i \bigl\{ x : u_j(x) > \delta_i \bigr\}) \to 0,$$  \hspace{1cm} (5.3)

when $i$ tends to $+\infty$. Indeed, by the definition of flat norm, we know that for every $\tilde{\varepsilon} > 0$ there exists $i = i(\tilde{\varepsilon})$ such that for each $i \geq i(\tilde{\varepsilon})$ there are $R_i \in \mathcal{P}_{m+1}(X, G)$ and $Q_i \in \mathcal{P}_m(X, G)$ with $\mathcal{M}(Q_i) + \mathcal{M}(R_i) \leq \tilde{\varepsilon}$ and $P - P_i = Q_i + \partial R_i$. Hence for every $\rho > 0$ we can write

$$P \bigl\{ x : u_j(x) > \rho \bigr\} - P_i \bigl\{ x : u_j(x) > \rho \bigr\} = (Q_i + \partial R_i) \bigl\{ x : u_j(x) > \rho \bigr\} = Q_i \bigl\{ x : u_j(x) > \rho \bigr\} + \partial R_i \bigl\{ x : u_j(x) > \rho \bigr\} + ((\partial R_i) \bigl\{ x : u_j(x) > \rho \bigr\} - \partial R_i \bigl\{ x : u_j(x) > \rho \bigr\}).$$

From here we see that for every $i \geq i(\tilde{\varepsilon})$ and for every $\rho > 0$ one has

$$\mathcal{F}(P \bigl\{ x : u_j(x) > \rho \bigr\} - P_i \bigl\{ x : u_j(x) > \rho \bigr\}) \leq \tilde{\varepsilon} + \mathcal{M}(R_i, u_j, \rho).$$  \hspace{1cm} (5.4)

On the other hand, referring to the slicing properties we infer that for every $i$ there exists $\delta_i \in (\delta/2, \delta)$ such that $\mathcal{M}(R_i, u_j, \delta_i) \leq 2\mathcal{M}(R_i)/\delta$. Passing to a subsequence if necessary, we may assume that the sequence $\{\delta_i\}_{i=1}^\infty$ converges towards some $\tilde{\delta}$. Note that

$$\lim_{i \to \infty} \|P\|\{x : \tilde{\delta} > u_j(x) \geq \delta_i\} = 0.$$  

Hence for the fixed $\tilde{\varepsilon}$ there exists number $M (= M(\tilde{\varepsilon}))$ such that $\|P\|\{x : \tilde{\delta} > u_j(x) \geq \delta_i\} \leq \tilde{\varepsilon}$ once $i > M$. According to the inequality (5.4) applied to $\rho = \delta_i$, for each $i > M$ there holds

$$\mathcal{F}(P \bigl\{ x : u_j(x) > \tilde{\delta} \bigr\} - P_i \bigl\{ x : u_j(x) > \delta_i \bigr\}) \leq \mathcal{F}(P \bigl\{ x : u_j(x) > \delta_i \bigr\} - P_i \bigl\{ x : u_j(x) > \delta_i \bigr\}) + \mathcal{F}(P \bigl\{ x : \tilde{\delta} > u_j(x) \geq \delta_i \bigr\}) \leq \tilde{\varepsilon} + \frac{2\mathcal{M}(R_i)}{\delta} + \|P\|\{x : \tilde{\delta} > u_j(x) \geq \delta_i\} \leq \tilde{\varepsilon} + \frac{2\tilde{\varepsilon}}{\delta} + \tilde{\varepsilon},$$

and the claim (5.3) follows.

We further derive an upper bound on the mass of the chains $g_j[\sigma_j]$ first using the inequality (5.2) then the first case of the proof and finally the fact that $\delta_i \geq \delta/2$:

$$\mathcal{M}(g_j[\sigma_j]) \leq \varepsilon + \mathcal{M}(P \bigl\{ x : u_j(x) > \tilde{\delta} \bigr\}) \leq \varepsilon + \liminf_{i \to +\infty} \mathcal{M}(P_i \bigl\{ x : u_j(x) > \delta/2 \bigr\}).$$

We continue our estimates by taking the sum over all indexes $j$ between 1 and $N$ in the previous line:

$$\mathcal{M}(P) = \sum_{j=1}^N \mathcal{M}(g_j[\sigma_j]) \leq N\varepsilon + \liminf_{i \to +\infty} \sum_{j=1}^N \mathcal{M}(P_i \bigl\{ x : u_j(x) > \delta/2 \bigr\}) \leq N\varepsilon + \liminf_{i \to +\infty} \mathcal{M}(P_i),$$
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where the last inequality follows from the fact that the sets \(\{x : u_j(x) > \delta/2\}\) are pairwise disjoint. The first implication will now follow after we let \(\varepsilon\) tend to 0.

\[ \Rightarrow \]

We argue by a contradiction. Suppose that there exist \(g[\sigma], g_1[\sigma_1], g_2[\sigma_2], \ldots, g_N[\sigma_n]\) such that

\[
\sum_{i=1}^{N} \mathcal{M}(g_i[\sigma_i]) < \mathcal{M}(g[\sigma])
\]

and \(\partial T = 0\) where

\[
T = \sum_{i=1}^{N} g_i[\sigma_i] - g[\sigma].
\]

Write \([\sigma] = [x_0, \ldots, x_m]\). For every positive natural number \(n\) we introduce the following set of multiindexes:

\[
I_n = \{k = (k_1, \ldots, k_m) \in \mathbb{N}^m : k_1 + \ldots + k_m \leq n - 1\}.
\]

Further, for each \(k \in I_n\) we call \(f_{n,k}\) the map

\[
f_{n,k}(x) := x_0 + \sum_{i=1}^{m} k_i \frac{x - x_0}{n} + \frac{x - x_0}{n}.
\]

Note that \(f_{n,k}(\sigma) \subseteq \sigma\) and that \(\text{Lip}(f_{n,k}) = 1/n\) for each multiindex \(k\). Next, we define polyhedral chains \(S_n\) by the following formula

\[
S_n := g[\sigma] + \sum_{k \in I_n} f_{n,k} T.
\]

Let us prove that \(S_n\) converge to \(g[\sigma]\) in flat norm. First note that since \(\partial T = 0\), referring to the cone construction, we infer that there exists \(U \in \mathcal{P}_{m+1}(X, G)\) such that \(\partial U = T\). Now we estimate

\[
\mathcal{F}(S_n - g[\sigma]) = \mathcal{F}(\partial \sum_{k \in I_n} f_{n,k}#U) \leq \sum_{k \in I_n} \mathcal{M}(f_{n,k}#U) \leq (\text{Lip}(f_{n,k}))^{m+1} n^m \mathcal{M}(U) = \frac{\mathcal{M}(U)}{n},
\]

and the convergence follows. Since \(\mathcal{M}\) is lower semicontinuous, we conclude that

\[
\limsup_{n \to \infty} \mathcal{M}(S_n) \leq \mathcal{M}(g[\sigma]). \quad (5.6)
\]

On the other hand,

\[
\mathcal{M}(S_n) = \mathcal{M}(g[\sigma]) + \sum_{k \in I_n} \left( \mathcal{M} \left( \sum_{i=1}^{N} f_{n,k}#g_i[\sigma_i] \right) - \mathcal{M}(f_{n,k}#g[\sigma]) \right) \leq
\]

\[
\mathcal{M}(g[\sigma]) + \frac{\text{card}(I_n)}{n^m} \left( \sum_{i=1}^{N} \mathcal{M}(g_i[\sigma_i]) - \mathcal{M}(g[\sigma]) \right). \quad (5.7)
\]
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Note that
\[
\frac{\text{card}(I_n)}{n^m} \to \frac{1}{2} \text{ as } n \to \infty.
\]
Hence the lines \([5.5]\) and \([5.7]\) now give \(\limsup_n M(S_n) < M(g[\sigma])\), which contradicts \([5.6]\).

**Corollary 5.1.1.** Let \((X, || . ||)\) be an \(n\)-dimensional Banach space and let \(G\) be a complete normed Abelian group. Then the mass \(M\) is lower semicontinuous with respect to flat convergence on the space \(P_{n-1}(X,G)\).

**Proof.** It suffices to recall Theorems 5.1.1 and 5.1.2.

**Remark 5.1.1.** We pay the reader’s attention to the fact that the present proof of Corollary 5.2.1 is not yet available in other codimensions since Theorem 5.1.1 is known only in codimension one.

### 5.2 The lower semicontinuity on the space of \((n-1)\)-rectifiable \(G\) chains.

The following theorem is the main result of this section and of the whole chapter.

**Theorem 5.2.1.** Let \((X, || . ||)\) be an \(n\)-dimensional Banach space and let \(G\) be a complete normed Abelian group. Then the mass \(M\) is lower semicontinuous with respect to flat convergence on \(R_{n-1}(X,G)\).

**Proof.** We start with the first step where we prove the theorem in case when the limit chain \(P\) is polyhedral.

**Lemma 5.2.1.** Let \(X\) and \(G\) be the same as in the theorem. If \(T_i \in R_{n-1}(X,G), P \in P_{n-1}(X,G), \text{ and } \lim_{i \to \infty} F(T_i - P) = 0\), then
\[
M(P) \leq \liminf_{i \to \infty} M(T_i).
\]

**Proof.** Thanks to the weak approximation theorem in Banach spaces (see [Pau14], Theorem 4.4, page 329), we know that for every natural number \(i\) there exists a polyhedral chain \(P_i \in P_{n-1}(X,G)\) such that \(F(T_i - P_i) \leq 1/i\) and \(M(P_i) \leq M(T_i) + 1/i\). Hence, since \(F(P_i - P) \to 0\), when \(i \to +\infty\), we can write, using Corollary 5.2.1
\[
M(P) \leq \liminf_{i \to +\infty} M(P_i) \leq \liminf_{i \to +\infty} \left( M(T_i) + \frac{1}{i} \right) = \liminf_{i \to +\infty} M(T_i),
\]
and the lemma follows.

The second step is the following version of the strong approximation theorem of Federer (see [Fed69], Theorem 4.2.20).
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chains.

**Theorem 5.2.2.** Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $G$ be a complete normed Abelian group. If $T \in \mathcal{R}_m(X, G)$ with $1 \leq m \leq n$ then for every $\varepsilon > 0$ there exists a polyhedral chain $P \in \mathcal{P}_m(X, G)$ and a $C^1$–smooth diffeomorphism $F : X \to X$ such that

- $||F(x) - x|| \leq \varepsilon$ for all $x \in X$ and $F(x) = x$ once $\text{dist}(x, \text{set}(T)) \geq \varepsilon$;
- $\max\{\text{Lip}(F), \text{Lip}(F^{-1})\} \leq 1 + \varepsilon$, where the Lipschitz constants are taken with respect to the norm $||\ldots||$;
- $\mathcal{M}(F # T - P) \leq \varepsilon$.

**Proof.** We first prove a lemma where we construct a very special parametrisation of pieces of smooth manifolds.

**Lemma 5.2.2.** Let $X$ be as in the formulation of the theorem, let $M \subset X$ be a $C^1$–smooth manifold and let $a \in M$. Fix $t \in (0, 1)$. There exists $r_0$ and a $C^1$–smooth diffeomorphism $F : X \to X$ such that for all $r \in (0, r_0)$ there holds

- for all $x \not\in \mathbb{U}(a, r)$ one has $F(x) = x$;
- $\max\{\text{Lip}(F), \text{Lip}(F^{-1})\} \leq 1/t$ where both Lipschitz constants are taken with respect to the norm $||\ldots||$;
- $\mathbb{U}(a, tr) \cap M = \mathbb{U}(a, tr) \cap F^{-1}(a + \text{Tan}(M, a))$.

We remind to the reader that by $\mathbb{U}(a, r)$ we mean an open Euclidean ball centred at $a$ with radius $r$.

**Proof.** Throughout this proof we write $C_n$ for a constant depending on $n$ only.

We present here a (little bit modified to our setting) proof, borrowed from H.Federer’s book (section 3.1.23). Suppose, as we can, that $a = 0$ and denote $W = \text{Tan}(M, 0)$. Let $P : W \to X$ be the orthogonal projection with the range $W$ (we mean the orthogonal projection with respect to the fixed Euclidean structure). Note that for every $\varepsilon$ there exists $r_1$ such that for all $\xi \in \mathbb{U}(0, r_1)$ there holds $|||D\psi(\xi) - p^*||| \leq \varepsilon$, where we denote

$$\psi := (P \upharpoonright (M \cap \mathbb{U}(0, r_1)))^{-1} : P(M \cap \mathbb{U}(0, r_1)) \to M \cap \mathbb{U}(0, r_1)$$

and $p^* := D\psi(0) : W \to X$ for brevity. We remind to the reader that we fix the symbol $\upharpoonright$ for restrictions of applications and that the operator norm $|||\ldots|||$ of a linear mapping $L : X \to X$ is defined here as $|||L||| := \sup\{|||L(h)||| : |||h||| \leq 1\}$. Define for $r \leq r_1$ a function $u : X \to [0, 1]$ as

$$u(x) = \gamma \left( \frac{|x|}{r} \right),$$

where $\gamma$ is any $C^\infty$ smooth function satisfying $\gamma(z) = 1$ for $z \in [0, t], \gamma(1) = 0$, $\text{Lip}(\gamma) \leq \frac{1}{t - 1}$ and $\gamma$ is strictly decreasing on $[t, 1]$. 

