Université Paris Diderot (Paris 7) Sorbonne Paris Cité
UFR de Mathématiques

Thèse

pour obtenir le titre de

Docteur de l'Université Paris Diderot (Paris 7)
Sorbonne Paris Cité

Spécialité : Mathématiques

présentée par

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Multisymplectic formalism for theories of super-fields and non-equivalent symplectic structures on the covariant phase space

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Soutenue publiquement le 7 Décembre 2016 devant le jury composé de :

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Remerciements

Desidero ringraziare il sistema dell'Istruzione Pubblica Italiana, che mi ha permesso di dedicare tre anni allo studio e alla ricerca necessari per elaborare questa mia tesi e che mi ha sostenuto economicamente durante questo periodo. In particolare ringrazio per il suo incoraggiamento e il suo sotegno la prof.ssa M. A. Ragone, dirigente scolastico della scuola presso cui insegno, il Liceo Classico, Linguistico e Artistico "G. e Q. Sella" di Biella.

Desidero ringraziare con tutto il cuore il mio direttore di tesi prof. Fédéric Hélein, per la sua disponibilità, per la sua gentilezza, per la generosità con cui ha sempre condiviso con me la sua profonda conoscenza della Matematica e della Fisica e per i suoi preziosi suggerimenti che mi sono serviti da guida negli anni di preparazione di questa tesi.

Merci Frédéric.

I also wish to thank the two referees who dedicated their time with patience to read my work : prof. Hovhannes Khudaverdian and prof. Gijs Tuynman. Thank you for your precious comments and suggestions.

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Résumé de la Thèse

Le Calcul des Variations et son interprétation géométrique ont toujours joué un rôle crucial en Physique Mathématique, que ce soit par le formalisme lagrangien, ou à travers les équations hamiltoniennes, qui sont profondément liés à l'approche lagrangienne. Ce rôle a été confirmé et renforcé au cours du siècle dernier, par exemple par le théorème de Noether, qui relie les symétries et les quantités conservées, ainsi que par l'utilisation du Calcul des Variations faites en Mécanique Quantique et en Théorie Quantique des Champs.

Pour les théories des champs, qui correspondent à des problèmes variationnels avec plusieurs variables spatio-temporelles, l'approche Lagrangienne est maintenant bien-fondée et largement utilisée. Cependant l'approche Hamiltonienne n'est sans doute pas si bien développée et utilisée dans toute sa puissance. Afin de parvenir à une description Hamiltonienne de la théorie des champs, qui amène à la quantification dite canonique, la plupart des physiciens utilisent un fractionnement de l'espace-temps comme produit cartésien d'une variété spatiale par une droite de temps. Ceci est bien sûr suffisant pour la plupart des calculs, mais il est clair que cela brise la covariance relativiste et donc contribue à obscurcir la relation entre la théorie quantique et la Relativité.

Plusieurs théories ont été développées pour remédier à cette situation : elles reposent toutes plus ou moins sur le fait que l'espace de toutes les solutions d'un problème variationnel est doté d'une structure symplectique, une observation qui remonte à Lagrange et qui fut précisée par J.-M. Souriau. C'est le message fondamental véhiculé par la théorie dite de l'*espace des phases covariant*. Le problème est de trouver les outils techniques les plus appropriés pour représenter cette structure symplectique et faire des calculs avec elle. Des différentes approches existent, par exemple : celle développée par Deligne et Freed dans [36] et basée sur la théorie de Takens,ou la théorie de Vinogradov, développée par exemple par Vitagliano dans [147], ou encore l'approche multisymplectique.

Le formalisme multisymplectique permet une description géométrique de dimension finie des théories de champ classiques vues d'un point de vue hamiltonien. La géométrie multisymplectique joue un rôle similaire à celui de la géométrie symplectique dans la description de la mécanique hamiltonienne classique. De plus, l'approche multisymplectique fournit un outil pour construire une structure symplectique sur l'espace des solutions de la théorie des champs et pour l'étudier.

Le formalisme multisymplectique

La généralisation des équations de Hamilton à un problème variationnel de premier ordre avec plusieurs variables remonte à deux articles de V. Volterra de 1890, [150, 151], dans lequel deux variantes différentes ont été proposées. Aujourd'hui, la première théorie proposée par Volterra est connue sous le nom de théorie de Donder-Weyl parce qu'une version a été exposée par H. Weyl en 1934, [160], et une autre par T. De Donder en 1935, dans [32], cette théorie est en fait d'ordre supérieur et est reliée aux travaux de H. Poincaré, [123], et E. Cartan, [27], sûr la théorie de "Invariants Intégraux", mettant en évidence son contenu géométrique. Des résultats

fondamentales ont été obtenues après par T. Lepage, [104], en 1936 et P. Dedecker en 1953, [33], voir aussi [34].

Dans une série d'articles au cours des années 70, J. Kijowski seul, [94, 95], puis avec W. Szczyrba [97, 98] et ensuite avec W. M. Tulczyjew [99] a donné naissance au point de vue multisymplectique sur ces théories. Le formalisme multisymplectique permet une géométrisation complète et fournit un point de vue covariant sur les théories de champs hamiltoniennes. Les idées de Kijowski permettent de construire une structure symplectique sur l'espace de toutes les solutions d'une théorie de champ (qui est souvent appelé aujourd'hui l'espace des phases covariant), d'une manière qui est dans une certaine mesure indépendante de tout choix de séparation de l'espace-temps dans l'espace et le temps et qui est donc entièrement covariant.

Des idées similaires sur la structure symplectique de l'espace des phases covariant étaient présentes, dans les années 50, dans les travaux pionniers de R. E. Peierls, voir [122], et I. Segal, voir [144], et sont apparues de nouveau dans les années 80 dans des articles de E. Witten [161], C. Crnkovic et Witten [28] et G. Zuckerman [164] qui ne connaissaient probablement pas les travaux de Kijowski et de l'école polonaise.

Les articles de Witten ont provoqué un nouvel intérêt pour les théories hamiltoniennes covariantes et sur l'approche de l'espace des phases covariant, tant dans la communauté des physiciens que dans le communauté des mathématiciens. A partir de la seconde moitié des années 80, le formalisme multisymplectique des théories de champs a été étudié et revisité par plusieurs auteurs et présenté dans de nombreuses variantes différentes. Des applications à la mécanique des fluides et à l'hydrodynamique ont été proposées. Des méthodes numériques covariantes pour les équations aux dérivées partielles, développées dans ou inspirées par le cadre multisymplectique, ont été introduites. L'intérêt pour les théories multisymplectiques de champ a donné naissance aussi à un certain nombre d'études sur la géométrie multisymplectique ou n-plectique. Dans l'introduction de la première partie de cette thèse, je donne une courte liste de quelques-uns des travaux les plus importants sur ces sujets.

Le lecteur peux consulter le papier de M. J. Gotay, J. Isenberg, J. Marsden, R. Montgomery, J. Sniatycki et P. B. Yasskin, [63, 64], pour une présentation du formalisme multisymplectique pour les théories de champ. On trouvera une introduction plus courte dans Román-Roy [135]. Pour une introduction avec une section sur l'histoire des idées autour du formalisme multisymplectique, leur origine et leur évolution, on peut lire Hélein [70].

Il est intéressant de noter que, jusqu'à aujourd'hui, presque aucune tentative d'adapter le formalisme multisymplectique aux théories des super-champs (comme par exemple les théories supersymétriques de champ) n'a pas été faite. Aussi les travaux qui pourraient être considérés comme préliminaires à cette tâche, comme les papiers sur la formulation géométrique de la supermécanique ou ceux sur la super-forme de Poincaré-Cartan pour les théories de super-champs (que je citerai ci-après), sont très peu nombreux.

Dans cette thèse, je m'intéresserai principalement au formalisme multisymplectique pour construire des théories de champ de premier ordre et j'espère pouvoir donner deux principales contributions originales :

– Je montrerai que, dans certaines situations, la structure symplectique de l'espace des phases covariant peut en effet dépendre du choix de la topologie du découpage de l'espace-temps en l'espace et en le temps ;
– Je construis une extension du formalisme multisymplectique aux théories de super-champs.

En tant que «sous-produit», je présenterai une autre contribution originale :

– Je définirai des formes fractionnaires sur des supervariétés avec leur calcul de Cartan.

Ces formes fractionnaires seront nécessaires pour construire le formalisme multisymplectique pour les théories de super-champs.

J'utiliserai une version du formalisme multisymplectique qui rend minimale la dimension des espaces impliqués, ce qui me semble approprié pour une première tentative d'extension aux théories de super-champs. J'appelle cette version le cadre minimal pour le formalisme multisymplectique. L'ingrédient principal de ce cadre sera l'espace de multimoments de dimension finie P, lequel correspond à ce que Forger et Romero appellent l'espace des multiphases ordinaire, [53] et ce que Román-Roy, [135], appelle le fibré des multimoments restreint.

Cette thèse est organisée en trois parties. Les principaux résultats originaux de la ***Prémière Partie*** sont contenus dans le chapitre 3 et consistent en l'étude des structures symplectiques non équivalentes sur l'espace des phases covariant.

Dans le ***Chapitre 1***, je présenterai un cadre géométrique standard pour l'approche Lagrangienne des théories de champ au premier ordre.

Si E, X et F sont des variétés C^∞ de dimension finie et (E, π, X, F) est le fibré différentiel d'espace totale E, de base X, de fibre type F et de projection C^∞ π, de sorte que nous avons la situation suivante :

$$\begin{array}{c} E \\ {\scriptstyle\pi}\big\downarrow \\ X \end{array}$$

nous appelons E le fibré des champs ou le fibré des configurations et un champ ϕ sur X est l'un de ses sections C^∞ : on peut donc appeler $\Gamma\big(E\big)$ l'espace des champs.

Si X est n-dimensionnelle, le Lagrangien $\mathcal{L}$ est une n-forme différentielle π-horizontale définie sur le premier espace des jets $J^1\pi \equiv J^1E$. Si $U \subset X$ est une sous-variété n-dimensionnelle (éventuellement avec un bord) de X, l'action A_U sur un champ ϕ est l'intégral de $\mathcal{L}$ sur la surface n-dimensionnelle $j^1\phi(U)$ est :

$$A_U(\phi) := A_U(j^1\phi) = \int_{j^1\phi(U)} \mathcal{L} = \int_U j^1\phi^*\mathcal{L} \tag{1}$$

L'espace des solutions de la théorie   est celui des points critiques.

Si, sur J^1E, nous utilisons les coordonnées locales $\big(x^a, q^i, \dot{q}^i_a\big)$, alors localement $\mathcal{L} = L\beta$, où $\beta := dx^1 \wedge \cdots \wedge dx^n$ et L est une fonction locale de $\big(x^a, q^i, \dot{q}^i_a\big)$.

Chaque solution $\phi \in \mathscr{E}$ vérifie les équations d'Euler-Lagrange :

$$\phi \text{ est une solution } \iff \text{ sur chaque } U,\ \forall x \in U,\ \frac{d}{dx^a}\frac{\partial L}{\partial \dot{q}^i_a}\big(j^1\phi(x)\big) - \frac{\partial L}{\partial q^i}\big(j^1\phi(x)\big) = 0 \tag{2}$$

Dans le ***Chapitre 2***, je présenterai ce que j'appelle le cadre minimal pour la description multisymplectique des théories des champs. Je définis l'espace des multimoments P, avec les coordonnées locales $\big(x^a, q^i, p^a_i\big)$ et la transformation de Legendre entre J^1E et P :

$$\mathbb{FL}: \quad \big(x^a, q^i, \dot{q}^i_a\big) \longmapsto \quad \big(x^a, q^i, p^a_i\big) = \left(x^a, q^i, \frac{\partial L}{\partial \dot{q}^i_a}\big(x^a, q^i, \dot{q}^i_a\big)\right)$$

L'hamiltonien H est alors défini par $H(x^a, q^i, p^a_i) := \dot{q}^i_a p^a_i - L\left(x^a, q^i, \dot{q}^i_a\right)$ et un champ ϕ est une solution de la théorie si et seulement si $z := \mathbb{FL} \circ j^1\phi$ vérifie le système covariant généralisé de Hamilton-Volterra :

$$\begin{cases} \dfrac{\partial q^i}{\partial x^a}\big(\mathbb{FL}j^1\phi(x)\big) = \dfrac{\partial H}{\partial p^a_i}\big(\mathbb{FL}j^1\phi(x)\big) \\ \dfrac{\partial p^a_i}{\partial x^a}\big(\mathbb{FL}j^1\phi(x)\big) = -\dfrac{\partial H}{\partial q^i}\big(\mathbb{FL}j^1\phi(x)\big) \end{cases} \tag{3}$$

Sur P il est possible de définir la *forme multisymplectique*, qui est la $n+1$-forme globale :

$$\omega := -dq^i \wedge dp_i^a \wedge \beta_a - dH \wedge \beta$$

où $\beta_a = \frac{\partial}{\partial x^a} \lrcorner \beta$.

Kijowski a montré dans [94] le résultat suivant :

Theorem 1. *Une section $z \in \Gamma(P)$ est l'image d'une solution des équations d'Euler-Lagrange* (2) *par la transformée de Legendre si et seulement si* $\forall u \in TP,\ z^*(u \lrcorner \omega) = 0$.

Un des principaux intérêts de la construction géométrique de dimension finie de l'espace des multimoments avec sa structure multisymplectique (n-plectique selon la terminologie la plus recente) est qu'il fournit un moyen de construire une structure symplectique sur l'espace des phases covariant (l'espace des solutions de la théorie des champs). Il existe alors un lien direct entre la théorie des champs multisympléctique et la formulation canonique classique de la théorie des champs. Ce lien relie la géométrie multisympléctique aux travaux des physiciens théoriciens sur la théorie canonique des champs, initiés par les travaux de Peierls, [122], et Segal, [144] et développés par B. DeWitt [40, 41, 42], par García et Pérez-Rendón in [56, 57, 58] et par Goldschmidt et Sternberg dans [62].

Ici, je suivrai [70] pour montrer comment à l'aide de la forme multisymplectique ω, il est possible de construire une forme symplectique Ω sur l'espace des n-courbes hamiltoniennes de la forme $\mathscr{G} = z(X)$ où z est solution de la théorie.

Appelons $\mathscr{G}$ l'espace des surfaces hamiltoniennes ; nous avons que $\mathscr{G} \cong \mathscr{E}$ et donc nous pouvons identifier ces deux espaces. Soit $G \in \mathscr{G}$ et soit $\delta_u G \in T_G\mathscr{G}$ un vecteur sur G : il est associé a un champ *champ de Jacobi* $u \in \Gamma(i^*(VP))$, id est une section sur G du tiré en arrière du fibré tangent vertical (par rapport à la projection π_P de l'espace total P sur la base X) VP par l'immersion $i\ :\ G \longrightarrow P$. La section u peut être vue intuitivement comme un champ vectoriel sur G, "suivant" lequel chaque point $g \in G$ est envoyé sur un point $g' \in G'$, où $G' \in \mathscr{G}$ est une autre n-courbe hamiltonienne. La n-courbe hamiltonienne G est ainsi déformée de façon infinitesimale par u dans une autre n-courbe hamiltonienne G'.

Soit Σ une sous-variété de co-dimension 1 dans P, avec la propriété que, pour tout n-courbe hamiltonienne $G \in \mathscr{G}$, l'intersection de Σ avec G est transversal. Alors, nous pouvons définir :

$$\Omega_\Sigma\big|_G(\delta_1 G, \delta_2 G) \ := \ \int_{\Sigma\cap G} u_1 \wedge u_2 \lrcorner \omega \tag{4}$$

et, dans certaines conditions de régularité du Lagrangien, Ω_Σ est une 2-forme symplectique sur $\mathscr{G}$.

Une question naturelle se pose alors : la forme symplectique Ω dépend-elle du choix de la sous-variété Σ ? Kijowski a déjà prouvé que si Σ et Σ' sont deux sous-variétés compactes dans la même classe d'homologie, alors $\Omega_\Sigma = \Omega_{\Sigma'}$.

Dans le ***Chapitre 3*** je montre, sur quelques exemples de théories de champ construites sur un tore bidimensionnel, que lorsque Σ et Σ' sont dans différentes classes d'homologie, il peut arriver que $\Omega_\Sigma \neq \Omega_{\Sigma'}$. Il semble que ce résultat n'ait pas encore été remarqué.

De plus, je présente quelques exemples où Σ et Σ' ne sont pas compacts et je montre que dans ce cas, la situation est certainement plus délicate. L'homologie standard des sous-variétés de P n'est plus appropriée pour déterminer la structure symplectique sur $\mathscr{G}$. J'étudie le champ scalaire libre et massif sur $\mathbb{R}^2$ et j'explique ce qui se passe quand on choisit Σ et Σ' tels qu'ils restent sur deux côtés différents du cône de lumière. Je vais montrer le résultat que, dans ce cas, $\Omega_\Sigma \neq \Omega_{\Sigma'}$.

Dans le ***Chapitre 4***, qui est le dernier de la première partie, je montrerai comment la structure symplectique sur l'espace des solutions est liée aux crochets de champs utilisés par les physiciens. En suivant J. Kijowski et W. Szczyrba [97, 98], je montrerai que la forme symplectique Ω peut être utilisée pour obtenir une structure de Poisson sur $\mathscr{E} \cong \mathscr{G}$; ceci fixera les bases pour l'extension de la même construction aux théories de super-champ.

Le formalisme multisymplectique n'a pas encore été appliqué à la description des théories de champs supersymétriques. La principale difficulté à laquelle nous devons faire face pour l'adapter aux super-champs est à mon avis le fait que la généralisation des formes différentielles au cadre supersymétrique conduit à deux types différents d'objets. Pour tirer pleinement parti du formalisme multisymplectique, on aimerait plutôt utiliser les mêmes objets pour pour l'intégration (comme dans (1) et (4)) et pour effectuer le calcul de Cartan (comme par exemple dans le théorème 1). A cet effet, je crois que les objets les plus appropriés à utiliser sont les superformes définies par Th. Voronov et A. Zorich dans leurs papiers à la fin des années 80, [155, 156, 157, 158]. Plus précisément, j'utiliserai une classe de superformes de Voronov-Zorich, que j'appelle les formes fractionnaires.

Dans la ***Deuxième Partie*** de ma thèse, je vais introduire les notions de formes fractionnaires, de coformes fractionnaires et de formes fractionnaires mixtes sur des supervariétés et je proposerai une nouvelle notation, adaptée aux calculs. Les formes fractionnaires seront un ingrédient essentiel pour la définition d'une théorie de super champs et pour le formalisme supermultisymplectique qui est le principal objet de la troisième partie de cette thèse.

Dans le ***Chapitre 5***, après avoir brièvement présenté l'approche concrète de Rogers-DeWitt aux supervariétés, je définirai les formes fractionnaires et je fixerai les règles pour exécuter avec elles un calcul de Cartan.

Les formes fractionnaires sont des exemples des $r|s$-formes introduites par Th. Voronov et A. Zorich comme analogues naturels sur les supervariétés des formes classiques sur les variétés classiques. En désignant par T_0X et T_1X respectivement l'espace tangent pair et impair d'une supervariété X, nous avons :

Definition 2 (Voronov and Zorich)**.** *Une forme de degré $r|s$ sur un point $x \in X$, supervariété de dimension $n|m$, est une fonction ω de classe G^∞ définie sur un ouvert O de $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$ et à valeurs dans $\mathbb{R}_S$, qui vérifie la suivante condition : $\forall v \in O \subset \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$:*

$$\forall g \in GL(r|s),\ \omega(g \cdot v) = \omega(v)\,\mathrm{Ber}_{r,s}(g) \tag{5}$$

$$\frac{\partial^2 \omega}{\partial v_G^B \partial v_F^A} + (-1)^{|G||F|+(|G|+|F|)|A|} \frac{\partial^2 \omega}{\partial v_F^B \partial v_G^A} = 0 \tag{6}$$

où $A, B = 1, \cdots, n+m$ sont les indices dans l'espace T_xX et donc aussi dans les deux espaces $T_{x,0}X$ et $T_{x,1}X$ avec leur degré habituel; v_F^A est la A-ième coordonnée de v_F dans la base locale $(\partial_A|_x)_A$; F va de 1 à $r+s$ et $v_F \in T_{x,|F|}X$, où nous posons $|F| = 0$ quand $F = 1, \cdots, r$ et $|F| = 1$ quand $F = r+1, \cdots, r+s$.

Dans la section 5.2.2, je présenterai une extension originale des $r|s$-formes, de sorte que le domaine de définition des premiers r arguments devienne tout l'espace tangent TX, au lieu d'être juste sa partie paire.

Puis, dans les sections 5.3 et 5.4, je donne une preuve directe que les fonctions θ et $\mu \wedge \theta$, définies comme suit :

$$\forall x \in U,\ \forall v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} :$$

$$\theta(v) := \mathrm{sdet}_{r,s} \begin{pmatrix} \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_r}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$$

et :

$$\forall x \in U,\ \forall w := (\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s};$$

$\forall \mu \in T_x^*U,$ si nous décomposons $\mu = dx^A \mu_A$:

$$\mu \wedge \theta(w) := \mathrm{sdet}_{r+1,s} \begin{pmatrix} \overline{v_1^A}\mu_A & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^A}\mu_A & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \overline{v_{r+1}^A\mu_A} & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\mu_s}} \\ \widetilde{v_1^A}\mu_A & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^A}\mu_A & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$$

sont en effet de superformes de Voronov-Zorich.

Ces exemples de superformes de Voronov-Zorich seront importants pour nous. Il est souhaitable d'avoir une notation plus compacte et intuitive pour les définir. Je propose dans cette thèse les notations suivantes :

$$\frac{dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}} := \theta$$
$$\mu \wedge \frac{dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}} := \mu \wedge \theta = \frac{\mu \wedge dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}}$$

Je vais aussi définir des objets comme :

$$\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$$

Où $\Theta^1 \dots \Theta^r$ sont des $1|0$-formes de parité générique et $\theta^1 \dots \theta^s$ sont des $1|0$-formes impaires.

Ce genre d'objets, que j'appelle des formes fractionnaires, était déjà utilisé dans la littérature mathématique et physique, avec une notation différente, mais, à ma connaissance, une preuve directe qu'ils sont en effet des superformes de Voronov-Zorich, n'avaient pas encore été publiée.

Dans la section 5.4 j'expliquerai les règles pour effectuer le calcul de Cartan avec des formes fractionnaires, y compris des produits intérieurs et extérieurs par vecteurs et covecteurs de toute parité, la dérivation extérieure et des combinaisons de ces opérations avec les commutateurs correspondants. Je donnerai également une formule originale utile pour des calculs efficaces utilisant des superdéterminants. Ceux-ci seront utilisés dans la troisième partie de la thèse ; en fait, le calcul de Cartan avec les formes fractionnaires s'avère plus commode que le calcul de Cartan pour les $r|s$-formes génériques.

Dans la section 5.5 je présenterai la théorie de l'intégration des superformes (et donc des formes fractionnaires) développée par Voronov et Zorich. Je proposerai aussi une petite modification de la définition standard de l'intégrale de Berezin, basée sur le concept de *corps immergé* d'une supervariété que je définirai. Cela sera utile pour les sujets traités au chapitre 9.

Dans le ***Chapitre 6*** je définirai les coformes fractionnaires et les formes mixtes fractionnaires du premier et du second type. Les coformes fractionnaires sont des exemples de ce que Voronov appelle *twisted covariant dual Lagrangians satisfying the fundamental equations* ou plus brièvement *twisted dual forms* dans [153] et [154], et ils sont à la base de la définition de ses formes stables. Les formes mixtes fractionnaires sont des exemples de ce que Voronov a appelé les formes mélangées, [153], [154].

Un exemple de coforme fractionnaires sera la coforme w localement écrite comme suit :

$$w = \frac{\partial_{A_{t+1}} \wedge \partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}}$$

ou la coforme u localement écrite comme suit :

$$u = \frac{v_1 \wedge \cdots \wedge v_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}}$$

où $v_1, \ldots, v_t$ sont des champs vectoriels de n'importe quelle parité et $\widetilde{v_1} \ldots \widetilde{v_q}$ sont des champs vectoriels impairs.

Un exemple d'une forme mixte fractionnaire sera :

$$\frac{v_1 \wedge \cdots \wedge v_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}} \lrcorner \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{7}$$

Je vais montrer comment effectuer un calcul de Cartan avec des coformes fractionnaires et des formes mixtes fractionnaires, ce qui me permettra de donner un sens aux formules comme (19) ou comme :

$$\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \tag{8}$$

aussi quand, dans (8), $l > t$ et $d < q$; ou quand, dans (7), $t > r$ et $q < s$.

Par exemple nous donnerons un sens à :

$$d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

et par symétrie à :

$$\frac{1}{\frac{\partial}{\partial \xi^1}} \lrcorner d\xi^1 := d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

ou encore à :

$$\frac{\partial}{\partial \xi^1} \lrcorner \frac{1}{d\xi^1}$$

où ξ_1 est une coordonnée locale impaire sur une supervariété.

Dans la section 6.3 je montre comment intégrer des coformes et des formes mixtes sur des supervariétés.

Le chapitre 6 est en effet indépendant du reste de cette thèse et, contrairement au chapitre 5, il n'est pas nécessaire de le lire pour comprendre la troisième partie de la thèse. Par conséquent, le matériel présenté ici n'est pas traité en détail. Il peut être considéré comme un complément naturel du chapitre 5 et comme un travail préliminaire pour des études futures, en particulier pour des études sur les théories de super-champs de Batalin-Vilkovisky et Bechi-Rouet-Stora-Tyutin.

Les principales contributions originales dans la ***Troisième Partie*** de ma thèse sont : la construction d'un formalisme multisymplectique pour les théories de super-champs ; la formulation du théorème de comparaison (entre la formulation en composantes et celle des super-champs,voir plus bas) avec les outils offerts par les formes fractionnaires et l'intégrale sur un *immersed body* ; la formulation du traitement géométrique des symétries et des supersymétries des théories de superchamps en utilisant le langage des formes fractionnaires et en exploitant la forme fractionnaire de Poincaré-Cartan.

Les théories des superchamps ont commencé à être étudiées abondamment à la fin des années 70 avec le développement de la supersymétrie en Physique.

Une théorie de champ supersymétrique peut être habituellement présentée de deux façons différentes : soit comme une théorie de champs sur une variété classique, avec des composantes fermioniques et bosoniques, soit comme une théorie définie sur une supervariété. La première approche est parfois appelée approche en composantes, tandis que la seconde est parfois appelée l'approche de superchamps propre. Dans les deux cas, la théorie est définie par les équations de la dynamique que les champs doivent satisfaire.

Quand on utilise l'approche en composantes, les équations des champs peuvent être dérivées à partir d'un principe variationnel avec une action définie comme l'intégrale sur la variété de base classique d'une densité lagrangienne. L'action implique évidemment l'usage à la fois de composantes bosoniques (commutantes) et fermioniques (anticommutantes) des champs, traitées selon les parités respectives.

Pour exprimer le principe d'action dans un langage géométrique, il est utile de formuler un calcul variationnel pour les densités lagrangiennes définies en termes de formes différentielles. Puisque la densité lagrangienne dépend en général de la dérivée des composantes des champs (qui peut être bosonique ou fermionique), même si la théorie est définie sur une variété classique (bosonique), il est clair qu'il est nécessaire de développer un calcul pour les formes différentielles valable également pour le secteur fermionique. Ce n'est pas du tout simple et cela a été fait par D. Hernández Ruipérez and J. Muñoz Masqué pendant les années 80, précisément dans le cas où la variété de base est classique. Dans [76, 77, 78, 118, 119] ils ont en effet développé un calcul variationnel gradué pour les densités lagrangiennes définies en termes de formes différentielles graduées de Kostant et ils ont obtenu le formalisme lagrangien correspondant (équations d'Euler-Lagrange, forme de Poincaré-Cartan, invariants de Noether, etc.).

Lorsque la variété de base n'est pas classique et qu'il s'agit d'une supervariété, comme dans l'approche des superchamps aux théories supersymétriques, alors la tâche est encore plus difficile. La théorie peut toujours être dérivée d'un principe variationnel, mais l'action dans ce cas est définie comme l'intégrale berezinienne (réalisée à l'aide d'une densité de volume berezinienne) d'une densité lagrangienne, qui doit donc être une densité de volume berezinienne.

En 1987, dans [80], Hernández Ruipérez et Muñoz Masqué notent : «l'absence d'une définition intrinsèque d'une notion appropriée de densités intermédiaires de Berezin avec leur calcul extérieur de Cartan, nous empêche de développer un formalisme lagrangien ...», ce qui signifie qu'il manque un formalisme lagrangien valide également pour le cas où la variété de base st une supervariété. Néanmoins, dans [79] et [80] ils sont parvenus à une formulation intrinsèque de la notion de densité lagrinienne-berezinienne et de sections critiques Bereziniennes. En outre, dans le cas où une théorie peut être exprimée à la fois ens composantes et avec l'approche des superchamps, ils ont montré comment relier les sections critiques de la densité lagrangienne-berezinienne, définie sur la supervariété, aux sections critiques du Lagrangien gradué correspondant, défini sur une variété bosonique avec des formes différentielles graduées. Ils ont obtenu ce résultat via une première version de ce que l'on appellera ultérieurement le Théorème de Comparaison.

En 1992 J. Monterde, dans [113], a montré que les sections critiques berezinienne, d'une action définie avec une densité larangienne-berezinienne, doivent satisfaire une version "super" des équations d'Euler-Lagrange. A l'époque, le formalisme des superformes défini par Voronov et Zorich existait déjà mais il ne l'a pas utilisé, peut-être en raison du fait que le système de notations qui était nécessaire à cet effet est un peu lourd. Malheureusement il semble que ses résultats n'aient pas été beaucoup exploités, ni par des mathématiciens ni par des physiciens.

Par la suite, pendant les années 90 et durant la première décennie du nouveau millénaire, certains auteurs ont travaillé à l'élaboration d'une approche géométrique des théories des superchamps d'un point de vue lagrangien et hamiltonien. Des idées importantes ont été recueillies dans les articles sur la mécanique mécanique de Monterde et Muñoz Masqué, [114, 115] ; par L. A. Ibort et J. Marín-Solano [83] ; par J. F. Cariñena et H. Figueroa [26] ; par Monterde and J. A. Vallejo, dans [117].

En 2006, Monterde, Muñoz Masqué et Vallejo ont publié un article [116], dans lequel ils ont proposé un formalisme de Hamilton-Cartan pour des problèmes variationnels berezinien de premier ordre valide pour des champs définis sur des supervariétés de dimension quelconque. Ils ont atteint leur objectif en étudiant, à l'aide du théorème de comparaison, un problème variationnel associé, d'ordre supérieur, défini sur une variété de base bosonique. Ils ont obtenu une super-forme de Poincaré-Cartan valable pour des théories sur des bases de n'importe quelle dimension. Cependant ils utilisent une notation qui ne me semble pas très adaptée aux preuves générales, ni aux calculs réels. Ils ont également obtenu un très beau et important résultat, qui est la généralisation du premier théorème de Noether aux théories de super champs (théorème 8.2 dans [116]) ; au prix d'une hypothèse plutôt technique que nécessaire.

À ma connaissance, aucun mathématicien n'a utilisé les résultats sur les super-formes de Poincaré-Cartan pour décrire les théories des superchamps avec l'approche multisymplectique.

Indépendamment des résultats obtenus par l'école espagnole, il n'y a eu, à ma connaissance, qu'une tentative pour étendre le formalisme multisymplectique à des super-champs. S. P. Hrabak dans [81, 82] a étudié la formulation de la symétrie BRST classique dans le cadre d'une théorie multisymplectique. Pour ce faire, il a eu besoin d'étendre le formalisme multisymplectique pour qu'il fonctionne aussi pour les théories de champ dont la base est une variété bosonique classique, mais dont l'espace des champs est une supervariété, avec secteurs bosonique et fermionique (en raison de la présence des fantômes). Il accomplit cette tâche dans [82]. Il n'a cependant pas montré comment étendre éventuellement le formalisme aussi au cas où la base elle-même est une supervariété.

Ici, dans la troisième partie de ma thèse, je présenterai une version multisymplectique complète des théories de superchamps valables pour toute dimension (pair et impair) de l'espace de base et de l'espace des champs. Mes résultats sont une généralisation de ceux obtenus par Hrabak dans [82] et ils sont fondés sur une pleine exploitation du potentiel de la théorie des superformes de Voronov et de Zorich. Elles peuvent aussi être considérées comme une généralisation naturelle des résultats obtenus dans les présentations géométriques de la supermécanique à dimension finie, qui comprennent l'utilisation d'une superforme symplectique, comme les présentations dans [83] et [26].

Si l'on veut construire une théorie de superchamps multisymplectique, on doit utiliser des objets (par exemple la forme multisymplectique) qui peuvent être intégrés sur une supervariété et qui, en même temps, peuvent être utilisés pour un calcul de Cartan, y compris une contraction par des supervecteurs, un produit extérieur par une 1-forme et une dérivée extérieure. C'est le point difficile. En fait, avant les articles de Voronov des années 90, [153, 154], aucun objet de ce genre n'existait. Avant l'apparition des superformes de Voronov et de Zorich, les meilleurs candidats pour jouer le rôle que jouent dans la théorie des champs classiques les formes différentielles étaient les formes de Kostant ou les formes pseudodifférentielles et intégrales. Malheureusement, les formes de Kostant ne peuvent être intégrées que sur des variétés de base pairs. D'autre part, les formes pseudodifférentielles et intégrales conviennent à l'intégration mais n'admettent pas une version simple et naturelle du calcul de Cartan. Afin de trouver un moyen de contourner cette difficulté fondamentale, Hernández Ruipérez, Muñoz Masqué, Monterde et Vallejo furent obligés dans leurs travaux de traiter les théories définies sur une superbase, en les rapportant à des théories correspondantes (d'ordre supérieur) qui peuvent être comprises comme définies sur une base paire.

Dans mon travail, j'utilise une approche différente. Je crois que les superformes de Voronov et Zorich sont les objets naturels à utiliser pour construire une théorie multisymplectique. En effet ils peuvent être intégrés et ils admettent un calcul de Cartan complet. De plus, j'essaie d'utiliser autant que possible des superformes fractionnaires. De cette manière, toutes les preuves et tous les calculs deviennent plus transparents et directement comparables avec ceux de la théorie des champs classiques.

Dans le ***Chapitre 7*** je vais montrer comment définir des théories de superchamps basées sur un principe d'action, quand le Lagrangien est une superforme berezinienne fractionnaire :

$$\mathcal{L} = L\left(x^A; q^I; \dot{q}^I_A\right) \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$$

J'obtiendrai la même super version des équations d'Euler-Lagrange que celle déjà obtenue dans [113] :

$$(-1)^{|A||I|} \frac{d}{dx^A} \frac{\partial L}{\partial \dot{q}^I_A} \left(j^1\Phi(x)\right) - \frac{\partial L}{\partial q^I} \left(j^1\Phi(x)\right) = 0 \tag{9}$$

Ils sont une généralisation de (2). En particulier nous verrons qu'il n'est pas nécessaire d'utiliser un Lagrangien d'ordre supérieur en composantes pour une théorie qui peut être décrite par un Lagrangien Berezinien de premier ordre.

Le ***Chapitre 8*** est la partie que je juge la plus importante de ma thèse : il contient les idées que je juge les plus originales et les principaux résultats de ce travail. Il consiste en la présentation de l'approche multisymplectique des théories de superchamps réalisée à l'aide de formes fractionnaires.

Dans la section 8.1, je définis l'espace des super-multimoments P comme un sous-fibré de $Hom_\pi(V_\pi E, B^{n-1|m}X)$; où $V_\pi E$ est le fibré tangent vertical du fibré de configurations E, $B^{n-1|m}X$ est un sous-fibré du fibré de $n-1|m$-formes sur la supervariété de base X et $Hom_\pi(V_\pi E, B^{n-1|m}X)$ est un fibré sur X dont la fibre sur un point $x \in X$ est la collection de toutes les fonctions $\mathbb{R}_S$-linéaires entre les supermodules $V_e E$ et $B^{n-1|m}_x X$, pour tout e tel que $\pi(e) = x$.

Sur $Hom_\pi(V_\pi E, B^{n-1|m}X)$ nous pouvons utiliser comme coordonnées locales $\left(x^A, q^I, \overline{p^A_I}, \widetilde{p^A_I}\right)$ et alors la version super de la transformée de Legendre est :

$$\mathbb{FL} : \quad \left(x^A, q^I, \dot{q}^I_A\right) \longmapsto \quad \left(x^A, q^I, \overline{p^A_I}, \widetilde{p^A_I}\right) = \left(x^A, q^I, (-1)^{|A|} \frac{\partial \overline{L}}{\partial \dot{q}^I_A}\left(x^A, q^I, \dot{q}^I_A\right), \frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\left(x^A, q^I, \dot{q}^I_A\right)\right)$$

Dans la section 8.2 je définis sur P le super-Hamiltonien :

$$H(x^A, q^I, p^A_I) := \dot{q}^I_A \widetilde{p^A_I} + (-1)^{|A|} \dot{q}^I_A \overline{p^A_I} - L\left(x^A, q^I, \dot{q}^I_A\right)$$

Puis je présente la version super des équations de Hamilton-Volterra, qui, quand $L = \underline{L}$ est pair prennent la forme :

$$\begin{cases} (-1)^{|I|} \dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \overline{H}}{\partial \overline{p_I^A}}(z(x)) \\ (-1)^{|A|}(-1)^{|A||I|} \dfrac{\partial \overline{p_I^A}}{\partial x^A}(z(x)) = -\dfrac{\partial \overline{H}}{\partial q^I}(z(x)) \end{cases} \tag{10}$$

et quand $L = \widetilde{L}$ est impair sont :

$$\begin{cases} \dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \widetilde{H}}{\partial \widetilde{p_I^A}}(z(x)) \\ (-1)^{|A||I|} \dfrac{\partial \widetilde{p_I^A}}{\partial x^A}(z(x)) = -\dfrac{\partial \widetilde{H}}{\partial q^I}(z(x)) \end{cases} \tag{11}$$

Ils sont une généralisation de : (3).