Choose \( \varepsilon = \varepsilon(t) := (1-t)(1/t-1)/(C_n(2-t)) \) and put \( r_0 = r_0(t) = r_1(\varepsilon(t)) \). We are now ready to define the desired function \( F \). Let \( F(x) = x \) for \( x \notin U(0, r) \) and let \( F(x) = x + u(x) \cdot [p^*(P(x)) - \psi(P(x))] \) otherwise. Let us check the assertions of the lemma. The first one follows easily from the construction of \( F \); the third one follows from the facts that \( u(x) = \gamma(|x|/r) = 1 \) for \( |x| \leq tr \leq r \) and that \( \psi(P(x)) = x \) once \( x \in M \cap U(0, r) \) and since the image of \( p^* \) is contained in \( W \). Hence it is left to estimate the Lipchitz constants of \( F \) and \( F^{-1} \). To this end, we first calculate the differential of \( F \):

\[
DF(x) = \text{id}_X + Du(x)[p^*(P(x)) - \psi(P(x))] + u(x)[p^* - D\psi(P(x))](P(x)).
\]

Next we estimate the norm of the difference between \( DF \) and \( \text{id}_X \):

\[
||DF(x) - \text{id}_X|| \leq ||Du(x)(p^*(P(x)) - \psi(P(x)))|| + ||u(x)(p^* - D\psi(P(x))) \circ P|| =: ||L_1|| + ||L_2||.
\]

We estimate the operators \( L_1 \) and \( L_2 \) separately. We first treat \( L_1 \). If \( h \in X \) such that \( ||h|| \leq 1 \) then

\[
||L_1(h)|| \leq |Du(x)(h)| \cdot ||p^*(P(x)) - \psi(P(x))|| \leq \frac{\text{Lip}(\gamma)||h||}{r} \leq \frac{\varepsilon C_n|x|}{(1-t)r},
\]

where the last inequality follows from the fact that \( P \) is orthogonal and since \( \text{Lip}(\gamma) = 1/(1-t) \). Next we proceed to the term \( L_2 \):

\[
||L_2|| = ||u||_{\infty} \cdot ||(p^* - D\psi(P(x))) \circ P|| \leq ||P|| \cdot ||p^* - D\psi|| \leq C_n \varepsilon.
\]

As a consequence we conclude that for all \( x \in B(0, r) \) there holds

\[
||DF(x) - \text{id}_X|| \leq \frac{(1-t)}{C_n(2-t)} \left( \frac{1}{t} - 1 \right) \left( C_n + \frac{C_n|x|}{(1-t)r} \right) \leq \left( \frac{1}{t} - 1 \right).
\]

We introduce one more abbreviation, for \( s \in (0, 1) \) and \( x, y \in X \) we write

\[
b(s) = F(x + s(y - x)) - s(y - x).
\]

Hence it follows that \( b'(s) = DF(x + s(y - x))(y - x) - \text{id}_X(y - x) \). Now we estimate for \( x, y \in B(0, r) \):

\[
||F(x) - F(y)|| - ||x - y|| \leq ||F(x) - x|| + ||F(y) - y|| = \int_0^1 b'(s)ds = \int_0^1 (DF(x + s(y - x)) - \text{id}_X)(y - x)ds \leq \left( \frac{1}{t} - 1 \right) ||y - x||.
\]
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where the last inequality holds true since \(x + s(y - x) = ys + x(1 - s) \in B(0, r_0)\). Hence

\[ ||F(x) - F(y)|| \leq \frac{1}{t} ||y - x||.\]

On the other hand, the last line implies that \(||F(x + v) - F(x) - v|| \leq (1/t - 1)||v||\). This estimate combined with the fact that \(||F(x + v) - F(x) - v|| \geq ||v|| - ||F(x + v) - F(x)||\) implies \(||F(x + v) - F(x)|| \geq 1/t\), which finishes the proof of the lemma. \(\square\)

Let us now derive the approximation theorem from Lemma 5.2.2. Since \(T \in R_m(X, G)\), we know that \(T = \theta(x)H^m \setminus M\), for some \(m\)-rectifiable set \(M\) which we will call set(\(T\)) and a Borel measurable function \(\theta : \text{set}(T) \to G\) such that \(\theta \in L^1(X, G)\) (which means that the function \(|\theta| \in L^1(H^m \setminus \text{set}(T))\)). Since set(\(T\)) is \(m\)-rectifiable, there exist countably many \(C^1\)-smooth manifolds \(\{M_i\}_{i=1}^{\infty}\) and a set \(M_0\) such that \(H^m(M_0) = 0\) satisfying

\[ \text{set}(T) \subseteq \bigcup_{i \geq 1} M_i \cup M_0.\]

Let us next consider a subset \(\widetilde{M} \subseteq \text{set}(T)\) such that for all \(a \in \widetilde{M}\) one has \(\Theta^m(H^m \setminus M, a) = \Theta^m(H^m \setminus \text{set}(T), a) = 1\) for some \(M = M(a) \in \{M_i\}_{i=1}^{\infty}\). Note that we can guarantee \(H^m(\text{set}(T) \setminus \widetilde{M}) = 0\). Next, for each \(a \in \widetilde{M}\) we define \(r = r(a)\) such that for all \(r \in (0, r(a))\) one has

\[ ||T||(B(x, r) \setminus (M(a) \cap B(a, r))) \leq \varepsilon||T||(B(a, r)), \]

where by \(||T||\) we, as always, denote the measure \(||T|| = ||\theta||H^m \setminus \text{set}(T)\). Let us next consider the following family of balls: \(B = \{B(a, r) : a \in \widetilde{M}, 0 < r < r(a)\}\). Note that \(B\) is a Vitali covering of the set \(\widetilde{M}\). According to the Vitali theorem, we can find a disjoint countable covering, say \(\{B(a_j, r_j) : a_j \in \widetilde{M}, 0 < r_j < r(a_j)\}\). We further renumber the family of manifolds \(\{M_i\}_{i=1}^{\infty}\) in a way that for each \(j \geq 1\) the point \(a_j\) belongs to the manifold \(M_j\). Let us now use the lemma: for each \(j \geq 1\), for each triplet \((a_j, M_j, B(a_j, r_j))\) and for each \(t \in (0, 1)\) there exists a \(C^1\)-diffeomorphism \(F_j : X \to X\) satisfying the claims of Lemma 5.2.2. Since set(\(T\)) = \(\bigcup_j (M_j \cap B(a_j, r_j))\) \(\cup N\) with \(H^m(N) = 0\), we can find a number \(N = N(\varepsilon)\) such that

\[ M\left(\sum_{j=N+1}^{\infty} T_j\right) \leq \varepsilon, \]

where we denote \(T_j = T \setminus (M_j \cap B(a_j, r_j))\) for sake of brevity.

The next step consists in proving that for every \(\delta > 0\) there exists a polyhedral chain \(\Psi_j \in P_m(X, G)\) such that

\[ M(F_{j#}T_j - \Psi_j) \leq \delta. \]

First of all, we know that \(||F_{j#}T_j|| = ||F_j^{-1} \circ \theta||H^m \setminus F_j(M_j \cap B(a_j, r_j))\). Since the function \(\theta\) is integrable on the set \(M_j \cap B(a_j, r_j)\), according to the Lusin theorem (contact \[Fed69\] Theorem 2.3.5) for each \(\delta > 0\) we can find a compact subset \(C \subset F_j(M_j \cap B(a_j, r_j))\) such that \((F_j^{-1} \circ \theta) \upharpoonright C\) is continuous and that \(||F_{j#}T_j||(F_j(M_j \cap B(a_j, r_j)) \setminus C) \leq \delta\). Suppose,
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as we can that $C \subseteq \{ x : \theta(x) \neq 0 \}$. Second of all, we decompose $C$ into a finite union of sets in a way that

$$C = \bigcup_{i=1}^{K} C_i \cup C_0$$

with $||F_j T_j||(C_0) \leq \delta$ and $\text{osc}(\theta \upharpoonright C_i) \leq \delta \inf_{C_i} |\theta|$. Third of all, for each natural $i$ between 1 and $K$ there exists $S_i \subset F_j (M_j \cap B(a_j, r_j))$, a finite union of dyadic $m$–semi–cubes such that $H^m(C_i \triangle S_i) \leq \delta H^m(C_i)$. Finally, for each $i \in [1, \ldots, K]$ we choose an element $\theta_i \in G$ that guarantees $|\theta_i - \theta(x)| \leq \delta \inf_{C_i} |\theta|$ for each $x \in C_i$. We define the desired polyhedral chain as

$$\Psi_j = \sum_{i=1}^{K} \theta_i [S_i].$$

Let us estimate the mass of the following difference

$$M(F_j \# T_j - \Psi_j) \leq \int_{F_j (M_j \cap B(a_j, r_j))} |\theta(x)| \, dH^m(x) - \sum_{i=1}^{K} |\theta_i| H^m(S_i) \leq$$

$$\delta + \sum_{i=1}^{K} \int_{C_i} |\theta(x)| \, dH^m(x) - \sum_{i=1}^{K} |\theta_i| H^m(S_i) \leq$$

$$\delta + \delta ||F_j T_j||(C) + \sum_{i=1}^{K} |\theta_i| (H^m(C_i) - H^m(S_i)) \leq \delta + (2 + \delta) ||F_j T_j||(C),$$

and the claim (5.10) follows.

Now we are ready to finish the proof of the theorem. We choose

$$F := F_1 \circ \ldots \circ F_N \text{ and } P := \sum_{j=1}^{N} \Psi_j.$$ 

By the construction we see that $F$ and $P$ satisfy the first two claims of the theorem. Let us check the third one:

$$M(F_\# T - P) = M(F_\# \sum_{j=1}^{\infty} T \subseteq B(a_j, r_j) - P) \leq M(\sum_{j=1}^{N} F_j \# T \subseteq B(a_j, r_j) - P) + \varepsilon \leq \ldots,$$

where the last inequality follows from the line (5.9). We continue the estimate, first using the estimate (5.8) and afterwards the inequality (5.10) with $\delta = \varepsilon/N$:

$$\ldots \leq M\left( \sum_{j=1}^{N} F_j \# T_j - \sum_{j=1}^{N} \Psi_j \right) + \varepsilon \sum_{j=1}^{N} ||T|| (B(a_j, r_j)) + \varepsilon \leq \frac{N \varepsilon}{N} + \varepsilon + \varepsilon M(T).$$

Since $\varepsilon > 0$ is arbitrary, the theorem follows. \qed
5.3 Solution to the Plateau problem whose support stays inside of the convex hull of the boundary.

The third step of the proof of Theorem \[5.2.1\] consists in derivation of this result from Theorem \[5.2.2\]. Let \( T_j \) be a sequence of \( m \)-rectifiable \( G \) chains, converging in flat norm to some \( T \in R_m(X,G) \). Fix \( \varepsilon > 0 \). According to the approximation theorem, there exist a \( P \in P_m(X,G) \) and \( F : X \to X \) such that \( P = F\#T + E \) with \( M(E) \leq \varepsilon \). Note that since \( F \) is Lipschitz, one has \( \lim_{j \to \infty} F(T_j - F\#T) = 0 \) and hence also \( \lim_{j \to \infty} F(E + F\#T_j - P) = 0 \) by the definitions of \( E \) and \( P \). Since \( P \) is polyhedral, we can use Lemma \[5.2.1\]:

\[
M(E + F\#T) = M(P) \leq \liminf_{j \to \infty} M(E + F\#T_j) \leq \varepsilon + \liminf_{j \to \infty} (\text{Lip}(F))^m M(T_j) \leq \varepsilon + (1 + \varepsilon)^m \liminf_{j \to \infty} M(T_j), \tag{5.11}
\]

where the last inequality follows from the fact that \( F \) is \((1 + \varepsilon)\)-Lipschitz. On the other hand, since \( \text{Lip}(F^{-1}) \leq 1 + \varepsilon \), we infer that

\[
M(E + F\#T) \geq M(F\#T) - M(E) \geq (1 + \varepsilon)^{-m} M(T) - \varepsilon. \tag{5.12}
\]

The lines (5.11) and (5.12) now give

\[
(1 + \varepsilon)^{-m} M(T) \leq (1 + \varepsilon)^m \liminf_{j \to \infty} M(T_j) + 2\varepsilon,
\]

and the theorem will follow after we let \( \varepsilon \) tend to zero.

**Corollary 5.2.1.** Let \((X,||\ldots||)\) be an \( n \)-dimensional Banach space. If \( T \in R_{n-1}(X,G) \) and if \( \pi_W : X \to W \) is a \( W \)-good projection then the following inequality holds true:

\[
M(\pi_W\#T) \leq M(T).
\]

**Proof.** Note that according to the strong approximation Theorem \[5.2.2\] there exists a sequence \( P_j \in P_{n-1}(X,G) \) such that \( F(T - P_j) \to 0 \) and \( M(T) = \lim_j M(P_j) \). Next, since \( \text{Lip}(\pi_W) < \infty \), we infer that \( F(\pi_W(T - P_j)) \to 0 \) and hence referring to Theorem \[5.2.1\] we write

\[
M(\pi_W\#T) \leq \liminf_j M(\pi_W\#P_j) \leq \liminf_j M(T),
\]

where the second inequality follows from Theorem \[4.3.5\].

5.3 Solution to the Plateau problem whose support stays inside of the convex hull of the boundary.

As a consequence of the lower semicontinuity theorem, we obtain the following existence result.

**Theorem 5.3.1.** Let \((X,||\ldots||)\) be an \( n \)-dimensional Banach space and let \( G \) be a locally compact complete normed Abelian group. Then the following Plateau problem admits a solution:

\[
\begin{cases}
\text{minimize} \ M(T) \\
T \in R_{n-1}(X,G) \text{ such that } \partial T = B,
\end{cases}
\]

\[
\]
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where $B \in \mathcal{R}_{n-2}(X, G)$ has a compact support and satisfies $\partial B = 0$. Moreover, among the solutions, there exists at least one, say $T_b$ satisfying $\text{spt}(T_b) \subseteq \text{conv}(\text{spt}(B))$.

**Proof.** We apply the direct method of calculus of variation. Take a minimizing sequence $T_j \in \mathcal{R}_{n-1}(X, G)$. By this we mean that $\partial T_j = B$ and $\mathbb{M}(T_j) \leq m + 1/j$, where $m$ is the infimum of the problem. In what follows we are going to construct another minimizing $Q$ Euclidean nearest point projection onto the convex set $B$ by $Q$ hyperplanes, by $Q$ constructions that $\text{spt}(\rho_i)$ minimize the $Q$ Euclidean nearest point projection onto the convex set $B$.

To this end, consider any bounded convex polyhedron $Q$ such that $B \subseteq Q$. Denote by $Q_1, \ldots, Q_N$ the $(n-1)$ dimensional facets of $Q$, by $W_1, \ldots, W_N$ the corresponding hyperplanes, by $K$ the set

$$K = Q \cup \bigcup_{i=1}^{N} W_i$$

and finally by $W_1^+, \ldots, W_N^+$ the halfspaces in $X$ satisfying $Q = \bigcap_i W_i^+$. Define mappings $\rho_i : X \to X$ as $\rho_i(x) = x$ if $x \in W_i^+$ and $\rho_i(x) = \pi_{W_i}(x)$, where $\pi_{W_i} = \text{GP}(W_i)$, otherwise. Write $\rho = \rho_1 \circ \cdots \circ \rho_N$. We claim that $\mathbb{M}(\rho_\#(T_j)) \leq \mathbb{M}(T_j)$. First, it is obviously sufficient to show that $\mathbb{M}(\rho_i\#(T_j)) \leq \mathbb{M}(T_j)$ for each $1 \leq i \leq N$. On the other hand,

$$\mathbb{M}(\rho_i\#(T_j)) = \mathbb{M}((T_j) \bigcap (X \setminus W_j^+)) + \mathbb{M}(\pi_{W_i}\#(T_j) \bigcap (W_j^+)) \leq \mathbb{M}(T_j),$$

where the last inequality here follows from Corollary [5.2.1]. Next, it follows from the construction that $\text{spt}(\rho_\#(T_j)) \subseteq K$ and that $\partial(\rho_\#T_j) = B$. Note we have obtained another minimizing sequence $\tilde{T}_j = \rho_\#T_j$ that has support in $K$. Hence it is left to “get rid” of the union of hyperplanes $\bigcup_i W_i$.

Fix an integer $i$ between 1 and $N$. Consider mappings $f_i : K \to X$ defined as follows. First, $f_i(x) = x$ if $x \in K \setminus W_i$ and $f_i(x) = P_i(x)$ otherwise, where $P_i : W_i \to Q_i$ is the Euclidean nearest point projection onto the convex set $Q_i$. Let us prove that the functions $f_i$ are Lipschitz. Note that it is sufficient to estimate $|f_i(x) - f_i(y)|$ when $x \in W_i$ and $y \in K \setminus W_i$. Note that by definition $f_i(y) = y$ and denote by $z = f_i(x)$. Suppose first that there holds $\langle a, b \rangle \leq 0$ where $a = x - z, b = y - z$. Hence

$$|x - y|^2 = |\langle a - b, a - b \rangle|^2 \geq |a|^2 + |b|^2 - 2\langle a, b \rangle \geq \frac{|x - z| + |z - y|)^2}{2} \geq \frac{|f_i(x) - f_i(y)|^2}{2}.$$ 

Hence we can assume that $\langle a, b \rangle \geq 0$. In this case there exists a constant $C = C_i \in (0, 1)$ depending on $W_i$ and $Q$ such that

$$1 - C \geq \frac{\langle a, b \rangle}{|a||b|} \geq 0.$$ 

Indeed, this follows from the fact that the angles between the facets $Q_j, j \neq i$ and $W_i$ are
bounded from below. Now we estimate:

\[
|x - y|^2 \geq |a|^2 + |b|^2 - 2(1 - C)|a||b| = \\
\frac{C^2}{4}(|a|^2 + |b|^2 + 2|a||b|) + \left(1 - \frac{C^2}{4}\right)(|a|^2 + |b|^2) - 2 \left(1 - \frac{C}{2}\right)^2 |a||b| = \\
\frac{C^2}{4}(|x - z| + |z - y|)^2 \geq \frac{C^2}{4} |f_i(x) - f_i(y)|^2,
\]

where the second inequality here follows from the fact that \( D := (1-C/2)^4 - (1-C^2/4)^2 \leq 0 \) when \( C \in (0,1) \). Next, extend according to the Kirszbraun theorem the functions \( f_i \) to the whole space \( X \) preserving the Lipschitz condition. The Lipschitz constants of \( f_i \) may grow, but this is not important. We call the extended functions \( f_i \) as well.

Consider further mappings \( g_i \) is defined as the Euclidean nearest point projections \( g_i : X \to Q_i \), onto the facet \( Q_i \). Consider now the chain \( f_i \# T_j \). Note that \( \text{spt}(f_i \# T_j) \subseteq Q \cup \bigcup_{k \neq i} W_k \). In order to estimate its mass, write

\[
\tilde{T}_j = R_0 + \sum_{k=1}^{N} R_k,
\]

where \( R_k = \tilde{T}_j \cap (W_k \setminus Q) \) and \( R_0 = \tilde{T}_j \cap Q \). Note that \( \text{spt}(R_k) \subset W_k \) and that \( f_i = g_i \) on \( W_i \) and hence \( f_i \# R_i = g_i \# R_i \). On the other hand, there exists a constant \( C = C(W_i) \) such that \( \mathbb{M}(A) = C \mathbb{M}_2(A) \) (where \( \mathbb{M}_2 \) stands for the Euclidean mass) for all chains \( A \) such that \( \text{spt}(A) \subset W_i \). Hence, since \( \text{Lip}(g_i) \leq 1 \) we infer that

\[
\mathbb{M}(f_i \# R_i) = C \mathbb{M}_2(g_i \# (R_i)) \leq C \mathbb{M}_2(R_i) = \mathbb{M}(R_i).
\]

We finally estimate the mass of the chain \( f_i \# \tilde{T}_j \):

\[
\mathbb{M}(f_i \# \tilde{T}_j) \leq \sum_{k=0}^{N} \mathbb{M}(f_i \# R_k) = \sum_{k \neq i}^{N} \mathbb{M}(R_k) + \mathbb{M}(f_i \# R_i) \leq \sum_{k=0}^{N} \mathbb{M}(R_k) = \mathbb{M}(\tilde{T}_j).
\]

Performing the same procedure for each \( i \) between 1 and \( N \) consequently, we obtain a minimizing sequence \( T_{i,Q} \) such that \( \text{spt}(T_{i,Q}) \subseteq Q \). Write \( C = \text{conv}(\text{spt}(B)) \). According to the Hahn–Banach theorem we know that \( C = \bigcap_{x \in \partial C} H_x^+ \), where \( H_x \) is a supporting hyperplane to \( C \) and \( H_x^+ \) is the corresponding halfspace. Let us prove that

\[
C = \bigcap_{i=1}^{\infty} H_i^+
\]

for a countable subset of halfspaces \( H_i^+ \). Indeed, the polar set \( C^* = X^* \cap \{ \alpha : \sup_{C} \alpha \leq 1 \} \) of the convex set \( C \) is separable, hence there exists a dense sequence \( \alpha_n \) in \( C^* \). Note that

\[
C = \bigcap_{\alpha \in C^*} \alpha^{-1}(-\infty, 1].
\]
We infer that $C = \bigcap_n \alpha_n^{-1}(-\infty, 1]$. Obviously, $C$ is contained in the intersection. We prove the reciprocal inclusion by contradiction. Take $c \notin C$, but $c \in \bigcap_n \alpha_n^{-1}(-\infty, 1]$. Hence there exists $\alpha \in C^0$ and $x \in C$ such that $\alpha(x) \geq 1 + \varepsilon$ for some $\varepsilon > 0$. Approximating $\alpha$ by $\alpha_n$ with $n$ large enough yields a contradiction. Next write

$$C_N = \bigcap_{i=1}^N H_i^+$$

with some $N$ such that $C_N$ is bounded. Apply the procedure described in the previous three paragraphs to the polyhedron $C_N$. Note that $T_{j,C_N} \in \mathcal{F}_{n-1}(X, G) \cap \{T : \mathbb{N}(T) \leq C, \text{supp}(T) \subseteq C_N\}$, which is $\mathcal{F}$–compact by Theorem 3.2.7. Hence there exists $T_N \in \mathcal{R}^{n-1}(X, G)$ such that $T_{j,C_N} \to T_N$. Since the mass is lower semicontinuous, we infer that

$$\mathcal{M}(T_N) \leq \liminf_j \mathcal{M}(T_{j,N}) \leq \liminf_j \left( m + \frac{1}{i} \right) = m,$$

and hence $\mathcal{M}(T_N) = m$. After a subsequence $T_{N_k} \to T_0$ and obviously $\partial T_0 = B$ and $\mathcal{M}(T_0) = m$. On the other hand, $\text{spt}(T_0) \subseteq \bigcap_k C_{N_k} = C$. \qed
Chapter 6

Gross measures and the second proof of the existence in codimension one.

In this chapter we are going to present our second proof of the existence of mass minimising \((n-1)\)-rectifiable \(G\) chains in \(n\)-dimensional Banach spaces.

**Remark 6.0.1.** The theorems proved in this chapter were first formulated and proved here by I. Vasilyev.

### 6.1 A little bit more about good projections.

The current section is the one where we establish a couple of useful properties of good projections.

**Remark 6.1.1.** Let \(X\) be a Banach space. We denote here by \(\text{Hom}(X, X)\) the space of bounded linear operators on \(X\) furnished with the norm topology (for \(f \in \text{Hom}(X, X)\) we define \(\|f\| := \sup\{|f(x)| : x \in X, \|x\| = 1\}\)).

**Proposition 6.1.1.** The set \(\text{Pr}\) of all pairs \((W, \pi_W)\), where \(W \in G(n-1, X)\) and \(\pi\) is a \(W\)-good projection, is a closed subset of the space \(G(n-1, X) \times \text{Hom}(X, X)\), furnished with the product topology.

**Proof.** Let \(\{W_i\}_{i=1}^\infty\) be a sequence of hyperplanes in \(X\) and for each \(i \geq 1\) let \(\pi_{W_i}\) be a \(W_i\)-good projection. Suppose that \((W_i, \pi_{W_i}) \to (W, \pi_W)\) in \(G(n-1, X) \times \text{Hom}(X, X)\). Then the goal is to prove that \(\pi_W\) is a \(W\)-good projection. Let \(V \in G(n-1, X)\) and let \(A \subset V\). Note that we are done once we prove that \(\mathcal{H}^{n-1}_{\|\cdot\|}(\pi_W(A)) \leq \mathcal{H}^{n-1}_{\|\cdot\|}(A)\). Since each \(\pi_{W_i}\) is \(W_i\)-good, we know that \(\mathcal{H}^{n-1}_{\|\cdot\|}(\pi_{W_i}(A)) \leq \mathcal{H}^{n-1}_{\|\cdot\|}(A)\). Next, thanks to Theorem 4.3.1, we know that

\[
\mathcal{H}^{n-1}_{\|\cdot\|}(\pi_{W_i}(A)) = \alpha_{n-1} \frac{\mathcal{H}^{n-1}(\pi_{W_i}(A))}{\mathcal{H}^{n-1}(B \cap W_i)} \quad \text{and} \quad \mathcal{H}^{n-1}_{\|\cdot\|}(A) = \alpha_{n-1} \frac{\mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(B \cap V)}.
\]
This allows us to conclude that
\[
\frac{\mathcal{H}^{n-1}(\pi_{W_i}(A))}{\mathcal{H}^{n-1}(B \cap W_i)} \leq \frac{\mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(B \cap V)}.
\]

Hence it is sufficient to establish the following two convergences:
\[
\mathcal{H}^{n-1}(\pi_{W_i}(A)) \rightarrow \mathcal{H}^{n-1}(\pi_W(A)) \quad \text{and} \quad \mathcal{H}^{n-1}(B \cap W_i) \rightarrow \mathcal{H}^{n-1}(B \cap W).
\]

The second convergence can be derived from Theorem 4.2.1. Indeed, this theorem tells
that the Busemann–Hausdorff density function is convex, hence it is also continuous (of
course one should also use an obvious fact that \(\mathcal{H}^{n-1}(B \cap \tilde{W}) \neq 0\) for all \(\tilde{W} \in G(n - 1, X)\)).

Let us prove the first convergence. Let \(\pi_{W_i} : V \rightarrow W\) be a mapping defined as the
restriction of the projection \(\pi_{W_i}\) to the hyperplane \(V\). Note that
\[
\mathcal{H}^{n-1}(\pi_{W_i}(A)) = J_{\pi_{W_i}} \cdot \mathcal{H}^{n-1}(A),
\]
where \(J_{\pi_{W_i}}\) is the Jacobian of the mapping \(\pi_{W_i}\). Fix \(w_{i,1}, \ldots, w_{i,n-1}\), orthonormal bases
of \(W_i\). Note that we can suppose that the bases \(w_{i,1}, \ldots, w_{i,n-1}\) converge towards some
basis \(v_1, \ldots, v_{n-1}\) of the hyperplane \(V\). Indeed, we can consequently, for \(j\) from 1 to \(n - 1\),
extract subsequences \(w_{j,i}\) converging to some \(v_j\). Since the scalar product is a continuous
operation the limiting system \(v_1, \ldots, v_{n-1}\) will be an orthonormal basis of \(V\). This might
force us to consider a subsequence of the sequence \(W_i\), but this changes nothing. Then for
all indexes \(1 \leq i \leq k \leq n - 1\) one has
\[
\pi_{W_i}(v_j) = \sum_{k=1}^{n-1} a_{i,j,k} w_{i,k}
\]
for some real coefficients \(a_{i,j,k}\). On the other hand, we know that
\[
\pi_{W_i}(v_j) = v_j - \frac{\langle v_j, \omega_i \rangle}{\langle v_j, u_i \rangle} u_i,
\]
where \(\omega_i\) is the Euclidean unit normal to the hyperplane \(W_i\) and \(u_i\) is the Euclidean unit
vector, spanning the (one–dimensional) kernel of the projection \(\pi_{W_i}\). Hence for all indexes
\(1 \leq k \leq n - 1\) one has
\[
a_{i,j,k} = \langle \pi_{W_i}(v_j), w_{i,k} \rangle = \langle v_j, w_{i,k} \rangle - \frac{\langle v_j, \omega_i \rangle}{\langle v_j, u_i \rangle} \langle u_i, w_{i,k} \rangle.
\]
Since \(J_{\pi_{W_i}} = \det \{a_{i,j,k}\}_{k,j=1}^{n-1}\), the desired convergence follows.

In what follows we are going to use the following two definitions.

**Definition 6.1.1.** (Polish space) A **Polish space** is a separable topological space that can
be metrized using a complete metric.
6.2 The Gross measures and another proof of the existence in codimension one.

Definition 6.1.2. (Suslin set) A subset of a Polish space $X$ is an analytic set if it is a continuous image of some Polish space.

The following result can now be derived from Theorem 6.1.1.

Corollary 6.1.1. Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space. Then there exists a universally measurable mapping $GP : G(n - 1, X) \to \text{Hom}(X, X)$ such that for every $W \in G(n - 1, X)$, $GP(W)$ is a $W$–good projection. We shall often write $\pi_W$ instead of $GP(W)$. We shall call such mapping $GP$ a good choice of good projections mapping.

Proof. Since every closed set is Suslin (analytic), the corollary follows from [Coh13], Theorem 8.5.3. □

6.2 The Gross measures and another proof of the existence in codimension one.

Following Federer ([Fed69], section 2.10.3) we define the Gross measures.

Definition 6.2.1. (Gross measure) Let $P : G(n - 1, X) \to \text{Hom}(X, X)$ be a map such that for every $W \in G(n - 1, X)$ the mapping $P(W)$ is a linear projection with range $W$. For a Borelian set $A \subseteq X$ denote

$$\zeta_{||\ldots||}^{n-1}P(A) := \sup_{W \in G(n-1, X)} H_{||\ldots||}^{n-1}(P_W(A)),$$

The Gross measure $G_{||\ldots||}^{n-1}P$ is defined by the following formula

$$G_{||\ldots||}^{n-1}P(S) := \sup_{\delta > 0} \inf \left\{ \sum_{j=1}^{\infty} \zeta_{||\ldots||}^{n-1}P(A_j) : A_j \subset X \text{ are Borelian, } \bigcup_{j=1}^{\infty} A_j \supset S, \text{ diam}(A_j) \leq \delta \right\}.$$

Remark 6.2.1. Notice that the measure $G_{||\ldots||}^{n-1}P$ is defined by the Caratheodory construction using a gauge defined on Borelian subsets of $X$. Hence it is automatically a Borel regular measure, see [Fed69] 2.10.4.

The following theorem shows that for rectifiable subsets of finite dimensional Banach spaces the Hausdorff and the Gross measures coincide.

Theorem 6.2.1. Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $GP : G(n - 1, X) \to \text{Hom}(X, X)$ be a good choice of good projections mapping. If $A$ is a Borelian $(n - 1)$–rectifiable subset of $X$, then

$$H_{||\ldots||}^{n-1}(A) = G_{||\ldots||}^{n-1}GP(A).$$
Gross measures and the second proof of the existence in codimension one.

**Proof.** Let us first prove that \( \mathcal{H}^{n-1}_{|| \cdot ||}(A) \geq \mathcal{G}^{n-1}_{|| \cdot ||,GP}(A) \). Fix \( \delta > 0 \). Let \( \{B_j\}_{j \in J} \) be a sequence of Borelian sets satisfying \( A = \bigcup_j B_j \) and \( \text{diam}_{|| \cdot ||}(B_j) \leq \delta \) and such that the sets \( B_j \) are pairwise disjoint. Indeed, one can take \( B_j = C_j \cap A \), where \( C_j \) is the system of the dyadic semi-cubes with edges of length \( \delta/\sqrt{n} \). Next, since each \( \pi_W = GP(W) \) is a \( W \)-good projection, thanks to Theorem 4.3.4 for each \( j \in J \) we have

\[
\zeta^{n-1}_{|| \cdot ||,GP}(B_j) \leq \mathcal{H}^{n-1}_{|| \cdot ||}(B_j).
\]

Let us take the sum in the previous line and use the fact that \( \mathcal{G}^{n-1}_{|| \cdot ||,GP} \) is a Borelian measure:

\[
\mathcal{G}^{n-1}_{|| \cdot ||,GP}(A) \leq \sum_{j \in J} \zeta^{n-1}_{|| \cdot ||,GP}(B_j) \leq \sum_{j \in J} \mathcal{H}^{n-1}_{|| \cdot ||}(B_j) = \mathcal{H}^{n-1}_{|| \cdot ||}(A),
\]

and the first inequality follows.

We proceed to the reciprocal inequality. Note that since \( A \) is \( (n-1) \)-rectifiable, there exists a countable union of \( C^1 \)-manifolds \( \{M_i\}_{i \in I} \) and a set \( M_0 \) such that \( \mathcal{H}^{n-1}(M_0) = 0 \) satisfying \( A \subseteq \bigcup_i M_i \cup M_0 \). Writing \( A \) as a disjoint union \( A = \bigcup_i A_i \cup M_0 \) where \( A_i \subseteq M_i \) are Borelian, we infer that it is sufficient to prove the inequality for Borelian subsets of \( C^1 \)-manifolds.