Dans la section 8.3 j'introduis la superforme de Poincaré-Cartan et la super forme multisymplectique :

$$\begin{aligned} \theta :=& dq^I \wedge p_I^A \beta_A - H\beta \\ \omega :=& - dq^I \wedge dp_I^A \beta_A - dH \wedge \beta \end{aligned}$$

Où $\beta = \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$; et je montre qu'ils sont globalement bien définis.

Ensuite je prouve ce qui suit, qui est une généralisation du théorème 1 :

Theorem 3. *Soit L une fonction lagrangienne régulière ou impaire-régulière sur J^1E et H sa fonction hamiltonienne correspondante sur l'espace des super-multimoments P, alors une section $z \in \Gamma\big(\mathbb{FL}(J^1\pi)\big)$, est une solution de la théorie si et seulement si $\forall U$ carte local de P, munie de la $n+1|m$-forme multisymplectique locale correspondante ω et $\forall u \in \Gamma(TU)$:*

$$z^*(u \lrcorner \widehat{\omega}) = 0$$

où $\widehat{\omega}$ est l'extension pour le premier argument à tout TU de la super-forme multisymplectique ω.

Dans la section 8.4 je construis une structure super-symplectique sur le super-espace des phases covariant (l'espace de solutions de la théorie), d'une manière qui est complètement analogue à celle utilisée pour le cas classique, intégrant la superforme multisymplectique étendue $\widehat{\omega}$ sur une sous-variété $\Sigma \subset P$ qui est une supervariété de codimension $1|0$. On obtient la forme symplectique $\widehat{\Omega_\Sigma}$, qui est une $2|0$-forme étendue sur l'espace des solutions $\mathscr{E} \equiv \mathscr{G}$ et qui, comme dans le cas classique, peut dépendre de la sousvariété Σ choisie.

Dans le ***Chapitre 9*** Je vais montrer comment le théorème de comparaison peut être vu du point de vue du formalisme introduit dans les deux chapitres précédents. L'approche concrète choisie permettra de clarifier les relations existant entre les théories dites en composantes et les théories de superchamps. Tout le traitement sera basé sur le diagramme suivant, qui est expliqué dans ce chapitre :

$$\begin{tikzcd}
 & J^{m+1}\pi \arrow[d, "j^{m+1}_{1|m}\pi"] & & & & & \\
 & J^{1|m}\pi \arrow[dl, "j^{1|m}_{0|m}\pi"'] \arrow[dr, "j^{1|m}_{1}\pi"] & & & & b^*\left(J^{m+1}\pi\right) \arrow[d] & \\
J^{0|m}\pi \arrow[dr, "j^{0|m}_{0}\pi"'] & & J^{1}\pi \arrow[dl, "j^{1}_{0}\pi"] \arrow[rrr, dotted] & & & b^*\left(J^{1|m}\pi\right) \arrow[dl] \arrow[dr] & \\
 & E \arrow[dd, "\pi"] \arrow[rrr, dotted] & & & b^*\left(J^{0|m}\pi\right) \arrow[dr] & & b^*\left(J^{1}\pi\right) \arrow[dl] \\
 & & & & & b^*\left(E\right) \arrow[d] & \\
 & X \arrow[rrrr, "\epsilon"] & & & & \underline{X} \arrow[llll, bend left, "b"] &
\end{tikzcd}$$

Dans la section 9.2, je regarde la Comparaison du point de vue hamiltonien et je ferai une première comparaison de structures symplectiques sur les espaces de solutions de théories exprimées dans le formalisme des superchamps et dans le formalisme en composantes.

Dans le ***Chapitre 10*** j'explique comment le formalisme supermultisymplectique peut être utilisé pour définir des super crochets de Poisson pour les superchamps. En particulier dans la section 10.1, je étudie plus en détail le cas le plus simple de la supermécanique ; je montrerai comment, sur l'espace des solutions d'une théorie de supermécanique, on définit naturellement une structure super-symplectique et je comparerai mes résultats aux résultats déjà publiés obtenus par Khudaverdian, voir [91], et par Monterde et Muñoz Masqué, [115].

Si $\mathscr{G}$ est l'espace des solutions de la théorie et si $f,\ g \in \mathcal{F}(\mathscr{G})$, alors nous définissons les champs vectoriels $u_f,\ u_g \in \Gamma\left(T\mathscr{G}\right)$ avec :

$$\widehat{\Omega_\Sigma}\left(\cdot, u_f\right) = df(\cdot) \tag{12}$$

nous posons

$$\{f, g\} := \widehat{\Omega_\Sigma}\left(u_f, u_g\right)$$

et nous obtenons :

$$\{f,g\} = (-1)^{(|I|+|L|)(|f|+1)}\,\frac{\partial f}{\partial p_I}\frac{\partial g}{\partial q^I} - (-1)^{(|I|+|L|)(|g|+1)}\,(-1)^{(|f|+|L|)(|g|+|L|)}\,\frac{\partial g}{\partial p_I}\frac{\partial f}{\partial q^I}$$

Ce crochet de Poisson est pair ou impair selon la parité du Lagrangien L.

Une autre question est de construire de façon covariante une structure super symplectique sur l'espace $\mathscr{G}$ des solutions d'une théorie de superchamps et généraliser ainsi les travaux de Monterde, Muñoz Masqué et Vallejo, cités ci-dessus, qui concernaient le cas spécial des variétés de base de dimension $1|1$ (qui donne lieu à une théorie de supermécanique). Dans la section 10.2, je montre comment les constructions exposées dans le chapitre 4 pour les théories de champs classiques, peuvent être directement étendues au cas super pour les super théories de champs définies sur une supervariété de base X de toute dimension paire et impaire . L'espace $\mathscr{G}$ acquiert ainsi une structure de super-espace des phases covariant.

Si A et B sont des fonctionnelles définies sur l'espace des phases covariant $\mathscr{G}$, nous avons :

$$\begin{aligned}\{A,B\}(G) = & \int_{\Sigma_X} \left[(-1)^{(|I|+|L|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta B}{\delta q^I}\right|_G (\vec{x}) + \right. \\ & \left. - (-1)^{(|I|+|L|)(|B|+1)} (-1)^{(|A|+|L|)(|B|+|L|)} \left.\frac{\delta B}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta A}{\delta q^I}\right|_G (\vec{x}) \right] d\vec{x} \\ = & \int_{\underline{\Sigma_X}} \left[(-1)^{(|I|+|\Lambda|+|\underline{L}|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta B}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}}) + \right. \\ & \left. - (-1)^{(|I|+|\Lambda|+|\underline{L}|)(|B|+1)} (-1)^{(|A|+|\underline{L}|)(|B|+|\underline{L}|)} \left.\frac{\delta B}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta A}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}}) \right] d\underline{\vec{x}}\end{aligned}$$

où Σ_X est une variété de Cauchy dans X de codimension $1|0$, $(\vec{x})$ sont la restriction des coordonnées sur la surface de Cauchy sur la supervariété $n-1|m$ dimensionnelle Σ_X, $d\vec{x}$ est la $n-1|m$-forme fractionnaire canonique de volume définie par les coordonnées sur la surface de Cauchy sur Σ_X ; $\underline{\Sigma_X}$ est le *body* de Σ_X, $(\underline{\vec{x}})$ sont les coordonnées sur la surface de Cauchy sur $\underline{\Sigma_X}$ et $d\underline{\vec{x}}$ est la $n-1$-forme volume canonique définie par les coordonnées sur la surface de Cauchy sur $\underline{\Sigma_X}$; et où des dérivées fonctionnelles sont utilisées et π_I est le moment canonique associé à la sous-variété Σ. Notez que si Σ est définie par l'équation $x^1 = 0$, alors $\pi_I = p_I^1$. Les moments π_I^Λ seront définis par intégration sur la partie impaire de la supervariété. Notez que $\left|\pi_I^\Lambda\right| = |I| + l(\Lambda) + |L| + m$.

De la super structure de Poisson construite sur $\mathscr{G}$, je dérive les règles de super commutation auxquelles les champs Fermioniques et Bosoniques doivent obéir et je montre que ces règles sont exactement celles attendues d'un point de vue physique. Ceci donne une justification géométrique à l'utilisation de l'anticommutateur pour les champs fermioniques.

Dans le ***Chapitre 11*** j'étudierai les symétries et les supersymétries des théories de superchamps avec les techniques offertes par le formalisme des formes fractionnaires mixtes et du point de vue de l'approche super multisymplectique exposée dans les chapitres précédents.

Quelques auteurs ont déjà donné une version "super" du premier théorème de Noether valide pour la supermécanique : Ibort et Marín-Solano [83] et Cariñena et Figueroa [25].

L. Fatibene et M. Francaviglia, dans [48] et L. Fatibene, M. Ferraris, M. Francaviglia et R. G. McLenaghan, dans [47], ont exploré la manière de donner une interprétation géométrique des supersymétries en utilisant les versions classiques de la forme de Poincaré-Cartan et des champs vectoriels généralisés définis sur une variété bosonique pour des théories de champ dont les espaces des champs sont le produit des puissances extérieures de certains espaces vectoriels, de sorte que les spineurs anticommutants peuvent être inclus dans la théorie.

Comme déjà cité ci-dessus, en 2006, Monterde, Muñoz Masqué et Vallejo, [116], ont obtenu une version du premier théorème de Noether valable pour les super théories génériques, sous une hypothèse technique supplémentaire.

Ici, dans la section 11.2, je montrerai que mon approche permet d'avoir une version super du théorème de Noether qui est rélativement plus naturelle et simple à prouver avec mon formalisme, et plus générale, puisqu'elle n'exige aucune hypothèse technique spécifique du genre utilisé dans [116].

Theorem 4. *Considérons une théorie de champs définie par un fibré des configurations E, avec, comme fibre-type, la supervariété F et avec base la supervariété X, et par une forme lagrangienne $\mathcal{L}$ qui est localement de la forme $\mathcal{L} = L\beta$. Soit $\mathscr{E}$ l'espace des solutions de la théorie des champs.*

Soit χ un champ vectoriel projetable sur E tel que :

$$\forall \Phi \in \mathscr{E} \;\; \exists \alpha_\Phi \in \Omega^{n-1|m} X : \;\forall U \subset X \;\; : \int_U j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = \int_U d\alpha_\Phi$$

alors on a :

$$\begin{aligned} &\forall U \subset X, \;\forall \Phi \in \mathscr{E} \;\; : \\ &\int_U d\left\{ j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)} c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \right\} = 0 \end{aligned}$$

Bien entendu, dans le théorème 4, des formes fractionnaires apparaissent.

Dans la section 11.3, je présenterai une extension super de la fonction multimoment introduite par Gotay, Isenberg, Marsden, Montgomery, Sniatycki et Yasskin dans [63].

Corollaire 5. *Soit G un supergroupe de Lie agissant sur P avec une action covariante relevée, c'est à dire de sorte que à chaque élément $k \in \mathfrak{g}$ (la superalgèbre), correspond une action correspondante sur E générée par le champ vectoriel χ, tel que* $\operatorname{Lie}_{\chi_P} \omega = 0$. *Supposons qu'il existe $J \in Hom\left(\mathfrak{g}, \Omega^{n-1|m}(P)\right)$, tel que pour chaque $k \in \mathfrak{g}$:*

$$\chi_P \lrcorner \omega = d\left[J(k)\right]$$

alors, $\forall \Phi \in \mathscr{E}$:

$$j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = d\left[J(k) + \chi_P \lrcorner \theta\right]$$

Dans ce cas nous disons que la fonction :

$$\begin{aligned} J \;:\; \mathfrak{g} &\longrightarrow \Omega^{n-1|m}(P) \\ k &\longrightarrow J(k) \end{aligned}$$

est la super fonction comoment covariante de l'action.

Le théorème super-Noether et la super fonction multimoment seront présentés avec une formulation qui se révélera très proche de celle correspondante pour les théories classiques.

Enfin, dans le ***Chapitre 12***, je vais présenter quelques exemples montrant comment toute la théorie peut être mise en œuvre pour certains Lagrangiens spécifiques. L'étude du superoscillateur dans la section 12.1 et celle de la superparticule dans un espace courbe dans la section 12.2 montreront comment les outils présentés dans cette thèse peuvent être efficaces pour étudier la supermécanique d'un point de vue qui n'était pas disponible auparavant, exploitant au mieux la potentialité du formalisme des superchamps. Dans la section 12.3 je présenterai le modèle σ 3-dimensionnel.

Je veux conclure cette introduction à ma thèse en commentant une remarque qui a paru dans un article très récent sur la quête d'un cadre géométrique pour la supermécanique Lagrangienne d'un point de vue catégorique, par A. J. Bruce, K. Grabowska et G. Moreno, [18]. Les auteurs écrivent : "*... À notre connaissance, le formalisme de la théorie des champs géométrique - structures multisymplectiques et ainsi de suite - n'a pas été appliqué à la théorie des champs supersymétriques. En partie, nous pensons que cela est dû au manque d'appréciation des méthodes catégorielles appliquées à la supergéométrie au sein de la communauté de la mécanique géométrique...* ". Je ne suis pas d'accord avec cette remarque. Certains auteurs, par exemple D. S. Freed dans certaines passages de [54], écrivent que l'utilisation de la notion de *foncteur de points* est nécessaire pour une présentation mathématique efficace d'une théorie des superchamps. Encore une fois je suis en désaccord, même si toutes ces notions mènent certainement à des points de vue intéressants. Dans toute cette thèse, j'utilise l'approche concrète à la supergéométrie, au sens de DeWitt-Rogers, travaillant toujours avec des supervariétés concrètes définies à partir de l'algèbre de Grassmann infiniment générée $\mathbb{R}_S$. Je ne fais pas usage de la notion de foncteur de points. Il est vrai qu'un point de vue catégorique des théories de superchamps peut aide à comprendre le fait qu'il n'y a probablement pas de distinction physique possible entre les théories définies à partir des différentes algèbres de Grassmann. Mais je ne pense pas que le point clé pour développer une théorie géométrique, multisymplectique, des superchamps significative soit l'adoption d'un point de vue catégorique, ni d'un point de vue algébro-géométrique. Comme je l'ai écrit plus haut, la difficulté cruciale à surmonter est de trouver des objets qui peuvent être intégrés et qui permettent à la fois un calcul de Cartan. Cette tâche peut être entreprise aussi bien avec une approche concrète et c'est exactement ce que je veux présenter ici. De plus, à mon avis, le choix de l'approche concrète a l'avantage d'aider à une compréhension plus intuitive du cadre géométrique, en évitant certains niveaux d'abstraction qui ne sont pas strictement nécessaires.

Introduction

The Calculus of Variations and its geometric interpretation always played a key role in Mathematical Physics, either through the Lagrangian formalism, or through the Hamiltonian equations, which are deeply linked with the Lagrangian. This role has been confirmed and reinforced during the last century, for example by the Noether theorem, which connects symmetries and conserved quantities, and by the use of the Calculus of Variations made in Quantum Mechanics and in Quantum Field Theory.

For fields theories, which correspond to variational problems with several space-time variables, the Lagrangian approach is now well-founded and widely used. However the Hamiltonian approach is not so well developed and used. In order to achieve a Hamiltonian description of fields theory, which is the road to the so-called canonical quantization, most physicists use a splitting of the space-time as a Cartesian product of a space manifold by a time line. This is of course sufficient for most purpose, but it clearly breaks the relativistic covariance and hence contributes to obscure the relationship between the quantum theory and Relativity.

Several theories have been developed to care this situation: they all rest more or less on the fact that the space of all solutions of a variational problem is endowed with a symplectic structure, an observation which somehow goes back to Lagrange. This is the fundamental message carried by the so-called *covariant phase space* theory. The problem is to find the most suitable technical tools to represent this symplectic structure and to do computations with it. Some approaches exist, for example: the one developed by Deligne and Freed in [36] and based on Takens' theory, or Vinogradov's theory, as developed for example by Vitagliano in [147], or the multisymplectic approach.

The multisymplectic formalism allows a finite dimensional geometric description of classical field theories seen from an Hamiltonian point of view. Multisymplectic geometry plays the same role played by symplectic geometry in the description of classical Hamiltonian mechanics. Moreover the multisymplectic approach provides a tool for building a symplectic structure on the space of solutions of the field theory and for investigating it.

The generalization of Hamilton equations to a first order variational problem with several variables dates back to two papers of V. Volterra of 1890, [150, 151], in which two different variants were proposed. Today the first theory proposed by Volterra is mostly known as De Donder-Weyl theory because one version was expounded by H. Weyl in 1934, [160], and another one by T. De Donder in 1935, in [32], where he extended it to higher order theories and related it to H. Poincaré, [123], and E. Cartan, [27], theory of "Invariants Intégraux", giving it a strong geometrical flavor. Fundamental insights were obtained subsequently by T. Lepage [104] in 1936 and P. Dedecker in 1953 [33], see also [34].

In a series of papers during the 70's, J. Kijowski, [94, 95], with W. Szczyrba [97, 98] and then with W. M. Tulczyjew [99] gave birth to the multisymplectic point of view on all these theories. The multisymplectic formalism allows a complete geometrization and provides a full covariant point of view on Hamiltonian field theories. Kijowski ideas allows to build a symplectic structure on the space of all solutions of a field theory (which is then called the *covariant phase space*) in a way which is to some extent independent from any choice of splitting of spacetime in space and time and which is therefore fully covariant.

Similar ideas on the symplectic structure of the covariant phase space were present, during the 50's, in the pioneering works of R. E. Peierls, see [122], and I. Segal, see [144], and appeared again in 80's in papers by E. Witten [161], C. Crnkovic and Witten [28] and G. Zuckerman [164] who probably did not know about the works of Kijowski and the Polish school.

Witten papers gave birth to a new interest on the then called covariant Hamiltonian theories and on the covariant phase space approach, both in the physical and in the mathematical communities. From the second half of 80's on, the multisymplectic formalism for field theories was studied and revisited by several authors and presented in many different variants. Applications to continuum Mechanics and hydrodynamics have been proposed. Covariant numerical methods for partial differential equations, developed in or inspired by the multisymplectic framework, have been introduced. The interest in multisymplectic field theories gave birth also to a number of studies on the so called multisymplectic or n-plectic geometry. In the introduction to the first part of this thesis I will give a short list of some of the most important works on these subjects.

The paper of M. J. Gotay, J. Isenberg, J. E. Marsden, R. Montgomery, J. Śniatycki and P. B. Yasskin, [63, 64], can be read for a presentation of the multisymplectic formalism for field theories. For a shorter introduction one can read Román-Roy [135]. For an introduction with a report on the history on how the ideas around the multisymplectic formalism originated and evolved, one can read Hélein [70].

It is worth noting that, until today, almost any attempt to adapt the multisymplectic formalism to theories of super-fields (like for example supersymmetric field theories) hasn't been made. Also the works which could be considered preliminary to this task, like the papers on the geometrical formulation of supermechanics or those on the super Poincaré-Cartan form for super-field theories (which I'll quote in the following), have been very few.

In this thesis I will be mainly concerned with the multisymplectic formalism to build first order field theories and I hope to give two main original contributions:

- I will show that, in some situations, the symplectic structure on the *covariant phase space* may indeed depend from the choice of splitting of spacetime in space and time;
- I will extend the multisymplectic formalism to superfield theories.

As a "byproduct", I will present another original contribution:

- I will define fractional forms on supermanifolds with their relative Cartan Calculus.

These fractional forms will be necessary to build the multisymplectic formalism for superfield theories.

I will use a version of the multisymplectic formalism which minimizes the dimension of the spaces involved, which seems suitable to me for a first attempt of extending to super-field theories. I call it the minimal setting for the multisymplectic formalism. The main ingredient of this setting will be the finite dimensional multimomenta space P, which corresponds to what Forger and Romero call the ordinary multiphase space, [53] and what Román-Roy, [135], calls the restricted multimomentum bundle.

This thesis is organized in three parts. The main original results of the ***First Part*** are contained in Chapter 3 and consist in the study of non equivalent symplectic structures on the covariant phase space.

In ***Chapter 1*** I will introduce a standard geometrical framework for the Lagrangian approach to first order field theories.

If E, X and F are finite dimensional C^∞ manifolds, we let (E,π,X,F) be a differential fiber bundle with total space E, base X, type-fiber F and bundle C^∞ projection π, so that we have the following situation:

$$\begin{array}{c} E \\ {\scriptstyle\pi}\big\downarrow \\ X \end{array}$$

We call E the fields bundle or the configurations bundle and a field ϕ over X is one of its C^∞ sections: so we can call $\Gamma\big(E\big)$ the space of fields.

If X is n-dimensional, the Lagrangian $\mathcal{L}$ is a π-horizontal differential n-form defined on the first jet space $J^1\pi \equiv J^1E$. The action A_U over a field ϕ is the integral of $\mathcal{L}$ on the n-dimensional surface $j^1\phi(U)$, where $U \subset X$ is a n-dimensional submanifold (possibly with boundary) of X:

$$A_U(\phi) := A_U(j^1\phi) = \int_{j^1\phi(U)} \mathcal{L} = \int_U j^1\phi^*\mathcal{L} \tag{13}$$

The space of solutions of the theory $\mathscr{E}$ is determined by the principle of critical action.

If on J^1E we use the local coordinates $\left(x^a, q^i, \dot{q}^i_a\right)$, then locally $\mathcal{L} = L\beta$, where $\beta := dx^1 \wedge \cdots \wedge dx^n$. Every solution $\phi \in \mathscr{E}$ satisfies the Euler-Lagrange equations:

$$\phi \text{ is a solution } \iff \text{ on every } U,\ \forall x \in U,\ \frac{d}{dx^a}\frac{\partial L}{\partial \dot{q}^i_a}\left(j^1\phi(x)\right) - \frac{\partial L}{\partial q^i}\left(j^1\phi(x)\right) = 0 \tag{14}$$

In ***Chapter 2*** I will introduce what I call the minimal setting for the multisymplectic description of field theories. I define the multimomenta space P, with local coordinates $\left(x^a, q^i, p^a_i\right)$ and the Legendre transform between J^1E and P:

$$\mathbb{FL}: \quad \left(x^a, q^i, \dot{q}^i_a\right) \longmapsto \quad \left(x^a, q^i, p^a_i\right) = \left(x^a, q^i, \frac{\partial L}{\partial \dot{q}^i_a}\left(x^a, q^i, \dot{q}^i_a\right)\right)$$

The Hamiltonian H is then defined by $H(x^a, q^i, p^a_i) := \dot{q}^i_a p^a_i - L\left(x^a, q^i, \dot{q}^i_a\right)$ and a field ϕ is a solution of the theory if and only if $z := \mathbb{FL} \circ j^1\phi$ satisfies the generalized covariant Hamilton-Volterra system:

$$\begin{cases} \dfrac{\partial q^i}{\partial x^a}\left(\mathbb{FL}j^1\phi(x)\right) = \dfrac{\partial H}{\partial p^a_i}\left(\mathbb{FL}j^1\phi(x)\right) \\ \dfrac{\partial p^a_i}{\partial x^a}\left(\mathbb{FL}j^1\phi(x)\right) = -\dfrac{\partial H}{\partial q^i}\left(\mathbb{FL}j^1\phi(x)\right) \end{cases} \tag{15}$$

On P it is possible to define the *multisymplectic form*, which is the global $n+1$-form:

$$\omega := -dq^i \wedge dp^a_i \wedge \beta_a - dH \wedge \beta$$

where $\beta_a = \frac{\partial}{\partial x^a} \lrcorner \beta$.

Kijowski proved in [94] the following:

Theorem 6. *A section $z \in \Gamma(P)$ is a solution of the theory if and only if $\forall u \in TP$, $z^*(u \lrcorner \omega) = 0$.*

One of the main interests of the finite dimensional geometric construction of the multimomenta space with its multisymplectic (n-plectic in the most up to date terminology) structure is that it provides a way to build a symplectic structure on the covariant phase space (the space of solutions of the field theory). There is then a direct link between the multisymplectic field theory and the classical canonical formulation of the field theory. This link connects the works on multisymplectic geometry to the works of Physics and Mathematical Physics communities on canonical field theory, pioneered by the papers of Peierls, [122], and Segal, [144] (already cited above), continued and developed by B. DeWitt [40, 41, 42], by García and Pérez-Rendón in [56, 57, 58] and by Goldschmidt and Sternberg in [62].

Here I will follow [70] to show how with the help of the multisymplectic form ω, it is possible to build a symplectic form Ω on the space of Hamiltonian surfaces $\mathscr{G} = z(X)$ with z solution of the theory.

Let's call $\mathscr{G}$ the space of Hamiltonian surfaces; we have that $\mathscr{G} \cong \mathscr{E}$ so we can identify these two spaces. Let be $G \in \mathscr{G}$ and $\delta_u G \in T_G\mathscr{G}$ be a vector over G: it is associated to a so called *Jacobi field* $u \in \Gamma(i^*(VP))$, id est a section over G of the pull-back image of the vertical (with respect to the projection π_P of the total space P onto the base X) tangent bundle VP by the embedding map $i : G \longrightarrow P$. The section u can be seen as a vector field on G, "following" which each point $g \in G$ is sent to a point $g' \in G'$, where $G' \in \mathscr{G}$ is another Hamiltonian n-curve. An Hamiltonian n-curve G is deformed by u into another Hamiltonian n-curve G'.

Let Σ be a slice of co-dimension 1 in P, with the property that for any Hamiltonian n-curve $G \in \mathscr{G}$ the intersection of Σ with G is transverse. Then we can define:

$$\Omega_\Sigma\big|_G(\delta_1 G, \delta_2 G) := \int_{\Sigma \cap G} u_1 \wedge u_2 \lrcorner\, \omega \tag{16}$$

and, under certain regularity conditions of the Lagrangian, Ω_Σ is a symplectic 2-form on $\mathscr{G}$.

A natural question then arises: does the symplectic form Ω depend on the choice of the slice Σ? Kijowski already proved that if Σ and Σ' are two compact slices in the same homology class, then $\Omega_\Sigma = \Omega_{\Sigma'}$.

In ***Chapter 3*** I show, with some examples of field theories built on a 2-dimensional torus, that when Σ and Σ' are in different homology classes, then it may happen that $\Omega_\Sigma \neq \Omega_{\Sigma'}$. It seems that this result has not been noticed yet.

Moreover I study some examples where Σ and Σ' are non compact and I show that in that case the situation is definitely more delicate. The standard homology of submanifolds of P is not anymore suitable to determine the symplectic structure on $\mathscr{G}$. I study the free and the massive scalar field on $\mathbb{R}^2$ and I explain what happen when one chooses Σ and Σ' such that they stay on two different sides of the light-cone. I will prove the original result that also in that case $\Omega_\Sigma \neq \Omega_{\Sigma'}$.

In ***Chapter 4***, which is the last one of the First Part, I will show how the symplectic structure on the space of solutions is linked with the field brackets used by physicists. Following J. Kijowski and W. Szczyrba [97, 98], I will show that the symplectic form Ω can be used to obtain a Poisson structure on $\mathscr{E} \equiv \mathscr{G}$; this will fix the bases for the extension of the same construction to super-field theories.

The multisymplectic formalism has not yet been applied to the description of supersymmetric field theories. The main difficulty one has to face, to adapt it to super fields, is in my opinion the fact that the differential forms can be naturally integrated in classical geometry, whereas in supergeometry integral forms and differential forms are usually defined as two different kind of objects. To take fully advantage of the multisymplectic formalism, one would like instead to use the same objects to integrate (as in (13) and (16)) and to perform Cartan calculus (as for example in theorem 6). To this purpose I believe that the most suitable objects to use are the superforms defined by Th. Voronov and A. Zorich in their papers during late '80-s, [155, 156, 157, 158]. More precisely I will use a subclass of Voronov-Zorich superforms, which I call fractional forms.

In the ***Second Part*** of my thesis I'll introduce the notions of fractional forms, fractional coforms and fractional mixed forms on supermanifolds and I will propose a new notation, suitable for computations. The fractional forms will be an essential ingredient for the definition of a superfield theory and for the supermultisymplectic formalism which is the main object of the third part of this thesis.

In ***Chapter 5***, after having very briefly introduced the Rogers-DeWitt concrete approach to supermanifolds, I will define fractional forms and I will set the rules to perform with them a Cartan calculus.

Fractional forms are good examples of the $r|s$-forms introduced by Th. Voronov and A. Zorich as the natural analogous on supermanifold of classical forms on classical manifolds. Denoting by T_0X and T_1X respectively the even and the odd tangent space of a supermanifold X, we have that:

Definition 7 (Voronov and Zorich)**.** *A form of degree $r|s$ over a point $x \in X$, supermanifold of dimension $n|m$, is a G^∞ map $\omega : \; O \subset \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s} \longrightarrow \mathbb{R}_S$, which satisfies the following: $\forall v \in O$, open subset of $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$:*

$$\forall g \in GL(r|s), \; \omega(g \cdot v) = \omega(v)\,\mathrm{Ber}_{r,s}(g) \tag{17}$$

$$\frac{\partial^2 \omega}{\partial v_G^B \partial v_F^A} + (-1)^{|G||F|+(|G|+|F|)|A|} \frac{\partial^2 \omega}{\partial v_F^B \partial v_G^A} = 0 \tag{18}$$

where $A, B = 1, \cdots, n+m$ are the indices in the space T_xX and so also in both spaces $T_{x,0}X$ and $T_{x,1}X$ with their usual degree; v_F^A is the A-th coordinates of v_F in the local base $(\partial_A|_x)_A$; F runs from 1 to $r+s$ and we have $v_F \in T_{x,|F|}X$, where we set $|F| = 0$ when $F = 1, \cdots, r$ and $|F| = 1$ when $F = r+1, \cdots, r+s$.

In section 5.2.2 I will present an original extension of $r|s$-forms, such that the domain of the first r arguments become all the tangent space TX, instead of its even part.

Then, in sections 5.3 and 5.4, I will give a direct proof that the maps θ and $\mu \wedge \theta$, defined as follows:

$$\forall x \in U, \; \forall v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} :$$

$$\theta(v) := \mathrm{sdet}_{r,s} \begin{pmatrix} \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_r}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$$

and:

$$\forall x \in U,\ \forall w := (\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s};$$

$$\forall \mu \in T_x^*U,\ \text{if we decompose } \mu = dx^A \mu_A :$$

$$\mu \wedge \theta(w) := \mathrm{sdet}_{r+1,s} \begin{pmatrix} \overline{v_1^A}\mu_A & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^A}\mu_A & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \overline{v_{r+1}^A}\mu_A & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\mu_s}} \\ \widetilde{v_1^A}\mu_A & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^A}\mu_A & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$$

are indeed Voronov-Zorich superforms.

These examples of Voronov-Zorich superforms will be important for us. It is desirable to have available a more compact and intuitive notation for defining them. I propose in this thesis the following notations:

$$\frac{dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}} := \theta$$

$$\mu \wedge \frac{dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}} := \mu \wedge \theta = \frac{\mu \wedge dx^{A_1} \wedge \cdots \wedge dx^{A_r}}{dx^{\alpha_1} \odot \cdots \odot dx^{\alpha_s}}$$

I will also define objects like:

$$\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$$

Where $\Theta^1 \dots \Theta^r$ are $1|0$-forms of generic parity and $\theta^1 \dots \theta^s$ are odd $1|0$-forms.

These kind of objects, which I call fractional forms, were already used in the Mathemathical and Physical literature, with a different notation, but, to my knowledge, a direct proof that they are indeed Voronov-Zorich superforms, had not been published yet.

In section 5.4 I will explicit the rules to perform the Cartan calculus with fractional forms, including interior and exterior products by vectors and covectors of any parity, exterior derivation and combinations of these operations with the corresponding commutators. I will also give some original useful formula for effective computations using superdeterminants. These will be used in the third part of the thesis; in fact Cartan calculus on fractional forms turns out to be more manageable than the Cartan calculus for generic $r|s$-forms.

In section 5.5 I will present the theory of integration of superforms (and hence fractional forms), developed by Voronov and Zorich. I will also propose a little modification to the standard definition of Berezin integral, based on the concept of *immersed body* of a supermanifold, which I will define. This will be useful for the matter treated in Chapter 9.

In ***Chapter 6*** I will define fractional coforms and fractional mixed forms of the first and of the second type. Fractional coforms are examples of what Voronov calls *twisted covariant dual Lagrangians satisfying the fundamental equations* or shortly *twisted dual forms* in [153] and [154], and they are the basis for the definition of his stable forms. Fractional mixed forms are examples of what Voronov called mixed forms, [153], [154].

An example of a fractional coform will be the coform w locally written as:

$$w = \frac{\partial_{A_{t+1}} \wedge \partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}}$$

or the coform u locally written as:

$$u = \frac{v_1 \wedge \cdots \wedge v_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}}$$

where $v_1, \ldots, v_t$ are vector fields of any parity and $\widetilde{v_1} \ldots \widetilde{v_q}$ are odd vector fields.

An example of a fractional mixed form will be:

$$\frac{v_1 \wedge \cdots \wedge v_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}} \lrcorner \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{19}$$

I will show how to perform a Cartan Calculus with fractional coforms and fractional mixed forms, and this will allow me to give sense to formula like (19) or like:

$$\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \tag{20}$$

also when, in (20), $l > t$ and $d < q$; or when, in (19), $t > r$ and $q < s$.

For example I can give sense to:

$$d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

and by symmetry:

$$\frac{1}{\frac{\partial}{\partial \xi^1}} \lrcorner d\xi^1 := d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

or:

$$\frac{\partial}{\partial \xi^1} \lrcorner \frac{1}{d\xi^1}$$

where ξ_1 is an odd local coordinate on a supermanifold.

In section 6.3 I show how to integrate coforms and mixed forms on supermanifolds.

Chapter 6 is indeed independent of the rest of this thesis and, unlike Chapter 5, it is not necessary to read it in order to understand the third part of the thesis. Therefore the material presented there is not treated in detail. It can be considered as a natural complement of Chapter 5 and as a preliminary work for future studies, especially in the direction of Batalin-Vilkovisky and Bechi-Rouet-Stora-Tyutin superfield theories.

The main original contributions in the ***Third Part*** of my thesis are: the construction of a multisymplectic formalism for super-field theories; the formulation of the Comparison Theorem with the tools offered by the fractional forms and the integral over an immersed body; the formulation of the geometrical treatment of symmetries and supersymmetries of superfield theories using the language of fractional forms and exploiting the super Poincaré-Cartan fractional form.

The theories of superfields began to be studied extensively at the end of 70's with the developement of supersymmetry in Physics.

A supersymmetric field theory can be usually presented in two different ways: as a field theory on a classical manifold, with fermionic and bosonic components of the field, or as a theory defined on a supermanifold. The first approach is sometime referred to as the components approach, whereas the second one is sometime referred to as the proper superfield approach. In both cases the theory is defined by the equations which the fields have to satisfy.

When one uses the components approach, the field equations can be derived by a variational principle with an action defined as the integral on the classical base manifold of a Lagrangian density. The action obviously involves both the bosonic (commuting) and the fermionic (anticommuting) components of the fields, treating them accordingly to the respective parities.

To express the action principle in a geometric language requires the formulation of a variational calculus for Lagrangian densities defined in terms of differential forms. Since the Lagrangian density typically depends on the derivative of the components of the fields (which can be bosonic or fermionic), even if the theory is defined on a classical (bosonic) manifold, it is clear that it is necessary to develop a differential forms calculus valid also for the fermionic sector. This is not simple at all and it has been done by D. Hernández Ruipérez and J. Muñoz Masqué during 80's, exactly for the case when the base manifold is classical. In [76, 77, 78, 118, 119] they have indeed developed a graded variational calculus for Lagrangian densities defined in terms of graded Kostant differential forms and they have obtained the corresponding Lagrangian formalism (Euler-Lagrange equations, Poincaré-Cartan forms, Noether invariants, etc.).

When the base manifold is not classical and it is a supermanifold, like in the superfield approach to supersymmetric theories, then the task is even more difficult. The theory can still be derived by a variational principle, but the action in this case is defined as the Berezinian integral (performed with the help of a Berezinian volume density) of a Lagrangian density, which must be therefore a Berezinian volume density.

In 1987 in [80] Hernández Ruipérez and Muñoz Masqué recognize: "the lack of an intrinsic definition of a suitable notion of intermediate Berezinian densities with its Cartan exterior calculus, prevents us from developing a Lagrangian formalism...", meaning a full Lagrangian formalism valid also for the case when the base manifold is a supermanifold. Nonetheless, in [79] and [80] they arrived to an intrinsic formulation of the notion of Berezinian Lagrangian density and Berezinian critical sections. Moreover, in the case where a theory can be expressed both with the components and with the superfield approach, they showed the way to relate the critical sections of the Berezinian Lagrangian density, defined on the supermanifold, to the critical sections of the corresponding graded-Lagrangian, defined on a bosonic manifold with graded differential forms. They did this via a first version of what will be then called the Comparison Theorem.

In 1992 J. Monterde, in [113], showed that the Berezinian critical sections of an action defined with a Berezinian Lagrangian density must satisfy a super version of the Euler-Lagrange equations. When he wrote his paper, he had at his disposal the formalism of superforms defined by Voronov and Zorich during 80's. Nevertheless, perhaps because of the fact that the system of notations which was needed for this purpose is (to my opinion) a bit heavy, unfortunately it seems that his results have not been exploited much nor by mathematicians neither by physicists.

After that, during the 90's and during the first decade of the new millennium, some other steps had been made by some authors in building a geometric approach to superfield theories from a Lagrangian and from an Hamiltonian point of view. Important insights were in the papers on supermechanics by Monterde and Muñoz Masqué, [114, 115]; by L. A. Ibort and J. Marín-Solano [83]; by J. F. Cariñena and H. Figueroa [26]; by Monterde and J. A. Vallejo, [117].

In 2006 Monterde, Muñoz Masqué and Vallejo published a paper, [116], in which they proposed a Hamilton-Cartan formalism for first-order Berezinian variational problems valid for fields defined on supermanifolds of any dimension. They achieved their purpose by studying, with the help of the Comparison Theorem, an associated higher-order graded variational problem, defined on a bosonic base manifold. They obtained a super Poincaré-Cartan form valid for theories on bases of any dimension. However they chose a notation which I judge not very much adapted to general proofs, neither to actual calculations. They also obtained a very beautiful and important result, which is the generalization of first Noether theorem to super field theories (theorem 8.2 in [116]); but they obtained it using a rather technical assumption needed in the hypothesis.

To my knowledge neither Monterde, Muñoz Masqué or Vallejo, nor any other mathematician, tried to use the results on super Poincaré-Cartan forms to describe the general superfield theories with the multisymplectic approach.

Independently from the results obtained by the Spanish school, there has been, to my knowledge, only one attempt to extend the multisymplectic formalism to superfields. S. P. Hrabak in [81, 82] initiated a study of the formulation of the classical BRST symmetry within the framework of a multisymplectic theory. To do so, he needed to extend the multisymplectic formalism so that it works also for field theories whose base is a classical bosonic manifold, but whose space of fields is a supermanifold, with bosonic and fermionic (because of the presence of the ghosts) sectors. He accomplished his task in [82]. He didn't however show how to eventually extend the formalism also to the case when the base itself is a supermanifold.

Here in the third part of my thesis I will present a full multisymplectic version of the superfield theories valid for any dimension (even and odd) of the base space and of the space of fields. My results are a generalization of those obtained by Hrabak in [82] and they are based on a full exploitation of the potential of the theory of superforms of Voronov and Zorich. They can also be considered a natural generalization of the results obtained in the geometrical presentations of finite dimensional supermechanics, which include the use of a super symplectic form, like the presentations in [83] and [26].

If one wants to build a multisymplectic superfield theory, he has to use objects (for example the multisymplectic form) which can be integrated on a supermanifold and which in the same time can be used for a Cartan calculus, including contraction by supervectors, external product by one forms and external derivation. This is the difficult point. In fact, before the articles of Voronov of the 90's, [153, 154], no such object did exist. Before the appearance of superforms of Voronov and Zorich, the best candidates to play the role which in classical field theory is played by differential forms were Kostant forms or pseudodifferential and integral forms. Unfortunately Kostant forms can be integrated only on even base manifolds. On the other hand, pseudodifferential and integral forms are suitable for integration but do not admit a simple and natural version of Cartan calculus. In order to to find a way to bypass this fundamental difficulty, Hernández Ruipérez, Muñoz Masqué, Monterde and Vallejo were obliged, in their works, to treat theories defined on a superbase, relating them to corresponding theories (of higher order) which can be understood as defined on an even base.

In my work, I use a different approach. I believe that Voronov Zorich superforms are the natural objects to use to build a multisymplectic theory. Indeed they can be integrated and they admit a full Cartan calculus. Moreover, I try and use, as far as possible, only fractional superforms. In this way all the proofs and calculations become more transparent and directly comparable to the corresponding ones of classical field theory.

In ***Chapter 7*** I will show how to found superfield theories based on an action principle, when the Lagrangian is a Berezinian fractional superforms:

$$\mathcal{L} = L\left(x^A; q^I; \dot{q}^I_A\right) \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$$

I will obtain the same super version of Euler-Lagrange equations already obtained in [113]:

$$(-1)^{|A||I|} \frac{d}{dx^A} \frac{\partial L}{\partial \dot{q}^I_A} \left(j^1\Phi(x)\right) - \frac{\partial L}{\partial q^I} \left(j^1\Phi(x)\right) = 0 \tag{21}$$

They are a generalization of (14) and I will obtain them using a notation lighter than the one used in [113] and a formalism which I judge more natural and which allows simpler and shorter proofs. In particular I will show that there is no need to use an higher order Lagrangian in components for a theory which can be described by a first order Berezinian Lagrangian.

Chapter 8 is the most important part of my thesis: it contains the ideas that I judge the most original and the main results of this work. It consists in the presentation of the multisymplectic approach to superfield theories made with the help of fractional forms.

In section 8.1 I define the super-multimomenta space P as a subbundle of $Hom_\pi(V_\pi E, B^{n-1|m}X)$; where $V_\pi E$ is the vertical tangent bundle of the configurations bundle E, $B^{n-1|m}X$ is a subbundle of the bundle of $n-1|m$-forms over the base supermanifold X and $Hom_\pi(V_\pi E, B^{n-1|m}X)$ is a fiber bundle over X whose fiber over a point $x \in X$ is the collection of all $\mathbb{R}_S$-linear maps between the supermodules $V_e E$ and $B_x^{n-1|m}X$, for all e such that $\pi(e) = x$.

On $Hom_\pi(V_\pi E, B^{n-1|m}X)$ we can use as local coordinates $\left(x^A, q^I, \overline{p_I^A}, \widetilde{p_I^A}\right)$ and then the super version of the Legendre transform is:

$$\mathbb{FL}: \quad \left(x^A, q^I, \dot{q}_A^I\right) \longmapsto \left(x^A, q^I, \overline{p_I^A}, \widetilde{p_I^A}\right) = \left(x^A, q^I, (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right), \frac{\partial \widetilde{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right)\right)$$

In section 8.2 I define on P the super Hamiltonian:

$$H(x^A, q^I, p_I^A) := \dot{q}_A^I \widetilde{p_I^A} + (-1)^{|A|}\dot{q}_A^I \overline{p_I^A} - L\left(x^A, q^I, \dot{q}_A^I\right)$$

Then I present the super version of the Hamilton-Volterra equations, which, when $L = \underline{L}$ is even are:

$$\begin{cases} (-1)^{|I|}\dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \overline{H}}{\partial \overline{p_I^A}}(z(x)) \\ (-1)^{|A|}(-1)^{|A||I|}\dfrac{\partial \overline{p_I^A}}{\partial x^A}(z(x)) = -\dfrac{\partial \overline{H}}{\partial q^I}(z(x)) \end{cases} \tag{22}$$

and when $L = \widetilde{L}$ is odd, are:

$$\begin{cases} \dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \widetilde{H}}{\partial \widetilde{p_I^A}}(z(x)) \\ (-1)^{|A||I|}\dfrac{\partial \widetilde{p_I^A}}{\partial x^A}(z(x)) = -\dfrac{\partial \widetilde{H}}{\partial q^I}(z(x)) \end{cases} \tag{23}$$

They are a generalization of (15).

In section 8.3 I introduce the super Poincaré-Cartan form and the super multisymplectic form:

$$\begin{aligned} \theta :=& dq^I \wedge p_I^A \beta_A - H\beta \\ \omega :=& -dq^I \wedge dp_I^A \beta_A - dH \wedge \beta \end{aligned}$$

where $\beta = \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$; and I show that they are globally well defined.

Then I prove the following:

Theorem 8. *Let L be an even-regular or an odd-regular Lagrangian function on J^1E and H be its corresponding Hamiltonian function on the super-multimomenta-space P, then a section $z \in \Gamma\big(\mathbb{FL}(J^1\pi)\big)$, is a solution of the theory if and only if $\forall U$ local chart of P, with corresponding local super-multisymplectic $n+1|m$-form ω, and $\forall u \in \Gamma(TU)$:*

$$z^*(u \lrcorner \widehat{\omega}) = 0$$

where $\widehat{\omega}$ is the extension in the first argument to all TU of the super multisymplectic form ω.

It is a generalization of theorem 6 and and I consider this result as the most important theorem proved in this thesis.

In section 8.4 I build a super symplectic structure on the super covariant phase space (the space of solutions of the theory), in a way that is completely analogous to the one used for the classical case, integrating the extended super multisymplectic form $\widehat{\omega}$ over a slice $\Sigma \subset P$ which is a supermanifold of codimension $1|0$. I obtain the symplectic form $\widehat{\Omega_\Sigma}$, which is an extended $2|0$-form on the space of solutions $\mathscr{E} \equiv \mathscr{G}$ and which, as in the classical case, may depend on the chosen slice Σ.

In ***Chapter 9*** I will show how the Comparison Theorem can be seen from the perspective of the formalism introduced in the two previous chapters. The chosen concrete approach hopefully will clarify the relations existing between the so called components theories and the so called superfield theories. All the treatment will be based on the following diagram, which is explained in this chapter:

$$
\begin{tikzcd}
 & J^{m+1}\pi \arrow[d, "j^{m+1}_{1|m}\pi"] & & & & \\
 & J^{1|m}\pi \arrow[dl, "j^{1|m}_{0|m}\pi"'] \arrow[dr, "j^{1|m}_{1}\pi"] & & & b^*\left(J^{m+1}\pi\right) \arrow[d] & \\
J^{0|m}\pi \arrow[dr, "j^{0|m}_{0}\pi"'] & & J^{1}\pi \arrow[dl, "j^{1}_{0}\pi"] \arrow[rr, dotted] & & b^*\left(J^{1|m}\pi\right) \arrow[dl] \arrow[dr] & \\
 & E \arrow[rr, dotted] \arrow[dd, "\pi"] & & b^*\left(J^{0|m}\pi\right) \arrow[dr] & & b^*\left(J^{1}\pi\right) \arrow[dl] \\
 & & & & b^*\left(E\right) & \\
 & X \arrow[rrr, "\epsilon"] & & & \underline{X} \arrow[u] \arrow[lll, bend left, "b"] &
\end{tikzcd}
$$

In section 9.2, I will look at the comparison from the Hamiltonian point of view and I will make a first comparison of symplectic structures on the spaces of solutions of theories expressed in the superfield and in the components formalisms. These results are, up to my knowledge, original.

In ***Chapter 10*** I will explain how the supermultisymplectic formalism can be used to define super Poisson brackets for super fields. In particular in section 10.1 I will study in more detail the simplest case of supermechanics; I will show how on the space of solutions of a supermechanic theory is naturally defined a super symplectic structure and I will relate my results to the already published results obtained by Khudaverdian, see [91], and by Monterde and Muñoz Masqué, [115].

If $\mathscr{G}$ is the space of solutions of the theory and if $f,\ g \in \mathcal{F}(\mathscr{G})$, then we define the vector fields $u_f,\ u_g \in \Gamma\left(T\mathscr{G}\right)$ with:

$$\widehat{\Omega_\Sigma}\left(\cdot, u_f\right) = df(\cdot) \tag{24}$$

and we pose:

$$\{f, g\} := \widehat{\Omega_\Sigma}\left(u_f, u_g\right)$$

obtaining:

$$\{f,g\} = (-1)^{(|I|+|L|)(|f|+1)} \frac{\partial f}{\partial p_I}\frac{\partial g}{\partial q^I} - (-1)^{(|I|+|L|)(|g|+1)} (-1)^{(|f|+|L|)(|g|+|L|)} \frac{\partial g}{\partial p_I}\frac{\partial f}{\partial q^I}$$

This Poisson bracket is even or odd depending on the parity of the Lagrangian L.

To my knowledge, nobody has tried yet to build covariantly a super symplectic structure on the space $\mathscr{G}$ of solutions of a superfield theory; excepted Monterde, Muñoz Masqué and Vallejo, in [115] and [117] quoted above, who did it for the special case of base manifold of dimension $1|1$, which give rise to a supermechanics theory. In section 10.2 I show how the constructions expounded in chapter 4 for classical field theories, can be directly extended to the super case for super field theories defined on base supermanifold X of any even and odd dimension. The space $\mathscr{G}$ becomes then a truly super covariant phase space.

If A and B are functionals defined on the covariant phase space $\mathscr{G}$, we have:

$$\begin{aligned}
\{A,B\}(G) = \int_{\Sigma_X} \Bigg[& (-1)^{(|I|+|L|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta B}{\delta q^I}\right|_G (\vec{x}) + \\
& - (-1)^{(|I|+|L|)(|B|+1)} (-1)^{(|A|+|L|)(|B|+|L|)} \left.\frac{\delta B}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta A}{\delta q^I}\right|_G (\vec{x}) \Bigg] d\vec{x} \\
= \int_{\underline{\Sigma_X}} \Bigg[& (-1)^{(|I|+|\Lambda|+|\underline{L}|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta B}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}}) + \\
& - (-1)^{(|I|+|\Lambda|+|\underline{L}|)(|B|+1)} (-1)^{(|A|+|\underline{L}|)(|B|+|\underline{L}|)} \left.\frac{\delta B}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta A}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}}) \Bigg] d\underline{\vec{x}}
\end{aligned}$$

where Σ_X is a Cauchy slice in X of codimension $1|0$, $(\vec{x})$ are the restriction of the Cauchy coordinates on the $n-1|m$ dimensional supermanifold Σ_X, $d\vec{x}$ is the canonical volume fractional $n-1|m$-form defined by the Cauchy coordinates on Σ_X; $\underline{\Sigma_X}$ is the body of Σ_X, $(\underline{\vec{x}})$ are the Cauchy coordinates on $\underline{\Sigma_X}$ and $d\underline{\vec{x}}$ is the canonical volume $n-1$-form defined by the Cauchy coordinates on $\underline{\Sigma_X}$; and where functional derivatives are used and π_I is the canonical momentum associated to the slice Σ. Note that if Σ is defined by the equation $x^1 = 0$, then $\pi_I = p_I^1$. The momenta π_I^Λ will be defined by integration over the odd part of the supermanifold. Note that $\left|\pi_I^\Lambda\right| = |I| + l(\Lambda) + |L| + m$.

From the super Poisson structure built on $\mathscr{G}$, I derive the super commutation rules to which Fermionic and Bosonic fields have to obey and I demonstrate that these rules are exactly those expected from a physical point of view. This will give a justification, in a natural way, to the use of anticommutator for Fermionic fields.

In ***Chapter 11*** I will study the symmetries and supersymmetries of super field theories with the techniques offered by the formalism of fractional mixed forms and from the point of view of the super multisymplectic approach expounded in the previous chapters.

Some authors have already given a "super" version of the first Noether theorem valid for supermechanics: Ibort and Marín-Solano [83] and Cariñena and Figueroa [25].

L. Fatibene and M. Francaviglia, in [48] and L. Fatibene, M. Ferraris, M. Francaviglia and R. G. McLenaghan, in [47], have explored the way to give a geometric interpretation of supersymmetries using the classical tools of classical Poincaré-Cartan form and generalized vector fields defined over a bosonic manifold for field theories whose field spaces are product of exterior powers of some vector spaces, such that anticommuting spinors can be included in the theory.

As already quoted above, in 2006 Monterde, Muñoz Masqué and Vallejo, [116], obtained a version of the first Noether theorem valid for generic super field theories, but with the help of a rather technical assumption needed in the hypothesis.

Here in section 11.2, I will show that my approach allows to have a super version of Noether theorem which is quite natural, simple to prove with my formalism, and quite general since it does not require any specific technical assumption of the kind used in [116].

Theorem 9. *Consider a field theory defined by a configurations bundle E with fiber type the supermanifold F over a base supermanifold X and by a Lagrangian form $\mathcal{L}$ which locally is $\mathcal{L} = L\beta$. Let $\mathscr{E}$ be the space of solutions of the field theory. Let χ be a projectable vector field on E such that:*

$$\forall \Phi \in \mathscr{E} \ \exists \alpha_\Phi \in \Omega^{n-1|m} X : \ \forall U \subset X \ : \int_U j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = \int_U d\alpha_\Phi$$

then we have that:

$$\forall U \subset X, \ \forall \Phi \in \mathscr{E} \ :$$
$$\int_U d\left\{ j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)} c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \right\} = 0$$

Of course in theorem 9 fractional forms do appear.

In section 11.3, I will present a super extension of the multimomentum map introduced by Gotay, Isenberg, Marsden, Montgomery, Śniatycki and Yasskin in [63].

Corollary 10. *Let G be a Lie supergroup acting on P with a lifted covariant action, id est so that for each element $k \in \mathfrak{g}$ (the superalgebra), there exists a corresponding action on E generated by the projectable vector field χ, such that* $\operatorname{Lie}_{\chi_P} \omega = 0$. *Suppose there exists $J \in Hom\left(\mathfrak{g}, \Omega^{n-1|m}(P)\right)$, such that for each $k \in \mathfrak{g}$:*

$$\chi_P \lrcorner \omega = d\left[J(k)\right]$$

then, $\forall \Phi \in \mathscr{E}$:

$$j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = d\left[J(k) + \chi_P \lrcorner \theta\right]$$

In this case we say that the map:

$$\begin{aligned} J \ : \ \mathfrak{g} &\longrightarrow \Omega^{n-1|m}(P) \\ k &\longrightarrow J(k) \end{aligned}$$

is the super covariant comomentum map of the action.

Both the super Noether theorem and the super multimomentum map will be presented with a formulation which will reveal to be very close to the corresponding one for classical theories.

Finally, in ***Chapter 12***, I will present some examples showing how all the theory can be implemented for some specific Lagrangians. The study of the superoscillator in section 12.1 and the one of the superparticle in a curved space in section 12.2 will show how the tools presented in this thesis can be useful to investigate supermechanical theories from a point of view which wasn't available before, exploiting at best the potentiality of the superfield formulation. In section 12.3 I will work on the 3-dimensional super σ-model.

I conclude this introduction to my thesis by commenting a remark which appeared in a very recent paper on a geometric framework for Lagrangian supermechanics from a categorical point of view, by A. J. Bruce, K. Grabowska and G. Moreno, [18]. The authors write: "*...To our knowledge, the framework of geometric field theory - multisymplectic structures and so on - has not been applied to supersymmetric field theory. In part, we think that this is due to* *the lack of appreciation of categorical methods applied to supergeometry within the geometric mechanics community...* ". I disagree with this remark. Some authors, for example D. S. Freed in some lines of [54], write that the use of the notion of *functor of points* is necessary for a mathematical completely meaningful presentation of a superfield theory. Again I disagree, even if all these notions certainly lead to interesting viewpoints. Throughout all this thesis, I use a concrete approach to super theories, in the sense of DeWitt-Rogers, always working with concrete supermanifolds defined starting from the infinitely generated Grassmann algebra $\mathbb{R}_S$. I do not make use of the notion of functor of points. It is true that a categorical point of view of superfield theories may go better with the fact that probably there is no physical possible distinction between theories defined starting from different Grassmann algebras. But I don't think that the key point for developing a meaningful geometric, multisymplectic, super-field theory is the adoption of a categorical point of view, neither of a geometro algebraic one. As I have written above, the crucial difficulty to overcome is to find objects which can be integrated and which allow a Cartan calculus. This task can be undertaken as well with a concrete approach and this is exactly what I want to present here. Moreover, in my opinion, the choice of the concrete approach has the advantage to help a more intuitive insight on the geometrical framework, avoiding all the levels of abstraction which are not strictly necessary.

The principal aim of the third part of my thesis is precisely to show that fractional superforms are the natural objects to use when one wants to give a geometric formulation of a super field theory and that they are the key ingredient in doing so, and specifically in building the super version of the multisymplectic formalism.

Part I

Multisymplectic formalism and nonequivalent symplectic structures on the covariant phase space

Introduction to Part I

The multisymplectic formalism allows a finite dimensional geometric description of classical field theories seen from an Hamiltonian point of view. Multisymplectic geometry plays the role played by symplectic geometry in the description of classical Hamiltonian mechanics. Moreover the multisymplectic approach provides a tool for building a symplectic structure on the space of solution of the field theory and to investigate it.

After the early works of V. Volterra in 1890, [150, 151], it was in the first half of last century that the Hamiltonian approach to field theories was developed by H. Weyl, in [160], by T. De Donder, in [32], and then by T. Lepage, in [104] and P. Dedecker in 1953 [33], see also [34].

In a series of papers during the 70's, J. Kijowski, [94, 95], with W. Szczyrba [97, 98] and then with W. M. Tulczyjew [99] proposed the multisymplectic point of view on all these theories. Kijowski ideas allows to build a symplectic structure on the space of solution of a field theory, which become the *covariant phase space*. This symplectic structure is to some extent independent from any choice of splitting of spacetime in space and time. Therefore the multisymplectic formalism provides a fully covariant point of view on Hamiltonian field theories.

Similar ideas on the symplectic structure of the covariant phase space were present, during the 50's, in the pioneering works of R. E. Peierls, see [122], and I. Segal, see [144], and appeared again in 80's in papers by E. Witten [161], C. Crnkovic and Witten [28] and G. Zuckerman [164].

A new interest flourished on the then called covariant Hamiltonian theories and on the covariant phase space approach, both in the physical and in the mathematical communities. The multisymplectic formalism for field theories was studied and revisited by different authors and presented in different variants; for example by J. F. Cariñena, M. Crampin and L. A. Ibort in [24], by I. Kanatchikov in [85, 86], by M. Forger and H. Römer in [52], by Forger, C. Paufler and Römer in [50, 51], by Forger and S. V. Romero in [53], by A. Echeverría-Enríquez, M. Muñoz-Lecanda and N. Román-Roy in [46], by M. de León, D. Martín de Diego and A. Santamaría-Merini in [29], by F. Hélein in [71], by Hélein and D. Vey in [75] and, under the Lagrangian-Hamiltonian Unified Formalism label, for example by A. Echeverría-Enríquez, C. López, J. Marín Solano, M. Muñoz-Lecanda and N. Román-Roy in [44].

The multisymplectic approach has been the starting point for a very original attempt of quantization made by I. Kanatchikov: between his earliest papers on the subject, see [87]; between his most recent ones, see [88].

Similar formalisms appeared: under the n-symplectic label in L. K. Norris [120] and in M. McLean and L. K. Norris [111]; under the polisymplectic label for example in C. Günter [66], in G. Sardanashvily [140], in G. Giachetta, L. Mangiarotti and Sardanashvily [59] and [60]; under the k-symplectic label for example in A. Rey, Román-Roy and M. Salgado in [127]; under the k-cosymplectic label for example in M. de León, E. Merino, J. A. Oubiña, P. Rodrigues and M. Salgado [31], Rey, Román-Roy, Salgado and S. Vilariño in [128]. The mutual relations existing between some of these formalisms and with the multisymplectic formalism are explained in the review paper of M. de León, M. McLean, K. L. Norris, A. Rey-Roca and M. Salgado [30].

Hélein and J. Kouneiher, [72, 73, 74], studied in the multisymplectic framework a formalism, more general than the one used in the De Donder-Weyl theory, based on the works of Lepage and Dedecker.

The multisymplectic formalism, born for first order field theories, has been generalized to higher order theories: see for example S. Kourambaeva and S. Shkoller [102], L. Vitagliano [148], P. D. Prieto-Martínez and Román-Roy [124] and J. Kijowski and G. Moreno [96].

The formalism has been used for theories defined on base manifolds with boundary: see L. A. Ibort and A. Spivak [84].

Applications of the formalism to continuum Mechanics and hydrodynamics have been proposed; see for example: J. E. Marsden and S. Shkoller [108], the already quoted above [102] and J. E. Marsden, S. Pekarsky, S. Shkoller and M. West [107]. Covariant numerical methods for partial differential equations, developed in or inspired by the multisymplectic framework, have been introduced: see for example J. E. Marsden, G. W. Patrick and S. Shkoller [106], Kourambaeva and Shkoller [102], F. Demoures, F. Gay-Balmaz and T. S. Ratiu [38] and Demoures, Gay-Balmaz, M. Kobilarov and Ratiu [39].

The interest in multisymplectic field theories gave birth also to a number of studies on the so called multisymplectic geometry. Unfortunately a general accepted definition of multisymplectic manifold has not been settled. For some of the results obtained on this subject, one can see F. Cantrijn, L. A. Ibort and M. de León [22] and [23], Forger and L. Gomes [49] and the more recent works on higher symplectic geometry, see for example J. C. Baez, A. E. Hoffnung and C. L. Rogers [3], Baez and C. L. Rogers [4], C. L. Rogers [134], M. Richter [129, 130, 131] where the terminology of n-plectic geometry has been introduced.

The important work of M. J. Gotay, J. Isenberg, J. E. Marsden, R. Montgomery, J. Śniatycki and P. B. Yasskin, [63, 64], can be read for a presentation of the multisymplectic formalism for field theories. For a shorter introduction one can read Román-Roy [135]. For an introduction with a report on the history on how the ideas around the multisymplectic formalism originated and evolved, one can read Hélein [70].

In this thesis I will be mainly concerned with the multisymplectic formalism to build first order field theories. I will use a version of the multisymplectic formalism which minimizes the dimension of the spaces involved, which seems suitable to me for a first attempt of extending to super-field theories. I call it the minimal setting for the multisymplectic formalism. The main ingredient of this setting will be the finite dimensional multimomenta space P, which corresponds to what Forger and Romero call the ordinary multiphase space, [53] and what Román-Roy, [135], calls the restricted multimomentum bundle. For a review of other possible settings for the formulation of multisymplectic field theories and for a list of the various fiber bundles more often used in that context, see Echeverría-Enríquez, Muñoz-Lecanda and Román-Roy [45].

The main interest of the finite dimensional geometric construction of the multimomenta space with its multisymplectic (n-plectic in the most up to date terminology) structure is that it provides a way to build a symplectic structure on the covariant phase space. There is then a direct link between the multisymplectic field theory and the classical canonical formulation of the field theory. This link connects the works on multisymplectic geometry to the works of Physics and Mathematical Physics communities on canonical field theory, pioneered by the papers of Peierls and Segal already cited above, continued and developed by B. DeWitt [40, 41, 42], by García and Pérez-Rendón in [56, 57, 58] and by Goldschmidt and Sternberg in [62].

Here I will follow [70] to show how, with the help of the multisymplectic form ω, it is possible to build a symplectic form Ω on the space of Hamiltonian surfaces $\mathscr{G}$. For more details one can read Hélein and Kouneiher [73] and the papers of Kijowski, Szczyrba and Tulczyjew [97, 98, 99]. Then I will very shortly explain how the symplectic structure obtained on $\mathscr{G}$ is related to the Poisson brackets used by physicists in field theory.

Throughout all this thesis I will deal with theories arising from regular Lagrangians. The multisymplectic approach, with some adaptations, can be used also for theories arising from irregular Lagrangians, possibly with gauge symmetries and many of the results here presented can be extended to that case: it is possible to built a symplectic structure on a space obtained by reducing $\mathscr{G}$. For an introduction to the subject one can see [63, 64] and [135] and the bibliography therein.

In chapter 1 I will introduce a geometrical framework for the Lagrangian approach to first order field theories, defining the Action and the space of solutions of the theory.

In chapter 2 I will introduce what I call the minimal setting for the multisymplectic description of field theories. In section 2.1 I will define the multimomenta space and the Legendre transform. In section 2.2 I will expose the Volterra theorem and I will show the Hamilton-Volterra equations. In section 2.3 I'll introduce the multisymplectic form. In section 2.4 I'll explain the techniques used to build a symplectic structure on covariant phase space.

Chapter 3 contains the main original contributions of this first part of my thesis. Starting from the multisymplectic form, we can build a symplectic structure on the space of solutions $\mathscr{G}$ of a field theory. The constructions however depends on the choice of a surface Σ of codimension 1 in the base manifold of the theory. It is then natural to ask oneself if that choice really affects the symplectic structure or if the symplectic structure is in fact independent from that choice. I will address this question and I will show that the choice of the surface of codimension 1 may indeed influence the symplectic structure on the space of solutions. In section 3.1 I will show, with the help of very simple examples, that two surfaces Σ and Σ' belonging to different homology classes may give birth to nonequivalent symplectic structures. In section 3.2, again with simple examples, I will show that, when the base manifold cannot be split in a time-line times a compact space (id est when the space is non compact), then again the choice of the surface of codimension 1 does influence the symplectic structure: some subtle phenomena occurs in this case and I will investigate them.

In chapter 4 I will show how the symplectic structure on the space of solutions is linked with the field brackets used by physicists.

Chapter 1

Lagrangian formulation of classical field theories

In this section I'll fix the notation used in this work to formulate a classical first order Lagrangian field theory. I'll list a number of different ways to identify the solutions of a Lagrangian theory and I'll recall some standard facts. Most of the results presented can be extended, with some modifications, to higher order theories. For a less synthetic, more accurate and more rigorous presentation of the notions here treated, one can see a text book on classical field theory like for example [48] and [61] or, with a slightly different approach, [36] and [54].

The aim of this section is also to present some of the definitions and of the results which I try to extend to the super-field theories in chapter 5.

Let E, X and F be finite dimensional C^∞ manifolds, E and X being connected and X being also oriented; let (E, π, X, F) be a differential fiber bundle with total space E, base X, type-fiber F and bundle C^∞ projection π, so that we have the following situation:

$$\begin{array}{c} E \\ \pi \big\downarrow \\ X \end{array}$$

I will sometime denote the fiber bundle with its total space E, when there is no risk of confusion, or with its projection π.

A field ϕ over X is a C^∞ section of the fiber bundle π and we write $\phi \in \Gamma\big(E\big)$. As often done in the literature, we can call $\Gamma\big(E\big)$ the space of fields. From now on and throughout all this section and the following one of this paper, all the maps between manifolds will be considered C^∞ if not otherwise stated.
The first order jet space of sections of E is denoted by $J^1\pi$, and $j^1\phi$ denotes the first order jet of the section ϕ, which is a lift of the section ϕ to $J^1\pi$.

The jet space $J^1\pi$ has the structure of a differential fiber bundle over the base X with projection $j^1\pi$ and of an affine fiber bundle over the base E with projection $j^1_0\pi$. The two projections are linked by $j^1\pi = \pi \circ j^1_0\pi$. We can call j^1 the lifting map from $\Gamma\big(E\big)$, the space of section of E, to $\Gamma\big(J^1\pi\big)$, the space of section of $J^1\pi$, seen as fiber bundle over the base X.

$$j^1 \;:\; \Gamma\big(E\big) \;\longrightarrow\; j^1\Gamma\big(E\big) \subset \Gamma\big(J^1\pi\big)$$

On $J^1\pi$, on E and on X do exist adapted atlases of charts, so that, if on an open chart U of X we use the local coordinates x^a, $a = 1, \cdots n$, where n is the dimension of X, then on the open chart V of E it is possible to use, with the little abuse of notation $x^a = x^a \circ \pi$, the coordinates $\left(x^a, q^i\right)$ with $i = 1, ..., m$, m being the dimension of the fiber F, and on the open chart W of $J^1\pi$ we similarly use the coordinates $\left(x^a, q^i, \dot{q}^i_a\right)$.

Sometime I'll denote by ∂_a the operator $\frac{\partial}{\partial x^a}$.

The field ϕ will read in coordinates:

$$\phi \ : \ x^a \longrightarrow \left(x^a, q^i\left(\phi(x)\right)\right)$$

A section $s \in \Gamma\left(J^1\pi\right)$ will read:

$$s \ : \ x^a \longrightarrow \left(x^a, q^i(s(x)), \dot{q}^i_a(s(x))\right)$$

Sometime, for brevity, I'll write:

$$\phi \ : \ x^a \longrightarrow q^i\left(x^a\right)$$

$$s \ : \ x^a \longrightarrow \left(q^i(x^a), \dot{q}^i_a(x^a)\right)$$

Then $\forall x \in X,\ \forall \phi \in \Gamma\left(E\right)$, it will be $\dot{q}^i_a\left(j^1\phi\left(x\right)\right) = \partial_a q^i(\phi(x))$.

On a local chart of $J^1\pi$ we can define the local contact one forms $c^i \in \Omega^1_{\mathrm{loc}} J^1\pi$, by the formulas in local coordinates:

$$c^i := dq^i - \dot{q}^i_a dx^a$$

We can identify the sections in $j^1\Gamma\left(E\right)$ by the following proposition:

$$s \in \Gamma\left(J^1\pi\right) \text{ belongs to } j^1\Gamma\left(E\right) \iff \forall i,\ \forall v \in TX,\ c^i(s_* v) = 0 \tag{1.1}$$

The Lagrangian $\mathcal{L}$ of our field theory is a n-dimensional differential form on $J^1\pi$ which is horizontal with-respect to the projection $j^1\pi$. On a local chart it is always possible to write it as:

$$\mathcal{L} = L\left(x^a, q^i, \dot{q}^i_a\right)\beta$$

Where $\beta = dx^1 \wedge \cdots \wedge dx^n$ is the canonical n-form on the chart and where the coordinates x^a are seen as coordinates on $J^1\pi$. Note that L does not define an intrinsic (*id est* independent of coordinates) function on $J^1\pi$; in fact, when changing chart, L transforms as an n-density: it can therefore be called the Lagrangian density.

The action A_U is the integral of $\mathcal{L}$ on the n-dimensional surface $s(U)$, where $U \subset X$ is a n-dimensional submanifold (possibly with boundary) of X, s is a section of $J^1\pi$ and so $s(U) \subset J^1\pi$ is a a n-dimensional submanifold (possibly with boundary) of $J^1\pi$. The value of A is always well defined for every section s if U is compact. A particularly interesting case occur when $s = j^1\phi$ is the lift of a local section ϕ of E. In this case, with a little abuse of notation, we define:

$$A_U(\phi) := A_U(j^1\phi) = \int_{j^1\phi(U)} \mathcal{L} = \int_U j^1\phi^* \mathcal{L} \tag{1.2}$$

The space $\Gamma\left(E\right)$ of all possible sections of E is not in general a finite dimensional manifold. The same can be said for the space of their lifts to $J^1\pi$, $j^1\Gamma\left(E\right)$, for the space $\Gamma\left(J^1\pi\right)$ of all sections of $J^1\pi$ and for their subspaces. Nevertheless, it is possible to give them a differential structure, for example obtaining an infinite dimensional manifold modeled on some infinite dimensional linear space, or, choosing a different way, obtaining a diffeological space, or, following Kijowsky with Szczyrba in one of their original works on multisymplectic field theory [98], considering them as inductive differential manifolds. I will not go trough the technical problems involved in building such differential structures, one can see [126] for a possible approach to this subject. What is important for the rest of this paper is to know that it is possible to define the notion of differentiable paths on these spaces, the notion of differentiable functions and so the notions of tangent and cotangent spaces. In those contexts, A is a differentiable function on one of those spaces.

Let's call $\mathscr{U}$ a space of sections of E which share the same values on ∂U: $\mathscr{U} \subset \Gamma\big(E\big)$. I call $j^1\mathscr{U}$ the space of their lifts which are sections of $J^1\pi$: $j^1\mathscr{U} \subset j^1\Gamma\big(E\big)$. There is one such space $\mathscr{U}$ for each assignment of values on ∂U. Chosen U, I call $\mathscr{U}_\phi$ the space of sections of E which share the same values with ϕ on ∂U. It could be called $\mathscr{U}_{(U,\phi)}$ to stress its dependence both on ϕ and on U, but I'll mostly drop the first index for sake of brevity and to lighten a little bit the notations.

A section s of $J^1\pi$ which is not a lift doesn't belong to any of the spaces $j^1\mathscr{U}$ described above. Let's denote by $A_U\big|_{\mathscr{U}}$ the function A_U restricted to a set $\mathscr{U}$. A field ϕ is said to be a solution, or a critical point, of our theory if, for every U compact submanifold with boundary, ϕ is an extremal point for A_U in $\mathscr{U}_\phi$; that is ϕ is a point where the differential dA_U vanishes in $\mathscr{U}_\phi$. So we have:

$$\phi \text{ is a solution } \iff \forall U,\ d\left(A_U\big|_{\mathscr{U}_\phi}\right)(\phi) = 0 \iff \forall U\ d\left(A_U\big|_{j^1\mathscr{U}_\phi}\right)(j^1\phi) = 0 \tag{1.3}$$

Where, as in (1.2), we have indicated with the same name A_U two different functions defined on different spaces. There should not be any risk of confusion.

I call $\mathscr{E}$ the set of all solutions, id est of all critical points, of the theory. $\mathscr{E} \subset \Gamma\big(E\big)$. Again, as all other subset of $\Gamma\big(E\big)$, $\mathscr{E}$ is not in general a finite dimensional manifold, but it can be endowed in many cases (perhaps in any case) with a differential structure.