Let \( A \subseteq M \) be a Borelian subset of manifold \( M \). Fix \( \varepsilon > 0 \). First, we claim that for every \( x \in A \) there exists \( \delta = \delta(x) \) such that for all \( r < \delta \) there holds

\[
\mathcal{H}^{n-1}_{|| \cdot ||}(A \cap B(x, r)) \leq (1 + \varepsilon) \mathcal{H}^{n-1}_{|| \cdot ||}(\pi_W(A \cap B(x, r))),
\]

where by \( W \) we denote the tangent plane \( \text{Tan}(M, x) \) to the manifold \( M \) at the point \( x \) and \( \pi_W = GP(W) \). Indeed note that we are done once we show that the function \( \pi_W | (A \cap B(x, r)) \) is \( (1 + \varepsilon) \)-Lipschitz with respect to the norm \( || \cdot || \). To this end, write \( \pi_W(y) = y + \omega g(y) \), where \( \omega \) is the Euclidean unit vector, spanning the (one-dimensional) kernel of the projection \( \pi_W \) and \( g \) is a real valued function. Hence

\[
||\pi_W(y) - \pi_W(z)|| \leq ||y - z|| + ||\omega|| \cdot |g(y) - g(z)|
\]

and it remains to show that the function \( g : A \cap B(x, r) \to \mathbb{R} \) is \( \varepsilon \)-Lipschitz for \( r \) small enough. To prove this, consider the orthogonal projection \( P_W : X \to W \) given by the formula \( P_W(y) = y + e f(y) \), where \( e \) is the Euclidean unit normal vector to the hyperplane \( W \) and \( f \) is a real valued function. Since \( M \) is a manifold, it follows that there exists \( \delta = \delta(x) \) such that for all \( r \leq \delta \) the function \( f | (M \cap B(x, r)) \) is \( \varepsilon \)-Lipschitz. Note that for all \( z, y \in X \) one has \( g(z) f(y) = g(y) f(z) \). Hence for all \( z, y \in M \cap B(x, r) \) and all \( r \leq \delta \) one has

\[
|g(y) - g(z)| \leq |g(y)| \cdot \left| 1 - \frac{g(z)}{g(y)} \right| \leq \left| \frac{g(y)}{f(y)} \right| \cdot |f(y) - f(z)| \leq C \varepsilon ||y - z||,
\]

and our claim follows.
6.2 The Gross measures and another proof of the existence in codimension one.

Second, we infer that for every Borelian subset \( E \subseteq X \) there holds
\[
\zeta^{n-1}_{|| \cdot ||}(E) \leq \mathcal{G}^{n-1}_{|| \cdot ||}(E). \tag{6.1}
\]
Indeed, suppose \( E \subseteq \bigcup_{j \in J} B_j \), where \( B_j \) are Borelian. For each \( V \in G(n-1,X) \) one obviously has
\[
\mathcal{H}^{n-1}_{|| \cdot ||}(\pi_V(E)) \leq \sum_{j \in J} \mathcal{H}^{n-1}_{|| \cdot ||}(\pi_V(B_j)),
\]
where \( \pi_V = \text{GP}(V) \). Hence
\[
\zeta^{n-1}_{|| \cdot ||}(E) = \sup_{V \in G(n-1,X)} \mathcal{H}^{n-1}_{|| \cdot ||}(\pi_V(E)) \leq \sup_{V \in G(n-1,X)} \sum_{j \in J} \mathcal{H}^{n-1}_{|| \cdot ||}(\pi_V(B_j)) \leq \sum_{j \in J} \sup_{V \in G(n-1,X)} \mathcal{H}^{n-1}_{|| \cdot ||}(\pi_V(B_j)) = \sum_{j \in J} \zeta^{n-1}_{|| \cdot ||}(B_j).
\]
Since \( \{B_j\}_{j \in J} \) is arbitrary, the inequality follows.

Finally, consider the following collection of balls \( B = \{B(x,r) : x \in A, 0 < r < \delta(x)\} \) which is a covering of the set \( A \). Referring to the Besicovitch covering theorem \( 3.2.9 \), we infer that there exists a countable disjoint collection of balls \( \{B(x_i,r_i)\}_{i \in I} \subset B \) such that \( \mathcal{H}^{n-1}_{|| \cdot ||}(A\setminus \bigcup_i B(x_i,r_i)) = 0 \). Hence we can estimate
\[
\mathcal{H}^{n-1}_{|| \cdot ||}(A) = \sum_{i \in I} \mathcal{H}^{n-1}_{|| \cdot ||}(A \cap B(x_i,r_i)) \leq (1 + \varepsilon) \sum_{i \in I} \zeta^{n-1}_{|| \cdot ||}(A \cap B(x_i,r_i)) \leq (1 + \varepsilon) \mathcal{G}^{n-1}_{|| \cdot ||}(A),
\]
and the theorem follows. \( \Box \)

**Remark 6.2.2.** We would like to emphasise that in the previous theorem the first inequality (namely \( \mathcal{H}^{n-1}_{|| \cdot ||}(A) \geq \mathcal{G}^{n-1}_{|| \cdot ||,\text{GP}}(A) \)) does not require the rectifiability of \( A \).

Next we give one more definition in connection with the Gross measures.

**Definition 6.2.2.** (Gross mass) Let \( (X,|| \cdot ||) \) be an \( n \)-dimensional Banach space and let \( G \) be a complete normed Abelian group. Suppose that \( T \in \mathcal{R}_m(X,G) \) with \( T = \theta_T(x) \mathcal{H}^m \ast \text{set}(T) \). We define the **Gross mass** \( \mathbb{M}_G \) by the following formula
\[
\mathbb{M}_G(T) := \int_{\text{set}(T)} \theta_T(x) d\mathcal{G}^{n-1}_{|| \cdot ||,\text{GP}}(x).
\]
We shall use Theorem \( 6.2.1 \) in our second proof of the lower semicontinuity result in codimension one.

**Theorem 6.2.2.** Let \( (X,|| \cdot ||) \) be an \( n \)-dimensional Banach space. Then the mass \( \mathbb{M} \) is lower semicontinuous with respect to flat convergence on \( \mathcal{R}_{n-1}(X,G) \).
Proof. Define an auxiliary mass denoted as $\zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}$ and defined for $T \in \mathcal{R}_{n-1}(X, G)$ as

$$
\zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}(T) := \sup_{W \in G(n-1, X)} \mathbb{M}(\pi_W(T)),
$$

where $\pi_W = \text{GP}(W)$.

Note that, thanks to Theorem 6.2.1, it is sufficient to show that the Gross mass $\mathbb{M}_G$ is lower semicontinuous with respect to flat convergence. The rest of the proof is devoted to this statement. Let us trace briefly the plan of our proof. First, we prove a lemma where we show that the mass $\zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}$ is itself $\mathcal{F}$–lower semicontinuous. Second, we prove that the mass $\zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}$ is majorized by the mass $\mathbb{M}_G$ (by the way, this step requires a variant of Crofton’s formula). We finally derive the desired lower semicontinuity from the first two steps using a covering argument which depends on a formula for the Gross density.

**Lemma 6.2.1.** If $\{T_i\}$ is a sequence of $(n-1)$–rectifiable $G$ chains such that $\mathcal{F}(T_i - T) \to 0$ for some $T \in \mathcal{R}_{n-1}(X, G)$, then

$$
\zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}(T) \leq \liminf_i \zeta_{\ell_1,\ldots,\ell_{n-1}}^{n-1,\text{GP}}(T_i).
$$

**Proof.** Given $W \in G(n-1, X)$, since $\pi_W$ is Lipschitz, one has $\mathcal{F}(\pi_W(T) - \pi_W(T_i)) \to 0$. Note that on the space $\mathcal{R}_{n-1}(W, G)$ flat norm $\mathcal{F}$ is equivalent to the mass $\mathbb{M}$. Now we estimate:

$$
\lim_i |\mathbb{M}(\pi_W(T)) - \mathbb{M}(\pi_W(T_i))| \leq \lim_i \mathbb{M}(\pi_W(T) - \pi_W(T_i)) \leq C(n) \lim_i \mathcal{F}(\pi_W(T) - \pi_W(T_i)) \leq C(n) \lim_i \mathcal{F}(T - T_i).
$$

Hence for each $W$ the application $T \to \mathbb{M}(\pi_W(T))$ is continuous. Since supremum of continuous functions is lower semicontinuous, the proof is complete. $\square$

In what follows we will need a variant of Crofton’s inequality, recall 2.2. The desired version resembles the inequality from Proposition 5.5.1 in [BP15]. In order to prove this result, we first give a number of definitions from [BP15].

**Definition 6.2.3.** (Multiplicity function) Given $A \subset X$, its **multiplicity function** $N_A$ is defined as follows

$$
N_A: G(n-1, X) \times X \to \mathbb{R}_+ \cup \{\infty\}
$$

$$(W, y) \mapsto \text{card}(A \cap \pi_W^{-1}(y)),
$$

where $\pi_W = \text{GP}(W)$.

**Definition 6.2.4.** (Universally measurable set) Given a Polish space $Z$ (see 6.1.1) with Borelian sigma–algebra $\Omega$, a subset $A \subset Z$ is called a **universally measurable set** if it lies in the completion of the $\sigma$–algebra $\Omega$ with respect to every finite measure $\mu$ on $(Z, \Omega)$.
6.2 The Gross measures and another proof of the existence in codimension one.

Definition 6.2.5. (Universally measurable function) We call a mapping \( f : Y \to Z \) where \( Y \) and \( Z \) are two measurable spaces a universally measurable function if \( f^{-1}(A) \subset Y \) is a universally measurable set, once the set \( A \subset Z \) is Borelian.

Remark 6.2.3. It is worth mentioning that universally measurable sets form a \( \sigma \)-algebra.

With each set \( A \subset X \) we associate the set \( \chi(A) \) defined as

\[
\chi(A) = \{(W, y) \in G(n - 1, X) \times X : y \in \pi_W(A)\}.
\]

We also fix the following notation: denote by \( Q_j \) a system of dyadic semi-cubes with edges of the length \( 2^{-j} \). Let us prove the following lemma which gives a formula for the function \( N_A \).

Lemma 6.2.2. Let \( A \subset X \) and let \( B_j = \{Q \in Q_j : A \cap Q \neq \emptyset\} \). It follows that

\[
N_A = \lim_j \sum_{B \in B_j} 1_{\chi(B)}.
\]

Proof. The proof is identical to that of the Proposition 5.4.1 of the paper [BP15]. Nevertheless we present here a proof for the reader’s convenience. Fix \((W, y) \in G(X, m) \times X\). For each integer \( j \) define \( G_j = B_j \cap \{B : 1_{\chi(B)}(W, y) = 1\} \). Next, one easily defines an injective map \( G_j \to G_{j+1} \). Thus the sequence \( N_{A,j}(W, y) \) is nondecreasing and therefore has a limit in \( \mathbb{R}^+ \cup \{\infty\} \).

Our next claim is that

\[
N_A(W, y) \geq N_{A,j}(W, y)
\]

for every integer \( j \). Let \( k \in \mathbb{Z}_+ \) be such that \( N_{A,j}(W, y) \geq k \). This means that there are distinct \( B_1, \ldots, B_k \in B_j \) such that \((W, y) \in \chi(B_k), k = 1, \ldots k \). In particular there are \( x_k \in B_k \) so that \( y = \pi_W(x_k) \), therefore also \( x_k \in A \cap \pi_W^{-1}(y) \). Since the sets \( B_k \) are distinct so are the \( x_k \) and we conclude that \( N_A(W, y) \geq k \). Since \( k \) is arbitrary the proof of the claim is complete.

In order to finish the proof of the proposition it remains to show that

\[
N_A(W, y) \leq \lim_j N_{A,j}(W, y).
\]

Let \( F \subset A \cap \pi_W^{-1}(y) \) be a finite set. Put \( \delta := \min\{|x - x'| : x, x' \in F \text{ are distinct}\} \). Making the side lengths of the cubes small enough, we infer that there is an integer \( j_0 \) such that \( \text{diam}(B) < \delta \) whenever \( B \in B_j \) and \( j \leq j_0 \). Fix such \( j \). Then each \( x \in F \) belongs to some \( B \in B_j \) and two distinct \( x, x' \in F \) cannot belong to the same \( B \in B_j \) . Therefore \( N_{A,j}(W, y) \leq \text{card}(F) \). If \( N_A(W, y) < \infty \) then we simply let \( F = A \cap \pi_W^{-1}(y) \) and the lemma is established. Otherwise the argument shows that \( \lim_j N_{A,j}(W, y) = \infty \) and the assertion of the lemma follows in this case as well.

We further infer that given a hyperplane \( W \in G(n - 1, X) \) and a universally measurable set \( A \) the function

\[
y \to N_A(W, y)
\]

is a Borelian.
is universally measurable. Indeed, this follows from the previous lemma and from the fact that the set \( \pi_W(B) \) is universally measurable for each \( B \in \mathcal{B}_j \), since \( \pi_W \) is continuous. As a consequence of this fact we observe that the function

\[
g_A: G(n - 1, X) \to \mathbb{R}_+ \cup \{\infty\}
\]

\[
W \mapsto \int_X N_A(W, y) d \left( \mathcal{H}^{n-1}_{||\cdot||} \llcorner W \right)(y)
\]

is well defined. Now we are ready to state the needed variant of Crofton’s inequality:

**Theorem 6.2.3.** Let \( A \subset X \) be a Borelian set. Then the following holds

\[
\sup_{W \in G(n-1, X)} g_A(W) \leq G^{n-1}_{||\cdot||, GP}(A).
\]

**Proof.** Let \( \mathcal{B}_j \) be as above. Thus by Lemma 6.2.2 we infer that

\[
N_A(W, y) = \lim_j \sum_{B \in \mathcal{B}_j} 1_{\chi(B)}(W, y)
\]

for every \((W, y) \in G(n - 1, X) \times X\). Thanks to the Monotone Converge Theorem, for each \( W \) we have

\[
g_A(W) = \int_X N_A(W, y) d \left( \mathcal{H}^{n-1}_{||\cdot||} \llcorner W \right)(y) = \lim_j \sum_{B \in \mathcal{B}_j} \int_X 1_{\chi(B)}(W, y) d \left( \mathcal{H}^{n-1}_{||\cdot||} \llcorner W \right)(y) = \lim_j \sum_{B \in \mathcal{B}_j} \mathcal{H}^{n-1}_{||\cdot||}(\pi_W(B)).
\]

Next, referring to basic properties of the supremum we get

\[
\sup_{W \in G(n-1, X)} g_A(W) \leq \liminf_j \sum_{B \in \mathcal{B}_j} \sup_{W \in G(n-1, X)} \mathcal{H}^{n-1}_{||\cdot||}(\pi_W(B)) = \liminf_j \sum_{B \in \mathcal{B}_j} \zeta^{n-1}_{||\cdot||}(B) \leq G^{n-1}_{||\cdot||, GP}(A),
\]

where the last inequality here follows from the inequality (6.1). \( \square \)

As a consequence of this theorem we obtain the following proposition.

**Proposition 6.2.1.** Let \( \theta : X \to \mathbb{R}_+ \) be a Borelian function. Then for every \( W \in G(n - 1, X) \) holds

\[
\int_X \sum_{y \in \pi_W^{-1}(x)} \theta(y) d (\mathcal{H}^{n-1}_{||\cdot||} \llcorner W)(x) \leq \int_X \theta d G^{n-1}_{||\cdot||, GP}.
\]
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Proof. Note that the case \( \theta = 1_A \) corresponds to the previous theorem. Next, if \( \theta = \sum_{j \in J} a_j \mathbb{1}_{A_j} \), then we use the additivity of the integral and the fact that \( \sup(f + g) \leq \sup(f) + \sup(g) \). The general case (i.e. when \( \theta \) is Borel) is treated via regarding a nondecreasing sequence of step functions converging pointwise to the function \( \theta \) and using the Monotone Convergence Theorem.