A path in $\mathscr{U}_\phi$ trough ϕ is a function p from an interval $I \subset \mathbb{R}$ containing the point 0 to the set $\mathscr{U}_\phi$, which is differentiable according to the differential structure given to $\mathscr{U}_\phi$ and so that $p(0) = \phi$. Let's parametrize I with the coordinate ℓ; we have then:

$$\phi \text{ is a solution of the theory } \iff \forall U,\ \forall\ p \text{ in } \mathscr{U}_\phi \text{ trough } \phi,\ \left.\frac{dA_U}{dl}\right|_{\ell=0} = 0 \tag{1.4}$$

Condition (1.4) is equivalent to condition (1.3). We can see this condition from some other points of view. For example, if we consider $\partial_\ell\big|_{\ell=0}$ which is a vector tangent to I at $\ell = 0$, then $p_*\partial_\ell\big|_{\ell=0}$ is a vector belonging to $T\mathscr{U}_\phi$ over the point ϕ. Condition (1.4) is then equivalent to:

$$\phi \text{ is a solution } \iff \forall U,\ \forall\ p \text{ in } \mathscr{U}_\phi \text{ trough } \phi,\ dA_U\big|_{\mathscr{U}_\phi}(\phi)\cdot p_*\partial_\ell\big|_{\ell=0} = 0 \tag{1.5}$$

Condition (1.4) can be written in another form. Let $V_\pi E \subset TE$ be the fiber-bundle of vectors on E vertical with respect to the projection π. If ϕ is a section of E, we define $V_\pi E\big|_{\phi(X)} := \phi^* V_\pi E$: it is a vector bundle with base X. We call $\mathscr{V}_{\phi(U)}$ the set of all sections of $V_\pi E\big|_{\phi(X)}$ null on ∂U, id est the set of vertical vector fields on $\phi(X)$ whose value on $\phi(\partial U)$ is 0.

The tangent space $T_\phi\mathscr{U}_\phi$ is then in a one-to-one natural correspondence with $\mathscr{V}_{\phi(U)}$. Let's set $\phi_\ell := p(\ell)$. The vector $p_*\partial_\ell\big|_{\ell=0} \in T_\phi\mathscr{U}_\phi$ corresponds to the vector field $u \in \mathscr{V}_{\phi(U)}$ defined by:

$$\forall x \in X,\ u(x) = \left.\frac{\partial q^i(\phi_\ell(x))}{\partial \ell}\right|_{\ell=0} \frac{\partial}{\partial q^i}$$

Conversely, if the closure $\overline{U}$ of U is compact, it can be shown that $\forall\ u \in \mathscr{V}_{\phi(U)}$, it exists a path p trough ϕ in $\mathscr{U}_\phi$ such that $p_*\partial_\ell\big|_{\ell=0} \in T_\phi\mathscr{U}_\phi$ corresponds to u.

Since j^1 sends a section in $\mathscr{U}_\phi$ to a section in $j^1\mathscr{U}_\phi$, it induces the map $j^1_* : T_\phi\mathscr{U}_\phi \longrightarrow T_{j^1\phi}j^1\mathscr{U}_\phi$. The map $j^1_* : T\mathscr{U}_\phi \longrightarrow Tj^1\mathscr{U}_\phi$, is then a diffeomorphism when the appropriate differential structures are used.

Let $V_{j^1\pi}J^1\pi \subset TJ^1\pi$ be the fiber-bundle of vectors on $J^1\pi$ vertical with respect to the projection $j^1\pi$. If s is a section of $J^1\pi$, we define $V_{j^1\pi}J^1\pi\Big|_{s(X)} := s^*V_{j^1\pi}J^1\pi$: it is also a vector fiber bundle with base X.

We can represent each $p_*\partial_\ell\big|_{\ell=0} \in T_\phi\mathscr{U}_\phi$ by a section w of $V_{j^1\pi}J^1\pi\Big|_{j^1\phi(X)}$: let's call K the map which associates w to $p_*\partial_\ell\big|_{\ell=0} \in T_\phi\mathscr{U}_\phi$. K is defined as follow:

$$\forall x \in X,\ w(x) = K(p_*\partial_\ell\big|_{\ell=0})(x) := \frac{\partial q^i(j^1\phi_\ell(x))}{\partial \ell}\Bigg|_{\ell=0}\frac{\partial}{\partial q^i} + \frac{\partial \dot{q}^i_a(j^1\phi_l(x))}{\partial \ell}\Bigg|_{\ell=0}\frac{\partial}{\partial \dot{q}^i_a}$$

The map K is injective and we can set $\mathscr{V}_{j^1\phi(U)} := K\left(T_\phi\mathscr{U}_\phi\right)$. For it's definition $\mathscr{V}_{j^1\phi(U)}$ is diffeomorph to $\mathscr{V}_{\phi(U)}$, to $T_\phi\mathscr{U}_\phi$ and to $T_{j^1\phi}j^1\mathscr{U}_\phi$ and for its construction $\forall\ w \in \mathscr{V}_{j^1\phi(U)}$, it exists a path p trough ϕ in $\mathscr{U}_\phi$ so that $p_*\partial_l\big|_{l=0} \in T_\phi\mathscr{U}_\phi$ corresponds to w.
It is important to note that $w \in \mathscr{V}_{j^1\phi(U)}$ does not imply that $w(\partial U) = 0$. The components of w in $\frac{\partial}{\partial q^i}$ are indeed equal to 0 on ∂U and the tangential projection of $\frac{\partial}{\partial \dot{q}^i_a}$ is equal to 0 on ∂U, however the normal projection does not necessarily vanish.

Fixed a path p trough ϕ in $\mathscr{U}_\phi$, we call ∂U_ℓ the n-dimensional surface in $J^1\pi$ spanned by the image of ∂U by the map $j^1\phi_\ell$ when l goes from 0 to ℓ. In general $\partial U_\ell \neq \emptyset$ precisely because w, corresponding to $p_*\partial_\ell$, is in general different from 0 when calculated in $x \in \partial U$. Nonetheless we have that $\int_{\partial U_\ell}\mathcal{L} = 0$ because w is vertical with respect to $j^1\pi$ whereas $\mathcal{L}$ is horizontal. We call V_l the $n+1$-dimensional surface in $J^1\pi$ spanned by the image of U by the map $j^1\phi_\ell$ when ℓ goes from 0 to l. We have that $\partial V_\ell = j^1\phi_\ell(U) + \partial U_\ell - j^1\phi(U)$, if the suitable orientations are used.

Keeping all this in mind, we can write:

$$\begin{aligned}
\frac{dA_U}{d\ell}\bigg|_{\ell=0} &= \lim_{\ell\to 0}\left\{\frac{1}{\ell}\left[\int_{j^1\phi_\ell(U)}\mathcal{L} - \int_{j^1\phi(U)}\mathcal{L}\right]\right\} \\
&= \lim_{\ell\to 0}\left\{\frac{1}{\ell}\left[\int_{j^1\phi_\ell(U)}\mathcal{L} - \int_{j^1\phi(U)}\mathcal{L}\right]\right\} + \lim_{\ell\to 0}\left\{\frac{1}{\ell}\int_{\partial U_\ell}\mathcal{L}\right\} \\
&= \lim_{\ell\to 0}\left\{\frac{1}{\ell}\left[\int_{j^1\phi_\ell(U)}\mathcal{L} - \int_{j^1\phi(U)}\mathcal{L} + \int_{\partial U_\ell}\mathcal{L}\right]\right\} \\
&= \lim_{\ell\to 0}\left\{\frac{1}{\ell}\left[\int_{V_\ell} d\mathcal{L}\right]\right\} = \int_{j^1\phi(U)} w\lrcorner d\mathcal{L} = \int_{j^1\phi(U)} dL(w)\beta
\end{aligned} \tag{1.6}$$

Where the last equality holds on a local chart.

So (1.4) is equivalent to the following:

$$\phi \text{ is a solution of the theory} \iff \forall U,\ \forall w \in \mathscr{V}_{j^1\phi(U)},\ \int_{j^1\phi(U)} w\lrcorner d\mathcal{L} = 0 \tag{1.7}$$

Let $U \subset X$ be included in an chart of an adapted atlas with coordinates as in the beginning of this section, then, with classical arguments, it can be shown that:

$$\phi \text{ is a solution} \iff \text{on every } U,\ \forall x \in U,\ \frac{d}{dx^a}\frac{\partial L}{\partial \dot{q}^i_a}\left(j^1\phi(x)\right) - \frac{\partial L}{\partial q^i}\left(j^1\phi(x)\right) = 0 \tag{1.8}$$

where the condition in (1.8) is the classical Euler-Lagrange system of equations for first order Lagrangian field theories.

Chapter 2

Multisymplectic formulation of classical field theories.

This chapter is devoted to give a brief introduction to the multisymplectic formalism (in the minimal setting). In section 2.1 I define the multimomenta space and the Legendre transform. In section 2.2 I introduce the Hamiltonian and I show the Hamilton-Volterra equations. In section 2.3 I introduce the multisymplectic form and in section 2.4 I show how, out of it, a symplectic structure can be built in the covariant phase space.

2.1 The multimomenta space and the Legendre transform.

Let's consider the space $V_\pi E \subset TE$, introduced in the last section: it can be seen as the total space of a vector fiber-bundle over E with the canonical projection, τ, but it can be also considered as the total space for a (non vectorial in general) fiber bundle over X with projection equal to $\pi \circ \tau$. Let $e \in E$, then we can call $V_e E$ the space of all points of $V_\pi E$ who are projected by τ on e (it is the fiber over e of $(V_\pi E, \tau, E)$). It is possible then to build, with standard techniques, a fiber bundle with base X, and with fiber over $x \in X$ the collection of all linear maps, from all the spaces $V_e E$ with $\pi(e) = x$, to $\Lambda^{n-1} T^*_x X$; where $\Lambda^{n-1} T^* X$ is the fiber-bundle of $(n-1)$-forms on X. The fiber-type will then result to be a finite dimensional manifold. I will call this new fiber bundle $Hom_\pi(VE, \Lambda^{n-1} T^* X)$. I will not exhibit here the co-cycle of the transition functions of this fiber bundle, being it the most natural one that one can think of in this context. When $E = X \times F$ is the trivial fiber bundle with its canonical projection, then $Hom_\pi(VE, \Lambda^{n-1} T^* X) = E \times_X Hom\left(X \times TF, \Lambda^{n-1} T^* X\right) \simeq E \times_X \left(\Lambda^{n-1} T^* X \otimes T^* F\right) = \left\{(x, n, m) | (x, n) \in E, (x, m) \in Hom\left(T_n F, \Lambda^{n-1} T^*_x X\right)\right\}$.

Let's call P the total space of this new fiber-bundle: it will be our multimomenta space and I will call it sometime also the minimal multiphase space. For simplicity we will often denote with P the fiber bundle itself, so that we have the following:

Definition 11. *The multimomenta space of a field theory is the fiber-bundle* $P = (P, \pi_P, X) := Hom_\pi(VE, \Lambda^{n-1} T^* X)$ *with total space* P*, base* X *and projection* π_P*.*

Then, on a chart of an adapted atlas, a point $p \in P$ will be denoted by the coordinates (x^a, q^i, p^a_i). If $e \in E$ has coordinates (x^a, q^i), $x = \pi(e) \in X$ has coordinates (x^a) and $v \in V_e E$ is $v = v^i \frac{\partial}{\partial q^i}\Big|_e$, then $p(v) \in \Lambda^{n-1} T^*_x X$ can be locally decomposed: $p(v) = v^i p^a_i \beta_a$, where $\beta_a =$

$\frac{\partial}{\partial x^a} \lrcorner dx^1 \wedge \cdots \wedge dx^n$. This provides us with the local coordinates p_i^a for $p \in Hom_\pi \left(V_e E, \Lambda^{n-1} T_x^* X\right)$. When one changes coordinates naturally on X, E, VE and $\Lambda^{n-1} T^* X$, it can be easily shown that this formula remains unchanged and univocally defines an element of $\Lambda^{n-1} T^* X$.

As suggested by the local coordinates structure, it can be shown that P can be considered also as a fiber-bundle over E and it can also be shown that it is a vector fiber-bundle over E.

This is not a very elegant construction, but it has the advantage of minimizing the number of variables needed to use the multisymplectic formalism; which is sufficient when you think to work on local coordinates (as I'll mainly do in the following) and when you aim to find an extension to a wider framework (the one of super-fields).

We call Legendre transform the map $\mathbb{FL}$ between $J^1\pi$ and P, defined on local charts by:

$$\mathbb{FL}\left(x^a, q^i, \dot{q}_a^i\right) = \left(x^a, q^i, \frac{\partial L}{\partial \dot{q}_a^i}\left(x^a, q^i, \dot{q}_a^i\right)\right) \tag{2.1}$$

We say that the Lagrangian $\mathcal{L}$ is regular if $J^1\pi$ is diffeomorphic to $\mathbb{FL}\left(J^1\pi\right) \subset P$ [108]. We say that $\mathcal{L}$ is hyperregular if $J^1\pi$ is diffeomorphic to P [53].

2.2 The Hamiltonian and Hamilton-Volterra equations.

On $\mathbb{FL}(J^1\pi)$ it is possible to define the Hamiltonian "function" H by:

$$H(x^a, q^i, p_i^a) := \dot{q}_a^i p_i^a - L\left(x^a, q^i, \dot{q}_a^i\right) \tag{2.2}$$

where we assume that $\dot{q}_a^i$ is a solution of

$$\frac{\partial L}{\partial \dot{q}_a^i}\left(x^a, q^i, \dot{q}_a^i\right) = p_i^a \tag{2.3}$$

H is not in fact a function on P and not even a coefficient of a form defined on P, but a slightly more complicated object. Formula (2.2) is indeed locally well defined on every chart of P. It is well defined even when the Lagrangian is not regular provided that (2.3) has at least one solution and that the set of its solutions is connected, see [73]; this is because the value of H does not depend on the choice of the particular solution $\dot{q}_a^i$. I will not treat here the regularity conditions needed on L in order for H to be smooth.

When one changes coordinates on E and passes from coordinates (x^a, q^i) to coordinates $(x^{a'}, q^{i'})$, so that

$$\begin{aligned} x^{a'} &= x^{a'}(x^a) \\ q^{i'} &= q^{i'}(x^a, q^i) \end{aligned} \tag{2.4}$$

then corresponding natural changes of coordinates hold on $J^1\pi$ and on P, so that:

$$\begin{aligned} \dot{q}_{a'}^{i'} &= \frac{\partial x^a}{\partial x^{a'}}(x^a)\dot{q}_a^i \frac{\partial q^{i'}}{\partial q^i}(x^a, q^i) + \frac{\partial x^a}{\partial x^{a'}}(x^a)\frac{\partial q^{i'}}{\partial x^a}(x^a, q^i) \\ p_{i'}^{a'} &= \det\left(\frac{\partial x}{\partial x'}\right)(x^a)\frac{\partial x^{a'}}{\partial x^a}(x^a)\frac{\partial q^i}{\partial q^{i'}}(x^a, q^i)p_i^a \end{aligned} \tag{2.5}$$

As a consequence, if we call H' the Hamiltonian "function" defined with the new coordinates, we have that:

$$\begin{aligned} H'\left(x^{a'}, q^{i'}, \dot{q}^{i'}_{a'}\right) =& \det\left(\frac{\partial x}{\partial x'}\right)(x^a)\dot{q}^i_a p^a_i - \det\left(\frac{\partial x}{\partial x'}\right)(x^a) L\left(x^a, q^i, \dot{q}^i_a\right) + \\ &+ \det\left(\frac{\partial x}{\partial x'}\right)(x^a)\frac{\partial q^{i'}}{\partial x^a}(x^a, q^i)\frac{\partial q^i}{\partial q^{i'}}(x^a, q^i) p^a_i = \\ =& \det\left(\frac{\partial x}{\partial x'}\right)(x^a) H\left(x^a, q^i, \dot{q}^i_a\right) + \\ &+ \det\left(\frac{\partial x}{\partial x'}\right)(x^a)\frac{\partial q^{i'}}{\partial x^a}(x^a, q^i)\frac{\partial q^i}{\partial q^{i'}}(x^a, q^i) p^a_i \end{aligned} \tag{2.6}$$

Formula (2.6) shows that H is not a global function on P.

In the paper of Cariñena, Camprin and Ibort [24], one finds that the Hamiltonian can be naturally seen as the section of a fiber bundle Z (which we can call the multiphase space) over P (the minimal multiphase space); see also Marsden and Shkoller [108], Forger and Romero [53] and Hrabak [82] to understand the nature of H. Note however that Hrabak call P 'covariant-phase-space'. For the definition of the multiphase space one can also look at [63]. Note that the authors there call Z also covariant-phase-space, which introduces an ambiguity with the terminology used here, since I call covariant phase space the space of solutions of the field theory. Forger and Romero, [53] call Z the extended multiphase space, and call P the ordinary multiphase space; Román-Roy, [135], call Z the extended multimomentum bundle and P the restricted multimomentum bundle; as seen above, I prefer to call P minimal multiphase space, or multimomenta space.

The space Z is a sub-bundle of the bundle $\Lambda^n(E)$ and it is the space where the construction of the Volterra-De Donder-Weyl theory is geometrically more natural. Formula (2.6) defines the transition functions of the fiber bundle Z over P and so implicitly defines what a sections of Z is. Since the construction of Z is not essential for what follows, and it is not in the spirit of the minimal setting that I am using, I will not explicit it here. In what follows formula (2.2) and (2.6) will be sufficient to define all the geometric objects needed.

When L is hyperregular, (2.3) has one and only one solution for every $p \in P$: then H is defined for every $p \in P$.

When L is regular, the Hessian determinant of L with respect to the variables $\dot{q}^i_a$ is different than 0 for each point $r \in J^1\pi$; id est: $\forall r \in J^1\pi, det\left|\frac{\partial^2 L}{\partial \dot{q}^i_a \partial \dot{q}^j_b}\right|(r) \neq 0$. Then it can be shown that $\dot{q}^i_a\left(x^a, q^i, p^a_i\right)$ is actually equal to $\frac{\partial H}{\partial p^a_i}$.

In the following I will not deal with regularity problems, so, if not otherwise stated, I will always suppose L hyperregular and $P = \mathbb{FL}\left(J^1\pi\right)$. When L is regular but not hyperregular, all the results exposed here for P are still valid for $\mathbb{FL}\left(J^1\pi\right)$.

For more information about regularity conditions on $\mathbb{FL}$ one can see [45].

On a local chart, with canonical horizontal n-form β, one can define the local Hamiltonian n-form $\mathcal{H} := H\beta$ but this in general doesn't lead to a well defined Hamiltonian n-form on P. As shown in [108], to have a globally defined Hamiltonian n-form on P, it is necessary first to chose a connection on E; see also [24]. I will not deal with this, since it is not essential for what follows.

Using $\mathbb{FL}$ we can associate to every section $s \in \Gamma\big(J^1\pi\big)$ a section $\mathbb{FL}(s)$ of P:

$$\mathbb{FL}(s)(x) := \mathbb{FL} \circ s\ (x)$$

I will call $\mathbb{FL}$ the corresponding map between $\Gamma\big(J^1\pi\big)$ and $\Gamma\big(P\big)$, with a little abuse of notations. When L is regular, this correspondence is invertible.

With the help of the map j^1 defined above, we can also establish a correspondence between section of E and sections of P (not invertible because j^1 is not one-to-one):

$$\phi \in \Gamma\big(E\big) \ \longrightarrow \ j^1\phi \in \Gamma\big(J^1\pi\big) \ \longrightarrow \ \mathbb{FL} \circ j^1\phi \in \Gamma\big(P\big)$$

Note that, on a local chart, $p_i^a\left(\mathbb{FL}j^1\phi(x)\right) = \frac{\partial L}{\partial q_a^i}\left(j^1\phi(x)\right), \forall x \in X$.

Since 1890, [150], it was proved the following:

Theorem 12 (Volterra 1890)**.** *Let L be a regular Lagrangian density on $J^1\pi$ and let H be its corresponding Hamiltonian density on P, then $\forall U$ local chart and $\forall x \in U \subset X$, a field $\phi \in \Gamma(\pi)$ is a solution of the Euler-Lagrange system of equations in* (1.8)*, id est it is a solution of the Lagrangian field theory, if and only if, $\forall U$ local chart and $\forall x \in U \subset X$, $\mathbb{FL} \circ j^1\phi$ satisfies the generalized covariant Hamilton system:*

$$\begin{cases} \dfrac{\partial q^i}{\partial x^a}\left(\mathbb{FL}j^1\phi(x)\right) = \dfrac{\partial H}{\partial p_i^a}\left(\mathbb{FL}j^1\phi(x)\right) \\ \dfrac{\partial p_i^a}{\partial x^a}\left(\mathbb{FL}j^1\phi(x)\right) = -\dfrac{\partial H}{\partial q^i}\left(\mathbb{FL}j^1\phi(x)\right) \end{cases} \tag{2.7}$$

I will prove in section 8.2 a super version of it.

Note that in the following I will sometime call the system of equations (2.7) as Hamilton-Volterra system. In the literature it is sometime called Hamilton-De Donder-Weyl system, see for example [46], sometime the Hamilton-De Donder system, see for example [60].

2.3 The multisymplectic form

The image of a section $z \in \Gamma\big(P\big)$ is a n-dimensional surface in the total space P. It is then natural to look for a geometric condition on n-surfaces in P to be the images of sections which themselves are correspondents of sections ϕ of E which are solutions of our theory. Let's rephrase it this way: if we have a n-dimensional submanifold $G \subset P$, when can we say that there exist $\phi \in \mathscr{E}$ so that $G = \mathbb{FL} \circ j^1\phi(X)$?

The answer is that G has to satisfy four conditions:

1. G has to be the image of a section $z \in \Gamma\big(P\big)$, so: $\exists z \in \Gamma\big(P\big)$ so that $G = z(X)$
2. z has to be the image trough $\mathbb{FL}$ of a section $s \in \Gamma\big(J^1\pi\big)$, so: $\exists s \in \Gamma\big(J^1\pi\big)$ such that $z = \mathbb{FL}(s)$;
3. s must belong to $j^1\Gamma\big(E\big)$, so: $\exists \phi \in \Gamma\big(E\big)$ such that $s = j^1\phi$;
4. ϕ must be a solution of the theory, so: $\phi \in \mathscr{E}$; or, which is equivalent, ϕ has to satisfy one of the conditions in (1.3), (1.4), (1.5), (1.7) or (1.8).

We call Hamiltonian a n-submanifold of P who satisfies the above four conditions; we call $\mathscr{G}$ the space of all Hamiltonian submanifolds. Then $\mathscr{G}$ and $\mathscr{E}$ are in one-to-one correspondence and they are indeed diffeomorphic if a suitable differential structure is put on them.

Condition 1 can be translated in the following geometric one:

Proposition 13. *Let U be a local chart of an adapted atlas of P and let β be the horizontal non degenerate n-form on U defined in by $\beta = dx^1 \wedge \cdots \wedge dx^n$, let $G \subset P$ be a n-dimensional submanifold of P, let $p \in G$ be one of its points, and let $T_pG \subset T_pP$ be the tangent space to G in p, then the n-dimensional submanifold $G \subset P$ is the image of a section $z \in \Gamma\big(P\big)$ if and only if $\forall U$, $\forall p \in G$, $\forall v_1, \cdots, v_n \in T_pG$ linear independent, $\beta(v_1, \cdots, v_n) \neq 0$.*

This condition can be linked to the Hamiltonian local forms $\mathcal{H}$ provided some assumptions are made on their zeros. For example we have:

Proposition 14. *Let U_α be the elements of the atlas of charts of P, let $\mathcal{H}_\alpha$ be the Hamiltonian local n-form on U_α, and assume that all $\mathcal{H}_\alpha$ are everywhere different than 0; let $G \subset P$ be a n-dimensional submanifold of P, let $p \in G$ be one of its points, and let $T_pG \subset T_pP$ be the tangent space to G in p; then the n-dimensional submanifold $G \subset P$ is the image of a section $z \in \Gamma\big(P\big)$ if and only if $\forall p \in G$, $\forall v_1, \cdots, v_n \in T_pG$ linear independent and $\forall U_\alpha$ containing p, $\mathcal{H}_\alpha(v_1, \cdots, v_n) \neq 0$. Equivalently, calling $i : G \longrightarrow P$ the immersion of G in P: $G \subset P$ is the image of a section $z \in \Gamma\big(P\big)$ if and only if $\forall \alpha\ i^*\mathcal{H}_\alpha \neq 0$.*

Condition 2 is automatically satisfied if L is hyperregular.

To translate condition 3 in a geometrical condition on G, we first note that, if L is regular, we can push-forward the local contact forms c^i defined in the previous section, and we have in coordinates: $g^i := \mathbb{FL}_* c^i = dq^i - \frac{\partial H}{\partial p^a_i} dx^a$, where $g^i \in \Omega^1_{\text{loc}} P$ are local differential one-forms on P that we can call contact forms without risk of confusion.

If we call section-submanifolds those n-submanifolds $G \subset P$ which satisfy condition 1 , and we call lifted-submanifolds those n-submanifolds $G \subset P$ which satisfy conditions 1 , 2 and 3, then we have the following:

Proposition 15. *Let L be a regular Lagrangian density on $J^1\pi$ and H be its corresponding Hamiltonian function on the covariant-phase-space P, let g^i the local contact forms on P defined by $g^i = dq^i - \frac{\partial H}{\partial p^a_i} dx^a$; then a section-submanifold $G \subset P$ with $G = z(X)$ for $z \in \Gamma\big(P\big)$ is a lifted-submanifold of P if and only if $\forall\ g^i :\ z^* g^i = 0$.*

Note that this condition is equivalent to the first equation in the Hamilton-Volterra system 2.7.

The most interesting condition that G has to satisfy is the fourth one 4, which is the only dynamical one. As it was seen by Kijowski [94] and by his coauthors [98], [99], condition 4 can be translated in a geometric one; see also [70].

On a local chart U let's set $\beta_a := \partial_a \lrcorner \beta$ where β is the n-form of Proposition 13 and remember that $\partial_a \in TP$ is the vector field on U defined in local coordinate by $\partial_a = \frac{\partial}{\partial x^a}$. Let's also fix the following important definition:

Definition 16. *If L is a Lagrangian density on $J^1\pi$ and H is its corresponding Hamiltonian function on $\mathbb{FL}(J^1\pi)$, I call the* multisymplectic form *of $\mathbb{FL}(J^1\pi)$, the $(n+1)$-form $\omega \in \Omega^{n+1}P$ defined on local charts of an adapted atlas by the formula: $\omega := dp^a_i \wedge dq^i \wedge \beta_a - dH \wedge \beta$.*

The fact that this is a good definition relies on the proof that H is locally well defined in each point of each chart of P, on formula (2.6) and on the fact that the form $\theta := dq^i \wedge p^a_i \beta_a - H\beta$ is globally well defined; because then: $\omega = d\theta$. Let's see why the form θ is globally well defined. As we recalled before, P is a natural bundles with respect to the bundle E and so are the bundles T^*P, $\Lambda^{n-1}T^*P$, $\Lambda^n T^*P$ and $\Lambda^{n+1}T^*P$. If one changes coordinates on E as in (2.4), then:

$$\begin{aligned} \beta' &= \det\left(\frac{\partial x'}{\partial x}\right)\beta \\ \beta_{a'} &= \det\left(\frac{\partial x'}{\partial x}\right)\frac{\partial x^a}{\partial x^{a'}}(x^a)\beta_a \\ dq^{i'} &= dq^i \frac{\partial q^{i'}}{\partial q^i}(x^a, q^i) + dx^a \frac{\partial q^{i'}}{\partial x^a}(x^a, q^i) \end{aligned} \tag{2.8}$$

and, putting together (2.5), (2.6) and (2.8), we have that:

$$\theta'\left(x^{a'}, q^{i'}, p_{i'}^{a'}\right) = \theta\left(x^a, q^i, p_i^a\right) \tag{2.9}$$

Note that ω is closed and it can be also proved to be non degenerate, so throughout all this thesis I call it the multisymplectic form of $\mathbb{FL}(J^1\pi)$, although this name does not agree with all the definitions of multisymplectic forms given in the literature: see for example [49] where some other conditions (generally not satisfied by my ω) are required to define a multisymplectic form. The ω used here is a n-plectic form, using the terminology of Baez and C. L. Rogers, [4, 134], and of Richter [129]. However, since in the contest of field theories the term n-plectic has not become standard yet, I will keep calling ω multisymplectic.

Then condition 4 become:

Proposition 17. *Let ω be the multisymplectic form on the multimomenta space P, then a section-submanifold $G \subset P$, with $G = z(X)$ for $z \in \Gamma\big(P\big)$, is a Hamiltonian submanifold of P if and only if $\forall u \in TP,\ z^*(u \lrcorner \omega) = 0$.*

The proof of Proposition 17 can be done by directly calculating on local coordinates the value of $\omega(u, z_*\partial_1, \cdots, z_*\partial_n)$ using (2.7).

Note that in Proposition 17 it is requested in the hypothesis that G is a section-submanifold: the fact that G ends out to be also a lifted-submanifold is a consequence of the condition imposed on it. This can be seen easily if one notes that ω can also be written as

$$\omega = dp_i^a \wedge g^i \wedge \beta_a - \frac{\partial H}{\partial q^i} dq^i \wedge \beta$$

If a section-submanifold G satisfies the condition requested in proposition 17, then the latter condition can be applied to $u = \frac{\partial}{\partial p_j^b}$ and this yields that $\forall b, \forall j,\ z^*(g^j \wedge \beta_b) = 0$ which in turns yields that $\forall j,\ z^*(g^j) = 0$.

2.4 The symplectic structure of the covariant phase space

Let's study the tangent space of $\mathscr{G}$ on a point $G \in \mathscr{G}$. Let $\delta_u G \in T_G\mathscr{G}$ be a vector over G: it is associated to a so called *Jacobi field* $u \in \Gamma\left(i^*\left(VP\right)\right)$, id est a section over G of the pull-back image of the vertical (with respect to the projection π_P of the total space P onto the base X) tangent bundle VP by the embedding map $i\ :\ G \longrightarrow P$. The section u can be seen as a vector field on G, "following" which, each point $g \in G$ is sent to a point $g' \in G'$, where $G' \in \mathscr{G}$ is another Hamiltonian n-curve. An Hamiltonian n-curve G is deformed by u into an other Hamiltonian n-curve G'. We could write symbolically $\delta_u = \int u$. Note that for example u could be the restriction to G of a vector field $\bar{u} \in \Gamma\left(VP\right)$ defined on all P and satisfying the condition $L_{\bar{u}}\omega = 0$. Such a field is called infinitesimal symplectomorphism and since it preserves the multisymplectic form ω, its integral flux sends Hamiltonian n-curves to Hamiltonian n-curves and it so induces a path in $\mathscr{G}$, whose tangent vector in G would be exactly $\delta_u G$. If $u \in \Gamma\left(i^*\left(VP\right)\right)$ is not a restriction of a field $\bar{u}$ defined on all P, nonetheless, by using the fact that $G \in \mathscr{G}$, it is possible to give sense to and calculate $L_u\omega\left(u', X_1, \cdots, X_n\right)\big|_g\ \forall g \in G\ \forall u' \in \Gamma\left(VP\right),\ \forall X_1, \cdots, X_n \in \Gamma\left(TG\right)$, see Hélein []. The condition ensuring that u corresponds to a $\delta_u G$ tangent to G in $\mathscr{G}$ is precisely that

$$\forall g \in G\ ,\ \forall u' \in \Gamma\left(VP\right),\ \forall X_1, \cdots, X_n \in \Gamma\left(TG\right), L_u\omega\left(u', X_1, \cdots, X_n\right)\big|_g = 0 \tag{2.10}$$

See again the original papers of Kijowski for more details.

It is important to specify that there may be different u and $\tilde{u}$ corresponding to the same $\delta_u G$, so that $\delta_{\tilde{u}} = \delta_u = \int u = \int \tilde{u}$. This happens whenever the vector field $u - \tilde{u} \in \Gamma(TG)$: in other words when the vector field $u - \tilde{u}$ is in every point tangent to G. This by the way ensure that $u - \tilde{u}$ satisfies (2.10). We can always chose u to be in every point vertical with respect to π_P, id est without components tangent to the n-curve G.

On P we consider a slice Σ of co-dimension 1 with the property that for any Hamiltonian n-curve $G \in \mathscr{G}$ the intersection of Σ with G is transverse. It will be then $\pi_P(\Sigma) = \Sigma_X$ with Σ_X a $n-1$-submanifold of X. We also assume that $\forall G \in \mathscr{G},\ \Sigma \cap G = z(\Sigma_X)$ is an oriented manifold of dimension $n-1$ and we suppose that Σ_X and $z(\Sigma_X)$ have suitable coorientations with respect to the orientations of X and P, id est we suppose that the bundle $i_\Sigma^*(TP)/T\Sigma$ is oriented (where i_Σ is the immersion of Σ in P), see [73]. In the rest of this work, I will consider only slices which are the lift of slices on the base manifold X, $\Sigma = \pi^{-1}\Sigma_X$. We can then define Ω_Σ to be a functional acting on couples of vectors on $\mathscr{G}$ and sending them to $\mathbb{R}$ in this way:

Definition 18. *Let be $\delta_1 G,\ \delta_2 G \in T_G\mathscr{G}$ two vectors on $G \in \mathscr{G}$, and let be $u_1, u_2 \in \Gamma(i^*(VP))$ the corresponding vector fields over G, then we pose:*

$$\Omega_\Sigma\big|_G (\delta_1 G, \delta_2 G) \ := \ \int_{\Sigma\cap G} u_1 \wedge u_2 \lrcorner \omega \tag{2.11}$$

This hence defines Ω_Σ, our symplectic 2-form on $\mathscr{G}$.

Thanks to (2.10), Ω_Σ is well defined and it is antisymmetric. If L satisfies certain conditions of regularity, Ω_Σ is non degenerate, the proof is given by Hélein in a paper in preparation. For the proof that Ω_Σ is closed one can see [97]. It is a symplectic 2-form on $\mathscr{G}$: it was first introduced by Kijowski and Szczyrba in [97] and [98].

The natural question arises if the Ω actually depends on the choice of the slice Σ. We address this question in chapter 3.

A symplectic form Ω on $\mathscr{G}$ can be pull-back on $\mathscr{E}$. Remember that if $G \in \mathscr{G}$, then G is a lifted-submanifold of P, so it is the image of a section $z \in \Gamma\big(P\big)$. By the Legendre transform $\mathbb{FL}$ we can pull back it to a section $s \in \Gamma\big(J^1\pi\big)$ which is the lift by j^1 of a section $\phi \in \Gamma\big(E\big)$ which is a solution of the theory. If we call $\mathbb{GL}$ the inverse of one-to-one map just described, we have that

$$\begin{aligned} \mathbb{GL} \ &: \ \mathscr{E} \ \longrightarrow \ \mathscr{G} \\ \mathbb{GL}^{-1} \ &: \ \mathscr{G} \ \longrightarrow \ \mathscr{E} \end{aligned}$$

and if $\Omega \in \Omega^2\mathscr{G}$, then $\mathbb{GL}^*\Omega \in \Omega^2\mathscr{E}$.

If L is not regular, the construction just described doesn't work. We can still build $\omega \in \Omega^{n+1}\mathbb{FL}(J^1\pi)$, but in general ω will not be defined on all P. Nevertheless, we can still pull-back ω by $\mathbb{FL}$ on $J^1\pi$.

Let's call o this pull-back: $o := \mathbb{FL}^*\omega$. On a local chart we have that:

$$o := \mathbb{FL}^*\omega = dp_i^a \wedge dq^i \wedge \beta_a - dH \wedge \beta \tag{2.12}$$

where p_i^a and H are now considered as local function of the variables $x^a,\ q^i$ and $\dot{q}_a^i$. Remembering that the contact forms are $c^i = dq^i - \dot{q}_a^i dx^a$ and considering that $\dot{q}_b^i dx^b \wedge p_i^a \beta_a = \dot{q}_a^i p_i^a \beta$, we have that:

$$o = d\left[c^i \wedge p_i^a\beta_a + L \wedge \beta\right] = -p_i^a d\dot{q}_a^i \wedge \beta + dL \wedge \beta + dp_i^a \wedge c^i \wedge \beta_a \tag{2.13}$$

Note that $d\mathcal{L} - o = -d\left(c^i \wedge p_i^a\beta_a\right)$.

We can repeat on $J^1\pi$ the construction made before on P. On $J^1\pi$ we consider a slice Σ of co-dimension 1 with the property that, for any $s = \Gamma(J^1\pi)$, the intersection of Σ with $s(X)$ is transverse. An important example is $\Sigma = j^1\pi^{-1}(\Sigma_X)$ with Σ_X a $(n-1)$-submanifold of X. If we call i the embedding map $i \;:\; s(X) \;\hookrightarrow J^1\pi$, we can then define O_Σ to be a functional acting on couples of vectors on $s \in \Gamma(J^1\pi)$ and sending them to $\mathbb{R}$ in this way:

Definition 19. *Let be $\delta_1 s,\ \delta_2 s \in T_s\Gamma(J^1\pi)$ two vectors on $s \in \Gamma(J^1\pi)$, and let be $u_1, u_2 \in \Gamma\left(i^*\left(V_{j^1\pi}J^1\pi\right)\right)$ the corresponding vertical vector fields over $s(X)$, then we pose:*

$$O_\Sigma\big|_s(\delta_1 s, \delta_2 s) \;:=\; \int_{\Sigma\cap s(X)} u_1 \wedge u_2 \lrcorner o \tag{2.14}$$

This gives us a 2-form O_Σ on $\Gamma(J^1\pi)$.

O_Σ can be restricted to $j^1\mathscr{E} \subset j^1\Gamma(E) \subset \Gamma(J^1\pi)$. Let be $i \;:\; j^1\phi(X) \;\hookrightarrow J^1\pi$ the embedding map of $j^1\phi(X)$. Let be $\delta_1 j^1\phi,\ \delta_2 j^1\phi \in T_{j^1\phi}j^1\mathscr{E}$ two vectors on $j^1\phi \in j^1\mathscr{E}$, and let be $u_1,\ u_2 \in \Gamma\left(i^*\left(V_{j^1\pi}J^1\pi\right)\right)$ the corresponding vertical vector fields over $j^1\phi(X)$, then we pose:

$$O_\Sigma\big|_{j^1\phi}\left(\delta_1 j^1\phi, \delta_2 j^1\phi\right) \;:=\; \int_{\Sigma\cap j^1\phi(X)} u_1 \wedge u_2 \lrcorner o \tag{2.15}$$

On a local chart U, we have:

$$\begin{aligned} O_{\Sigma,U}\big|_{j^1\phi}\left(\delta_1 j^1\phi, \delta_2 j^1\phi\right) \;:=\; & \int_{\Sigma\cap j^1\phi(U)} u_1 \wedge u_2 \lrcorner o = \\ = & \int_{\Sigma\cap j^1\phi(U)} u_1 \wedge u_2 \lrcorner dp_i^a \wedge dq^i \wedge \beta_a = \\ = \int_{\Sigma\cap j^1\phi(U)} u_1 \wedge u_2 \lrcorner & \left[\frac{\partial^2 L}{\partial q^j \partial \dot{q}_a^i} dq^j \wedge dq^i \wedge \beta_a + \frac{\partial^2 L}{\partial \dot{q}_b^j \partial \dot{q}_a^i} d\dot{q}_b^j \wedge dq^i \wedge \beta_a\right] \end{aligned} \tag{2.16}$$

Finally the restriction of O_Σ to $j^1\mathscr{E}$ can be pull-back by j^1 to $\mathscr{E}$. We have the following:

Definition 20. *Let be $\delta_1\phi,\ \delta_2\phi \in T_\phi\mathscr{E}$ two vectors on $\phi \in \mathscr{E}$, and let be $u_1,\ u_2 \in \Gamma\left(\phi^* V_\pi E\right)$ the corresponding vertical vector fields over $\phi(X)$, then we pose:*

$$\Omega_{\Sigma_X}\big|_\phi(\delta_1\phi, \delta_2\phi) \;:=\; \int_{\Sigma\cap j^1\phi(X)} j^1_* u_1 \wedge j^1_* u_2 \lrcorner o \tag{2.17}$$

and $\Omega_{\Sigma_X} \in \Omega^2\mathscr{E}$ is our symplectic form on $\mathscr{E}$.

It can be shown that if L is regular, then Ω_{Σ_X} and $\mathbb{GL}^*\Omega_\Sigma$ coincide.

I have to note that calling O_Σ and Ω_{Σ_X} symplectic forms is an abuse. They are indeed 2-forms on their, possibly infinite dimensional, manifolds of definition, but they may not be non-degenerate, unless some regularity conditions on L are assumed.

Chapter 3

Nonequivalent symplectic structures on covariant phase space

At the end of last chapter I have shown how one can build a symplectic structure Ω_Σ on the covariant phase space starting from the multisymplectic form ω defined on the multiphase space P and from a cooriented slice Σ of P. What happens if one makes the same construction starting from a different slice $\bar{\Sigma}$?

Hélein in [70] states that $\Omega_\Sigma = \Omega_{\bar{\Sigma}}$ if for every Hamiltonian surface G, $\Sigma \cap G$ and $\bar{\Sigma} \cap G$ are compact and if Σ and $\bar{\Sigma}$ are in the same homology class.

The same results holds when $\Sigma \cap G$ is not compact, provided that we consider a subset of $\mathscr{I} \subset \mathscr{G}$ such that all $G \in \mathscr{I}$ satisfy some specific decay conditions at infinity.

The extended proofs of the corresponding theorems were privately communicated to me by the author and will be soon available in a paper in preparation.

It remains to verify if Ω_Σ is independent from Σ when one of the above conditions are not satisfied.

In this chapter I will show that indeed Ω_Σ, with Σ compact, does depend on the class of homology of Σ for certain field theories. In section 3.1 I will study some simple field theories defined on 2-dimensional tori. Those simple examples will be enough to fix the result. In section 3.2 I will instead consider the case of Σ non compact, studying simple theories on $\mathbb{R}^2$. I will then discuss how the results of Hélein for non compact Σ can be interpreted.

3.1 Symplectic structures on spaces of solutions of field theories over tori

In this section I consider bundles whose bases X are tori.

I note by $\mathbb{T}_{a,b}$ the Torus obtained as a quotient of the plane $\mathbb{R}^2$ by $a\mathbb{Z} \times b\mathbb{Z}$, with $a, b \in \mathbb{R}^+$.

3.1.1 Space of scalar fields over two dimensional tori.

Let's consider the bundle $E = \mathbb{T}_{a,b} \times \mathbb{R}$. We will have then:

$$\begin{array}{c} \mathbb{T}_{a,b} \times \mathbb{R} \\ {\scriptstyle\pi}\Big\downarrow \\ \mathbb{T}_{a,b} \end{array}$$

Being this bundle trivial, a section ϕ corresponds to a smooth function $\phi \ : \ \mathbb{T}_{a,b} \longrightarrow \mathbb{R};\ x^\mu \longrightarrow q = \phi(x^\mu)$.

On an chart $U \subset \mathbb{T}_{a,b}$, the local coordinates will be x^μ, $\mu = 0,1$ with $x^0 \in (0,a)$ and $x^1 \in (0,b)$. On $\mathbb{R}$ I will use the coordinate q. On $J^1\pi$ I will use the coordinates $(x^\mu, q, \dot{q}_\mu)$, so that $\forall x \in \mathbb{T}_{a,b},\ \forall \phi \in \Gamma\big(\mathbb{T}_{a,b} \times \mathbb{R}\big),\ \dot{q}_\mu \left(j^1\phi\left(x\right)\right) = \partial_\mu \phi$.

In local coordinates we have $\mathcal{L} : (x^\mu, q, \dot{q}_\mu) \longrightarrow L\left(x^\mu, q, \dot{q}_\mu\right)\beta,\ \beta = dx^0 \wedge dx^1$ being the standard volume 2-form on $\mathbb{T}_{a,b}$ and L, the Lagrangian density, being a local function between j^1U and $\mathbb{R}$.

We have that $P = Hom_\pi(VE, \Lambda^{n-1}T^*X) = \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} Hom\left(\mathbb{T}_{a,b} \times T\mathbb{R}, \Lambda^{n-1}T^*\mathbb{T}_{a,b}\right) \simeq \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} \left(\Lambda^{n-1}T^*\mathbb{T}_{a,b} \otimes T^*\mathbb{R}\right) \simeq \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} T^*\mathbb{T}_{a,b}$ because we identify the real line $\mathbb{R}$ with its tangent and with its cotangent. So we define the local Hamiltonian function:

$$\begin{aligned} H \ : \ \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} T^*\mathbb{T}_{a,b} \quad &\longrightarrow \mathbb{R} \\ (x^\mu, q, p^\mu) \qquad\qquad &\longmapsto p^\mu \dot{q}_\mu - L\left(x^\mu, q, \dot{q}_\mu\right) \end{aligned}$$

and we assume that $\dot{q}_\mu$ is the unique solution of $\frac{\partial L}{\partial \dot{q}_\mu}\left(x^\mu, q, \dot{q}_\mu\right) = p^\mu$.

As in section 2.2, to every section $\phi \in \Gamma\left(\mathbb{T}_{a,b} \times \mathbb{R}\right)$ we associate a section z of P, which in this case will be the section $z : \mathbb{T}_{a,b} \longrightarrow P$ defined by: $p^\mu\left(x^\nu\right) := \frac{\partial L}{\partial \dot{q}_\mu}\left(x^\nu, \phi, \partial_\mu\phi\right), \forall x \in \mathbb{T}_{a,b}$.

The Hamilton-Volterra system (2.7) become then:

$$\begin{cases} \dfrac{\partial \phi}{\partial x^\mu} = \dfrac{\partial H}{\partial p^\mu}\left(x^\nu, \phi, p^\nu\right) \\ \dfrac{\partial p^\mu}{\partial x^\mu} = -\dfrac{\partial H}{\partial \phi}\left(x^\nu, \phi, p^\nu\right) \end{cases} \tag{3.1}$$

The section $z \in \Gamma\left(P\right)$ reads:

$$z \ : \ x^\nu \longrightarrow \left(x^\mu, \phi, p^\mu\right)$$

and its image G is a submanifold of P. Since z depends on ϕ, we can associate to any ϕ section of $\left(\mathbb{T}_{a,b} \times \mathbb{R}\right) \xrightarrow{\pi} \mathbb{T}_{a,b}$, a submanifold G_ϕ of $P = \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} T^*\mathbb{T}_{a,b}$.

The field ϕ satisfies (3.1) if and only if G_ϕ is an hamiltonian 2-curve, id est satisfies the condition in proposition (17), which now is:

$$\forall X_1, X_2 \in \Gamma\left(TG_\phi\right), \forall \xi \in \Gamma\left(T\left(\mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} T^*\mathbb{T}_{a,b}\right)\right)$$

$$\left(dp^0 \wedge d\phi \wedge \beta_0 + dp^1 \wedge d\phi \wedge \beta_1 - dH \wedge \beta\right)\left(\xi, X_1, X_2\right) = 0$$

or:

$$\omega\left(\xi, X_1, X_2\right) = 0 \tag{3.2}$$

where:

$$\begin{aligned} \omega :=& dp^0 \wedge d\phi \wedge \beta_0 + dp^1 \wedge d\phi \wedge \beta_1 - dH \wedge \beta \\ =& dp^0 \wedge d\phi \wedge dx^1 - dp^1 \wedge d\phi \wedge dx^0 - dH \wedge dx^0 \wedge dx^1 \end{aligned}$$

We call again $\mathscr{E}$ the spaces of solution of our field theory for some Lagrangian. If the Lagrangian is regular, $\mathscr{E}$ will be isomorphic to the space of all Hamiltonian 2-curves, satisfying (3.2). For studying $\mathscr{E}$ it is useful to note that any smooth function $\phi \ : \ \mathbb{T}_{a,b} \longrightarrow \mathbb{R}$ can be developed in Fourier series. To make the following calculations easier, I will consider complex functions instead of real one (I complexify the space $\mathbb{R}$) and then I will again come back to real spaces at the end of calculations. So for every ϕ we will have:

$$\phi\left(x^0, x^1\right) = \sum_{k,j\in\mathbb{Z}} \phi^{kj} e^{i2\pi\left(\frac{k}{a}x^0 + \frac{j}{b}x^1\right)} \tag{3.3}$$

with $\phi^{kj} \in \mathbb{C}$. This development is useful since we are going to study Lagrangians which lead to Euler-Lagrange equations and Hamilton equations, linear with-respect to the field ϕ.

3.1.2 Free scalar field over a two dimensional torus.

Let's consider the simple Lagrangian density $L : J^1\pi \longrightarrow \mathbb{R}$ defined by:

$$L\left(x^\mu, q, \dot{q}_\mu\right) = \frac{1}{2}\left(\dot{q}_0\right)^2 - \frac{1}{2}\left(\dot{q}_1\right)^2 \tag{3.4}$$

To each field $\phi\left(x^\mu\right)$ we associate the momenta $p^\mu\left(x^\nu\right) := \frac{\partial L}{\partial \dot{q}_\mu}\left(x^\nu, \phi, \partial_\mu \phi\right)$, and we have:

$$p^0 = \partial_0 \phi \tag{3.5}$$

$$p^1 = -\partial_1 \phi \tag{3.6}$$

the Hamiltonian $H : \mathbb{T}_{a,b} \times \mathbb{R} \times_{\mathbb{T}_{a,b}} T^*\mathbb{T}_{a,b} \longrightarrow \ \mathbb{R}$ is then:

$$H\left(x^\mu, q, p^\mu\right) = \frac{1}{2}\left(p^0\right)^2 - \frac{1}{2}\left(p^1\right)^2 \tag{3.7}$$

From now on I write $x^0 = t$ and $x^1 = x$. The Euler-Lagrange equation is then the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{3.8}$$

A field ϕ developed in Fourier series as in (3.3) is a solution of our theory if and only if $\phi^{kj} = 0$ whenever k and j don't satisfy the following condition:

$$\frac{k^2}{a^2} - \frac{j^2}{b^2} = 0 \tag{3.9}$$

This Diophantin equation has an infinite number of solutions whenever the ratio $\frac{a}{b}$ is a rational number. If $\frac{a}{b}$ is not a rational number, then the space of solutions reduce to the one dimensional vector space of constants over $\mathbb{T}_{a,b}$, which is of no interest for our discussion. We consider the case when $\frac{a}{b} \in \mathbb{Q}$: then the space of all solution $\mathscr{E}$ is an infinite dimensional vector space and therefore an infinite dimensional manifold.

Every solution ϕ can be written as:

$$\phi\left(t, x\right) = \sum_{k\in\mathbb{Z}_0 \text{ so that } k\frac{b}{a}\in\mathbb{Z}_0} \left(\phi^{k+} e^{i2\pi\left(\frac{k}{a}t + \frac{k}{a}x\right)} + \phi^{k-} e^{i2\pi\left(\frac{k}{a}t - \frac{k}{a}x\right)}\right) + \phi^0 \tag{3.10}$$

where ϕ^{k+}, ϕ^{k-} and ϕ^0 are numbers and $\mathbb{Z}_0 \equiv \mathbb{Z} \setminus \{0\}$.

Without lack of generality we can suppose $a = 1$. Indeed studying the theory with Lagrangian (3.4) on the torus $\mathbb{T}_{a,b}$ is completely equivalent to study the theory with a Lagrangian of the same form on the torus $\mathbb{T}_{1,\frac{b}{a}}$.

Hence we set $a = 1$ and we study (3.4) on the torus $\mathbb{T}_{1,b}$. The condition (3.9) then become:

$$k^2 - \frac{j^2}{b^2} = 0 \tag{3.11}$$

which implies that b must be a rational number, and the solutions of the theory can be written as:

$$\phi\left(t,x\right) = \sum_{k\in\mathbb{Z}_0 \text{ so that } kb\in\mathbb{Z}_0} \left(\phi^{k+} e^{i2\pi(kt+kx)} + \phi^{k-} e^{i2\pi(kt-kx)}\right) + \phi^0 \tag{3.12}$$

where $t\in(0,1)$ and $x\in(0,b)$.

It is sometime useful to think of $\mathscr{E}$ as the direct sum of finite dimensional vector spaces:

$$\mathscr{E} = \left(\oplus_{k\in A}\mathscr{E}_k\right) \oplus \mathscr{E}_0 \tag{3.13}$$

where $\mathscr{E}_0$ is the one dimensional vector space generated by the function $\boldsymbol{\phi_0} = 1$ constant on $\mathbb{T}_{a,b}$ and $\mathscr{E}_k$ is the 4 dimensional vector space spanned by the functions $\boldsymbol{\phi_{k\nwarrow}} = e^{i2\pi(kt+kx)}$, $\boldsymbol{\phi_{k\nearrow}} = e^{i2\pi(kt-kx)}$, $\boldsymbol{\phi_{k\swarrow}} = e^{i2\pi(-kt+kx)}$ and $\boldsymbol{\phi_{k\searrow}} = e^{i2\pi(-kt-kx)}$; A is the infinite subset of $\mathbb{N}$ determined by the condition (3.11) which we translate into to the condition $kb \in \mathbb{N}\setminus\{0\}$. Then every $\phi \in \mathscr{E}$ can be decomposed as:

$$\phi = \phi^0\boldsymbol{\phi_0} + \phi^{k\nwarrow}\boldsymbol{\phi_{k\nwarrow}} + \phi^{k\nearrow}\boldsymbol{\phi_{k\nearrow}} + \phi^{k\searrow}\boldsymbol{\phi_{k\searrow}} + \phi^{k\swarrow}\boldsymbol{\phi_{k\swarrow}} \tag{3.14}$$

where Einstein convention is used for repeated indexes and k runs over $\mathbb{N}_0 \equiv \mathbb{N}\setminus\{0\}$. The numbers ϕ^0, $\phi^{k\nwarrow}$, $\phi^{k\nearrow}$, $\phi^{k\searrow}$ and $\phi^{k\swarrow}$ are coordinates on $\mathscr{E}$. Note that, to be more precise, the vector space in (3.13) is the complexified of the space $\mathscr{E}$ of solutions that we want to study. We should name it ${}^{\mathbb{C}}\mathscr{E}$, and the same we should do for its subspaces and for their tangent spaces, but we won't do so to avoid a heavier notation. We will come back to the real vector spaces at the end of this section.

In this situation it is easy to explicitly identify $T\mathscr{E}$. If G is an hamiltonian 2-curve, i.e. an element of $\mathscr{E}$, then it corresponds to a ϕ as in (3.14). A vector $\delta_u G \in T_G\mathscr{E}$ must be of the form:

$$\begin{aligned}\delta_u G &= \sum_{k\in\mathbb{Z}_0 \text{ so that } kb\in\mathbb{Z}_0} \left(u^{k+}\frac{\partial}{\partial\phi^{k+}} + u^{k-}\frac{\partial}{\partial\phi^{k-}}\right) + u^0\frac{\partial}{\partial\phi^0} \\ &= \sum_{k\in\mathbb{N}_0 \text{ so that } kb\in\mathbb{N}_0} \left(u^{k\nearrow}\frac{\partial}{\partial\phi^{k\nearrow}} + u^{k\nwarrow}\frac{\partial}{\partial\phi^{k\nwarrow}} + u^{k\searrow}\frac{\partial}{\partial\phi^{k\searrow}} + u^{k\swarrow}\frac{\partial}{\partial\phi^{k\swarrow}}\right) + u^0\frac{\partial}{\partial\phi^0}\end{aligned} \tag{3.15}$$

where u^{k+}, u^{k-}, $u^{k\nearrow}$, $u^{k\nwarrow}$, $u^{k\searrow}$, $u^{k\swarrow}$ and u^0 are numbers. When $k > 0$, we have that $u^{k+} = u^{k\nwarrow}$ and $u^{k-} = u^{k\nearrow}$; when $k < 0$, we have that $u^{k+} = u^{-k\swarrow}$ and $u^{k-} = u^{-k\searrow}$.

On the other hand $\delta_u G$ must correspond to a suitable $u \in \Gamma\left(i^*V\left(\mathbb{T}_{a,b}\times\mathbb{R}\times_{\mathbb{T}_{a,b}}T^*\mathbb{T}_{a,b}\right)\right)$, where i is the embedding map of G in $\mathbb{T}_{a,b}\times\mathbb{R}\times_{\mathbb{T}_{a,b}}T^*\mathbb{T}_{a,b}$. If on G we use t and x as local coordinates, and remembering (3.5) and (3.6), we have:

$$i\ :\ (t,x) \longrightarrow \left(t,x,\phi\left(t,x\right),\partial_0\phi\left(t,x\right),-\partial_1\phi\left(t,x\right)\right)$$

As explained in the previous chapter the correspondence between $\delta_u G$ and u is 1 to 1 if u is chosen to be vertical as in this case. The 'Jacobi field' representing $\delta_u G$ is then the vector field

u defined on G in this way:

$$\begin{aligned}
u(t,x) = & \sum_{k\in\mathbb{Z}_0 \text{ so that } kb\in\mathbb{Z}_0} \left(u^{k+}e^{i2\pi(kt+kx)} + u^{k-}e^{i2\pi(kt-kx)}\right)\frac{\partial}{\partial\phi} + u^0\frac{\partial}{\partial\phi} \\
& + \sum_{k\in\mathbb{Z}_0 \text{ so that } kb\in\mathbb{Z}_0} i2\pi k\left(u^{k+}e^{i2\pi(kt+kx)} + u^{k-}e^{i2\pi(kt-kx)}\right)\frac{\partial}{\partial p^0} \\
& - \sum_{k\in\mathbb{Z}_0 \text{ so that } kb\in\mathbb{Z}_0} i2\pi k\left(u^{k+}e^{i2\pi(kt+kx)} - u^{k-}e^{i2\pi(kt-kx)}\right)\frac{\partial}{\partial p^1}
\end{aligned}$$

where u^{k+}, u^{k-} and u^0 are the same we had in (3.15). This vector field has no component parallel to the 2-curve G.

As a base on the vector space $T_G\mathscr{E}$ we can chose the vector $\delta_0 G = \frac{\partial}{\partial\phi^0}$ with the families of vectors $\delta_{k+}G = \frac{\partial}{\partial\phi^{k+}}$ and $\delta_{k-}G = \frac{\partial}{\partial\phi^{k-}}$. The corresponding Jacobi fields are the vector fields:

$$\begin{aligned}
\xi_0(t,x) =& \frac{\partial}{\partial\phi} \\
\xi_{k+}(t,x) =& e^{i2\pi(kt+kx)}\left[\frac{\partial}{\partial\phi} + i2\pi k\left(\frac{\partial}{\partial p^0} - \frac{\partial}{\partial p^1}\right)\right] \\
\xi_{k-}(t,x) =& e^{i2\pi(kt-kx)}\left[\frac{\partial}{\partial\phi} + i2\pi k\left(\frac{\partial}{\partial p^0} + \frac{\partial}{\partial p^1}\right)\right]
\end{aligned} \tag{3.16}$$

with $k\in\mathbb{Z}_0$ so that $kb\in\mathbb{Z}_0$. Note that their expressions (3.16) do not depend in fact from G.

Alternatively, as a base on the vector space $T_G\mathscr{E}$, we can chose the vector $\delta_0 G = \frac{\partial}{\partial\phi^0}$ with the families of vectors $\delta_{k\nearrow}G = \frac{\partial}{\partial\phi^{k\nearrow}}$, $\delta_{k\nwarrow}G = \frac{\partial}{\partial\phi^{k\nwarrow}}$, $\delta_{k\searrow}G = \frac{\partial}{\partial\phi^{k\searrow}}$ and $\delta_{k\swarrow}G = \frac{\partial}{\partial\phi^{k\swarrow}}$; with $k\in\mathbb{N}_0$, such that $kb\in\mathbb{N}_0$.

We now chose two slices Σ and $\bar\Sigma$ of $(\mathbb{T}_{1,b}\times\mathbb{R})\oplus T\mathbb{T}_{1,b}$ which do not belong to the same homology class. We take Σ to be the slice defined in coordinates by the equation $t=0$, and $\bar\Sigma$ to be the one defined by $x=0$. We can now compute the two symplectic structures arising from these choices and then we can compare them.

We first compute Ω_G defined by:

$$\Omega_G(\delta_1 G, \delta_2 G) := \int_{\Sigma\cap G} \xi_1\wedge\xi_2 \lrcorner\,\omega$$

We set $\Omega_{k+,k'-} := \Omega_G(\delta_{k+}G, \delta_{k'-}G)$ and similarly for $\Omega_{k+,k'+}$ and $\Omega_{k-,k'-}$. In order to make the computation, we fix coordinates on $\Sigma\cap G$. This is easy since $\Sigma\cap G$ is the submanifold of G defined in coordinates t and x by the equation $t=0$. We have then:

$$\begin{aligned}
\Omega_{k+,k'-} &= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'-}\lrcorner\,\omega \\
\Omega_{k+,k'+} &= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'+}\lrcorner\,\omega \\
\Omega_{k-,k'-} &= \int_{\Sigma\cap G} \xi_{k-}\wedge\xi_{k'-}\lrcorner\,\omega
\end{aligned}$$

And since:

$$\begin{aligned}
\xi_{k+}\wedge\xi_{k'-}\lrcorner\,\omega &= \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\wedge dx - dp^1\wedge d\phi\wedge dt - dH\wedge dt\wedge dx\right) \\
&= \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\wedge dx - dp^1\wedge d\phi\wedge dt\right) \\
&= \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\right)dx - \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^1\wedge d\phi\right)dt
\end{aligned} \tag{3.17}$$

and analogously for $\Omega_{k+,k'+}$ and $\Omega_{k-,k'-}$; we have:

$$\begin{aligned}
\Omega_{k+,k'-} &= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'-}\lrcorner\omega \\
&= \int_{\Sigma\cap G} \left[\xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\right)dx - \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^1\wedge d\phi\right)dt\right] \\
&= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\right)dx \\
&= \int_0^b \left[i2\pi k e^{i2\pi kx} e^{i2\pi\left(-k'x\right)} - i2\pi k' e^{i2\pi\left(-k'x\right)} e^{i2\pi kx}\right] dx
\end{aligned} \tag{3.18}$$

Remembering that k, k', kb and $k'b$ are integer, we finally have:

$$\Omega_{k+,k'-} = i2\pi b\delta_{kk'}\left(k-k'\right) = 0 \tag{3.19}$$

where $\delta_{ab}=1$ if $a=b$ and $\delta_{ab}=0$ if $a\neq b$.

In the same way we compute $\Omega_{k+,k'+}$ and $\Omega_{k-,k'-}$, and we have:

$$\begin{aligned}
\Omega_{k+,k'+} &= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'+}\lrcorner\omega \\
&= \int_{\Sigma\cap G} \left[\xi_{k+}\wedge\xi_{k'+}\lrcorner\left(dp^0\wedge d\phi\right)dx - \xi_{k+}\wedge\xi_{k'+}\lrcorner\left(dp^1\wedge d\phi\right)dt\right] \\
&= \int_{\Sigma\cap G} \xi_{k+}\wedge\xi_{k'+}\lrcorner\left(dp^0\wedge d\phi\right)dx \\
&= \int_0^b \left[i2\pi k e^{i2\pi kx} e^{i2\pi k'x} - i2\pi k' e^{i2\pi k'x} e^{i2\pi kx}\right] dx \\
&= i2\pi b\delta_{k(-k')}\left(k-k'\right) = i4\pi bk\delta_{k(-k')}
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\Omega_{k-,k'-} &= \int_{\Sigma\cap G} \xi_{k-}\wedge\xi_{k'-}\lrcorner\omega \\
&= \int_{\Sigma\cap G} \left[\xi_{k-}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\right)dx - \xi_{k-}\wedge\xi_{k'-}\lrcorner\left(dp^1\wedge d\phi\right)dt\right] \\
&= \int_{\Sigma\cap G} \xi_{k-}\wedge\xi_{k'-}\lrcorner\left(dp^0\wedge d\phi\right)dx \\
&= \int_0^b \left[-i2\pi k e^{-i2\pi kx} e^{i2\pi k'x} - i2\pi k' e^{-i2\pi k'x} e^{-i2\pi kx}\right] dx \\
&= i2\pi b\delta_{k(-k')}\left(k-k'\right) = i4\pi bk\delta_{k(-k')}
\end{aligned} \tag{3.21}$$

Finally, setting $\Omega_{0,\cdot} := \Omega_G\left(\delta_0 G,\cdot\right)$, it is easy to see that $\Omega_{0,\cdot}=0$.

It is important to note that Ω, which could depend a priori on G, in fact does not. This is because, in this simple case, $\mathscr{E}$, which is a vector space, and $T_G\mathscr{E}$, can be identified for every G and in fact Ω is the same on every $T_G\mathscr{E}$.

For every G we can therefore split $T_G\mathscr{E}$ so that:

$$T_G\mathscr{E} = \left(\oplus_{k\in A} T_{Gk}\mathscr{E}\right)\oplus T_{G0}\mathscr{E} \tag{3.22}$$

where A is the infinite subset of $\mathbb{N}$ determined by the condition $kb\in\mathbb{N}\setminus\{0\}$, $T_{G0}\mathscr{E}$ is the one dimensional vector space generated by $\delta_0 G$ and $T_{Gk}\mathscr{E}$ is the 4-dimensional vector space spanned by $\delta_{k\nearrow}G$, $\delta_{k\nwarrow}G$, $\delta_{k\searrow} =$ and $\delta_{k\swarrow}G$. On $T_{Gk}\mathscr{E}$ it may be useful to visualize the 2-form Ω using the matrix of its components:

$$\begin{pmatrix} \Omega_{k\nwarrow,k\nwarrow} & \Omega_{k\nwarrow,k\searrow} & \Omega_{k\nwarrow,k\nearrow} & \Omega_{k\nwarrow,k\swarrow} \\ \Omega_{k\searrow,k\nwarrow} & \Omega_{k\searrow,k\searrow} & \Omega_{k\searrow,k\nearrow} & \Omega_{k\searrow,k\swarrow} \\ \Omega_{k\nearrow,k\nwarrow} & \Omega_{k\nearrow,k\searrow} & \Omega_{k\nearrow,k\nearrow} & \Omega_{k\nearrow,k\swarrow} \\ \Omega_{k\swarrow,k\nwarrow} & \Omega_{k\swarrow,k\searrow} & \Omega_{k\swarrow,k\nearrow} & \Omega_{k\swarrow,k\swarrow} \end{pmatrix} = i4\pi bk \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We then compute $\bar{\Omega}_G$ defined by:

$$\bar{\Omega}_G\left(\delta_1 G, \delta_2 G\right) := \int_{\bar{\Sigma}\cap G} \xi_1 \wedge \xi_2 \lrcorner \omega$$

Again we set $\bar{\Omega}_{k+,k'-} := \bar{\Omega}_G\left(\delta_{k+}G, \delta_{k'-}G\right)$ and similarly for $\bar{\Omega}_{k+,k'+}$, $\bar{\Omega}_{k-,k'-}$ and $\bar{\Omega}_{0,\cdot}$. We have then:

$$\begin{aligned} \bar{\Omega}_{k+,k'-} &= \int_{\bar{\Sigma}\cap G} \xi_{k+} \wedge \xi_{k'-} \lrcorner \omega \\ &= \int_{\bar{\Sigma}\cap G} \left[\xi_{k+} \wedge \xi_{k'-} \lrcorner \left(dp^0 \wedge d\phi\right) dx - \xi_{k+} \wedge \xi_{k'-} \lrcorner \left(dp^1 \wedge d\phi\right) dt\right] \\ &= \int_{\bar{\Sigma}\cap G} -\xi_{k+} \wedge \xi_{k'-} \lrcorner \left(dp^1 \wedge d\phi\right) dt \\ &= \int_0^1 - \left[-i2\pi k e^{i2\pi kt} e^{i2\pi\left(k't\right)} - i2\pi k' e^{i2\pi(kt)} e^{i2\pi k't}\right] dt \\ &= -\delta_{k(-k')}\left[-i2\pi k - i2\pi k'\right] = 0 \end{aligned} \tag{3.23}$$

$$\begin{aligned} \bar{\Omega}_{k+,k'+} &= \int_{\bar{\Sigma}\cap G} \xi_{k+} \wedge \xi_{k'+} \lrcorner \omega \\ &= \int_{\bar{\Sigma}\cap G} \left[\xi_{k+} \wedge \xi_{k'+} \lrcorner \left(dp^0 \wedge d\phi\right) dx - \xi_{k+} \wedge \xi_{k'+} \lrcorner \left(dp^1 \wedge d\phi\right) dt\right] \\ &= \int_{\bar{\Sigma}\cap G} -\xi_{k+} \wedge \xi_{k'+} \lrcorner \left(dp^1 \wedge d\phi\right) dt \\ &= \int_0^1 - \left[-i2\pi k e^{i2\pi kt} e^{i2\pi k't} + i2\pi k' e^{i2\pi k't} e^{i2\pi kt}\right] dt \\ &= -\delta_{k(-k')}\left[-i2\pi k + i2\pi k'\right] \\ &= i4\pi k \delta_{k(-k')} \end{aligned} \tag{3.24}$$

$$\begin{aligned} \bar{\Omega}_{k-,k'-} &= \int_{\bar{\Sigma}\cap G} \xi_{k-} \wedge \xi_{k'-} \lrcorner \omega \\ &= \int_{\bar{\Sigma}\cap G} \left[\xi_{k-} \wedge \xi_{k'-} \lrcorner \left(dp^0 \wedge d\phi\right) dx - \xi_{k-} \wedge \xi_{k'-} \lrcorner \left(dp^1 \wedge d\phi\right) dt\right] \\ &= \int_{\bar{\Sigma}\cap G} -\xi_{k-} \wedge \xi_{k'-} \lrcorner \left(dp^1 \wedge d\phi\right) dt \\ &= \int_0^1 - \left[i2\pi k e^{i2\pi kt} e^{i2\pi k't} - i2\pi k' e^{i2\pi k't} e^{i2\pi kt}\right] dt \\ &= -\delta_{k(-k')}\left[i2\pi k - i2\pi k'\right] \\ &= -i4\pi k \delta_{k(-k')} \end{aligned} \tag{3.25}$$

and

$$\bar{\Omega}_{0,\cdot} = 0 \tag{3.26}$$

On $T_{Gk}\mathscr{E}$ the components of the 2-form $\bar{\Omega}$ form the matrix:

$$\begin{pmatrix} \overline{\Omega}_{k\nwarrow,k\nwarrow} & \overline{\Omega}_{k\nwarrow,k\searrow} & \overline{\Omega}_{k\nwarrow,k\nearrow} & \overline{\Omega}_{k\nwarrow,k\swarrow} \\ \overline{\Omega}_{k\searrow,k\nwarrow} & \overline{\Omega}_{k\searrow,k\searrow} & \overline{\Omega}_{k\searrow,k\nearrow} & \overline{\Omega}_{k\searrow,k\swarrow} \\ \overline{\Omega}_{k\nearrow,k\nwarrow} & \overline{\Omega}_{k\nearrow,k\searrow} & \overline{\Omega}_{k\nearrow,k\nearrow} & \overline{\Omega}_{k\nearrow,k\swarrow} \\ \overline{\Omega}_{k\swarrow,k\nwarrow} & \overline{\Omega}_{k\swarrow,k\searrow} & \overline{\Omega}_{k\swarrow,k\nearrow} & \overline{\Omega}_{k\swarrow,k\swarrow} \end{pmatrix} = i4\pi k \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It may be the moment to come back to real vector spaces of solutions. On the vector spaces $\mathscr{E}$, $\mathscr{E}_0$ and $\mathscr{E}_k$ in (3.13) we consider the usual operation $*$ of complex conjugation. It is easy to see that it acts so that:

$$\begin{aligned} \boldsymbol{\phi}^*_{k+} &= \boldsymbol{\phi}_{-k+} \\ \boldsymbol{\phi}^*_{k-} &= \boldsymbol{\phi}_{-k-} \end{aligned}$$

or equivalently:

$$\begin{aligned} \boldsymbol{\phi}^*_{k\nwarrow} &= \boldsymbol{\phi}_{k\searrow} \\ \boldsymbol{\phi}^*_{k\nearrow} &= \boldsymbol{\phi}_{k\swarrow} \end{aligned}$$

We then look for elements of $\mathscr{E}$ invariant for the complex conjugation: they represent the real fields solution of (3.8) and they form a subspace which is a real vector space. For simplicity of notation we will continue to call it $\mathscr{E}$ and we'll call $T\mathscr{E}$ its real tangent space using again the same name used for its complexified's one. $\mathscr{E}$ also splits in components as in (3.13). On the four dimensional components $\mathscr{E}_k$ we can consider as a base the elements:

$$\begin{aligned} \boldsymbol{\phi}_{k1} &= \frac{\boldsymbol{\phi}_{k\nwarrow} + \boldsymbol{\phi}_{k\searrow}}{2} = \cos\left[2\pi\left(kt + kx\right)\right] \\ \boldsymbol{\phi}_{k2} &= \frac{\boldsymbol{\phi}_{k\nwarrow} - \boldsymbol{\phi}_{k\searrow}}{2i} = \sin\left[2\pi\left(kt + kx\right)\right] \\ \boldsymbol{\phi}_{k3} &= \frac{\boldsymbol{\phi}_{k\nearrow} + \boldsymbol{\phi}_{k\swarrow}}{2} = \cos\left[2\pi\left(kt - kx\right)\right] \\ \boldsymbol{\phi}_{k4} &= \frac{\boldsymbol{\phi}_{k\nearrow} + \boldsymbol{\phi}_{k\swarrow}}{2} = \sin\left[2\pi\left(kt - kx\right)\right] \end{aligned}$$

and as real coordinates ϕ^{k1}, ϕ^{k2}, ϕ^{k3} and ϕ^{k4}, obviously defined.