We continue the proof of Theorem 6.2.2 with a lemma, which connects \( \mathbb{M}_G \) and \( \zeta_{||...||_G}^{n-1} \).

**Lemma 6.2.3.** For any \( T \in \mathcal{R}_{n-1}(X, G) \) there holds \( \zeta_{||...||_G}^{n-1}(T) \leq \mathbb{M}_G(T) \).

**Proof.** Note that for each \( W \in G(n-1, X) \) one has

\[
\theta_{\pi W}(x) \leq \sum_{y \in \pi W(x) \cap \text{set}(T)} \theta_T(y).
\]

Hence

\[
\mathbb{M}(\pi W) \leq \int_{\pi W(x) \cap \text{set}(T)} \theta_T(y) d\mathcal{H}_{||...||}^{n-1}(x).
\]

Now we take the supremum over all \( W \in G(n-1, X) \) and further use Proposition 6.2.1 in order to conclude that

\[
\zeta_{||...||_G}^{n-1}(T) \leq \int_X \theta_T(x) d\mathcal{G}_{||...||_G}^{n-1}(x) \leq \mathbb{M}_G(T),
\]

and the lemma follows.

We shall need one more lemma that gives a formula for the Gross density.

**Lemma 6.2.4.** Suppose that \( T \in \mathcal{R}_{n-1}(X, G) \). Then for \( \mathcal{H}^{n-1} \)-almost all \( x \in \text{set}(T) \) one has

\[
\lim_{r \to 0} \frac{\zeta_{||...||_G}^{n-1}(T \sqcup B(x, r))}{\alpha_{n-1} r^{n-1}} = \theta_T(x).
\]

**Proof.** First note that according to Lemma 6.2.3 one has:

\[
\lim_{r \to 0} \frac{\zeta_{||...||_G}^{n-1}(T \sqcup B(x, r))}{\alpha_{n-1} r^{n-1}} \leq \lim_{r \to 0} \frac{\mathbb{M}_G(T \sqcup B(x, r))}{\alpha_{n-1} r^{n-1}} = \lim_{r \to 0} \frac{1}{\alpha_{n-1} r^{n-1}} \int_{B(x,r) \cap \text{set}(T)} \theta_T(y) d\mathcal{H}_{||...||}^{n-1}(y) = \theta_T(x).
\]

Hence it is left to prove the reciprocal inequality. In what follows we shall require a notion of approximately continuous functions. We refer the reader to the book [EG15] for the definition and properties of these functions. Suppose first that set(\( T \)) \( \subseteq M \) where \( M \) is a smooth manifold. Take \( x \in \text{set}(T) \) such that \( \theta_T \) is approximately continuous at \( x \) and denote \( W_0 = \text{Tan}(M, x) \). Indeed, since \( \theta_T \in L^1(X, G) \) it is approximately continuous
$\mathcal{H}^{n-1} \nabla M$–almost everywhere. Fix $\varepsilon > 0$. There exists $\rho_1 = \rho_1(x)$ such that for all $r \leq \rho_1(x)$ there holds

$$\left| \frac{\mathcal{H}^{n-1}(\pi_W(M \cap B(x,r)))}{\alpha_{n-1}r^{n-1}} - 1 \right| \leq \varepsilon.$$ 

Indeed, this follows from the proof of Theorem 6.2.1 and also from the fact that $\Theta_{||i||}^{n-1}(\mathcal{H}_{||i||}^{n-1} \nabla M, x) = 1$. Next we choose $\rho_2$ such that $M \cap B(x, \rho_2) = F(\pi_W(M \cap B(x, \rho_2)))$ where $\pi_W = \text{GP}(W)$ and $F = \pi_W^{-1} \upharpoonright (M \cap B(x, \rho_2))$. Since $\theta$ is approximately continuous at $x$ we can choose a subset $E_r \subset M \cap B(x, r)$ such that $\theta_T(y) \leq \theta_T(x) - \varepsilon$ for all $y \in E_r$ and $\mathcal{H}_{||i||}^{n-1}(E_r) \leq \varepsilon \alpha_{n-1}r^{n-1}$. Now for $r \leq \min\{\rho_1, \rho_2\}$ we estimate

$$\frac{\zeta_{n-1}^{||i||, \text{GP}}(T \cap B(x,r))}{\alpha_{n-1}r^{n-1}} \geq \frac{1}{\alpha_{n-1}r^{n-1}} \int_{\pi_W(M \cap B(x,r)) \setminus E_r} \theta_T(\pi_W^{-1} \upharpoonright (M \cap B(x,r))(y) d\mathcal{H}_{||i||}^{n-1}(y) \geq \frac{\theta_T(\pi_W^{-1} \upharpoonright (M \cap B(x,r)) \setminus E_r)}{\alpha_{n-1}r^{n-1}} \geq \frac{\theta_T(\pi_W^{-1} \upharpoonright (M \cap B(x,r))) - \varepsilon \mathcal{H}_{||i||}^{n-1}(\pi_W(M \cap B(x,r)))}{\alpha_{n-1}r^{n-1}} - 2\varepsilon \geq \theta_T(x)(1 - \varepsilon) - 3\varepsilon.$$

Let us now treat the general case. Write

$$T = \sum_{i=1}^{\infty} T_i,$$

where $T_i = T \cap A_i$, with $A_i \subseteq M_i$– Borelian subsets of smooth manifolds. Note that for $\mathcal{H}^{n-1} \nabla \text{set}(T)$ almost all $x$ there exists $k \in \mathbb{N}$ such that there holds

$$\Theta_{||i||}^{n-1}(\mathcal{H}_{||i||}^{n-1} \nabla A_k, x) = 1 \text{ and } \Theta_{||i||}^{n-1}(\mathcal{H}_{||i||}^{n-1} \nabla A_j, x) = 0,$$

if $j \neq k, j \in \mathbb{N}$. Hence for such $x$ there holds

$$\frac{\zeta_{n-1}^{||i||, \text{GP}}(T \cap B(x,r))}{\alpha_{n-1}r^{n-1}} - \frac{\zeta_{n-1}^{||i||, \text{GP}}(T_k \cap B(x,r))}{\alpha_{n-1}r^{n-1}} \geq \frac{\zeta_{||i||, \text{GP}}((T - T_k) \cap B(x,r))}{\alpha_{n-1}r^{n-1}} \geq - \frac{\mathcal{M}((T - T_k) \cap B(x,r))}{\alpha_{n-1}r^{n-1}}.$$

Taking the limit with respect to $r \to 0$ in the last line yields the desired inequality and finishes the proof of the lemma. \qed
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We are now ready to finish the proof of the announced lower semicontinuity result. Fix \( \varepsilon > 0 \). For \( \mathcal{H}^{n-1} \)-almost every \( x \in \text{set}(T) \) there exists \( \rho_x > 0 \) such that
\[
\mathcal{M}_G(T \sqcup B(x, r)) \leq (1 + \varepsilon)\zeta_{\|\cdot\|,\|\cdot\|,G}^{n-1}(T \sqcup B(x, r))
\]
for all \( 0 < r \leq \rho_x \). This follows from the previous lemma and from B.Kirchheim’s theorem 3.2.3 that
\[
\lim_{r \to 0} \alpha_{n-1}^{-1} r^{-(n-1)} \mathcal{H}^{n-1}(\text{set}(T) \cap B(x, r)) = 1
\]
for \( \mathcal{H}^{n-1} \)-almost all \( x \in \text{set}(T) \). Indeed, taking once again a point \( x \in \text{set}(T) \) such that \( \theta_T \) is approximately continuous at \( x \) we infer that
\[
\mathcal{M}_G(T \sqcup B(x, r)) \leq (\varepsilon + \theta_T(x)) \mathcal{H}^{n-1}(\text{set}(T) \cap B(x, r)) \leq (1 + \varepsilon)\zeta_{\|\cdot\|,\|\cdot\|,G}^{n-1}(T \sqcup B(x, r)),
\]
where \( \hat{\varepsilon} \to 0 \) when \( \varepsilon \to 0 \). Moreover, for each such \( x \) almost all \( 0 < r < \rho_x \) have the additional property that \( \lim_{r \to 0} \mathcal{F}(T \sqcup B(x, r) - T \sqcup B(x, r)) = 0 \) according to [PH12], 5.2.3. The collection \( \mathcal{B} \) of balls having all the stated properties is a Vitali covering of a conegligible subset of \( \text{set}(T) \) (i.e. a subset of \( \text{set}(T) \) whose complement in \( \text{set}(T) \) has \( \mathcal{H}^{n-1} \) measure zero). According to the Besicovitch covering theorem 3.2.9, there exists a disjoint countable subcollection \( \{B(x_j, r_j) : j \in J\} \) of \( \mathcal{B} \) that covers \( \mathcal{H}^{n-1} \)-almost all of \( \text{set}(T) \). Hence we can estimate using the inequality (6.2):
\[
\mathcal{M}_G(T) = \sum_{j \in J} \mathcal{M}_G(T \sqcup B(x_j, r_j)) \leq (1 + \varepsilon) \sum_{j \in J} \zeta_{\|\cdot\|,\|\cdot\|,G}^{n-1}(T \sqcup B(x_j, r_j)) =
\]
\[
= (1 + \varepsilon) \sum_{j \in J} \liminf_{i \to \infty} \zeta_{\|\cdot\|,\|\cdot\|,G}^{n-1}(T_i \sqcup B(x_j, r_j)) \leq \ldots
\]
where the last equality follows from Lemma 6.2.1. We finish the estimate, now using Lemma 6.2.3
\[
\ldots \leq (1 + \varepsilon) \sum_{j \in J} \liminf_{i \to \infty} \mathcal{M}_G(T_i \sqcup B(x_j, r_j)) \leq (1 + \varepsilon) \liminf_{i \to \infty} \sum_{j \in J} \mathcal{M}_G(T_i \sqcup B(x_j, r_j)) \leq
\]
\[
\leq \liminf_{i \to \infty} \mathcal{M}_G(T_i),
\]
where the last equality follows from the fact that the balls \( B(x_i, r_i) \) are pairwise disjoint. Hence the theorem follows.

As a consequence of our lower semicontinuity theorem, we obtain the following existence result.

**Theorem 6.2.4.** Let \((X, ||\cdot||)\) be an \( n \)-dimensional Banach space and let \( G \) be a locally compact complete normed Abelian group. Then the following Plateau problem admits a solution
\[
\left\{ \begin{array}{l}
\text{minimize } \mathcal{M}(T) \\
T \in \mathcal{R}_{n-1}(X, G) \quad \text{such that } \partial T = B,
\end{array} \right.
\]
where $B \in \mathcal{R}_{n-2}(X, G)$ has a compact support and satisfies $\partial B = 0$. Moreover, among the solutions, there exists at least one, say $T_0$ satisfying $\text{spt}(T_0) \subseteq \text{conv}(\text{spt}(B))$.

**Proof.** It is sufficient to recall Theorem 5.3.1. \qed
Chapter 7

The existence in the two dimensional case.

We proceed to the sixth chapter, where we will establish the existence of mass minimising 2 rectifiable $G$ chains in finite dimensional Banach spaces.

7.1 Some preliminary definitions and remarks.

We start with a number of motivating definitions and remarks.

Definition 7.1.1. (Totally convex densities) We call a $k$–density function $\phi$ on an $n$–dimensional Banach space $X, n > k$ totally convex if for every $k$–dimensional linear subspace there exists a linear projection onto that subspace which does not increase the volume $Vol_{\phi}$ corresponding to the density $\phi$, see [4.1.1].

There exists an equivalent way of defining totally convex densities. We mean the following result discussed in [TA04].

Theorem 7.1.1. A $k$–density $\phi$ on an $n$–dimensional vector space $X, n > k$, is totally convex if and only if it is extendibly convex and moreover the following holds. If $\Phi : \Lambda_k(X) \to \mathbb{R}$ is a convex extension of $\phi$, then through every point of the unit sphere $S = \{x \in \Lambda_k(X) : \Phi(x) = 1\}$ there passes a supporting hyperplane of the form $\xi = 1$, where $\xi$ a simple $k$–vector in $\Lambda_k(X^*)$.

Remark 7.1.1. Note that, thanks to Theorem [4.3.2], in case when $m = n - 1$ each Busemann–Hausdorff density is totally convex.

The following remark shows that there exist densities which are convex but not totally convex. However, the author does not know, whether there exists a Busemann–Hausdorff density, that is not extendibly convex.

Remark 7.1.2. For $m \neq \{1, n - 1\}$ it is possible to construct an example of a $m$–volume density which is extendibly convex, but not totally convex (see [BS60]). The density, constructed in [BS60] is called the quadratic density.
Hence it seems natural to me that one should use an approach that is different from the linear projections machinery, developed in the third and forth chapters, while trying to attack the Plateau problem in the codimensions different from 1 and $(n - 1)$. In the case of codimension $(n - 2)$ it is possible to provide another successful method based on one theorem, proved in the paper [BI12] by Burago and Ivanov.

### 7.2 Burago–Ivanov’s theorem and some of its consequences.

We would like to emphasize that throughout this chapter we denote by $\tilde{\phi}_{BH}$ a function acting from the Grassmann cone $GC(2, X)$ to $\mathbb{R}_+$ and defined for a simple 2–vector $\sigma = v_1 \wedge v_2$ as

$$\tilde{\phi}_{BH}(\sigma) = |\sigma|_2 \cdot \phi_{BH}(\text{span}\{v_1, v_2\}).$$

The main result of this section is the following theorem which is nothing but the “triangle inequality” for the 2–cycles.

**Theorem 7.2.1.** (I.Vasilyev) Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space with the unit ball $B = \{x \in X : ||x|| \leq 1\}$. Let $P \in \mathcal{P}_2(X, G)$ be a 2–cycle (i.e. $\partial P = 0$) such that

$$P = \sum_{j=1}^{N} g_j[\sigma_j],$$

where $\sigma_j$ are non–overlapping 2–simplexes. Then

$$|g_1|H^2_{||\ldots||}(\sigma_1) \leq \sum_{j=2}^{N} |g_j|H^2_{||\ldots||}(\sigma_j).$$

**Remark 7.2.1.** This result was first formulated and proved in the present thesis by I.Vasilyev.

Our proof of Theorem 7.2.1 highly depends on the following result of Burago and Ivanov.