Similarly: for $T_G\mathscr{E}$ holds (3.22) and on $T_{Gk}\mathscr{E}$ we choose as a base:

$$\begin{aligned} \delta_{k1}G &= \frac{\delta_{k\nwarrow}G + \delta_{k\searrow}G}{2} = \frac{\partial}{\partial\phi^{k1}} \\ \delta_{k2}G &= \frac{\delta_{k\nwarrow}G - \delta_{k\searrow}G}{2i} = \frac{\partial}{\partial\phi^{k2}} \\ \delta_{k3}G &= \frac{\delta_{k\nearrow}G + \delta_{k\swarrow}G}{2} = \frac{\partial}{\partial\phi^{k3}} \\ \delta_{k4}G &= \frac{\delta_{k\nearrow}G - \delta_{k\swarrow}G}{2i} = \frac{\partial}{\partial\phi^{k4}} \end{aligned}$$

If we define $\Omega_{k1,k2} := \Omega_G(\delta_{k1}G, \delta_{k2}G)$ and similarly for other components of Ω and $\bar{\Omega}$, after a short computation we can compare the matrices:

$$\begin{pmatrix} \Omega_{k1,k1} & \Omega_{k1,k2} & \Omega_{k1,k3} & \Omega_{k1,k4} \\ \Omega_{k2,k1} & \Omega_{k2,k2} & \Omega_{k2,k3} & \Omega_{k3,k4} \\ \Omega_{k3,k1} & \Omega_{k3,k2} & \Omega_{k3,k3} & \Omega_{k3,k4} \\ \Omega_{k4,k1} & \Omega_{k4,k2} & \Omega_{k4,k3} & \Omega_{k4,k4} \end{pmatrix} = 2\pi bk \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{\Omega}_{k1,k1} & \bar{\Omega}_{k1,k2} & \bar{\Omega}_{k1,-k3} & \bar{\Omega}_{k1,-k4} \\ \bar{\Omega}_{k2,k1} & \bar{\Omega}_{k2,k2} & \bar{\Omega}_{k2,-k3} & \bar{\Omega}_{k2,-k4} \\ \bar{\Omega}_{k3,k1} & \bar{\Omega}_{k3,k2} & \bar{\Omega}_{k3,-k3} & \bar{\Omega}_{k3,k4} \\ \bar{\Omega}_{k4,k1} & \bar{\Omega}_{k4,k2} & \bar{\Omega}_{k4,-k3} & \bar{\Omega}_{k4,k4} \end{pmatrix} = 2\pi k \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We conclude that $\Omega \neq \bar{\Omega}$.

Choosing two slices Σ and $\bar{\Sigma}$ in two different homology classes, we obtain two different symplectic structures on $T_G\mathscr{E}$.

3.1.3 Massive scalar field over a two dimensional torus.

Using the same techniques of last section, we can study on $E = \mathbb{T}_{a,b} \times \mathbb{R}$ field theories with slightly more complicated Lagrangians. We then find that similar results hold for different theories.

For example, using the same notation of last section, we may consider the Lagrangian of the massive scalar field, defined by the Lagrangian density $L : J^1\pi \longrightarrow \mathbb{R}$:

$$L(x^\mu, q, \dot{q}_\mu) = \frac{1}{2}(\dot{q}_0)^2 - \frac{1}{2}(\dot{q}_1)^2 - \frac{1}{2}m^2 q^2 \tag{3.27}$$

Since $p^\mu := \frac{\partial L}{\partial \dot{q}_\mu}(x^\nu, \phi, \partial_\mu \phi)$, we have that:

$$p^0 = \partial_0 \phi \tag{3.28}$$
$$p^1 = -\partial_1 \phi \tag{3.29}$$

so that the Hamiltonian $H : P \longrightarrow \mathbb{R}$ is:

$$H(x^\mu, q, p^\mu) = \frac{1}{2}\left(p^0\right)^2 - \frac{1}{2}\left(p^1\right)^2 + \frac{1}{2}m^2 q^2 \tag{3.30}$$

Writing again $x^0 = t$ and $x^1 = x$, the Euler-Lagrange equation is:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0 \tag{3.31}$$

A field ϕ developed in Fourier series as in (3.3) is a solution of our theory if and only if $k, j \in \mathbb{Z}$ satisfy the following condition:

$$-4\pi^2\left(\frac{k^2}{a^2} - \frac{j^2}{b^2}\right) + m^2 = 0 \tag{3.32}$$

The equation (3.32) is more difficult to study than (3.9); and in general, fixed a, b and m, it has only a finite number of solution, when any: see [145] and references therein for an algorithm to find them. We notice that if the couple $(k,\ j)$ is a solution, then so it is for $(-k,\ -j)$,

$(k,\ -j)$ and $(-k,\ j)$. When $m \neq 0$, these are four distinct couples for any acceptable k except at most one $\tilde{k}$ for which they may reduce to two couples. We call $A \subset \mathbb{N}$ the subset of natural number k which satisfy (3.32) for some integer j. Then $k \in A$ imply $\frac{j}{b} = \pm\sqrt{\frac{k^2}{a^2} - \frac{m^2}{'\pi^2}}$. We have then the four distinct solutions $\left(k,\ \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}\right)$, $\left(-k,\ -\sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}\right)$, $\left(k,\ -\sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}\right)$ and $\left(-k,\ \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}\right)$ for every $k \in A$ except at most one special $k_0 \in A$. This special value exists when the equation $k^2 = \frac{a^2 m^2}{4\pi^2}$ has a solution in the set of natural numbers: in this case $k_0 = \left|\frac{am}{4\pi}\right|$ and we have the two solutions $(k_0, 0)$ and $(-k_0, 0)$. In all other cases, we label the solutions with the names $k\nwarrow$, $k\searrow$, $k\nearrow$ and $k\swarrow$.

The space $\mathscr{E}$ of solutions of our theory is again a vector space and we can write:

$$\mathscr{E} = \left(\bigoplus_{k \in A \subset \mathbb{N}} \mathscr{E}_k\right) \tag{3.33}$$

Where each $\mathscr{E}_k$ is a four dimensional vector space except at most one specific degenerate $\mathscr{E}_{k_0}$ which may be one or two dimensional. Every non degenerate $\mathscr{E}_k$ is spanned by the four linear independent elements:

$$\begin{aligned}
\boldsymbol{\phi}_{\boldsymbol{k}\nwarrow} &= e^{i2\pi\left(\frac{k}{a}t + \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}x\right)} \\
\boldsymbol{\phi}_{\boldsymbol{k}\searrow} &= e^{i2\pi\left(-\frac{k}{a}t - \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}x\right)} \\
\boldsymbol{\phi}_{\boldsymbol{k}\nearrow} &= e^{i2\pi\left(\frac{k}{a}t - \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}x\right)} \\
\boldsymbol{\phi}_{\boldsymbol{k}\swarrow} &= e^{i2\pi\left(-\frac{k}{a}t + \sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}x\right)}
\end{aligned}$$

where, we remember, k is a natural number. In the following, to make the notation lighter, we will often write $\frac{j}{b}$ instead of $\sqrt{\frac{k^2}{a^2} - \frac{m^2}{4\pi^2}}$, keeping in mind that $\frac{j}{b}$ is a positive function of k.

When there isn't any degeneracy of any $\mathscr{E}_{k_0}$, we write the solutions of our theory in this way:

$$\phi(t,x) = \sum_{k \in A \subset \mathbb{N}} \left(\phi^{k\nwarrow}\boldsymbol{\phi}_{\boldsymbol{k}\nwarrow} + \phi^{k\swarrow}\boldsymbol{\phi}_{\boldsymbol{k}\swarrow} + \phi^{k\nearrow}\boldsymbol{\phi}_{\boldsymbol{k}\nearrow} + \phi^{k}\searrow\boldsymbol{\phi}_{\boldsymbol{k}\searrow}\right) \tag{3.34}$$

where $\phi^k\nwarrow$, $\phi^{k\swarrow}$, $\phi^{k\nearrow}$, $\phi^{k\searrow}$ are numbers and constitute the coordinates on $\mathscr{E}$.

From this point on, we can argument in the same way we did in the previous section, except for the fact that the set $A \subset \mathbb{N}$ is a finite subset of $\mathbb{N}$ (when it is not empty), instead of being an infinite one; and it is in general difficult to find it. I will give some example for some specifics value of a, b and m at the end of this section.

If $G \in \mathscr{E}$, on $\mathbb{T}_G\mathscr{E}$ we take as base the vectors $\delta_{k\nwarrow}G = \frac{\partial}{\partial\phi^{k\nwarrow}}$, $\delta_{k\swarrow}G = \frac{\partial}{\partial\phi^{k\swarrow}}$, $\delta_{k\nearrow}G = \frac{\partial}{\partial\phi^{k\nearrow}}$

and $\delta_{k\searrow} G = \frac{\partial}{\partial \phi^{k\searrow}}$. Their correspondents Jacobi fields over the Hamiltonian curve G are:

$$\begin{aligned}
\xi_{k\nwarrow}(t,x) =& e^{i2\pi\left(\frac{k}{a}t+\frac{j}{b}x\right)}\left[\frac{\partial}{\partial\phi}+i2\pi\left(\frac{k}{a}\frac{\partial}{\partial p^0}-\frac{j}{b}\frac{\partial}{\partial p^1}\right)\right]\\
\xi_{k\searrow}(t,x) =& e^{i2\pi\left(-\frac{k}{a}t-\frac{j}{b}x\right)}\left[\frac{\partial}{\partial\phi}+i2\pi\left(-\frac{k}{a}\frac{\partial}{\partial p^0}+\frac{j}{b}\frac{\partial}{\partial p^1}\right)\right]\\
\xi_{k\nearrow}(t,x) =& e^{i2\pi\left(\frac{k}{a}t-\frac{j}{b}x\right)}\left[\frac{\partial}{\partial\phi}+i2\pi\left(\frac{k}{a}\frac{\partial}{\partial p^0}+\frac{j}{b}\frac{\partial}{\partial p^1}\right)\right]\\
\xi_{k\swarrow}(t,x) =& e^{i2\pi\left(-\frac{k}{a}t+\frac{j}{b}x\right)}\left[\frac{\partial}{\partial\phi}+i2\pi\left(-\frac{k}{a}\frac{\partial}{\partial p^0}-\frac{j}{b}\frac{\partial}{\partial p^1}\right)\right]
\end{aligned} \tag{3.35}$$

with $k \in A \subset \mathbb{N}$.

We can compute Ω and $\bar{\Omega}$ integrating on the slices Σ and $\bar{\Sigma}$ defined in the last section. Using the same notation we have:

$$\begin{aligned}
\Omega_{k\nwarrow,k'\nearrow} =& \int_{\Sigma\cap G}\xi_{k\nwarrow}\wedge\xi_{k'\nearrow}\lrcorner\,\omega =\\
=& \int_0^b\left[i2\pi\frac{k}{a}e^{i2\pi\frac{j}{b}x}e^{i2\pi(-\frac{j'}{b}x)}-i2\pi\frac{k'}{a}e^{i2\pi(-\frac{j'}{b}x)}e^{i2\pi\frac{j}{b}x}\right]dx
\end{aligned} \tag{3.36}$$

Remembering that k, k', j and j' are integer, we finally have:

$$\Omega_{k\nwarrow,k'\nearrow} = i2\pi b\delta_{jj'}\left(\frac{k}{a}-\frac{k'}{a}\right) = i2\pi b\delta_{kk'}\left(\frac{k}{a}-\frac{k'}{a}\right) = 0 \tag{3.37}$$

In the same way we compute $\Omega_{k\nwarrow,k'\swarrow}$ and $\Omega_{k\nearrow,k'\nearrow}$ we find $\Omega_{\nwarrow,k'\swarrow} = \Omega_{k\nearrow,k'\nearrow} = 0$.

On the other hand we have:

$$\begin{aligned}
\Omega_{k\nwarrow,k'\nwarrow} =& \int_{\Sigma\cap G}\xi_{k\nwarrow}\wedge\xi_{k'\nwarrow}\lrcorner\,\omega\\
=& \int_0^b\left[i2\pi\frac{k}{a}e^{i2\pi\frac{j}{b}x}e^{i2\pi\frac{j'}{b}x}-i2\pi\frac{k'}{a}e^{i2\pi\frac{j'}{b}x}e^{i2\pi\frac{j}{b}x}\right]dx = 0
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
\Omega_{k\nwarrow,k'\searrow} =& \int_{\Sigma\cap G}\xi_{k\nwarrow}\wedge\xi_{k'\searrow}\lrcorner\,\omega\\
=& \int_0^b\left[i2\pi\frac{k}{a}e^{i2\pi\frac{j}{b}x}e^{i2\pi\frac{-j'}{b}x}-i2\pi\frac{-k'}{a}e^{i2\pi\frac{-j'}{b}x}e^{i2\pi\frac{j}{b}x}\right]dx\\
=& i2\pi\delta_{jj'}b\left(\frac{k}{a}+\frac{k'}{a}\right) = i4k\pi\delta_{kk'}\frac{b}{a}
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
\Omega_{k\nearrow,k'\swarrow} =& \int_{\Sigma\cap G}\xi_{k\nearrow}\wedge\xi_{k'\swarrow}\lrcorner\,\omega\\
=& \int_0^b\left[i2\pi\frac{k}{a}e^{-i2\pi\frac{j}{b}x}e^{i2\pi\frac{j'}{b}x}+i2\pi\frac{k'}{a}e^{i2\pi\frac{j'}{b}x}e^{i2\pi\frac{-j}{b}x}\right]dx\\
=& i2\pi\delta_{jj'}b\left(\frac{k}{a}+\frac{k'}{a}\right) = i4k\pi\delta_{kk'}\frac{b}{a}
\end{aligned} \tag{3.40}$$

If for one specific k_0, $\mathscr{E}_{k_0}$ is degenerate and has dimension 2, then on $\mathscr{E}_{k_0}$ we have:

$$\boldsymbol{\phi_{k_0}} := \boldsymbol{\phi_{k_0\nearrow}} = \boldsymbol{\phi_{k_0\nwarrow}} = e^{i2\pi\left(\frac{k}{a}t\right)}$$

and

$$\boldsymbol{\phi_{-k_0}} := \boldsymbol{\phi_{k_0\searrow}} = \boldsymbol{\phi_{k_0\swarrow}} = e^{i2\pi\left(-\frac{k}{a}t\right)}$$

because $j_0 = 0$.

Also $T_{Gk_0}\mathscr{E}$ is degenerate and has dimension 2; we then have:

$$\Omega_{k_0,-k_0} = \int_{\Sigma\cap G} \xi_{k_0} \wedge \xi_{-k_0} \lrcorner\, \omega = i2\pi b\left(\frac{k_0}{a} + \frac{k_0}{a}\right) = i4k_0\pi\frac{b}{a} \tag{3.41}$$

For all non degenerate subspaces $T_{Gk}\mathscr{E}$, we have:

$$\begin{pmatrix} \Omega_{k\nwarrow,k\nwarrow} & \Omega_{k\nwarrow,k\searrow} & \Omega_{k\nwarrow,k\nearrow} & \Omega_{k\nwarrow,k\swarrow} \\ \Omega_{k\searrow,k\nwarrow} & \Omega_{k\searrow,k\searrow} & \Omega_{k\searrow,k\nearrow} & \Omega_{k\searrow,k\swarrow} \\ \Omega_{k\nearrow,k\nwarrow} & \Omega_{k\nearrow,k\searrow} & \Omega_{k\nearrow,k\nearrow} & \Omega_{k\nearrow,k\swarrow} \\ \Omega_{k\swarrow,k\nwarrow} & \Omega_{k\swarrow,k\searrow} & \Omega_{k\swarrow,k\nearrow} & \Omega_{k\swarrow,k\swarrow} \end{pmatrix} = i4\pi k\frac{b}{a}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

For $\bar{\Omega}$ we have:

$$\begin{aligned} \bar{\Omega}_{k\nwarrow,k'\nearrow} &= \int_{\bar{\Sigma}\cap G} \xi_{k\nwarrow} \wedge \xi_{k'\nearrow} \lrcorner\, \omega \\ &= \int_{\bar{\Sigma}\cap G} -\xi_{k\nwarrow} \wedge \xi_{k'\nearrow} \lrcorner \left(dp^1 \wedge d\phi\right) dt \\ &= \int_0^a -\left[-i2\pi\frac{j}{b}e^{i2\pi\frac{k}{a}t}e^{i2\pi(\frac{k'}{a}t)} - i2\pi\frac{j'}{b}e^{i2\pi(\frac{k'}{a}t)}e^{i2\pi\frac{k}{a}t}\right] dt = 0 \end{aligned} \tag{3.42}$$

Similarly: $\bar{\Omega}_{k\nwarrow,k'\swarrow} = \bar{\Omega}_{k\nearrow,k'\nearrow} = \bar{\Omega}_{k\nearrow,k'\searrow} = 0$. But:

$$\begin{aligned} \bar{\Omega}_{k\nwarrow,k'\searrow} &= \int_{\bar{\Sigma}\cap G} \xi_{k\nwarrow} \wedge \xi_{k'\searrow} \lrcorner\, \omega \\ &= \int_0^a -\left[-i2\pi\frac{j}{b}e^{i2\pi\frac{k}{a}t}e^{i2\pi(\frac{-k'}{a}t)} - i2\pi\frac{j'}{b}e^{i2\pi(\frac{-k'}{a}t)}e^{i2\pi\frac{k}{a}t}\right] dt \\ &= -\delta_{kk'}\left[-i2\pi\frac{ja}{b} - i2\pi\frac{j'a}{b}\right] \\ &= i4\pi\delta_{kk'}j\frac{a}{b} \end{aligned} \tag{3.43}$$

$$\begin{aligned} \bar{\Omega}_{k\nearrow,k'\swarrow} &= \int_{\bar{\Sigma}\cap G} \xi_{k\nearrow} \wedge \xi_{k'\swarrow} \lrcorner\, \omega \\ &= \int_0^a -\left[i2\pi\frac{j}{b}e^{i2\pi\frac{k}{a}t}e^{i2\pi(\frac{-k'}{a}t)} + i2\pi\frac{j'}{b}e^{i2\pi(\frac{-k'}{a}t)}e^{i2\pi\frac{k}{a}t}\right] dt \\ &= -\delta_{kk'}\left[i2\pi\frac{ja}{b} + i2\pi\frac{j'a}{b}\right] \\ &= -i4\pi\delta_{kk'}j\frac{a}{b} \end{aligned} \tag{3.44}$$

If there is a $\mathscr{E}_{k_0}$ degenerate of dimension 2, we have on $T_{Gk_0}\mathscr{E}$:

$$\begin{aligned}\bar{\Omega}_{k_0,-k_0} &= \int_{\bar{\Sigma}\cap G} \xi_{k_0} \wedge \xi_{-k_0} \lrcorner\, \omega \\ &= \int_0^a - \left[-i2\pi \frac{\tilde{j}}{b} e^{i2\pi \frac{\tilde{k}}{a} t} e^{i2\pi(\frac{-\tilde{k}}{a} t)} - i2\pi \frac{\tilde{j}}{b} e^{i2\pi(\frac{-\tilde{k}}{a} t)} e^{i2\pi \frac{\tilde{k}}{a} t} \right] dt = 0\end{aligned} \tag{3.45}$$

because $j_0 = 0$.

For all non degenerate $T_{Gk}\mathscr{E}$, we have:

$$\begin{pmatrix} \bar{\Omega}_{k\nwarrow,k\nwarrow} & \bar{\Omega}_{k\nwarrow,k\searrow} & \bar{\Omega}_{k\nwarrow,k\nearrow} & \bar{\Omega}_{k\nwarrow,k\swarrow} \\ \bar{\Omega}_{k\searrow,k\nwarrow} & \bar{\Omega}_{k\searrow,k\searrow} & \bar{\Omega}_{k\searrow,k\nearrow} & \bar{\Omega}_{k\searrow,k\swarrow} \\ \bar{\Omega}_{k\nearrow,k\nwarrow} & \bar{\Omega}_{k\nearrow,k\searrow} & \bar{\Omega}_{k\nearrow,k\nearrow} & \bar{\Omega}_{k\nearrow,k\swarrow} \\ \bar{\Omega}_{k\swarrow,k\nwarrow} & \bar{\Omega}_{k\swarrow,k\searrow} & \bar{\Omega}_{k\swarrow,k\nearrow} & \bar{\Omega}_{k\swarrow,k\swarrow} \end{pmatrix} = i4\pi j \frac{a}{b} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

As we did in the previous section, we come back now to real valued fields. On the four dimensional components $\mathscr{E}_k$ of our space of solutions $\mathscr{E}$ we can consider as a base the elements:

$$\begin{aligned} \phi_{k1} &= \frac{\phi_{k\nwarrow} + \phi_{k\searrow}}{2} = \cos\left[2\pi\left(\frac{k}{a}t + \frac{j}{b}x\right)\right] \\ \phi_{k2} &= \frac{\phi_{k\nwarrow} - \phi_{k\searrow}}{2i} = \sin\left[2\pi\left(\frac{k}{a}t + \frac{j}{b}x\right)\right] \\ \phi_{k3} &= \frac{\phi_{k\nearrow} + \phi_{k\swarrow}}{2} = \cos\left[2\pi\left(\frac{k}{a}t - \frac{j}{b}x\right)\right] \\ \phi_{k4} &= \frac{\phi_{k\nearrow} + \phi_{k\swarrow}}{2} = \sin\left[2\pi\left(\frac{k}{a}t - \frac{j}{b}x\right)\right] \end{aligned}$$

On the 4 dimensional real vector spaces $T_{Gk}\mathscr{E}$ we choose as a base:

$$\begin{aligned} \delta_{k1}G &= \frac{\delta_{k\nwarrow}G + \delta_{k\searrow}G}{2} = \frac{\partial}{\partial\phi^{k1}} \\ \delta_{k2}G &= \frac{\delta_{k\nwarrow}G - \delta_{k\searrow}G}{2i} = \frac{\partial}{\partial\phi^{k2}} \\ \delta_{k3}G &= \frac{\delta_{k\nearrow}G + \delta_{k\swarrow}G}{2} = \frac{\partial}{\partial\phi^{k3}} \\ \delta_{k4}G &= \frac{\delta_{k\nearrow}G - \delta_{k\swarrow}G}{2i} = \frac{\partial}{\partial\phi^{k4}} \end{aligned}$$

Defining $\Omega_{k1,k2} := \Omega_G\left(\delta_{k1}G, \delta_{k2}G\right)$ and similarly for other components of Ω and $\bar{\Omega}$ exactly as we did in the previous section, we have the following matrices:

$$\begin{pmatrix} \Omega_{k1,k1} & \Omega_{k1,k2} & \Omega_{k1,k3} & \Omega_{k1,k4} \\ \Omega_{k2,k1} & \Omega_{k2,k2} & \Omega_{k2,k3} & \Omega_{k3,k4} \\ \Omega_{k3,k1} & \Omega_{k3,k2} & \Omega_{k3,k3} & \Omega_{k3,k4} \\ \Omega_{k4,k1} & \Omega_{k4,k2} & \Omega_{k4,k3} & \Omega_{k4,k4} \end{pmatrix} = 2\pi k \frac{b}{a} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.46}$$

and

$$\begin{pmatrix} \bar{\Omega}_{k1,k1} & \bar{\Omega}_{k1,k2} & \bar{\Omega}_{k1,-k3} & \bar{\Omega}_{k1,-k4} \\ \bar{\Omega}_{k2,k1} & \bar{\Omega}_{k2,k2} & \bar{\Omega}_{k2,-k3} & \bar{\Omega}_{k2,-k4} \\ \bar{\Omega}_{k3,k1} & \bar{\Omega}_{k3,k2} & \bar{\Omega}_{k3,-k3} & \bar{\Omega}_{k3,k4} \\ \bar{\Omega}_{k4,k1} & \bar{\Omega}_{k4,k2} & \bar{\Omega}_{k4,-k3} & \bar{\Omega}_{k4,k4} \end{pmatrix} = 2\pi j \frac{a}{b} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.47}$$

Again we see that $\Omega \neq \bar{\Omega}$.

I will now give one explicit example for specific values of a, b and m. Let's fix $a = 1$, $b = 2$ and $m = 4\pi^2 \cdot 24$.

Then we can see that $\mathscr{E} = \mathscr{E}_5 \oplus \mathscr{E}_{10}$ is the direct sum of two 4-dimensional vector spaces spanned by the elements:

$$\begin{aligned}
\boldsymbol{\phi_{5\nwarrow}} &= e^{i2\pi\left(5t+\sqrt{25-24}x\right)} = e^{i2\pi(5t+x)} \\
\boldsymbol{\phi_{5\searrow}} &= e^{i2\pi\left(-5t-\sqrt{25-24}x\right)} = e^{i2\pi(-5t-x)} \\
\boldsymbol{\phi_{5\nearrow}} &= e^{i2\pi\left(5t-\sqrt{25-24}x\right)} = e^{i2\pi(5t-x)} \\
\boldsymbol{\phi_{5\swarrow}} &= e^{i2\pi\left(-5t+\sqrt{25-24}x\right)} = e^{i2\pi(-5t+x)}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\phi_{7\nwarrow}} &= e^{i2\pi(7t+5x)} \\
\boldsymbol{\phi_{7\searrow}} &= e^{i2\pi(-7t-5x)} \\
\boldsymbol{\phi_{7\nearrow}} &= e^{i2\pi(7t-5x)} \\
\boldsymbol{\phi_{7\swarrow}} &= e^{i2\pi(-7t+5x)}
\end{aligned}$$

On $T_{G5}\mathscr{E}$ we have:

$$\begin{pmatrix} \Omega_{k1,k1} & \Omega_{k1,k2} & \Omega_{k1,k3} & \Omega_{k1,k4} \\ \Omega_{k2,k1} & \Omega_{k2,k2} & \Omega_{k2,k3} & \Omega_{k3,k4} \\ \Omega_{k3,k1} & \Omega_{k3,k2} & \Omega_{k3,k3} & \Omega_{k3,k4} \\ \Omega_{k4,k1} & \Omega_{k4,k2} & \Omega_{k4,k3} & \Omega_{k4,k4} \end{pmatrix} = 2\pi \begin{pmatrix} 0 & -5\cdot 2 & 0 & 0 \\ 5\cdot 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5\cdot 2 \\ 0 & 0 & 5\cdot 2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{\Omega}_{k1,k1} & \bar{\Omega}_{k1,k2} & \bar{\Omega}_{k1,-k3} & \bar{\Omega}_{k1,-k4} \\ \bar{\Omega}_{k2,k1} & \bar{\Omega}_{k2,k2} & \bar{\Omega}_{k2,-k3} & \bar{\Omega}_{k2,-k4} \\ \bar{\Omega}_{k3,k1} & \bar{\Omega}_{k3,k2} & \bar{\Omega}_{k3,-k3} & \bar{\Omega}_{k3,k4} \\ \bar{\Omega}_{k4,k1} & \bar{\Omega}_{k4,k2} & \bar{\Omega}_{k4,-k3} & \bar{\Omega}_{k4,k4} \end{pmatrix} = 2\pi \begin{pmatrix} 0 & -2\cdot\frac{1}{2} & 0 & 0 \\ 2\cdot\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot\frac{1}{2} \\ 0 & 0 & -2\cdot\frac{1}{2} & 0 \end{pmatrix}$$

On $T_{G7}\mathscr{E}$ we have:

$$\begin{pmatrix} \Omega_{k1,k1} & \Omega_{k1,k2} & \Omega_{k1,k3} & \Omega_{k1,k4} \\ \Omega_{k2,k1} & \Omega_{k2,k2} & \Omega_{k2,k3} & \Omega_{k3,k4} \\ \Omega_{k3,k1} & \Omega_{k3,k2} & \Omega_{k3,k3} & \Omega_{k3,k4} \\ \Omega_{k4,k1} & \Omega_{k4,k2} & \Omega_{k4,k3} & \Omega_{k4,k4} \end{pmatrix} = 2\pi \begin{pmatrix} 0 & -7\cdot 2 & 0 & 0 \\ 7\cdot 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7\cdot 2 \\ 0 & 0 & 7\cdot 2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{\Omega}_{k1,k1} & \bar{\Omega}_{k1,k2} & \bar{\Omega}_{k1,-k3} & \bar{\Omega}_{k1,-k4} \\ \bar{\Omega}_{k2,k1} & \bar{\Omega}_{k2,k2} & \bar{\Omega}_{k2,-k3} & \bar{\Omega}_{k2,-k4} \\ \bar{\Omega}_{k3,k1} & \bar{\Omega}_{k3,k2} & \bar{\Omega}_{k3,-k3} & \bar{\Omega}_{k3,k4} \\ \bar{\Omega}_{k4,k1} & \bar{\Omega}_{k4,k2} & \bar{\Omega}_{k4,-k3} & \bar{\Omega}_{k4,k4} \end{pmatrix} = 2\pi \begin{pmatrix} 0 & -10\cdot\frac{1}{2} & 0 & 0 \\ 10\cdot\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 10\cdot\frac{1}{2} \\ 0 & 0 & -10\cdot\frac{1}{2} & 0 \end{pmatrix}$$

The same techniques, with similar results, can be used to study the field theory arsing from the Lagrangian defined by the Lagrangian density $L : J^1\pi \longrightarrow \mathbb{R}$:

$$L\left(x^\mu, q, \dot{q}_\mu\right) = \frac{1}{2}\left(\dot{q}_0\right)^2 + \frac{1}{2}\left(\dot{q}_1\right)^2 - \frac{1}{2}m^2 q^2 \tag{3.48}$$

which leads to the elliptic Euler-Lagrange equation:

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0 \tag{3.49}$$

Note that with this Lagrangian the case $m = 0$ would be trivial, being the space of solution the one dimensional vector space of constants on $\mathbb{T}$.

3.1.4 Theories on non trivial fiber bundles over a torus.

In this section I study the symplectic structures on the spaces of fields which are sections of some non-trivial fiber bundles over a torus. I will obtain results analogous to the ones obtained in the past two sections. This study is useful to show how the techniques described in the first chapter can be applied to field theories defined on non trivial bundles of configurations and whose fiber-type is not $\mathbb{R}$.

Let's consider the (vector) fiber bundles:

$$\begin{array}{c} E \\ {\scriptstyle \pi}\big\downarrow \\ \mathbb{T}_{a,b} \end{array}$$

with fiber $\mathbb{C}$, the set of complex numbers, and so that their sections ϕ can be written in local coordinates t and x as:

$$\phi(t,x) = e^{i2\pi\left(\frac{\alpha}{a}t + \frac{\beta}{b}x\right)} \sum_{k,j\in\mathbb{Z}} \phi^{kj} e^{i2\pi\left(\frac{k}{a}t + \frac{j}{b}x\right)} \tag{3.50}$$

where $\alpha,\ \beta \in \mathbb{R}$ are fixed numbers depending on the fiber bundle; $k,\ j \in \mathbb{Z}$ and $\phi^{kj} \in \mathbb{C}$.

If on $\mathbb{T}_{a,b}$ we follow the closed path $\gamma : [0,\ a] \longrightarrow \mathbb{T}_{a,b}$ defined in coordinates by $\tau :\longrightarrow (t(\tau),\ x(\tau)) = (\tau,\ 0)$ with $\tau \in [0,\ a]$, we see that the fiber $\mathbb{C}$ undertakes a continuous rotation. The total angle of the rotation reaches at the end of the path a value of $-2\pi\alpha$. Analogously if we follow the closed path $\gamma' : [0,\ b] \longrightarrow \mathbb{T}_{a,b}$ defined in coordinates by $\tau :\longrightarrow (t(\tau),\ x(\tau)) = (0,\ \tau)$ with $\tau \in [0,\ b]$, we see that the fiber $\mathbb{C}$ undertakes a continuous rotation which at the end of the path corresponds to an angle $-2\pi\beta$.

To cover the fiber bundle E we can use an atlas two of which charts are U_ε with coordinates $(t,\ x,\ q)$, $t \in (-\varepsilon, a-\varepsilon)$, $x \in (-\varepsilon,\ b-\varepsilon)$ and U'_ε with coordinates $(t',\ x',\ q')$, $t \in (a+\varepsilon,\ 2a+\varepsilon)$, $x \in (b+\varepsilon,\ 2b+\varepsilon)$ so that $t' = t+a$, $x' = x+b$, $q' = e^{-i2\pi(\alpha+\beta)}q\ \forall t \in (\varepsilon,\quad a-\varepsilon)$, $x \in (\varepsilon,\quad b-\varepsilon)$, $q \in \mathbb{C}$. For every fixed α and β we have a different bundle.

On $J^1\pi$ we use as coordinates $(t,\ x,\ q,\ \dot{q}_0,\ \dot{q}_1)$, where $q,\ \dot{q}_0,\ \dot{q}_1$ are complex and $q^r,\ \dot{q}_0^r,\ \dot{q}_1^r$ are their real components, $q^i,\ \dot{q}_0^i,\ \dot{q}_1^i$ their imaginary components. For every $\phi \in \Gamma(E)$ section of E, we have $\dot{q}_\mu\left(j^1\phi(x)\right) = \partial_\mu\phi$.

We study the theories defined by the Lagrangian densities L:

$$\begin{array}{rcl} L\ :\ J^1\pi & \longrightarrow & \mathbb{R} \\ (t,\ x,\ q,\ \dot{q}_0,\ \dot{q}_1) & \longmapsto & \frac{1}{2}\left|\dot{q}_0\right|^2 - \frac{1}{2}\left|\dot{q}_1\right|^2 - \frac{1}{2}m^2\left|q\right|^2 \end{array}$$

where m is a real parameter.

Note that L is well defined on $J^1\pi$, because its expression in local coordinates depends only on $|\phi|$ and it is not therefore affected by the rotation of the fiber $\mathbb{C}$ of the fiber bundle.

Again, for every section of E we have:

$$p^0 = \dot{q}_0 = \partial_0\phi \tag{3.51}$$
$$p^1 = -\dot{q}_1 = -\partial_1\phi \tag{3.52}$$

where p^0 and p^1 are in this case complex.

The Hamiltonian is:

$$\begin{aligned} H \ : \ Hom_\pi(VE, \Lambda^{n-1}T^*\mathbb{T}_{a,b}) \cong Hom_\pi(VE, T^*\mathbb{T}_{a,b}) \quad &\longrightarrow \mathbb{R} \\ (x^\mu, q, p^\mu) \quad &\longmapsto \frac{1}{2}\left|p^0\right|^2 - \frac{1}{2}\left|p^1\right|^2 + \frac{1}{2}m^2\left|\phi\right|^2 \end{aligned}$$

where we have used an identification between $T^*\mathbb{C}$ and $\mathbb{C}$.

Our multisymplectic form is:

$$\omega = dp^0_r \wedge d\phi^r \wedge dx + dp^0_i \wedge d\phi^i \wedge dx - dp^1_r \wedge d\phi^r \wedge dt - dp^1_i \wedge d\phi^i \wedge dt - dH \wedge dt \wedge dx \tag{3.53}$$

where the labels r and i indicate the real and imaginary components respectively.

The Euler-Lagrange equation is:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi = 0 \tag{3.54}$$

A field (3.50) is a solution of the theory if the integers k and j satisfy the following equation:

$$\frac{1}{a^2}(\alpha + k)^2 - \frac{1}{b^2}(\beta + j)^2 = \frac{m^2}{4\pi^2} \tag{3.55}$$

As one could imagine, the structure of the space of solutions $\mathscr{E}$ depends on the parameters α and β. We will study the special case when α and β are integers or semi-integers: we may then write $\alpha = n \cdot \frac{1}{2}$ and $\beta = l \cdot \frac{1}{2}$ with $l,\ n \in \mathbb{Z}$. This case is particularly interesting because it corresponds to fiber bundles with fiber $\mathbb{C}$ which have natural sub-bundles with fiber $\mathbb{R}$. We will see right after that in this case the structure of $\mathscr{E}$ is strictly analogous to the one we have seen in the previous sections. At the end of this sections I will say something about the more general case when α, β, or both, aren't a multiple of $\frac{1}{2}$. For the moment let's follow the scheme we used in the past sections.

If $\alpha = n \cdot \frac{1}{2}$ and $\beta = l \cdot \frac{1}{2}$ with $l,\ n \in \mathbb{Z}$, then the space $\mathscr{E}$ of solutions of our theory is a vector space and it can be written as:

$$\mathscr{E} = \left(\bigoplus_{k \in A \subset \mathbb{Z}} \mathscr{E}_k \right) \tag{3.56}$$

where A is the set of integer numbers $k \geq -\alpha$ such that it exists $j \in \mathbb{Z}$ which satisfies with k the equation (3.55); in the following and at the end of the section I will make some more comments on the nature of the space $A \subset \mathbb{Z}$. In formula(3.56), every $\mathscr{E}_k$ is a four dimensional complex vector space except at most one specific degenerate $\mathscr{E}_{k_0}$ which may be two dimensional, if $m \neq 0$, or one dimensional, if $m = 0$. Every non degenerate $\mathscr{E}_k$ is spanned by the four linear independent complex functions:

$$\boldsymbol{\phi}_{\boldsymbol{k}\nwarrow} = e^{i2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t+\sqrt{\frac{1}{a^2}(\alpha+k)^2-\frac{m^2}{4\pi^2}}x\right]} \tag{3.57}$$

$$\boldsymbol{\phi}_{\boldsymbol{k}\searrow} = e^{i2\pi\left[-\left(\frac{k}{a}+\frac{\alpha}{a}\right)t-\sqrt{\frac{1}{a^2}(\alpha+k)^2-\frac{m^2}{4\pi^2}}x\right]} \tag{3.58}$$

$$\boldsymbol{\phi_{k\nearrow}} = e^{i2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t-\sqrt{\frac{1}{a^2}(\alpha+k)^2-\frac{m^2}{4\pi^2}}x\right]} \tag{3.59}$$

$$\boldsymbol{\phi_{k\swarrow}} = e^{i2\pi\left[-\left(\frac{k}{a}+\frac{\alpha}{a}\right)t+\sqrt{\frac{1}{a^2}(\alpha+k)^2-\frac{m^2}{4\pi^2}}x\right]} \tag{3.60}$$

Note that, if $k \geq -\alpha,\ k \in \mathbb{Z}$ satisfies (3.55), then so does $k' = -k - 2\alpha$, which, substituted to k into the solution (3.57), gives the solution (3.60) and, substituted into the solution (3.59), gives the solution (3.58); and we have $k' < -\alpha$.

It is precisely the condition that $\alpha = n \cdot \frac{1}{2}$ with $n \in \mathbb{Z}$ which is necessary and sufficient for k' to be integer (and minor or equal to α) as well, and therefore for the functions (3.58) and (3.60) to be of the form (3.61) and therefore to be actual sections of our fiber bundles with parameters α and β. An analogous discussion can be done for j and β.

In the following, to make the notation lighter, I will often write $\left(\frac{\beta}{b}+\frac{j}{b}\right)$ instead of $\sqrt{\frac{1}{a^2}\left(\alpha+k\right)^2-\frac{m^2}{4\pi^2}}$, understanding that $\left(\frac{\beta}{b}+\frac{j}{b}\right)$ is a positive function of k; id est $j \geq -\beta$.

When there isn't any degeneracy of any $\mathscr{E}_{\tilde{k}}$, we may write the solutions of our theory in this way:

$$\phi\left(t,x\right) = \sum_{k\in A\subset\mathbb{Z}} \left(\phi^{k\nwarrow}\boldsymbol{\phi_{k\nwarrow}} + \phi^{k\searrow}\boldsymbol{\phi_{k\searrow}} + \phi^{k\nearrow}\boldsymbol{\phi_{k\nearrow}} + \phi^{k\swarrow}\boldsymbol{\phi_{k\swarrow}}\right) \tag{3.61}$$

where $\phi^{k\nwarrow}$, $\phi^{k\searrow}$, $\phi^{k\nearrow}$, $\phi^{k\swarrow}$ are complex numbers.

It is more useful thought, to separate the real from the imaginary part of ϕ and to use the functions:

$$\boldsymbol{\phi_{k1}} = cos2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]$$

$$\boldsymbol{\phi_{k2}} = sin2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]$$

$$\boldsymbol{\phi_{k3}} = cos2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]$$

$$\boldsymbol{\phi_{k4}} = sin2\pi\left[\left(\frac{k}{a}+\frac{\alpha}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]$$

Then we can write:

$$\phi\left(t,x\right) = \sum_{k\in A\subset\mathbb{Z}} \left(\phi^{k1}\boldsymbol{\phi_{k1}} + \phi^{k2}\boldsymbol{\phi_{k2}} + \phi^{k3}\boldsymbol{\phi_{k3}} + \phi^{k4}\boldsymbol{\phi_{k4}}\right) \tag{3.62}$$

where ϕ^{k1}, ϕ^{k2}, ϕ^{k3}, ϕ^{k4} are complex numbers.

And:

$$\begin{aligned}\phi\left(t,x\right) = &\sum_{k\in A\subset\mathbb{Z}} \left(\phi^{k1r}\boldsymbol{\phi_{k1}} + \phi^{k2r}\boldsymbol{\phi_{k2}} + \phi^{k3r}\boldsymbol{\phi_{k3}} + \phi^{k4r}\boldsymbol{\phi_{k4}}\right) + \\ &+ i\sum_{k\in A\subset\mathbb{Z}} \left(\phi^{k1i}\boldsymbol{\phi_{k1}} + \phi^{k2i}\boldsymbol{\phi_{k2}} + \phi^{k3i}\boldsymbol{\phi_{k3}} + \phi^{k4i}\boldsymbol{\phi_{k4}}\right)\end{aligned} \tag{3.63}$$

where ϕ^{k1r}, ϕ^{k2r}, ϕ^{k3r}, ϕ^{k4r}, ϕ^{k1i}, ϕ^{k2i}, ϕ^{k3i}, ϕ^{k4i} are real numbers and they are the real and imaginary components of the complex numbers ϕ^{k1}, ϕ^{k2}, ϕ^{k3}, ϕ^{k4}.

ϕ^{k1r}, ϕ^{k2r}, ϕ^{k3r}, ϕ^{k4r}, ϕ^{k1i}, ϕ^{k2i}, ϕ^{k3i}, ϕ^{k4i} are the real coordinates on the real vector space $\mathscr{E}$.

The set A depends on the parameters a and b which fix our torus, on the parameters α and β which fix the fiber bundle on the torus and on the parameter m which fix the chosen Lagrangian between the family of Lagrangians we are studying. If $m = 0$, A may be an infinite set, whereas if $m \neq 0$, A is a finite set. Of course A is empty for most of values of a, b, α, β and m. I will make some more comments on A at the end of the section.

We can proceed now in our study in the same way we did in the previous sections. For simplicity we will not consider in the following the case when there is a degeneracy of $\mathscr{E}_{\tilde{k}}$ for a particular $\tilde{k}$.

If $G \in \mathscr{E}$, on the eight dimensional real vector space $\mathbb{T}_G\mathscr{E}$ we take as base the vectors $\delta_{k1r}G = \frac{\partial}{\partial \phi^{k1r}}$, $\delta_{k2r}G = \frac{\partial}{\partial \phi^{k2r}}$, $\delta_{k3r}G = \frac{\partial}{\partial \phi^{k3r}}$, $\delta_{k4r}G = \frac{\partial}{\partial \phi^{k4r}}$ and $\delta_{k1i}G = \frac{\partial}{\partial \phi^{k1i}}$, $\delta_{k2i}G = \frac{\partial}{\partial \phi^{k2i}}$, $\delta_{k3i}G = \frac{\partial}{\partial \phi^{k3i}}$, $\delta_{k4i}G = \frac{\partial}{\partial \phi^{k4i}}$.

Their corresponding vector fields over the Hamiltonian curve G are:

$$\begin{aligned}
\xi_{k1r}(t,x) =& \cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial \phi^r} \\
&- 2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^0} \\
&+ 2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^1} \\
\xi_{k2r}(t,x) =& \sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial \phi^r} \\
&+ 2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^0} \\
&- 2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^1}
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\xi_{k3r}(t,x) =& \cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial \phi^r} \\
&- 2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^0} \\
&- 2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^1} \\
\xi_{k4r}(t,x) =& \sin 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial \phi^r} \\
&+ 2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^0} \\
&+ 2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\cos 2\pi \left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_r^1}
\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}
\xi_{k1i}(t,x) =& \cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial\phi^i}\\
&-2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^0}\\
&+2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^1}\\
\xi_{k2i}(t,x) =& \sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial\phi^i}\\
&+2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^0}\\
&-2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t+\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^1}
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
\xi_{k3i}(t,x) =& \cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial\phi^i}\\
&-2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^0}\\
&-2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^1}\\
\xi_{k4i}(t,x) =& \sin 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial\phi^i}\\
&+2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^0}\\
&+2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\cos 2\pi\left[\left(\frac{\alpha}{a}+\frac{k}{a}\right)t-\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\frac{\partial}{\partial p_i^1}
\end{aligned} \tag{3.67}$$

with $k \in A \subset \mathbb{Z}$.

We can compute Ω and $\bar{\Omega}$ integrating on the slices Σ and $\bar{\Sigma}$ defined as in the last section.

Equations (3.53), (3.64), (3.65), (3.66) and (3.67) tell us that the real and the imaginary section of the theory are dis-coupled.

We have:

$$\Omega_{kqr,k'q'i} = \bar{\Omega}_{kqr,k'q'i} = 0,\ \forall q,\ q' \in 1,2,3,4 \tag{3.68}$$

Moreover we have:

$$\Omega_{kqr,k'q'r} = \Omega_{kqi,k'q'i},\ \forall q,\ q' \in 1,2,3,4 \tag{3.69}$$

and

$$\bar{\Omega}_{kqr,k'q'r} = \bar{\Omega}_{kqi,k'q'i},\ \forall q,\ q' \in 1,2,3,4 \tag{3.70}$$

We can therefore calculate the values of the components of Ω and $\bar{\Omega}$ on the real section of the theory only. I exhibit the calculations in Appendix A.1.

The results are:

$$\begin{pmatrix}\Omega_{k1,k1} & \Omega_{k1,k2} & \Omega_{k1,k3} & \Omega_{k1,k4}\\ \Omega_{k2,k1} & \Omega_{k2,k2} & \Omega_{k2,k3} & \Omega_{k3,k4}\\ \Omega_{k3,k1} & \Omega_{k3,k2} & \Omega_{k3,k3} & \Omega_{k3,k4}\\ \Omega_{k4,k1} & \Omega_{k4,k2} & \Omega_{k4,k3} & \Omega_{k4,k4}\end{pmatrix} = 2\pi\frac{b}{a}\begin{pmatrix}0 & -(\alpha+k) & 0 & 0\\ (\alpha+k) & 0 & 0 & 0\\ 0 & 0 & 0 & -(\alpha+k)\\ 0 & 0 & (\alpha+k) & 0\end{pmatrix} \tag{3.71}$$

$$\begin{pmatrix} \bar{\Omega}_{k1,k1} & \bar{\Omega}_{k1,k2} & \bar{\Omega}_{k1,-k3} & \bar{\Omega}_{k1,-k4} \\ \bar{\Omega}_{k2,k1} & \bar{\Omega}_{k2,k2} & \bar{\Omega}_{k2,-k3} & \bar{\Omega}_{k2,-k4} \\ \bar{\Omega}_{k3,k1} & \bar{\Omega}_{k3,k2} & \bar{\Omega}_{k3,-k3} & \bar{\Omega}_{k3,k4} \\ \bar{\Omega}_{k4,k1} & \bar{\Omega}_{k4,k2} & \bar{\Omega}_{k4,-k3} & \bar{\Omega}_{k4,k4} \end{pmatrix} = 2\pi \frac{a}{b} \begin{pmatrix} 0 & -(\beta+j) & 0 & 0 \\ (\beta+j) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\beta+j) \\ 0 & 0 & -(\beta+j) & 0 \end{pmatrix} \tag{3.72}$$

As already noted, we obtain the same results if we calculate Ω and $\bar{\Omega}$ on the imaginary section of the theory.

We see that $\Omega \neq \bar{\Omega}$.

Note the correspondence existing between (3.71), (3.72) and (3.46), (3.47).

We now come back to the question of identifying the set A to which k belongs. As we said A depends on, a, b, α, β and m. We first consider $\alpha = n\frac{1}{2}$ and $\beta = l\frac{1}{2}$ with $l,\ n \in \mathbb{Z}$. Then the equation (3.55) become:

$$\frac{1}{a^2}\left(n\frac{1}{2}+k\right)^2 - \frac{1}{b^2}\left(l\frac{1}{2}+j\right)^2 = \frac{m^2}{4\pi^2} \tag{3.73}$$

We study the case when $m = 0$. Then we have:

$$\frac{1}{a^2}\left(n\frac{1}{2}+k\right)^2 - \frac{1}{b^2}\left(l\frac{1}{2}+j\right)^2 = 0 \tag{3.74}$$

It is clear that (3.74) has solutions only if $\frac{a}{b}$ is rational. Without lack of generality we can suppose that a and b are integer without common divisor so that $\frac{a}{b}$ is reduced. It is easy to see that if both n and l are even, then there is a natural 1 to 1 correspondence between the solutions of the theory with $n = 0$, $l = 0$ and the solutions of the theory with n, l generic, the correspondence sending a solution labeled by k to a solution labeled by $k - \frac{n}{2}$. This correspondence let unchanged Ω and $\bar{\Omega}$.

If on the other hand one of the two numbers n and l is odd, then the other one must be even, since otherwise the set of solutions would be empty. Let's see why. We can rewrite (3.74) in this way:

$$|l + 2j| = \left|\frac{b}{a}(n + 2k)\right| \tag{3.75}$$

where all variables are integers. Suppose that n is odd: then a must be odd too, otherwise $\frac{b}{a}(n + 2k)$ could not be integer. But if a is odd, then b is even (because a and b have no divisors in common) and this imply that also $(l + 2j)$ must be even and so also l is even. A similar discussion can be done for the case when l is odd.

The conditions that n and a are odd while l and b are even or vice versa are sufficient to have a non empty, and in fact infinite, set A of solutions. We can show that those solutions are in a 1 to 1 correspondence with the solutions of the theory with $n = 1$ and $l = 0$ (or $n = 0$ and $l = 1$), which means $\alpha = \frac{1}{2}$ and $\beta = 0$ ($\alpha = 0$ and $\beta = \frac{1}{2}$). For the theory with $n = 1$ and $l = 0$ the set A is the set of natural numbers k so that $\frac{1+2k}{a}$ is an integer. This set is obviously infinite. Note that for this theory there is no $\tilde{k}$ for which $\mathscr{E}_{\tilde{k}}$ is degenerate. To a solution of this theory labeled by k we link the solution of the theory with generic odd n labeled by the integer $k - \frac{n}{2} + \frac{1}{2}$. The correspondence again leave Ω and $\bar{\Omega}$ unchanged as it can be easily seen by substitution in (3.71) and (3.72). A similar correspondence holds between the theory $n = 0$, $l = 1$ and the generic theory with n generic even number and l generic odd number.

These correspondences should not be unforeseen if one thinks of the geometrical meaning of the rotations with angles $2\pi\alpha$ and $2\pi\beta$.

If we study the case when $m \neq 0$, we find similar correspondences with similar arguments. We see that the real interesting cases to study are the one with $n = 1$ and $l = 0$, the one with

$n = 0$ and $l = 1$ and the one with $n = 1$ and $l = 1$ (an example for the case $n = 0$ and $l = 0$ has already been given in the previous section).

When α and β are not multiples of $\frac{1}{2}$, the situation becomes a bit more complicated. It does not make sense anymore to speak about a "real" and an "imaginary" section of the theory and there isn't anymore any sub-bundle with fiber $\mathbb{R}$. The formula (3.56) loses of its interest, and in fact, if both α and β were different than a multiple of $\frac{1}{2}$ then each $\mathscr{E}_k$ would be of complex dimension 1 instead of 4. Moreover it can be seen, with calculations similar to those shown here, that $\Omega_{k,k'}$ and $\bar{\Omega}_{k,k'}$ may be different than 0 even when $k \neq k'$.

Still in general it would be $\Omega \neq \bar{\Omega}$.

3.2 Symplectic structures on spaces of solutions of field theories over $\mathbb{R}^2$

The proof given by Hélein of the equivalence of the symplectic structures built from different slices belonging to the same homology class remains valid for non-compact slices if we restrict the space of solutions on which we want to build the symplectic structure. The point there is that, on a generic solution G, the symplectic form may not be defined; let's see why. When $G \in \mathscr{E}$ is an hamiltonian surface, id est the image in P of a solution of the theory, the symplectic form Ω_Σ is built by integration of $\xi_1 \wedge \xi_2 \lrcorner \omega$ over the surface obtained intersecating the slice Σ with G (where ω is the multisymplectic form and ξ_1 and ξ_2 are Jacobi vertical vector fields on G). For the integral to be well defined, ξ_1 and ξ_2 must go to zero fast enough at the infinity of $\Sigma \cap G$, otherwise the integral diverge. If we want to compare Ω_Σ with $\Omega_{\bar{\Sigma}}$, we have then to consider a subset $\mathscr{I} \subset \mathscr{E}$ of the whole space of solutions, such that, chosen the slices Σ and $\bar{\Sigma}$, for every $G \in \mathscr{I}$ and for every $\delta_1 G,\ \delta_2 G \in T_G\mathscr{I}$, the integrals $\int_{\Sigma\cap G} \xi_1 \wedge \xi_2 \lrcorner \omega$ and $\int_{\bar{\Sigma}\cap G} \xi_1 \wedge \xi_2 \lrcorner \omega$ (where ξ_1 and ξ_2 are Jacobi fields corresponding to the vectors $\delta_1 G$ and $\delta_2 G$) are well defined. Hélein prove then that Ω_Σ are $\Omega_{\bar{\Sigma}}$ are indeed equal, provided that some more conditions are satisfied. First of all it must exist a smooth one parameter family of slices Σ_t such that $\Sigma_0 = \Sigma$ and $\Sigma_1 = \bar{\Sigma}$ and such that $\forall t \in [0,1]$, $\forall \delta_1 G,\ \delta_2 G \in T_G\mathscr{I}$, $\int_{\bar{\Sigma}_t\cap G} \xi_1 \wedge \xi_2 \lrcorner \omega$ is well defined (where ξ_1 and ξ_2 are Jacobi fields corresponding to the vectors $\delta_1 G$ and $\delta_2 G$); then the family must satisfy some more conditions which ensure a good behavior at the infinity of $\Sigma_t \cap G$ for every t and for every G. Hélein uses a special kind of homology between Σ and $\bar{\Sigma}$. What happens if Σ and $\bar{\Sigma}$ are not homological in that sense?

In the next three sections I will study two examples of the situation described above. I will consider two simple field theories over $\mathbb{R}^2$ and I will compare Ω_Σ with $\Omega_{\bar{\Sigma}}$ on a class of hamiltonian surfaces for which they are well defined but such that Σ and $\bar{\Sigma}$ are not homological in the above sense (although they are homological for the standard homology).

To compare different symplectic structures I will use an integral Fourier development of the Jacobi fields ξ.

I will show in this way that, when Σ is non compact, some subtle phenomena can occur and in general standard homology is not sufficient to ensure the independence of Ω_Σ from Σ.

3.2.1 Space of scalar fields over $\mathbb{R}^2$.

Let's consider the fiber bundle $E = \mathbb{R}^2 \times \mathbb{R}$:

$$\begin{array}{c} \mathbb{R}^2 \times \mathbb{R} \\ \pi \big\downarrow \\ \mathbb{R}^2 \end{array}$$

A section ϕ of the bundle is a smooth function $\phi\ :\ \mathbb{R}^2 \longrightarrow \mathbb{R}$.

We have a chart $U = \mathbb{R}^2$ which cover all the base space $\mathbb{R}^2$. The coordinates on U will be x^μ, $\mu = 0, 1$ with $x^0 \in (-\infty, +\infty)$ and $x^1 \in (-\infty, +\infty)$. I will often use the following notation: $x^0 = t$ and $x^1 = x$. On $V = \phi(U) = \mathbb{R}$ we use ϕ as coordinate.

We call $J^1\pi$ the first order jet fiber bundle of E, which is isomorphic to $\mathbb{R}^5$. On $J^1\pi$ we use as coordinates $(x^\mu, q, \dot{q}_\mu)$, so that $\forall x \in \mathbb{R}^2,\ \forall \phi \in \Gamma\big(\mathbb{R}^2 \times \mathbb{R}\big),\ \dot{q}_\mu \left(j^1\phi(x)\right) = \partial_\mu \phi$.

In local coordinates we have the Lagrangian $\mathcal{L} : (x^\mu, q, \dot{q}_\mu) \longrightarrow L(x^\mu, q, \dot{q}_\mu)\,\beta,\ \beta = dx^0 \wedge dx^1 = dt \wedge dx$ being the standard volume 2-form on $\mathbb{R}^2$ and L, the Lagrangian density, being a function between $J^1\pi$ and $\mathbb{R}$.
The Euler-Lagrange equation is:

$$\frac{d}{dx^\mu}\left(\frac{\partial L}{\partial \dot{q}_\mu}(x, \phi, \partial\phi)\right) = \frac{\partial L}{\partial q}(x, \phi, \partial\phi) \tag{3.76}$$

Let $q \in \mathbb{R},\ x \in \mathbb{R}^2$: since $Hom\left(T_q\mathbb{R}, T_x\mathbb{R}^2\right) \simeq Hom\left(\mathbb{R}, \mathbb{R}^2\right) \simeq \mathbb{R}^2$, because we identify the real line $\mathbb{R}$ with its tangent and with its cotangent and the real plane $\mathbb{R}^2$ with its tangent, we have the Hamiltonian:

$$\begin{aligned} H\ &:\ \left(\mathbb{R}^2 \times \mathbb{R}\right) \times \mathbb{R}^2 && \longrightarrow \mathbb{R} \\ H\ &:\ (x^\mu, q, p^\mu) && \longrightarrow p^\mu \dot{q}_\mu - L(x^\mu, q, \dot{q}_\mu) \end{aligned}$$

where we assume that $\dot{q}_\mu$ is the unique solution of $\frac{\partial L}{\partial \dot{q}_\mu}(x^\mu, q, \dot{q}_\mu) = p^\mu$. To every section $\phi \in \Gamma\left(\mathbb{R}^2 \times \mathbb{R}\right)$ we associate a section p of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, which will be the section $p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ defined by: $q(x^\nu) := \phi(x^\nu)$ and $p^\mu(x^\nu) := \frac{\partial L}{\partial \dot{q}_\mu}(x^\nu, \phi, \partial_\mu\phi), \forall x \in \mathbb{R}^2$.

The Hamilton system is then:

$$\begin{cases} \dfrac{\partial \phi}{\partial x^\mu} = \dfrac{\partial H}{\partial p^\mu}(x^\nu, \phi, p^\nu) \\ \dfrac{\partial p^\mu}{\partial x^\mu} = -\dfrac{\partial H}{\partial \phi}(x^\nu, \phi, p^\nu) \end{cases} \tag{3.77}$$

We interpret (ϕ, p^μ) as a section of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$

$$(\phi, p^\mu)\ :\ x^\mu \longrightarrow (x^\mu, \phi, p^\mu)$$

and its image $G(x^\mu)$ is a submanifold of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, and so we can associate to any ϕ section of the bundle $\left(\mathbb{R}^2 \times \mathbb{R}\right) \xrightarrow{\pi} \mathbb{R}^2$, a submanifold G of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$.

The field ϕ satisfies (3.76) if and only if the section (ϕ, p^μ) satisfy (3.77) which happens if and only if $G(x^\mu)$ is an Hamiltonian 2-curve, id est satisfies the condition:

$$\forall X_1, X_2 \in \Gamma(TG), \forall \xi \in \Gamma\left(T\left(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2\right)\right)$$

$$\left(dp^0 \wedge d\phi \wedge \beta_0 + dp^1 \wedge d\phi \wedge \beta_1 - dH \wedge \beta\right)(\xi, X_1, X_2) = 0$$

or:

$$\omega(\xi, X_1, X_2) = 0 \tag{3.78}$$

where:

$$\begin{aligned} \omega :=& dp^0 \wedge d\phi \wedge \beta_0 + dp^1 \wedge d\phi \wedge \beta_1 - dH \wedge \beta \\ =& dp^0 \wedge d\phi \wedge dx^1 - dp^1 \wedge d\phi \wedge dx^0 - dH \wedge dx^0 \wedge dx^1 \\ =& dp^0 \wedge d\phi \wedge dx - dp^1 \wedge d\phi \wedge dt - dH \wedge dt \wedge dx \end{aligned}$$

We call $\mathscr{E}$ the space of solutions of our field theory for some Lagrangian and we identify it with the spaces of all Hamiltonian 2-curves, satisfying (3.78).

3.2.2 Free scalar field over $\mathbb{R}^2$

In this section we study the field theory defined by the Lagrangian density $L : J^1\pi \longrightarrow \mathbb{R}$:

$$L\left(x^\mu, q, \dot{q}_\mu\right) = \frac{1}{2}\left(\dot{q}_0\right)^2 - \frac{1}{2}\left(\dot{q}_1\right)^2 \tag{3.79}$$

To each field $\phi\left(x^\mu\right)$ we associate the momenta $p^\mu\left(x^\nu\right) := \frac{\partial L}{\partial \dot{q}_\mu}\left(x^\nu, \phi, \partial_\mu \phi\right)$, and we have:

$$p^0 = \partial_0 \phi \tag{3.80}$$

$$p^1 = -\partial_1 \phi \tag{3.81}$$

The Hamiltonian $H : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is then:

$$H\left(x^\mu, q, p^\mu\right) = \frac{1}{2}\left(p^0\right)^2 - \frac{1}{2}\left(p^1\right)^2 \tag{3.82}$$

The Euler-Lagrange equation is the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{3.83}$$

The space of solutions of the theory is the infinite dimensional real vector space $\mathscr{E}$. As we did in the previous sections of this chapter, we will identify $\mathscr{G}$ with $\mathscr{E}$. Being $\mathscr{E}$ a vector space, we have $\forall G:\ T_G\mathscr{E} \simeq \mathscr{E}$.

A generic solution of the theory ϕ can be written as

$$\phi\left(t, x\right) = \phi_+(t + x) + \phi_-(t - x) \tag{3.84}$$

where $\phi_+,\ \phi_- \in C^\infty\left(\mathbb{R}, \mathbb{R}\right)$ are generic smooth real functions in one variable.

It is easy to see that if G is an Hamiltonian surface in $\mathscr{E}$, then a generic vector $\delta_u G \in T_G\mathscr{E}$ is represented by a 'Jacobi field' $\xi \in \Gamma\left(i^*\left(T\left(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2\right)\right)\right)$, where i is the embedding map of G in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$. If on G we use t and x as local coordinates, we have:

$$i\ :\ (t, x) \longrightarrow \left(t, x, \phi\left(t, x\right), \partial_0 \phi\left(t, x\right), -\partial_1 \phi\left(t, x\right)\right)$$

and ξ is defined on G in this way:

$$\xi(t, x) = \left[u_+(t + x) + u_-(t - x)\right] \frac{\partial}{\partial \phi} + \left[u'_+(t + x) + u'_-(t - x)\right] \frac{\partial}{\partial p^0} + \left[-u'_+(t + x) + u'_-(t - x)\right] \frac{\partial}{\partial p^1} \tag{3.85}$$

where $u_+,\ u_- \in C^\infty\left(\mathbb{R}, \mathbb{R}\right)$ are generic smooth real functions in one variable and $u'_+ + u'_-$ are their first derivative.

In the following we will consider the two vectors $\delta_{u_+}G,\ \delta_{u_-}G \in T_G\mathscr{E}$, represented by the Jacobi fields:

$$\begin{aligned} \xi_+(t, x) =& u_+(t + x)\frac{\partial}{\partial \phi} + u'_+(t + x)\frac{\partial}{\partial p^0} - u'_+(t + x)\frac{\partial}{\partial p^1} \\ \xi_-(t, x) =& u_-(t - x)\frac{\partial}{\partial \phi} + u'_-(t - x)\frac{\partial}{\partial p^0} + u'_-(t - x)\frac{\partial}{\partial p^1} \end{aligned} \tag{3.86}$$

Note that the expressions in (3.86) do not depend in fact from G, which is consistent with the fact that $\mathscr{E}$ is a vector space and not a curved space.

On $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ we consider a slice Σ of codimension 1 with the property that for any Hamiltonian n-curve $G \in \mathscr{E}$ the intersection of Σ with G is transverse. We also assume that $\forall G \in \mathscr{E}$, $\Sigma \cap G$ is an oriented manifold of dimension $2 - 1 = 1$. We can then define the symplectic form Ω_Σ: let be $\delta_1 G, \delta_2 G \in T_G\mathscr{E}$ two vectors over $G \in \mathscr{E}$, and let be u_1 and u_2 the corresponding Jacobi fields over G, then:

$$\Omega_\Sigma\left(\delta_1 G, \delta_2 G\right) := \int_{\Sigma\cap G} u_1 \wedge u_2 \lrcorner \omega \tag{3.87}$$

The form Ω_Σ is not defined on all $T\mathscr{E}$ because the integral in (3.87) may diverge. This is a common situation one has to deal with when working with field theories defined on non compact base manifolds. The problem is well known and it can be addressed in different ways. For example Kijowski and Szczyrba in [98] suggested to consider only couples of vectors $\delta_1 G, \delta_2 G$ such that at least one between the corresponding Jacobi fields over G, u_1 and u_2, is compactly supported when restricted to any of the admissible slices Σ. This unfortunately would be a by far too strong condition for the theory we are studying, unless we strongly limit the choice of Σ. This is not what I want to do here. Let's put aside for a while the problem of the convergence of the integral in (3.87): we will come back to it at the end of this subsection.

Suppose that Ω_Σ is well defined, then it will not depend on the point G over which we calculate it. It may depends instead on the slice Σ: this dependence is precisely what I want to study in the following.

We choose the slice Σ_{cd} of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ to be the hyperspace of dimension 4 defined by the equation $x = ct + d$ with $c, d \in \mathbb{R}$ being two parameters. All Σ_{cd} clearly have the characteristics described above. For every G, $\Sigma_{cd} \cap G$ has dimension 1 and it can be parametrized with a parameter τ in this way:

$$\tau \longrightarrow \left(\tau, c\tau + d, \phi\left(\tau, c\tau + d\right), \partial_0\phi\left(\tau, c\tau + d\right), -\partial_1\phi\left(\tau, c\tau + d\right)\right)$$

Choosing this parametrization, we have also automatically chosen a naturally associated orientation of Σ_{cd}.

Let's set $\Omega_{cd} := \Omega_{\Sigma_{cd}}$. I want now to compute:

$$\Omega_{cd\ ++} := \Omega_{cd}\left(\delta_{u_+}G, \delta_{\tilde{u}_+}G\right) := \int_{\Sigma_{cd}\cap G} \xi_+ \wedge \widetilde{\xi_+} \lrcorner \omega \tag{3.88}$$

$$\Omega_{cd\ --} := \Omega_{cd}\left(\delta_{u_-}G, \delta_{\tilde{u}_-}G\right) := \int_{\Sigma_{cd}\cap G} \xi_- \wedge \widetilde{\xi_-} \lrcorner \omega \tag{3.89}$$

$$\Omega_{cd\ +-} := \Omega_{cd}\left(\delta_{u_+}G, \delta_{\tilde{u}_-}G\right) := \int_{\Sigma_{cd}\cap G} \xi_+ \wedge \widetilde{\xi_-} \lrcorner \omega \tag{3.90}$$

We have that:

$$\begin{aligned}
\Omega_{cd\ ++} &= \int_{\Sigma_{cd}\cap G} \xi_+ \wedge \widetilde{\xi_+} \lrcorner \omega \\
&= \int_{\Sigma_{cd}\cap G} \left[u'_+(t+x)\widetilde{u_+}(t+x) - u_+(t+x)\widetilde{u'_+}(t+x)\right] dx \\
&\quad + \int_{\Sigma_{cd}\cap G} \left[u'_+(t+x)\widetilde{u_+}(t+x) - u_+(t+x)\widetilde{u'_+}(t+x)\right] dt \\
&= \int_{-\infty}^{\infty} \left[u'_+(\tau + c\tau + d)\widetilde{u_+}(\tau + c\tau + d) - u_+(\tau + c\tau + d)\widetilde{u'_+}(\tau + c\tau + d)\right] (cd\tau + d\tau)
\end{aligned} \tag{3.91}$$

so, if $c=-1$, then:

$$\Omega_{cd\ ++} \ = \ 0$$

if $c \neq -1$, then:

$$\Omega_{cd\ ++} = \operatorname{sgn}(c+1)\int_{-\infty}^{\infty}\left[u'_+(z)\widetilde{u_+}(z) - u_+(z)\widetilde{u'_+}(z)\right]dz \tag{3.92}$$

For $\Omega_{cd\ --}$ we have:

$$\begin{aligned}
\Omega_{cd\ --} \ &= \int_{\Sigma_{cd}\cap G} \xi_- \wedge \widetilde{\xi_-} \lrcorner \omega \\
&= \int_{\Sigma_{cd}\cap G}\left[u'_-(t-x)\widetilde{u_-}(t-x) - u_-(t-x)\widetilde{u'_-}(t-x)\right]dx \\
&\quad - \int_{\Sigma_{cd}\cap G}\left[u'_-(t-x)\widetilde{u_-}(t-x) - u_-(t-x)\widetilde{u'_-}(t-x)\right]dt \\
&= \int_{-\infty}^{\infty}\left[u'_-(\tau - c\tau - d)\widetilde{u_-}(\tau - c\tau - d) - u_-(\tau - c\tau - d)\widetilde{u'_-}(\tau - c\tau - d)\right](cd\tau - d\tau)
\end{aligned} \tag{3.93}$$

so, if $c=1$, then:

$$\Omega_{cd\ --} \ = \ 0$$

if $c \neq 1$, then:

$$\Omega_{cd\ --} = \operatorname{sgn}(c-1)\int_{-\infty}^{\infty}\left[u'_-(z)\widetilde{u_-}(z) - u_-(z)\widetilde{u'_-}(z)\right]dz \tag{3.94}$$

And finally we have:

$$\begin{aligned}
\Omega_{cd\ +-} &= \int_{\Sigma_{cd}\cap G} \xi_+ \wedge \widetilde{\xi_-} \lrcorner \omega \\
&= \int_{\Sigma_{cd}\cap G}\left[u'_+(t+x)\widetilde{u_-}(t-x) - u_+(t+x)\widetilde{u'_-}(t-x)\right]dx \\
&\quad + \int_{\Sigma_{cd}\cap G}\left[u'_+(t+x)\widetilde{u_-}(t-x) + u_+(t+x)\widetilde{u'_-}(t-x)\right]dt \\
&= \int_{-\infty}^{\infty}\left\{(c+1)\left[u'_+(\tau + c\tau + d)\widetilde{u_-}(\tau - c\tau - d)\right] - (c-1)\left[u_+(\tau + c\tau + d)\widetilde{u'_-}(\tau - c\tau - d)\right]\right\}d\tau \\
&= \int_{-\infty}^{\infty}\left[\frac{df}{d\tau}(\tau)g(\tau) + f(\tau)\frac{dg}{d\tau}(\tau)\right]d\tau = f(\infty)g(\infty) - f(-\infty)g(-\infty)
\end{aligned} \tag{3.95}$$

where $f(\tau) := u_+(\tau + c\tau + d)$ and $g(\tau) := \widetilde{u_-}(\tau - c\tau - d)$. Note that $\Omega_{cd\ +-} \ = 0$ if u_+ or $\widetilde{u_-}$ goes to 0 at infinity.

Looking at the results obtained with (3.92), (3.94) and (3.95), we can conclude that Ω does not depend on the choice of the slice Σ_{cd} as long as we take in consideration the orientation of such slices and as far as we do not cross the slices corresponding to the parameters $c=1$ and $c=-1$. To be more precise: if we limit ourselves to slices of space type, which entirely lie inside the light cone, which means to slices corresponding to $c<-1$ and $c>1$, then it is possible to choose a continuum orientation in the space of slices Σ_{cd}, so that Ω does not depend on the choice of c and d. The same if we limit ourselves to slices with parameters $-1 < c < 1$. On the other hand the value of Ω may change if we calculate it on 2 different slices lying in the 2 different sectors and there is no possible choice of a continue orientation in the space of slices which can completely avoid this. Finally, if we consider the special slices with $c = -1$ or $c = 1$, we can see that the corresponding Ω shows a special degeneracy.

We can also observe that Ω never depends on the parameter d.

The symplectic structure change when we pass trough an ipersurface of light type.

Before to close this subsection, let's come back to the problem of the convergence of the integral (3.87). We saw above, equation (3.84), that every element ϕ of the space $\mathscr{E}$ can be written as:

$$\phi(t, x) = \phi_+(t + x) + \phi_-(t - x)$$

where $\phi_+,\ \phi_- \in C^\infty(\mathbb{R}, \mathbb{R})$.

Let's now consider the space $I \subset C^\infty(\mathbb{R}, \mathbb{R})$ of real smooth functions defined by:

$$I := H^1(\mathbb{R}) \cap C^\infty(\mathbb{R}, \mathbb{R}) \tag{3.96}$$

where $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ is the Sobolev space of functions $f \in L^2(\mathbb{R})$ such that their weak derivatives f' are also in $L^2(\mathbb{R})$.

Let's consider the subspace $\mathscr{I} \subset \mathscr{E}$ defined by:

$$\mathscr{I} := \{\phi \in \mathscr{E} | \phi = \phi_+ + \phi_- \text{ with } \phi_+, \phi_- \in I\}$$

Then it is $T\mathscr{I} \cong \mathscr{I}$. Suppose $G \in \mathscr{I}$ and $\delta_u \in T_G\mathscr{I}$, then δ_u is represented by a 'Jacobi field' ξ such that:

$$\xi(t, x) = [u_+(t + x) + u_-(t - x)] \frac{\partial}{\partial \phi} + [u'_+(t + x) + u'_-(t - x)] \frac{\partial}{\partial p^0} + [-u'_+(t + x) + u'_-(t - x)] \frac{\partial}{\partial p^1}$$

with $u_+,\ u_- \in I$.

Then, remembering equations (3.92), (3.94) and (3.95), it is easy to see that the integral (3.87) is always well defined when $u_1, u_2 \in T\mathscr{I}$. Therefore Formula (3.87) defines a symplectic form on $\mathscr{I} \subset \mathscr{E}$. The comparison between non-equivalent symplectic structures made in this subsection makes sense if we compare symplectic forms Ω and Ω' defined on $\mathscr{I}$.