**Theorem 7.2.2.** (Burago–Ivanov) Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space. Suppose that the unit ball $B = \{x \in X : ||x|| \leq 1\}$ is a bounded convex polyhedron. For every $W \in G(2, X)$ let $M = M_W$ be the polygon defined as the intersection $B \cap W$, $M = \text{conv}\{a_1, \ldots, a_d\}$ and let $F_i$ be a linear supporting function to the set $B$ such that $F_i$ equals to one on the segment $[a_i, a_{i+1}]$. Define coefficients $p_i$ by $p_i = 2H^2(\triangle 0a_ia_{i+1})/H^2(B \cap W)$ and a function $\alpha = \alpha_W : A_2(X) \to \mathbb{R}_+$ as

$$\alpha(\sigma) = \pi \sum_{1 \leq k < l \leq d} p_k p_l (F_k \wedge F_l, \sigma).$$

(7.1)
7.2 Burago–Ivanov’s theorem and some of its consequences. 83

Then for all $\sigma \in GC_2(X)$ the function $\alpha$ satisfies

$$
\alpha(\sigma) \leq \tilde{\phi}_{BH}(\sigma),
$$

(7.2)

and if $\sigma = v_1 \land v_2$, with $v_1, v_2 \in W$ linearly independent vectors, then

$$
\alpha(\sigma) = \tilde{\phi}_{BH}(\sigma).
$$

(7.3)

**Proof.** We shall first prove our Theorem 7.2.1. For each $\omega$ and further define $\varepsilon > \text{polyhedron}$. Indeed, we first note that for every $\varepsilon$, and if $B$ exists a 2–dimensional bounded convex polyhedron $P_{\varepsilon,j}$ satisfying $P_{\varepsilon,j} \subseteq B \cap W_j$ and $\mathcal{H}^2((B \cap W_j) \setminus P_{\varepsilon,j}) \leq \varepsilon$. Next we define by $P_{\varepsilon}$ the polyhedron, obtained as the convex combination of $P_{\varepsilon,j}$,

$$
P_{\varepsilon} = \text{conv} \left( \bigcup_{j=1}^{N} P_{\varepsilon,j} \right)
$$

and by $\|\ldots\|_\varepsilon$ the corresponding norm. Note that the norm $\|\ldots\|_\varepsilon$ is well defined. Indeed, we can suppose that $P_{\varepsilon}$ is symmetric by substituting it with its symmetrization $P_{\varepsilon}^{\text{sym}}$ if needed. Since $B$ is symmetric we will have $P_{\varepsilon} \subseteq P_{\varepsilon}^{\text{sym}} \subseteq B$ and hence $P_{\varepsilon} \cap W_j \subseteq B \cap W_j$ and $\mathcal{H}^2((B \cap W_j) \setminus (P_{\varepsilon} \cap W_j)) \leq \varepsilon$. In principle, $P_{\varepsilon}$ can be of dimension $n_1 < n$, but this changes nothing. Now we estimate

$$
\sum_{j=2}^{N} |g_j| \mathcal{H}^2_{W_j}(\sigma_j) = \sum_{j=2}^{N} |g_j| \mathcal{H}^2(\sigma_j) \tilde{\phi}_{BH}(W_j) = \sum_{j=2}^{N} |g_j| \frac{\mathcal{H}^2(\sigma_j)}{\mathcal{H}^2(B \cap W_j)} \geq 
$$

$$
\sum_{j=2}^{N} |g_j| \frac{\mathcal{H}^2(\sigma_j)}{\mathcal{H}^2(P_{\varepsilon,j})} + \varepsilon \geq \sum_{j=2}^{N} |g_j| \frac{\mathcal{H}^2(\sigma_j)}{\mathcal{H}^2(P_{\varepsilon,j})} \left( 1 - \frac{\tilde{C}_1 \varepsilon}{\mathcal{H}^2(P_{\varepsilon,j})} \right) \geq |g_1| \mathcal{H}^2_{W_1}(\sigma_1) - \tilde{C}_1 \varepsilon \geq \ldots,
$$

where the constants $\tilde{C}$ and $\tilde{C}_1$ do not depend on $\varepsilon$. On the other hand,

$$
|g_1| \mathcal{H}^2_{W_1}(\sigma_1) = |g_1| \frac{\mathcal{H}^2(\sigma_1)}{\mathcal{H}^2(P_{\varepsilon,1})} \geq |g_1| \frac{\mathcal{H}^2(\sigma_1)}{\mathcal{H}^2(B \cap W_1)}.
$$

We finish the estimate:

$$
\ldots \geq |g_1| \frac{\mathcal{H}^2(\sigma_1)}{\mathcal{H}^2(B \cap W_1)} - \tilde{C}_1 \varepsilon = |g_1| \mathcal{H}^2_{W_1}(\sigma_1) - \tilde{C}_1 \varepsilon,
$$

and our claim will follow after letting $\varepsilon$ tend to 0.

Therefore we can assume that $B$ is a bounded convex polyhedron. Let us use the theorem by Burago and Ivanov and keep the notations from there. Let $\alpha = \alpha_{W_1}$ be the
function corresponding to the plane \( W_1 \) (the one constructed in Theorem 7.2.2). Using the definition of the Hausdorff–Busemann density, the inequality (7.2) and the formula (7.1) we write:

\[
\sum_{j=2}^{N} |g_j| \mathcal{H}_{\|\cdot\|}(\sigma_j) = \sum_{j=2}^{N} |g_j| \mathcal{H}^2(\sigma_j) \dot{\phi}_{BH}(\omega_j) \geq \\
\sum_{j=2}^{N} |g_j| \mathcal{H}^2(\sigma_j) \alpha_{W_1}(\omega_j) = \sum_{j=2}^{N} \sum_{1 \leq k < l \leq d} |g_j| \mathcal{H}^2(\sigma_j) p_k p_l \langle F_k \wedge F_l, \omega_j \rangle = \ldots \quad (7.4)
\]

Following Burago and Ivanov, we define for all \( k < l \) from 1 to \( d \) mappings \( F_{k,l} : X \to \mathbb{R}^2 \) by the formula

\[
F_{k,l}(x) = (F_k(x), F_l(x)).
\]

It follows now from the paper \cite{BI12} (page 636 lines 1–3) that

\[
\mathcal{H}^2(\sigma_j) \langle F_k \wedge F_l, \omega_j \rangle = \mathcal{H}^2(F_{k,l}(\sigma_j)) \quad (7.5)
\]

for all \( k < l \) from 1 to \( n \) and all \( j \) from 1 to \( N \). This identity is an easy consequence of the area formula. Indeed, the area \( \mathcal{H}^2(F_{k,l}(\sigma_j)) \) of the image of the linear mapping \( F_{k,l} \) is equal to the area the pre–image \( \mathcal{H}^2(\sigma_j) \times \) its Jacobian, which is equal to the absolute value of the determinant of the matrix

\[
\begin{pmatrix}
F_k(v_1) & F_l(v_1) \\
F_k(v_2) & F_l(v_2).
\end{pmatrix}
\]

Next, note that since \( \partial P = 0 \), polyhedral 2 chain \( Q = F_{k,l}(P) \) satisfies \( \partial Q = 0 \). Since \( Q \) has compact support, the constancy theorem tells us that \( Q = 0 \) and hence

\[
\mathcal{M}(g_1[\sigma_1]) \leq \sum_{j=2}^{N} \mathcal{M}(g_j[\sigma_j]).
\]

Using this and the line (7.5) we continue the estimate (7.4):

\[
\ldots = \sum_{j=2}^{N} \sum_{1 \leq k < l \leq d} |g_j| p_k p_l \mathcal{H}^2(F_{k,l}(\sigma_j)) = \sum_{1 \leq k < l \leq d} p_k p_l \sum_{j=2}^{N} |g_j| \mathcal{H}^2(F_{k,l}(\sigma_j)) \geq \\
\sum_{1 \leq k < l \leq d} p_k p_l |g_1| \mathcal{H}^2(F_{k,l}(\sigma_1)) = \ldots \quad (7.6)
\]

We continue the estimate (7.6) now with help of the formulae (7.5), (7.1) and (7.3) and the definition of the Hausdorff–Busemann density:

\[
\ldots = |g_1| \mathcal{H}^2(\sigma_1) \sum_{1 \leq k < l \leq d} p_k p_l \langle F_k \wedge F_l, \omega_1 \rangle = |g_1| \mathcal{H}^2(\sigma_1) \alpha_{W_1}(\omega_1) = \\
|g_1| \mathcal{H}^2(\sigma_1) \dot{\phi}_{BH}(\omega_1) = |g_1| \mathcal{H}_{\|\cdot\|}(\sigma_1),
\]

and Theorem 7.2.1 follows. \( \square \)
Lemma 7.2.1. \( \leq \) Let \( \sum \) equalities. 

Proof. The proof splits into three elementary lemmata. 

\[ \sum_{1 \leq i < j \leq n} p_i p_j f_i \wedge f_j \leq \sum_{1 \leq i < j \leq n} p_i p_j |f_i \wedge f_j| \leq \frac{1}{\mathcal{H}^2(K)}. \]

In addition, if \( K \) is a convex \( 2n \)-gon \( a_1a_2 \ldots a_{2n} \), \( f_i \) are supporting functions of \( K \) (that is, \( f_i = 1 \) on \( [a_i, a_{i+1}] \)) and \( p_i = \mathcal{H}^2(\triangle 0a_ia_{i+1})/\mathcal{H}^2(K) \) then the above inequalities turn into equalities.

Proof. The proof splits into three elementary lemmata.

Lemma 7.2.2. Let \( K = a_1a_2 \ldots a_{2n} \) be a symmetric \( 2n \)-gon in \( \mathbb{R}^2 \). Let \( v_i = \overrightarrow{a_ia_{i+1}} \) for \( i = 1, \ldots, n \). Then

\[ \mathcal{H}^2(K) = \sum_{1 \leq i < j \leq n} |v_i \wedge v_j| = \sum_{1 \leq i < j \leq n} v_i \wedge v_j. \]

Proof. The second identity follows from the fact that all pairs \( (v_i, v_j), 1 \leq i < j \leq n \) are of the same orientation. To prove the first one, observe that

\[ \mathcal{H}^2(K) = 2\mathcal{H}^2(a_1a_2 \ldots a_{n+1}) = 2 \sum_{j=2}^{n} \mathcal{H}^2(\triangle a_1a_ja_{j+1}), \quad (7.7) \]

because \( K \) is symmetric. Further

\[ \mathcal{H}^2(\triangle a_1a_ja_{j+1}) = \frac{1}{2}|\overrightarrow{a_ia_j} \wedge \overrightarrow{a_ja_{j+1}}| = \frac{1}{2} \sum_{i=1}^{j} v_i \wedge v_j, \quad (7.8) \]

since \( \overrightarrow{a_ja_{j+1}} = v_1 + v_2 + \ldots + v_{j-1} \) and since all pairs \( (v_i, v_j), i < j \) are of the same orientation. The result now follows from identities \((7.7)\) and \((7.8)\). \( \square \)

Lemma 7.2.2. Let \( K = a_1a_2 \ldots a_{2n} \) be a symmetric \( 2n \)-gon in \( \mathbb{R}^2 \). Let \( v_i = \overrightarrow{a_ia_{i+1}} \) for \( i = 1, \ldots, n \), \( f_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) a linear function such that \( f_i = 1 \) on \( [a_i, a_{i+1}] \) and \( p_i = \mathcal{H}^2(\triangle 0a_ia_{i+1})/\mathcal{H}^2(K) \). Then

\[ p_i p_j |f_i \wedge f_j| = \frac{1}{\mathcal{H}^2(K)^2} |v_i \wedge v_j|, \]

for all \( i, j \), and therefore

\[ \sum_{1 \leq i < j \leq n} p_i p_j f_i \wedge f_j = \sum_{1 \leq i < j \leq n} p_i p_j f_i \wedge f_j = \frac{1}{\mathcal{H}^2(K)}. \]
The last sum is the two-dimensional mixed volume \( V \), and hence \( p_i = S_i/\mathcal{H}^2(K) \). Define a linear isometry \((\mathbb{R}^2)^* \to \mathbb{R}^2\) as \( J(S_i f_i) = v_i \). Note that

\[
S_i S_j |f_i \wedge f_j| = |S_i f_i \wedge S_j f_j| = |v_i \wedge v_j|.
\]

Therefore

\[
p_i p_j |f_i \wedge f_j| = \frac{1}{\mathcal{H}^2(K)^2} S_i S_j |f_i \wedge f_j| = \frac{1}{\mathcal{H}^2(K)^2} |v_i \wedge v_j|.
\]

To prove the second assertion, observe that all pairs \((f_i, f_j), 1 \leq i < j \leq n\) are of the same orientation, hence

\[
\left| \sum_{1 \leq i < j \leq n} p_i p_j f_i \wedge f_j \right| = \sum_{1 \leq i < j \leq n} p_i p_j |f_i \wedge f_j| = \frac{1}{\mathcal{H}^2(K)} \sum_{1 \leq i < j \leq n} |v_i \wedge v_j| = \frac{1}{\mathcal{H}^2(K)},
\]

where the last inequality follows from the previous lemma.

\[\square\]

**Lemma 7.2.3.** Let \( K = a_1 a_2 \ldots a_{2n} \) be a symmetric \( 2n \)-gon in \( \mathbb{R}^2 \). For each \( i = 1, \ldots, n \) let \( f_i : \mathbb{R}^2 \to \mathbb{R} \) be a linear function such that \( f_i = 1 \) on \([a_i, a_{i+1}]\) and let \( p_i \) be non–negative real numbers satisfying \( \sum p_i = 1 \). Then

\[
\sum_{1 \leq i < j \leq n} p_i p_j |f_i \wedge f_j| \leq \frac{1}{\mathcal{H}^2(K)}.
\]

**Proof.** Denote \( v_i = a_i a_{i+1}, q_i = 2\mathcal{H}^2(\triangle 0 a_i a_{i+1})/\mathcal{H}^2(K) \) and \( \lambda_i = p_i/q_i \). By the previous lemma we have that

\[
q_i q_j |f_i \wedge f_j| = \frac{1}{\mathcal{H}^2(K)} |v_i \wedge v_j|.
\]

Let \( v'_i = \lambda_i v_i \) for \( i \) from 1 to \( n \). Consider a symmetric \( 2n \)-gon \( K' = a'_1, \ldots, a'_{2n} \) such that \( a'_i, a'_{i+1} = v'_i \). Then

\[
\sum_{1 \leq i < j \leq n} |p_i p_j f_i \wedge f_j| = \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j q_i q_j f_i \wedge f_j| =
\]

\[
\frac{1}{\mathcal{H}^2(K)^2} \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j |v_i \wedge v_j| = \frac{1}{\mathcal{H}^2(K)^2} \sum_{1 \leq i < j \leq n} |v'_i \wedge v'_j| = \frac{\mathcal{H}^2(K')}{\mathcal{H}^2(K)^2}.
\]

Denote \( l_i = |v_i| \) and \( l'_i = |v'_i| = \lambda_i |v_i| \). Let \( h_i \) denote the distance from the origin to the line containing the side \([a_i, a_{i+1}]\). Then \( \mathcal{H}^2(\triangle 0, a_i, a_{i+1}) = h_i l_i \) hence \( q_i = h_i l_i/\mathcal{H}^2(K) \). Therefore

\[
1 = \sum p_i = \sum \lambda_i q_i = \frac{1}{\mathcal{H}^2(K)} \sum \lambda_i h_i l_i = \frac{1}{\mathcal{H}^2(K)} h_i l'_i.
\]

The last sum is the two-dimensional mixed volume \( V(K, K') \), thus \( V(K, K') = \mathcal{H}^2(K) \). By the Minkowski inequality we have that \( V(K, K')^2 \geq \mathcal{H}^2(K) \mathcal{H}^2(K') \). Therefore \( \mathcal{H}^2(K') \leq \mathcal{H}^2(K) \), and the lemma follows.

\[\square\]
To complete the proof of the proposition, it suffices to prove the previous lemma in a slightly more general setting, namely when $K$ is a symmetric polygon (not necessarily with $2n$ sides) and $f_1, \ldots, f_n$ are arbitrary linear functions such that $f_i \upharpoonright K \leq 1$. This condition means that $f_i$ belongs to the polar polygon $K^* \subset (\mathbb{R}^2)^*$. Consider the left hand side in the inequality (7.9) with others staying fixed. This function is convex (sum of absolute values of linear functions) therefore it attains its maximum at $K^*$ at a vertex of $K^*$. The vertices of $K^*$ are supporting linear functions to $K$ at its sides. So it is sufficient to consider the case when each $f_i$ equals 1 on one of the sides of $K$.

If two functions $f_i$ and $f_j$ coincide (with no loss of generality, $f_1 = f_n$) one reduces the problem to a smaller number of functions as follows: drop $f_n$ from the list of the functions and replace $p_1, p_2, \ldots, p_{n-1}$ by $p_1 + p_n, p_2, \ldots, p_{n-1}$. Also note that the changing of sign of one of the functions does not change the left–hand side of (7.9). Thus is suffices to consider the case when $f_i \neq f_j$ and $f_i \neq -f_j$ for all $i \neq j$. If $n = 1$ the left–hand side in (7.9) is zero so the inequality is trivial. If $n > 1$ we apply the previous lemma to the polygon $K' = \bigcup_{i=1}^{n} \{ x : |f_i| \leq 1 \}$ yields that

$$
\sum_{1 \leq i < j \leq n} p_i p_j |f_i \wedge f_j| \leq \frac{1}{\mathcal{H}^2(K')} \leq \frac{1}{\mathcal{H}^2(K)},
$$

since $K' \subseteq K$.

Let us now derive the theorem from the proposition. Let $\sigma = v_1 \wedge v_2$ where $v_1, v_2$ are linearly independent vectors and consider a linear embedding $I : \mathbb{R}^2 \to X$ that takes the standard basis of $\mathbb{R}^2$ to $v_1$ and $v_2$. Let $K = I^{-1}(B)$, then $\varphi_{BH}(\sigma) = \pi/\mathcal{H}^2(K)$. On the other hand,

$$
\alpha(\sigma) = \pi \sum_{1 \leq k < t \leq d} p_k p_t |f_k \wedge f_t|,
$$

where $f_i = F_i \circ I$. Recall that $F_i \leq 1$ on the set $B$ hence $f_i \leq 1$ on $K$ for all $i$. By the proposition we conclude that $\alpha(\sigma) \leq \pi/\mathcal{H}^2(K)$.

**Corollary 7.2.1.** Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $G$ be a complete normed Abelian group. Then the mass $\mathbb{M}$ is lower semicontinuous with respect to flat convergence on the space $\mathcal{P}_2(X, G)$.

**Proof.** It suffices to recall Theorems 5.1.2 and 7.2.1.

**7.3 The existence for the Plateau problem.**

In this subsection we finish the proof of the existence result for the Plateau problem in case of two–dimensional rectifiable $G$ chains. The following theorem is the main result of this chapter.

**Theorem 7.3.1.** Let $(X, ||\ldots||)$ be an $n$–dimensional Banach space and let $G$ be a complete normed Abelian group. Then the mass $\mathbb{M}$ is lower semicontinuous with respect to flat convergence on $\mathcal{R}_2(X, G)$. 


Proof. We start with a first step where we prove the theorem in case when the limiting chain $P$ is polyhedral.

Lemma 7.3.1. Let $X$ and $G$ be as in the formulation of the theorem. If $T_i \in \mathcal{R}_2(X, G)$, $P \in \mathcal{P}_2(X, G)$, and $\lim_{i \to \infty} \mathcal{F}(T_i - P) = 0$, then

\[ M(P) \leq \liminf_{i \to \infty} M(T_i). \]

Proof. Thanks to the weak approximation theorem in Banach spaces (see [Pau14], Theorem 4.4, page 329), we know that for every natural number $i$ there exists a polyhedral chain $P_i \in \mathcal{P}_2(X, G)$ such that $\mathcal{F}(T_i - P_i) \leq 1/i$ and $M(P_i) \leq M(T_i) + 1/i$. Next since $\mathcal{F}(P_i - P) \to 0$, when $i \to +\infty$, we estimate with help of Corollary 7.2.1

\[ M(P) \leq \liminf_{i \to +\infty} M(P_i) \leq \liminf_{i \to +\infty} \left( M(T_i) + \frac{1}{i} \right) = \liminf_{i \to +\infty} M(T_i), \]

and the lemma follows. \qed

Theorem 7.3.1 can now be proved with help of Lemma 7.3.1 and the strong approximation Theorem 5.2.1 exactly in the same way as Theorem 5.2.2. \qed

As a consequence of our lower semicontinuity theorem, we obtain the following existence result.

Theorem 7.3.2. Let $(X, || \ldots ||)$ be an $n$–dimensional Banach space and let $G$ be a locally compact complete normed Abelian group. Then the following Plateau problem admits a solution:

\[
\begin{align*}
\{ \text{minimize } & M(T) \\
T & \in \mathcal{R}_2(X, G) \\
\text{such that } & \partial T = B,
\end{align*}
\]

where $B \in \mathcal{R}_1(X, G)$ has a compact support and satisfies $\partial B = 0$.

Proof. We use the direct method of calculus of variations. Take a minimizing sequence, say $T_j$ such that $\inf M(T) = \lim_j M(T_j)$. Let choose $R_0$ such that $B \subset B(0, R_0)$. Let us prove that there exists a sequence of radii $r_k \to \infty$ such that $r_k \leq r_{k+1}$ and that

\[ \limsup_{k \to \infty} \left( M(T_j \sqcup B(0, r_k)) + M(\partial(T_j \sqcup B(0, r_k))) \right) = 0. \]

Denote $S^k_j = T_j \sqcup B(0, r_k)$ and $\beta_k = \sup_j (M(T_j \sqcup B(0, r_k)) + M(\partial(T_j \sqcup B(0, r_k))))$. Note that $M(\partial S^k_j) = 0$ once $r_k > R_0$, and hence it is left to prove that $\lim_k \alpha_k = 0$ where $\alpha_k = \sup_j M(S^k_j)$. This can be proved exactly in the same way as in the paper [AS13] of Ambrosio and Schmidt.

Next, since for each $j$ we have that $\text{spt}(S^k_j) \subseteq B(0, r_k)$ using the diagonal argument, we infer that there exists a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that for each $k$ there holds $\mathcal{F}(S^k_{\phi(j)} - S^k) \to 0$. Note that $S^k \sqcup (B(0, r_k)) = S^{k+1} \sqcup (B(0, r_k))$. We are going to
show that $S^k$ converge towards some chain $S$. We start with showing that $S^k$ is $\mathcal{F}$–Cauchy. Indeed since the mass is lower semicontinuous

$$
\mathbb{M}(S^k - S^{k+l}) = \mathbb{M}(S^k - S^{k+l} \cap (B(0, r_k))^c) \leq \\
\liminf_j \mathbb{M}(S^k_{\phi(j)} \cap (B(0, r_k))^c) + \liminf_j \mathbb{M}(S^k_{\phi(j)} \cap (B(0, r_k))^c) \leq \beta_k + \beta_{k+l}.
$$

Let us now show that $T_{\phi(j)}$ $\mathcal{F}$–converges to $S$. Fix $\varepsilon > 0$ and find $K$ such that if $k > K$, then $\mathcal{F}(S \cap (B(0, r_k)) - S) \leq \varepsilon$. Hence

$$
\mathcal{F}(T_{\phi(j)} - S) = \mathcal{F}(T_{\phi(j)} \cap (B(0, r_k)) - T_{\phi(j)}) + \\
\mathcal{F}(T_{\phi(j)} \cap (B(0, r_k)) - S \cap (B(0, r_k))) + \mathcal{F}(S \cap (B(0, r_k)) - S) \leq 2\varepsilon + \beta_k,
$$

and the theorem follows. \qed
The existence in the two dimensional case.
Appendices
Appendix A

Lower semicontinuity and semiellipticity in the RNP spaces.

Throughout this appendix $G$ denotes a locally compact normed Abelian group.

Definition A.0.1. Let $X$ be an arbitrary Banach space. Call a function $M : \mathcal{P}_m(X, G) \to \mathbb{R}_+$ a mass if it satisfies

- $M(T + S) \leq M(T) + M(S)$
- $M(T + S) \leq M(T) + M(S)$, if $\mathcal{H}^m(\text{set}(T) \cap \text{set}(S)) = 0$.
- $M(T) \leq CM(T)$, where $M$ is the Hausdorff mass and $C$ depends only on $X$.
- $M(F \# T) \leq \text{Lip}(F)^m M(T)$ for any piecewise linear mapping $F : X \to X$ and $C$ depends only on $X$.

Remark A.0.1. Note that $M(T)$ is well defined for any $T \in \mathcal{R}_m(X, G)$.

Definition A.0.2. Mass $M$ satisfies triangle inequality if for any oriented simplexes $\sigma, \sigma_1, \ldots, \sigma_N$ and elements $g, g_1, \ldots, g_N \in G$ there holds

$$\partial(g[\sigma]) = \partial(\sum_{i=1}^{N} g_i[\sigma_i]) \Rightarrow M(g[\sigma]) \leq \sum_{i=1}^{N} M(g_i[\sigma_i]).$$

Theorem A.0.3. Let $X$ be a Banach space with Radon–Nykodym property. If mass $M$ satisfies triangle inequality, then $M$ is lower semi continuous on the space $\mathcal{R}_m(X, G)$ with respect to the flat norm topology.

Remark A.0.2. This theorem gives a partial answer to a question, raised in the paper by Ambrosio and Kirchheim ([KA00] Appendix C).

Proof. The proof resembles that of Theorem 5.2.1.
Lemma A.0.2. Let $X$ be an arbitrary Banach space. If mass $\mathcal{M}$ satisfies triangle inequality, then it is lower semi continuous on the space $\mathcal{P}_m(X, G)$ with respect to the $F$–topology.

**Proof.** We start with a very special case when $P = g[\sigma]$, i.e. when the chain $P$ consists of only one simplex. Suppose that $\mathcal{F}(P - P_i) \to 0$ as $i$ tends to $\infty$ for some $P_i \in \mathcal{P}_m(X, G)$. By the definition of flat convergence there exists a sequence of $m$–chains $\{Q_i\}_{i=1}^\infty$ and a sequence of $(m+1)$–chains $\{R_i\}_{i=1}^\infty$ such that $P - P_i = Q_i + \partial R_i$ and $\mathbb{M}(Q_i) + \mathbb{M}(R_i) \to 0$ when $i$ tends to $\infty$. Note that $\partial(P - P_i - Q_i) = \partial(\partial R_i) = 0$, so we can apply the second condition to the chain $(P - P_i - Q_i)$ and obtain

$$\mathcal{M}(P) \leq \liminf_i \mathcal{M}(P_i + Q_i) \leq \liminf_i \mathcal{M}(P_i) + C\mathbb{M}(Q_i) \leq \liminf_i \mathbb{M}(P_i),$$

and hence the first case follows.

We pass to the general case. Suppose that $P_i \to P$ in flat norm for a sequence $P_i \in \mathcal{P}_m(X, G)$. For every $j$ between 1 and $N$ denote by $W_j$ the $m$–plane containing the simplex $\sigma_j$. Fix further a linear continuous projection $P_{W_j} : X \to W_j$.

We claim that there exist $(m+1)$–convex polyhedrons $F_j$ of the following form: $F_j = \text{conv}\{a_{1,j}, \ldots, a_{m+1,j}, v_{n,j}, -v_{n,j}\}$ where

$$v_{n,j} = \frac{1}{n} \sum_{k=1}^{m+1} a_{k,i} + \eta \omega_j,$$

for some (fixed) nonzero vector $\omega_j \in \text{Ker}(P_{W_j})$ and a small number $\eta$ chosen in a way that $F_i$ and $F_j$ do not overlap. Indeed, this follows from the finite dimensional case, Theorem 5.1.2 simply by regarding the linear subspace $Y \subset X$ defined as $Y = \text{span}\{W_1, \ldots, W_n, \omega_1, \ldots, \omega_N\}$.

Next, let $u_j$ be a function defined as

$$u_j(x) = \min\{\text{dist}_\infty(P_{W_j}(x), \sigma_j^\circ), \text{dist}_\infty(x, F_j^\circ)\},$$

where $P_{W_j}$ is the fixed projection onto $W_j$, $S^\circ$ stands for the complement of a set $S$ and $\text{dist}_\infty$ is defined in the following way:

$$\text{dist}_\infty(x, S) = \inf_{z \in S} ||x - z||_{l^\infty(Y)},$$

where $l^\infty(Y)$ is the supremum norm with respect to some Euclidean structure in $Y$. Note that it follows from the construction of the polyhedrons $F_j$ that the sets $\{x : u_j(x) > \kappa\}$ are pairwise disjoint whenever $\kappa > 0$.

The rest of the proof is almost identical to that of Theorem 5.1.2.

We infer that for every $j \in [1, \ldots, N]$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r \in (0, \delta)$ one has

$$\mathcal{M}(g_j[\sigma_j] - g_j[\sigma_j \cap \{x : u_j(x) > r\}]) \leq \varepsilon. \quad (A.1)$$

We claim that for every $j \in [1, \ldots, N]$ there exists $\tilde{\delta} \in [\delta/2, \delta]$ and a sequence $\delta_i \in (\tilde{\delta}/2, \delta)$ such that $\delta_i \to \tilde{\delta}$

$$\mathcal{F}(P \sqcup \{x : u_j(x) > \tilde{\delta}\} - P_i \sqcup \{x : u_j(x) > \delta_i\}) \to 0, \quad (A.2)$$
when \( i \) tend to \(+\infty\). Indeed, by the definition of flat norm, we know that for every \( \tilde{\varepsilon} > 0 \) there exists \( i = i(\tilde{\varepsilon}) \) such that for each \( i \geq i(\tilde{\varepsilon}) \) there are \( R_i \in \mathcal{P}_{m+1}(X,G) \) and \( Q_i \in \mathcal{P}_m(X,G) \) with \( \mathbb{M}(Q_i) + \mathbb{M}(R_i) \leq \tilde{\varepsilon} \) and \( P - P_i = Q_i + \partial R_i \). Hence for every \( \rho > 0 \) we can write

\[
P \sqcap \{ x : u_j(x) > \rho \} - P_i \sqcap \{ x : u_j(x) > \rho \} = (Q_i + \partial R_i) \sqcap \{ x : u_j(x) > \rho \} =
Q_i \sqcap \{ x : u_j(x) > \rho \} + \partial (R_i \sqcap \{ x : u_j(x) > \rho \}) + ((\partial R_i) \sqcap \{ x : u_j(x) > \rho \} - \partial (R_i \sqcap \{ x : u_j(x) > \rho \})).
\]

From here we see that for every \( i \geq i(\tilde{\varepsilon}) \) and for every \( \rho > 0 \) one has

\[
\mathcal{F}(P \sqcap \{ x : u_j(x) > \rho \} - P_i \sqcap \{ x : u_j(x) > \rho \}) \leq \tilde{\varepsilon} + \mathbb{M}(R_i, u_j, \rho).
\]  
(A.3)

On the other hand, referring to the slicing properties we infer that for every \( \delta \) such that \( \mathbb{M}(R_i, u_j, \delta) \leq 2\mathbb{M}(R_i)/\delta \). Passing to a subsequence if necessary, we may assume that the sequence \( \{\delta_i\}_{i=1}^\infty \) converges towards some \( \tilde{\delta} \). Note that

\[
\lim_{i \to \infty} ||P||(\{ x : \tilde{\delta} > u_j(x) \geq \delta_i \}) = 0.
\]

Hence for the fixed \( \tilde{\varepsilon} \) there exists number \( M(\tilde{\varepsilon}) \) such that \( ||P||(\{ x : \tilde{\delta} > u_j(x) \geq \delta_i \}) \leq \tilde{\varepsilon} \) once \( i > M \). According to the inequality (A.3) applied to \( \rho = \delta_i \), for each \( i > M \) there holds

\[
\mathcal{F}(P \sqcap \{ x : u_j(x) > \tilde{\delta} \} - P_i \sqcap \{ x : u_j(x) > \delta_i \}) \leq
\mathcal{F}(P \sqcap \{ x : u_j(x) > \delta_i \} - P_i \sqcap \{ x : u_j(x) > \delta_i \}) + \mathcal{F}(P \sqcap \{ x : \tilde{\delta} > u_j(x) \geq \delta_i \}) \leq
\tilde{\varepsilon} + \frac{2\mathbb{M}(R_i)}{\delta} + ||P||(\{ x : \tilde{\delta} > u_j(x) \geq \delta_i \}) \leq \tilde{\varepsilon} + \frac{2\varepsilon}{\delta} + \tilde{\varepsilon},
\]

and the claim follows.

We further derive an upper bound on the mass of the chains \( g_j[\sigma_j] \) first using the inequality (A.1) then the first case of the proof and finally the fact that \( \delta_i \geq \delta/2 \):

\[
\mathcal{M}(g_j[\sigma_j]) \leq \varepsilon + \mathcal{M}(P \sqcap \{ x : u_j(x) > \tilde{\delta} \}) \leq \varepsilon + \lim_{i \to +\infty} \mathcal{M}(P_i \sqcap \{ x : u_j(x) > \frac{\delta}{2} \}).
\]

We continue our estimates by taking the sum over all indexes \( j \) between 1 and \( N \) in the previous line:

\[
\mathcal{M}(P) = \sum_{j=1}^{N} \mathcal{M}(g_j[\sigma_j]) \leq N\varepsilon + \lim_{i \to +\infty} \sum_{j=1}^{N} \mathcal{M}(P_i \sqcap \{ x : u_j(x) > \frac{\delta}{2} \}) \leq N\varepsilon + \lim_{i \to +\infty} \mathcal{M}(P_i),
\]

where the last inequality follows from the fact that the sets \( \{ x : u_j(x) > \delta/2 \} \) are pairwise disjoint. The first implication will now follow after we let \( \varepsilon \) tend to 0. \( \square \)
Lemma A.0.3. Let $X$ be an arbitrary Banach space and let $M \subset X$ be a $C^1$–smooth manifold of dimension $m$ with $a \in M$. Fix $t > 0$. There exists $r_0$ such that for all $r < r_0$ there is a Lipschitz mapping $F : X \to X$ such that

- $F(x) = x$ once $x \not\in U(a, r)$
- $F(M \cap B(a, tr)) \subseteq W$ where $W = \text{Tan}(a, M)$
- $\text{Lip}(F), \text{Lip}(F^{-1}) \leq 1/t$.

Proof. Suppose as we can that $a = 0$. Choose, according to the Hahn–Banach theorem, a linear continuous projection $P = P_W : X \to W$ such that $||P|| \leq C(m)$. Note that there exists $\rho > 0$ such that $M \cap B(0, \rho) \subseteq \psi(W \cap B(0, \rho))$, where we write $\psi := (P | (M \cap U(0, r)))^{-1} : P(M \cap U(0, r)) \to M \cap U(0, r)$ for brevity. Take $\varepsilon = \varepsilon(t)$ to be defined by the end of the proof and choose $r_0 \leq \rho$ that guarantees $|||D\psi(\xi) - \text{id}_{W \cap B(0, r_0)}||| \leq \varepsilon$ for all $\xi \in B(0, r_0)$. Write Taylor’s formula for $\psi$: for $z_1, z_2 \in W \cap B(0, r)$ there holds

$$\psi(z_1) = \psi(z_2) + D\psi(z_2)(z_1 - z_2) + o(||z_1 - z_2||) = \psi(z_2) + (z_1 - z_2) + (D\psi(z_2) - \text{id}_{W \cap B(0, r_0)})(z_1 - z_2) + o(||z_1 - z_2||).$$

Hence for some $r_1$, for all $r < r_1$ and $z_1, z_2 \in W \cap B(0, r)$ one has

$$||z_1 - \psi(z_1) - z_2 + \psi(z_2)|| \leq \varepsilon||z_1 - z_2||.$$

Using almost the same argumentation, one obtains that there exists $r_2$ such that for all $z \in B(0, r_2)$

$$||\psi(z) - z|| \leq \varepsilon||z||.$$

Define further for $r < r_0$ and $x \in X$ a function $u : X \to [0, 1]$ as

$$u(x) = \gamma \left( \frac{||x||}{r} \right),$$

where $\gamma$ is the piecewise linear function satisfying $\gamma(z) = 1$ for $z \in [0, t]$ and $\gamma(1) = 0$.

We are now ready to define the desired function $F$. Let $F(x) = x$ for $x \not\in U(0, r)$ and let $F(x) = x + u(x) \cdot |P(x) - \psi(P(x))|$ otherwise. Let us check the assertions of the lemma. The first and the third are clear, let us estimate the Lipschitz constants of $F$ and $F^{-1}$. If $x, y \in B(0, r)$ then we estimate using the inequalities that we have derived from Taylor’s formula:

$$||F(x) - F(y)|| \leq ||x - y|| + ||u(x)(P(x) - \psi(P(x))) + u(y)(\psi(P(y)) - P(y))|| \leq ||x - y|| + u(x) - u(y) \cdot ||P(y) - \psi(P(y))|| + ||P(x) - \psi(P(x)) + \psi(P(y)) - P(y)|| \leq \leq ||x - y|| + \frac{||x - y||}{1 - t} \varepsilon||P(y)|| + \varepsilon||P(x) - P(y)|| \leq \frac{1}{t} ||x - y||.$$
with the last inequality valid for \( \varepsilon \) small enough. If \( y \in B(0, r) \) and \( x \notin B(0, r) \) then we estimate in the following way:

\[
\|F(x) - F(y)\| \leq \|x - y\| + |u(y)| \cdot \|P(y) - \psi(P(y))\| \leq
\]

\[
\|x - y\| + \frac{1 - |u(y)|}{1 - t} \varepsilon \|P(y)\| \leq \|x - y\| \left( 1 + \frac{C(m)\varepsilon}{r(1 - t)} \right) \leq \frac{1}{t} \|x - y\|,
\]

where the last inequality holds for \( \varepsilon \) small enough. The function \( F^{-1} \) can be treated analogously.

\[\square\]

**Theorem A.0.4.** Let \( X \) be a Banach space with the Radon Nykodim property and let \( G \) be a complete normed Abelian group. If \( T \in \mathcal{R}_m(X, G) \) with \( 1 \leq m \leq n \) then for every \( \varepsilon > 0 \) there exists a polyhedral chain \( P \in \mathcal{P}_m(X, G) \) and a Lipschitz mapping \( F : X \to X \) such that

- \( \|F(x) - x\| \leq \varepsilon \) for all \( x \in X \) and \( F(x) = x \) once \( \text{dist}(x, \text{set}(T)) \geq \varepsilon \);
- \( \max\{\text{Lip}(F), \text{Lip}(F^{-1})\} \leq 1 + \varepsilon \), where the Lipschitz constants are taken with respect to the norm \( \|\ldots\| \);
- \( \mathcal{M}(F \# T - P) \leq \varepsilon \).

**Proof.** Let us derive the approximation theorem [A.0.4] from the previous lemma. Since \( T \in \mathcal{R}_m(X, G) \), we know that \( T = \theta(x)\mathcal{H}^m \setminus M \), for some \( m \)-rectifiable set \( M \) which we will call \( \text{set}(T) \) and a Borel measurable function \( \theta : \text{set}(T) \to G \) such that \( \theta \in L^1(X, G) \) (which means that the function \( |\theta| \in L^1(\mathcal{H}^m \setminus \text{set}(T)) \)). Since \( \text{set}(T) \) is \( m \)-rectifiable and since \( X \) has Radon Nykodim property, there exist countably many \( C^1 \)-smooth manifolds \( \{M_i\}_{i=1}^\infty \) and a set \( M_0 \) such that \( \mathcal{H}^m(M_0) = 0 \) satisfying

\[
\text{set}(T) \subseteq \bigcup_{i \geq 1} M_i \cup M_0.
\]

Let us next consider a subset \( \widetilde{M} \subseteq \text{set}(T) \) such that for all \( a \in \widetilde{M} \) one has \( \Theta^m(\mathcal{H}^m \setminus M, a) = \Theta^m(\mathcal{H}^m \setminus \text{set}(T), a) = 1 \) for some \( M = M(a) \in \{M_i\}_{i=1}^\infty \). Note that we can guarantee \( \mathcal{H}^m(\text{set}(T) \setminus M) = 0 \). Next, for each \( a \in \widetilde{M} \) we define \( r = r(a) \) such that for all \( r \in (0, r(a)) \) one has

\[
\|T\| (B(x, r) \setminus (M(a) \cap B(a, r))) \leq \varepsilon \|T\| (B(a, r)),
\]

where by \( \|T\| \) we, as always, denote the measure \( \|T\| = |\theta|\mathcal{H}^m \setminus \text{set}(T) \). Let us next consider the following family of balls: \( \mathcal{B} = \{B(a, r) : a \in \widetilde{M}, 0 < r < r(a)\} \). Note that \( \mathcal{B} \) is a Vitali covering of the set \( \widetilde{M} \). According to the Vitali theorem, we can find a disjoint countable covering, say \( \{B(a_j, r_j) : a_j \in M, 0 < r_j < r(a_j)\} \). We further renumber the family of manifolds \( \{M_i\}_{i=1}^\infty \) in a way that for each \( j \geq 1 \) the point \( a_j \) belongs to the manifold \( M_j \). Let us now use the lemma: for each \( j \geq 1 \), for each triplet \( (a_j, M_j, B(a_j, r_j)) \) and for each \( t \in (0, 1) \) there exists a Lipschitz mapping \( F_j : X \to X \) satisfying the claims.
of the previous lemma. Since set \( (T) = \bigcup_j (M_j \cap B(a_j, r_j)) \cup N \) with \( \mathcal{H}^m(N) = 0 \), we can find a number \( N = N(\varepsilon) \) such that

\[
\mathbb{M} \left( \sum_{j=N+1}^{\infty} T_j \right) \leq \varepsilon, \tag{A.5}
\]

where we denote \( T_j = T \setminus (M_j \cap B(a_j, r_j)) \) for sake of brevity.

The next step consists in proving that for every \( \delta > 0 \) there exists a polyhedral chain \( \Psi_j \in \mathcal{P}_m(X, G) \) such that

\[
\mathbb{M}(F_j \# T_j - \Psi_j) \leq \delta. \tag{A.6}
\]

First of all, we know that \( ||F_j \# T_j|| = (F_j^{-1} \circ \theta) \mathcal{H}^m \cap F_j(M_j \cap B(a_j, r_j)) \). Since the function \( \theta \) is integrable on the set \( M_j \cap B(a_j, r_j) \), according to the Lusin theorem for each \( \delta > 0 \) we can find a compact subset \( C \subset F_j(M_j \cap B(a_j, r_j)) \) such that \( (F_j^{-1} \circ \theta) \upharpoonright C \) is continuous and that \( ||F_j \# T_j|||(F_j(M_j \cap B(a_j, r_j)) \setminus C) \leq \delta \). Suppose, as we can that \( C \subseteq \{ x : \theta(x) \neq 0 \} \).

Second of all, we decompose \( C \) into a finite union of sets in a way that

\[
C = \bigcup_{i=1}^{K} C_i \cup C_0
\]

with \( ||F_j \# T_j|||(C_0) \leq \delta \) and \( \text{osc}(\theta \upharpoonright C_i) \leq \delta \inf C |\theta| \). Third of all, for each natural \( i \) between 1 and \( K \) there exists \( S_i \subset F_j(M_j \cap B(a_j, r_j)) \), a finite union of dyadic \( m \)-semi–cubes such that \( \mathcal{H}^m(C_i \setminus S_i) \leq \delta \mathcal{H}^m(C_i) \). Finally, for each \( i \in [1, \ldots, K] \) we choose an element \( \theta_i \in G \) that guarantees \( |\theta_i - \theta(x)| \leq \delta \inf C_i |\theta| \) for each \( x \in C_i \). We define the desired polyhedral chain as

\[
\Psi_j = \sum_{i=1}^{K} \theta_i [S_i].
\]

Let us estimate the mass of the following difference

\[
\mathbb{M}(F_j \# T_j - \Psi_j) \leq \int_{F_j(M_j \cap B(a_j, r_j))} |\theta(x)| d\mathcal{H}^m(x) - \sum_{i=1}^{K} |\theta_i| \mathcal{H}^m(S_i) \leq
\]

\[
\delta + \sum_{i=1}^{K} \int_{C_i} |\theta(x)| d\mathcal{H}^m(x) - \sum_{i=1}^{K} |\theta_i| \mathcal{H}^m(S_i) \leq
\]

\[
\delta + \delta ||F_j \# T_j|||(C) + \sum_{i=1}^{K} |\theta_i| (\mathcal{H}^m(C_i) - \mathcal{H}^m(S_i)) \leq \delta + \delta (2 + \delta) ||F_j \# T_j|||(C),
\]

and the claim \( (A.6) \) follows.

Now we are ready to finish the proof of the theorem. We choose \( F := F_1 \circ \ldots \circ F_N \) and \( P := \sum_{j=1}^{N} \Psi_j \).
By the construction we see that $F$ and $P$ satisfy the first two claims of the theorem. Let us check the third one:

$$
M(F\#T - P) = M(F\# \sum_{j=1}^{\infty} T \ll B(a_j, r_j) - P) \leq M(\sum_{j=1}^{N} F_j\#T \ll B(a_j, r_j) - P) + \varepsilon \leq \ldots,
$$

where the last inequality follows from the line (A.5). We continue the estimate, first using the estimate (A.4) and afterwards the inequality (A.6) with $\delta = \varepsilon/N$:

$$
\ldots \leq M \left( \sum_{j=1}^{N} F_j\#T_j - \sum_{j=1}^{N} \Psi_j \right) + \varepsilon \sum_{j=1}^{N} ||T_j|| (B(a_j, r_j)) + \varepsilon \leq \frac{N\varepsilon}{N} + \varepsilon + \varepsilon M(T).
$$

Since $\varepsilon > 0$ is arbitrary, the theorem follows.

Let us now prove our lower semicontinuity result. Let $T_j$ be a sequence of $m$–rectifiable $G$ chains, converging in flat norm to some $T \in \mathcal{R}_m(X, G)$. Fix $\varepsilon > 0$. According to the strong approximation theorem, there exist a $P \in \mathcal{P}_m(X, G)$ and $F : X \to X$ such that $P = F\#T + E$ with $M(E) \leq \varepsilon$. Note that since $F$ is Lipschitz, one has $\lim_{j \to \infty} \mathcal{F}(F\#T_j - F\#T) = 0$ and hence also $\lim_{j \to \infty} \mathcal{F}(E + F\#T_j - P) = 0$ by the definitions of $E$ and $P$. Since $P$ is polyhedral, we can use the lemma:

$$
\mathcal{M}(E + F\#T) = \mathcal{M}(P) \leq \liminf_{j \to \infty} \mathcal{M}(E + F\#T_j) \leq C M(E) + \liminf_{j \to \infty} \mathcal{M}(F\#T_j)
$$

$$
C\varepsilon + \liminf_{j \to \infty} (\text{Lip}(F))^m \mathcal{M}(T_j) \leq C\varepsilon + (1 + \varepsilon)^m \liminf_{j \to \infty} \mathcal{M}(T_j), \quad (A.7)
$$

where the last inequality follows from the fact that $F$ is $(1 + \varepsilon)$–Lipschitz. On the other hand, since $\text{Lip}(F^{-1}) \leq 1 + \varepsilon$, we infer that

$$
\mathcal{M}(E + F\#T) \geq \mathcal{M}(F\#T) - \mathcal{M}(E) \geq (1 + \varepsilon)^{-m} \mathcal{M}(T) - C\varepsilon. \quad (A.8)
$$

The lines (A.7) and (A.8) now give

$$
(1 + \varepsilon)^{-m} \mathcal{M}(T) \leq (1 + \varepsilon)^m \liminf_{j \to \infty} \mathcal{M}(T_j) + 2C\varepsilon,
$$

and the theorem will follow after we let $\varepsilon$ tend to zero.
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