We could do better. A Fourier analysis would also show that the integral (3.87) is indeed always well defined if $u_1, u_2 \in T\mathscr{J}$; where:

$$\mathscr{J} := \{\phi \in \mathscr{E} | \phi = \phi_+ + \phi_- \text{ with } \phi_+, \phi_- \in J\}$$

with

$$J := H^{\frac{1}{2}}(\mathbb{R}) \cap C^\infty(\mathbb{R}, \mathbb{R}). \tag{3.97}$$

Remark 21. *Note that in Formula* (3.96) *and* (3.97)*, the intersection with* $C^\infty(\mathbb{R}, \mathbb{R})$ *is assumed, although it is not necessary to ensure the convergence of the integrals calculated in this section, because all the fields are from the beginning supposed to be smooth.*

3.2.3 Massive scalar field over $\mathbb{R}^2$.

On the same fiber bundle over $\mathbb{R}^2$ used in the last sections, we study now the field theory defined by the following Lagrangian density $L : J^1\pi \longrightarrow \mathbb{R}$:

$$L(x^\mu, q, \dot{q}_\mu) = \frac{1}{2}(\dot{q}_0)^2 - \frac{1}{2}(\dot{q}_1)^2 - \frac{1}{2}m^2 q^2 \tag{3.98}$$

This is a special case of the theory studied by Marsden and Shkoller in [108].
We will use the same notations used in the past sections.
The Hamiltonian $H : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is then:

$$H\left(x^\mu, q, p^\mu\right) = \frac{1}{2}\left(p^0\right)^2 - \frac{1}{2}\left(p^1\right)^2 + \frac{1}{2}m^2 q^2 \tag{3.99}$$

The Euler-Lagrange equation is the equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0 \tag{3.100}$$

The space of solutions of the theory is again a infinite dimensional real vector space $\mathscr{E}$. We study its complexified ${}^{\mathbb{C}}\mathscr{E}$ to make some calculations easier. We denote every $\phi \in {}^{\mathbb{C}}\mathscr{E}$ solution of (3.100) with its Fourier integral. We have then:

$$\phi\left(t,x\right) = \int_{-\infty}^{+\infty} \left[\phi^+(j) e^{i2\pi(kt+jx)} + \phi^-(j) e^{i2\pi(kt-jx)}\right] dj \tag{3.101}$$

where

$$k = \frac{j}{|j|}\sqrt{j^2 + \frac{m^2}{4\pi^2}} \tag{3.102}$$

is a real function of j with values always concordant with those of j.

We consider $\phi^+(j)$ and $\phi^-(j)$ as elements of a chosen functional space $I \subset C^\infty(\mathbb{R},\mathbb{R})$; in this way we restrict $\mathscr{E}$ to a subset $\mathscr{I}$

If $\phi \in {}^{\mathbb{C}}\mathscr{I}$, then it corresponds to an hamiltonian 2-curve G.

As in the previous section, if $\delta_u G \in {}^{\mathbb{C}}T_G\mathscr{I}$ is a vector over the point G, then $\delta_u G$ corresponds to a suitable $u \in \Gamma\left(i_*\left({}^{\mathbb{C}}T\left(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2\right)\right)\right)$. As a correspondent to $\delta_u G$ we take the vector field u defined on G by its Fourier integral in this way:

$$\begin{aligned} u\left(t,x\right) = &\int_{-\infty}^{+\infty} dj \left[u^+\left(j\right) e^{i2\pi(kt+jx)} + u^-\left(j\right) e^{i2\pi(kt-jx)}\right] \frac{\partial}{\partial \phi} + \\ &+ \int_{-\infty}^{+\infty} dj\ i2\pi k \left[u^+\left(j\right) e^{i2\pi(kt+jx)} + u^-\left(j\right) e^{i2\pi(kt-jx)}\right] \frac{\partial}{\partial p^0} \\ &- \int_{-\infty}^{+\infty} dj\ i2\pi j \left[u^+\left(j\right) e^{i2\pi(kt+jx)} - u^-\left(j\right) e^{i2\pi(kt-jx)}\right] \frac{\partial}{\partial p^1} \end{aligned} \tag{3.103}$$

where u^+, and u^- are elements of a functional space D determined by the choice of I, which in turns determines also $\mathscr{I}$ and ${}^{\mathbb{C}}\mathscr{I}$. The choice of I, and so of D, is made so that all the integral appearing in the following do converge. I will not show here how it is possible to chose a suitable I, but one can retains that a procedure analogous to the one adopted at the end of the previous subsection can be followed also in this case. A posteriori, one can verify (as in the case of free field treated in the previous session) that indeed a good choice of I is $I = H^{\frac{1}{2}}(\mathbb{R}) \cap C^\infty(\mathbb{R},\mathbb{R})$, which implies $D = I$.

We will distinguish again the vector fields of the type u^+ from those of the type u^-, where:

$$u^+\left(t,x\right) = \int_{-\infty}^{+\infty} dj \left[u^+\left(j\right) e^{i2\pi(kt+jx)}\right] \left[\frac{\partial}{\partial \phi} + i2\pi\left(k\frac{\partial}{\partial p^0} - j\frac{\partial}{\partial p^1}\right)\right] \tag{3.104}$$

and

$$u^-\left(t,x\right) = \int_{-\infty}^{+\infty} dj \left[u^-\left(j\right) e^{i2\pi(kt-jx)}\right] \left[\frac{\partial}{\partial \phi} + i2\pi\left(k\frac{\partial}{\partial p^0} + j\frac{\partial}{\partial p^1}\right)\right] \tag{3.105}$$

On $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ we use the same slices Σ_{cd} defined in the previous section and we parametrize them in the same way with the parameter τ.

We can calculate Ω for any $\delta_u G \in {}^{\mathbb{C}}T_G\mathscr{I}$. We compute:

$$\Omega_{cd\ ++} := \Omega_{cd}\left(\delta_{u^+}G, \delta_{\tilde{u}^+}G\right) := \int_{\Sigma_{cd}\cap G} u^+ \wedge \tilde{u}^+ \lrcorner \omega \tag{3.106}$$

$$\Omega_{cd\ --} := \Omega_{cd}\left(\delta_{u^-}G, \delta_{\tilde{u}^-}G\right) := \int_{\Sigma_{cd}\cap G} u^- \wedge \tilde{u}^- \lrcorner \omega \tag{3.107}$$

$$\Omega_{cd\ +-} := \Omega_{cd}\left(\delta_{u^+}G, \delta_{\tilde{u}^-}G\right) := \int_{\Sigma_{cd}\cap G} u^+ \wedge \tilde{u}^- \lrcorner \omega \tag{3.108}$$

We have then:

$$\begin{aligned}
\Omega_{cd\ ++} &= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^+(j') \\
&\left\{\int_{-\infty}^{+\infty} i2\pi\left(k-k'\right)\left[e^{i2\pi(k\tau+cj\tau+jd)}e^{i2\pi\left(k'\tau+cj'\tau+j'd\right)}\right] c\ d\tau\right. \\
&\left.+\int_{-\infty}^{+\infty} -i2\pi\left(-j+j'\right)\left[e^{i2\pi(k\tau+cj\tau+jd)}e^{i2\pi\left(k'\tau+cj'\tau+j'd\right)}\right] d\tau\right\} \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^+(j') \\
&\int_{-\infty}^{+\infty} i2\pi\left[c(k-k')+(j-j')\right] e^{i2\pi\left(j+j'\right)d} e^{i2\pi\left(k+k'+cj+cj'\right)\tau} d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^+(j')\ \ i2\pi\left[c(k-k')+(j-j')\right] e^{i2\pi\left(j+j'\right)d}\delta\left(k+k'+cj+cj'\right)
\end{aligned} \tag{3.109}$$

To study the value of this integral, it is necessary to study the value of the function $f = k + k' + cj + cj'$ which is a function of the variables j and j' depending on the parameter c (remembering that k and k' are defined by (3.102) as function of j and j'). In particular it is necessary to find the zeros of this function f.

One can see that if $c \geq 0$, then the function f has only one zero $j = -j'$. Moreover $\frac{df}{dj'} = \frac{ck'+j'}{k'}$, and, when $c \geq 0$, $\left|\frac{df}{dj'}\right| = \frac{df}{dj'}$

In (3.109) we can then perform the integration over j' and we find that, when $c \geq 0$:

$$\begin{aligned}
\Omega_{cd\ ++} &= \int_{-\infty}^{+\infty} dj\ u^+(j)\,\tilde{u}^+(-j)\ \ i4\pi\left[ck+j\right]\frac{k}{ck+j} \\
&= \int_{-\infty}^{+\infty} i4\pi k\ u^+(j)\,\tilde{u}^+(-j)\ \ dj
\end{aligned} \tag{3.110}$$

When $c < -1$, the function f has the only zero $j' = -j$ for $j > \frac{2c}{1-c^2}$ and $j < -\frac{2c}{1-c^2}$, but it has a second zero $j' = j_2$ for every $-\frac{2c}{1-c^2} < j < \frac{2c}{1-c^2}$, being j_2 a function of j and $j_2 \neq -j$. Moreover, if $c < -1$, then $\operatorname{sgn}(ck+j) = -\operatorname{sgn} k$ and so we have $\left|\frac{df}{dj'}\right| = \left|\frac{ck'+j'}{k'}\right| = -\frac{ck'+j'}{k'}$. We have then, when $c < -1$:

$$\begin{aligned}
\Omega_{cd\ ++} &= \int_{-\infty}^{+\infty} dj\ u^+(j)\,\tilde{u}^+(-j)\ \ i4\pi\,[ck+j]\left(-\frac{k}{ck+j}\right)\\
&+ \int_{-\frac{2c}{1-c^2}}^{\frac{2c}{1-c^2}} dj\ u^+(j)\,\tilde{u}^+(j_2)\ \ i2\pi\,[ck+j-ck_2-j_2]\,e^{i2\pi(j+j_2)d}\left(-\frac{k_2}{ck_2+j_2}\right)\\
&= \int_{-\infty}^{+\infty} -i4\pi k\ u^+(j)\,\tilde{u}^+(-j)\ \ dj\\
&+ \int_{-\frac{2c}{1-c^2}}^{\frac{2c}{1-c^2}} dj\ u^+(j)\,\tilde{u}^+(j_2)\ \ i2\pi\,[ck+j-ck_2-j_2]\,e^{i2\pi(j+j_2)d}\left(-\frac{k_2}{ck_2+j_2}\right)
\end{aligned} \tag{3.111}$$

where k_2 is a function of j_2 calculated with (3.102) and it is then a function of j.

To evaluate the second integral in (3.111) we must consider that $f(j' = j_2) = 0$ imply $k+cj = -(k_2+j_2)$. Moreover from (3.102) we obtain that $k^2 = j^2 + \frac{m^2}{4\pi^2}$ and $c^2j^2 = c^2k^2 - c^2\frac{m^2}{4\pi^2}$ and similarly for k_2^2 and j_2^2. So we have the following equivalent equalities:

$$\begin{aligned}
&k + cj = -(k_2 + cj_2)\\
&(k+cj)^2 = (k_2+cj_2)^2\\
&k^2 + 2cjk + c^2j^2 = k_2^2 + 2cj_2k_2 + c^2j_2^2\\
&j^2 + \frac{m^2}{4\pi^2} + 2cjk + c^2k^2 - c^2\frac{m^2}{4\pi^2} = j_2^2 + \frac{m^2}{4\pi^2} + 2cj_2k_2 + c^2k_2^2 - c^2\frac{m^2}{4\pi^2}\\
&(ck+j)^2 = (ck_2+j_2)^2\\
&(ck+j) = \pm(ck_2+j_2)
\end{aligned}$$

But the first of them taken with $(ck+j) = -(ck_2+j_2)$ would imply $j_2 = -j$; Since we know that $j_2 \neq -j$, then we conclude that $k+cj = -(k_2+cj_2)$ is equivalent to $ck+j = ck_2+j_2$ and therefore the second integral in (3.111) equals 0.

We then have, when $c < -1$:

$$\Omega_{cd\ ++} = \int_{-\infty}^{+\infty} -i4\pi k\ u^+(j)\,\tilde{u}^+(-j)\ \ dj \tag{3.112}$$

When $-1 < c < 0$ a similar situation occurs:

$$\begin{aligned}
\Omega_{cd\ ++} &= \int_{-\infty}^{+\infty} dj\ u^+(j)\,\tilde{u}^+(-j)\ \ i4\pi\,[ck+j]\left|\frac{k}{ck+j}\right| +\\
&+ \int_{-\frac{2c}{1-c^2}}^{\frac{2c}{1-c^2}} dj\ u^+(j)\,\tilde{u}^+(j_2)\ \ i2\pi\,[ck+j-ck_2-j_2]\,e^{i2\pi(j+j_2)d}\left(-\frac{k_2}{ck_2+j_2}\right)
\end{aligned} \tag{3.113}$$

We can repeat the reasoning made on j_2, which did not depend on the value of c, and conclude that the second integral equals 0.

In this case, however, the sign of the function $\frac{k}{ck+j}$ has a more complicated dependence on the variable j. We therefore have, when $-1 < c < 0$:

$$\Omega_{cd\ ++} = \int_{-\infty}^{+\infty} dj\ u^+(j)\,\tilde{u}^+(-j)\ \ i4\pi\,\mathrm{sgn}\,(ck+j)\,|k| \tag{3.114}$$

$\Omega_{cd\ ++}$ various continuously with the parameter c when $-1 < c < 0$.

When $c = -1$, again the equation $f = 0$ has only one solution $j' = -j$, and taking into consideration the sign of $\frac{k}{-k+j}$, we have:

$$\Omega_{cd\ ++} \ = \int_{-\infty}^{+\infty} -i4\pi k \ u^{+}(j)\,\tilde{u}^{+}(-j)\ \ dj \tag{3.115}$$

To complete our study, we may want to calculate Ω on a slice $\Sigma_{\infty d}$, defined by the equation $t = d$. In this case $\Sigma_{cd} \cup G$ can be parametrized with a parameter τ in this way:

$$\tau \longrightarrow (d, \tau, \phi(d,\tau), \partial_0 \phi(d,\tau), -\partial_1 \phi(d,\tau))$$

We then have:

$$\begin{aligned}
\Omega_{\infty d\ ++} \ &= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{+}(j)\,\tilde{u}^{+}(j') \int_{-\infty}^{+\infty} i2\pi(k-k') \left[e^{i2\pi(kd+j\tau)} e^{i2\pi\left(k'd+j'\tau\right)}\right] \ d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{+}(j)\,\tilde{u}^{+}(j') \int_{-\infty}^{+\infty} i2\pi\left[k-k'\right] e^{i2\pi\left(k+k'\right)d} e^{i2\pi\left(j+j'\right)\tau} d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{+}(j)\,\tilde{u}^{+}(j')\, i2\pi\left[k-k'\right] e^{i2\pi\left(k+k'\right)d} \delta(j+j') \\
&= \int_{-\infty}^{+\infty} i4\pi k \ u^{+}(j)\,\tilde{u}^{+}(-j)\ \ dj
\end{aligned} \tag{3.116}$$

In a similar way we can compute $\Omega_{cd\ --}$ and $\Omega_{\infty d\ --}$. The role of the special value $c = -1$ for the parameter c will be taken in this case by the value $c = 1$.

$$\begin{aligned}
\Omega_{cd\ --} \ &= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{-}(j)\,\tilde{u}^{-}(j') \\
&\left\{\int_{-\infty}^{+\infty} i2\pi(k-k') \left[e^{i2\pi(k\tau - cj\tau - jd)} e^{i2\pi\left(k'\tau - cj'\tau - j'd\right)}\right] c \ d\tau \right. \\
&\left. + \int_{-\infty}^{+\infty} -i2\pi(j-j') \left[e^{i2\pi(k\tau - cj\tau - jd)} e^{i2\pi\left(k'\tau - cj'\tau - j'd\right)}\right] d\tau \right\} \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{-}(j)\,\tilde{u}^{-}(j') \\
&\int_{-\infty}^{+\infty} i2\pi\left[c(k-k') - (j-j')\right] e^{i2\pi\left(-j-j'\right)d} e^{i2\pi\left(k+k'-cj-cj'\right)\tau} d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj' \ u^{-}(j)\,\tilde{u}^{-}(j') \ \ i2\pi\left[c(k-k') - (j-j')\right] e^{i2\pi\left(-j-j'\right)d} \delta(k+k'-cj-cj')
\end{aligned} \tag{3.117}$$

An argument similar and symmetric to the one we have undertaken before will take place to obtain the following results:

for $c \geq 1$

$$\Omega_{cd\ --} \ = \int_{-\infty}^{+\infty} 4\pi i \ k \ u^{-}(j)\,\tilde{u}^{-}(-j)\ \ dj \tag{3.118}$$

for $c \leq 0$

$$\Omega_{cd\ --} = -\int_{-\infty}^{+\infty} 4\pi i\ k\ u^-(j)\,\tilde{u}^-(-j)\ dj \tag{3.119}$$

for $0 < c < 1$

$$\Omega_{cd\ --} = -\int_{-\infty}^{+\infty} 4\pi i\ |k|\ \operatorname{sgn}(ck-j)\ u^-(j)\,\tilde{u}^-(-j)\ dj \tag{3.120}$$

and finally:

$$\Omega_{\infty d\ --} = \int_{-\infty}^{+\infty} 4\pi i\ k\ u^-(j)\,\tilde{u}^-(-j)\ dj \tag{3.121}$$

As for $\Omega_{cd\ +-}$, we have:

$$\begin{aligned}
\Omega_{cd\ +-} &= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^-(j') \\
&\left\{\int_{-\infty}^{+\infty} i2\pi\,(k-k')\left[e^{i2\pi(k\tau+cj\tau+jd)}e^{i2\pi\left(k'\tau-cj'\tau-j'd\right)}\right] c\ d\tau\right. \\
&\left. + \int_{-\infty}^{+\infty} -i2\pi\,(-j-j')\left[e^{i2\pi(k\tau+cj\tau+jd)}e^{i2\pi\left(k'\tau-cj'\tau-j'd\right)}\right] d\tau\right\} \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^-(j') \\
&\int_{-\infty}^{+\infty} i2\pi\left[c(k-k')+(j+j')\right] e^{i2\pi\left(j-j'\right)d}e^{i2\pi\left(k+k'+cj-cj'\right)\tau}d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^+(j)\,\tilde{u}^-(j')\ \ i2\pi\left[c(k-k')+(j+j')\right] e^{i2\pi\left(j-j'\right)d}\delta\left(k+k'+cj-cj'\right)
\end{aligned} \tag{3.122}$$

To show that this integral is always equal to 0, we can argue in this way: if there is a k' function of k so that $k+k'+cj-cj'=0$, then we have the following equivalent equalities:

$$\begin{aligned}
&k+cj = -(k'-cj') \\
&(k+cj)^2 = (k'-cj')^2 \\
&k^2+2cjk+c^2j^2 = k'^2-2cj'k'+c^2j'^2 \\
&j^2+\frac{m^2}{4\pi^2}+2cjk+c^2k^2-c^2\frac{m^2}{4\pi^2} = j'^2+\frac{m^2}{4\pi^2}{}'\,2cj'k'+c^2k'^2-c^2\frac{m^2}{4\pi^2} \\
&(ck+j)^2 = (ck'-j')^2 \\
&(ck+j) = \pm(ck'-j')
\end{aligned}$$

where one of the last two equalities hold.
But if $k+cj=-(k'-cj')$ and $(ck+j)=-(ck'-j')$ hold together, then $j=j'$, which is in contradiction with the first of the two (being $m \neq 0$), unless $c=\pm 1$. So, if $c \neq \pm 1$, then $k+k'+cj-cj'=0$ implies that $(ck+j)=(ck'-j')$, which implies that the integral in (3.124) equals 0.

If $c=\pm 1$, some algebra shows that $k+k'+cj-cj'=0$ would imply that $j'=j$ which in turns would imply $k'=k$ which together would be in contradiction with $k+k'+cj-cj'=0$, unless $k=0$, which is impossible if $m \neq 0$. So we conclude that if $c=\pm 1$, then $k+k'+cj-cj' \neq 0$ always and therefore again we have that the integral in (3.124) equals 0.

Therefore:

$$\Omega_{cd\ +-} = 0 \quad \forall c \in \mathbb{R} \tag{3.123}$$

Finally we have:

$$\begin{aligned}
\Omega_{\infty\ d\ +-} &= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^{+}(j)\,\tilde{u}^{-}(j') \int_{-\infty}^{+\infty} i2\pi\,(k-k')\left[e^{i2\pi(kd+j\tau)}e^{i2\pi\left(k'd-j'\tau\right)}\right]\ d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^{+}(j)\,\tilde{u}^{-}(j') \int_{-\infty}^{+\infty} i2\pi\,[k-k']\,e^{i2\pi(k+k')d}e^{i2\pi(j-j')\tau}\ d\tau \\
&= \int_{-\infty}^{+\infty} dj \int_{-\infty}^{+\infty} dj'\ u^{+}(j)\,\tilde{u}^{-}(j')\ \ i2\pi\,[k-k']\,e^{i2\pi(k+k')d}\delta\,(j-j') = 0
\end{aligned} \tag{3.124}$$

To come back to $\delta_u G \in T_G \mathscr{I}$ real vector space, it is enough in the formula (3.103), (3.104) and (3.105) to consider only $u^{+}(k)$ and $u^{-}(k)$ real and with the additional condition that $u^{+-}(k) = u^{+-}(-k)$; or $u^{+}(k)$ and $u^{-}(k)$ imaginary pure with the extra condition that $u^{+-}(k) = -u^{+-}(-k)$.

These conditions ensure that $\Omega_{cd\ ++}$, $\Omega_{\infty d\ ++}$, $\Omega_{cd\ --}$ and $\Omega_{\infty d\ --}$ have real value.

We can conclude that Ω does not depend on the parameter d.

If we choose only slices of space type, which entirely lie inside the light cone, which means that we choose slices corresponding to $c < -1$ and $c > 1$, then it is possible to choose a continuum orientation in the space of slices Σ_{cd}, so that Ω does not depend on the choice of c.

On the opposite, when c varies between the values -1 and 1, which means when we consider slices time-like, then Ω varies continuously with the parameter c.

There isn't any special phenomenon of degeneracy on the slices corresponding to $c = -1$ and $c = 1$ which delimit the light cone, but when such slices are crossed, then Ω begins to undertake changes.

Chapter 4

Observables and their Poisson brackets

In section 2.4 I showed how to build a symplectic structure on the covariant phase space $\mathscr{G}$ starting from the multisymplectic structure on the finite-dimensional multimomenta space P.

The space of functions on $\mathscr{G}$, let's call it $\mathscr{F}(\mathscr{G})$, or, for gauge theories, the space of functions on the quotient space, by the gauge group, of $\mathscr{G}$, is called the space of observables and it is of the highest interest from a physical point of view.

On $\mathscr{F}(\mathscr{G})$ physicists have introduced brackets, the "fields brackets", which in facts can be viewed as a Poisson structure on $\mathscr{G}$.

In this short chapter I want to show how to build a Poisson structure on $\mathscr{G}$, starting from its symplectic structure. I will then exhibit some formula for the Poisson brackets on $\mathscr{F}(\mathscr{G})$ and I will try to relate them to the brackets used by physicists. These results are known for multisymplectic field theories; in chapter 10 I will show how they can be extended to superfields theories.

Since in general $\mathscr{G}$ is infinite-dimensional, building on it a Poisson structure in a rigorous way would need an infinite-dimensional differential structure on $\mathscr{G}$, such that it would be possible to speak of the tangent space $T\mathscr{G}$, the cotangent space $T^*\mathscr{G}$ and the space of smooth functions $C^\infty(\mathscr{G})$. I will not explore here these questions and I will not go into the analytical subtleties of the matter. One can consult how Kijowsky and Szczyrba treated the matter in [97, 98]. I refer to K. A. Rejzner, [125, 126], for a possible approach to this subject, although other approaches may be equally valid. In this thesis I will treat $\mathscr{G}$, $\mathscr{F}(\mathscr{G})$, $T\mathscr{G}$ and $T^*\mathscr{G}$ as formal objects, meaning that all the considerations here done could be made rigorous whence a suitable differential structure on $\mathscr{G}$ is defined.

For more details on the use of a functional approach in studying Symplectic and Poisson structures on the covariant phase space, one can see M. Forger and S. V. Romero [53].

Let $G \in \mathscr{G}$ be a solution of the field theory: as seen in section 2.3 G can be seen as an Hamiltonian section-submanifold of P associated to the section $z \in \Gamma(P)$, so that $G = z(X)$. As we have seen in section 2.4, every vector $V_G \in \mathbb{T}_G\mathscr{G}$ corresponds to a vertical vector field, which here I will call with the same name V_G, over the Hamiltonian manifold G. We can write $V_G = V_G^{q^i}(z(x)) \frac{\partial}{\partial q^i}|_{z(x)} + V_G^{p_i^a}(z(x)) \frac{\partial}{\partial p_i^a}|_{z(x)}$. Remember that V_G must satisfy the Jacobi equations.

Let's suppose that the theory admits for the n-dimensional base manifold X a decomposition in a product of a $n-1$-dimensional space manifold and a one dimensional time manifold, such that every "space" manifold is a Cauchy surface (for a discussion on the Cauchy problem in field theory one can see the review paper of I. Khavkine [89] and the bibliography therein). On X, we can use local coordinates $(t, \vec{x}^b)$ with $b = 1, \ldots, n-1$, where $(\vec{x}^b)$ are coordinates on the space and such that, for every k constant, the equation $t = k$ defines a Cauchy surface: let's call them Cauchy coordinates. Let's call Σ_X the Cauchy surface of X corresponding to $t = 0$ and let's call Σ the restriction of P to Σ_X.

Then, if the Cauchy problem is well-posed and, in particular, in the absence of gauge symmetry, to fix a solution of the theory G, it is enough to know $z(\Sigma_X) = z(X) \cap \Sigma = G \cap \Sigma$ or, in other words, it is enough to know the values of the fields on Σ_X, $q^i(\vec{x}) := q^i(z(0, \vec{x}))$, together with the values of the first components of their multimomenta in the Cauchy coordinates on the Cauchy surface, id est $p_i^0(z(0, \vec{x}))$; let's call π_i that component: $\pi_i(\vec{x}) := p_i^0(z(0, \vec{x}))$. Inversely to each set of values $\left[q^i(\vec{x}), \pi_i(\vec{x})\right]$ on the Cauchy surface, it corresponds one solution of the theory.

In fact, to every vector $V_G \in \mathbb{T}_G\mathscr{G}$ corresponds a vector field, which here I again will call with the same name V_G, over the surface $\Sigma \cap G$ and of the form:

$$V_G(\vec{x}) = V_G^{q^i}(z(0, \vec{x})) \left.\frac{\partial}{\partial q^i}\right|_{z(0,\vec{x})} + V_G^{\pi_i}(z(0, \vec{x})) \left.\frac{\partial}{\partial \pi_i}\right|_{z(0,\vec{x})}$$

Let's consider a function $F \in \mathscr{F}(\mathscr{G})$: it's differential $dF \in \Gamma(T^*\mathscr{G})$ acts linearly on vector fields on $\mathscr{G}$ to give functions on $\mathscr{G}$. In other words $dF|_G$ acts linearly on $T_G\mathscr{G}$ to give a real number. Hence $dF|_G(V_G)$ can be written in this way:

$$dF|_G(V_G) = \int_{\Sigma_X} \left[V_G^{q^i}(\vec{x}) \left.\frac{\delta F}{\delta q^i}\right|_G (\vec{x}) + V_G^{\pi_i}(\vec{x}) \left.\frac{\delta F}{\delta \pi_i}\right|_G (\vec{x}) \right] d\vec{x}$$

where $d\vec{x}$ is the $n-1$-dimensional canonical volume form defined on Σ_x by the Cauchy coordinates, $V_G^{q^i}(\vec{x})$ and $V_G^{\pi_i}(\vec{x})$ are shortcut for $V_G^{q^i}(z(0, \vec{x}))$ and $V_G^{\pi_i}(z(0, \vec{x}))$ and where $\left.\frac{\delta F}{\delta q^i}\right|_G$ and $\left.\frac{\delta F}{\delta \pi_i}\right|_G$ are suitable distributions defined on Σ_X and depending on $\mathscr{G}$. In general it is not simple to calculate these distributions, but for certain classes of function on $\mathscr{G}$ it turns out that the calculation is easy.

On $\mathscr{G}$ it is possible to define a symplectic form Π_Σ in this way:

$$\Pi_\Sigma|_G(U_G, V_G) := \int_{\Sigma_X} -\left(U_G^{q^i}(\vec{x})V_G^{\pi_i}(\vec{x}) - V_G^{q^i}(\vec{x})U_G^{\pi_i}(\vec{x})\right) d\vec{x} = \int_{\Sigma\cap G} -(U_G \wedge V_G) \lrcorner\, dq^i \wedge d\pi_i \wedge d\vec{x} \tag{4.1}$$

where I denote by the same name $d\vec{x}$ the canonical volume $n-1$-form defined by the Cauchy coordinates on Σ_X and its lift on Σ.

Note that, a priori, Π_Σ depends not only on Σ but also on the chosen decomposition of X and on the chosen Cauchy coordinates, because the form $dq^i \wedge d\pi_i \wedge d\vec{x}$ depends on them. I don't keep track of this possible dependence because we will see below that indeed Π_Σ do not depend on anything else than Σ.

The, Peierls-like, fields brackets used by physicists are equivalent to the Poisson brackets which originates from the Symplectic structure on $\mathscr{G}$ defined by the symplectic form Π_Σ; see [89], chapter 3, for a discussion on this statement: we content ourselves to formally prove (4.5) below. Let's see briefly how to define these Poisson brackets with a procedure which mimics the one used on finite dimensional manifolds.

To every function $F \in \mathscr{F}(\mathscr{G})$ we can associate a vector field ${}^F V$ on $\mathscr{G}$ in this way:

$$dF(\cdot) = \Pi_\Sigma\left(\cdot, {}^F V\right)$$

then:

$$
{}^{F}V_G^{q^i} = \left.\frac{\delta F}{\delta \pi_i}\right|_G , \quad {}^{F}V_G^{\pi_i} = -\left.\frac{\delta F}{\delta q^i}\right|_G
$$

And we define:

$$
\{A, B\} := \Pi_\Sigma \left({}^{A}V, {}^{B}V\right) \tag{4.2}
$$

Then we see that:

$$
\begin{aligned}
\{A, B\}(G) &= \int_{\Sigma_X} -\left({}^{A}V_G^{q^i}(\vec{x})^{B}V_G^{\pi_i}(\vec{x}) - {}^{B}V_G^{q^i}(\vec{x})^{A}V_G^{\pi_i}(\vec{x})\right) d\vec{x} \\
&= \int_{\Sigma_X} \left(\left.\frac{\delta A}{\delta \pi_i}\right|_G (\vec{x}) \left.\frac{\delta B}{\delta q^i}\right|_G (\vec{x}) - \left.\frac{\delta B}{\delta \pi_i}\right|_G (\vec{x}) \left.\frac{\delta A}{\delta q^i}\right|_G (\vec{x}) \right) d\vec{x}
\end{aligned} \tag{4.3}
$$

Note that, if one changes the Cauchy decomposition of X, or if one changes the Cauchy coordinates, in general $\pi_i' \neq \pi_i$; so in general we have that: $dq^i \wedge d\pi_i \wedge d\vec{x} \neq dq^i \wedge d\pi_i' \wedge d\vec{x'}$. This means that the form $dq^i \wedge d\pi_i \wedge d\vec{x}$ is not defined in a covariant way. As a consequence also the fields brackets are not defined in a covariant way.

However: if, in order to define the Poisson structure, we replace the form Π_Σ with the Symplectic form Ω_Σ defined in section 2.4, following the ideas of Kijowski and Szczyrba, [98], then clearly we obtain a Poisson structure which do not depend on the full Cauchy decomposition nor on the coordinates chosen, but only, possibly, on the Cauchy surface Σ. We obtain then a covariantly defined Poisson structure on $\mathscr{G}$ which turn it into a true covariant phase space.

In fact, if one remembers the discussion made at the beginning of chapter 3, one see that Ω_Σ is even independent on Σ under certain assumptions (for example if we can chose Cauchy surfaces Σ which are all compact and belonging to the same homology class).

The situation is even better: it can be proven that, for every Σ, whatever is the global Cauchy decomposition of X and whatever are the Cauchy coordinates used to define Π_Σ, we have that:

$$
\Pi_\Sigma = \Omega_\Sigma
$$

The formal proof is very easy; see [64] for more details. By the definition of Π_Σ, π_i and by the definition of Cauchy coordinates given above and the definition of multisymplectic form ω and Symplectic form Ω_Σ, we have that:

$$
\begin{aligned}
\Pi_\Sigma|_G (U_G, V_G) &= \int_{\Sigma\cap G} -(U_G \wedge V_G) \lrcorner\, dq^i \wedge d\pi_i \wedge d\vec{x} = \\
&= \int_{\Sigma\cap G} -(U_G \wedge V_G) \lrcorner\, dq^i \wedge dp_i^0 \wedge \beta_0 = \\
&= \int_{\Sigma\cap G} (U_G \wedge V_G) \lrcorner\, \omega = \Omega_\Sigma|_G (U_G, V_G)
\end{aligned} \tag{4.4}
$$

So the Poisson structure covariantly built with Ω_Σ is equal to the Poisson structure built with Π_Σ. Equation (4.3) holds for both.

For a discussion on the equivalence of the Poisson structure built with Ω_Σ with the structure given by the Peierls brackets defined with the help of Green functions, see [53] and [89].

Let's now consider the functions $A = q^i(\vec{y})$ and $B = \pi_i(\vec{y})$ defined on $G \in \mathscr{G}$ in the following way:

$$
A(G) = q^i\left(z(0, \vec{y})\right), \quad B(G) = \pi_i\left(z(0, \vec{y})\right)
$$

Then it is easy to calculate:

$$\left.\frac{\delta A}{\delta q^j}\right|_G(\vec{x}) = \delta^i_j \delta^{n-1}(\vec{x}-\vec{y}), \quad \left.\frac{\delta A}{\delta \pi_j}\right|_G(\vec{x}) = 0$$
$$\left.\frac{\delta B}{\delta q^j}\right|_G(\vec{x}) = 0 \quad \left.\frac{\delta B}{\delta \pi_j}\right|_G(\vec{x}) = \delta^j_i \delta^{n-1}(\vec{x}-\vec{y})$$

So, from formula (4.3), we obtain:

$$\begin{aligned} &\left\{q^i(\vec{y}), q^j(\vec{y'})\right\} = 0 \\ &\left\{\pi_i(\vec{y}), q^j(\vec{y'})\right\} = \delta^j_i \delta^{n-1}(\vec{y}-\vec{y'}) \\ &\left\{\pi_i(\vec{y}), \pi_j(\vec{y'})\right\} = 0 \end{aligned} \tag{4.5}$$

Relations (4.5) are the fundamental commutators at equal times of physicists classical Bosonic field theory. We have then shown the link existing between the symplectic structure built on $\mathscr{G}$ with the multisymplectic techniques and the canonical commutators structure of Physics field theories.

Note that all the considerations made above make sense only under certain conditions on the field theory. For example, if the Lagrangian is gauge invariant under a certain gauge group, then the Cauchy problem has not an unique solution and the subject of Cauchy decompositions need to be treated more carefully, see [89]. Moreover, when the Lagrangian is not regular, the multisymplectic form ω can still be defined on P, but the symplectic form Ω_Σ may be, a priori, degenerate. In these cases the link between the multisymplectic form on P and the canonical structure of the field theory used in Physics is not anymore so simple to find. For a discussion on Yang-Mills theories in a multisymplectic framework see [71]; for a discussion on General Relativity and its possible extensions see [75]. For a discussion on Symplectic and Poisson structures on the spaces of solutions of field theories with constraints and gauge invariants see [89].

Relations (4.5) constitute the starting point and the main ingredient of canonical quantization. The other essential ingredients of canonical quantization are Green functions and field commutators at different times. I will not treat here the question of their covariant definitions.

It is worth noting that there have been some attempts of approaching fields quantization directly from the point of view of multisymplectic field theory, trying to exploit the fact that it is defined on the finite dimensional multimomenta space P and trying to exploit it's main resource, which is the existence of the multisymplectic differential form ω.

These attempts usually begin with a definition of the space of observable which is a drastic reduction of the space of all possible function on $\mathscr{G}$ and they continue by trying and build a Poisson-like structure on that space, defining somehow the brackets of observables in a covariant way. At this stage authors try and take advantage from the presence of the multisymplectic struture on the finite dimensional space P. Then the quantization programs possibly continue using the brackets defined for quantization.

All these approaches are naturally covariant and do not want to rely on a Cauchy decomposition (id est a time foliation) of the spacetime X.

For some examples of how the restricted space of observables with their Poisson-like brakets can be constructed, see Kijowski [94, 95], Kijowski and Szczyrba [97], Hélein [74], Forger and Römer [52], Forger, Paufler and Römer [50, 51], M. O. Salles [139] and the last chapter of Forger and Romero [53]. See also Baez, Hoffnung and C. L. Rogers [3] and Baez and Rogers [4], where on the space of observables of a string theory is built a Lie 2-algebra structure exploiting the

2-plectic (multisymplectic in my terminology) structure. Richter in [131] build on a certain space of observables a structure of homotopy Poisson-n algebra.

With his work, see for example [85, 86, 87, 88], I. Kanatchikov carries forward the program, until a quantized theory is built.

Part II

Fractional mixed forms on supermanifolds

Introduction to Part II

In this second part of my thesis I introduce the notions of fractional forms, fractional coforms and fractional mixed forms on supermanifolds. The fractional forms will be an essential ingredient for the definition of a superfield theory and for the supermultisymplectic formalism which is the main object of the third part of this thesis.

Fractional forms are good examples of the forms introduced by Th. Voronov and A. Zorich in their papers during late '80-s, [155, 156, 157, 158]. Voronov-Zorich forms were introduced as the natural analogous on supermanifold of classical forms on classical manifolds. They can be integrated over supermanifolds, they present a natural pairing with supervectors and they admit a Cartan calculus. The fractional forms that I define here in Chapter 5 are a specific class of Voronov-Zorich forms which can be defined through the use of superdeterminants. It is possible to set up rules for a Cartan calculus on fractional forms and I do it in section 5.4; this calculus turns out to be more manageable than the Cartan calculus for generic superforms.

When building a superfield theory, we can decide to restrict ourselves, as long as it is possible, to the use of fractional forms rather than using the more general but more complicated superforms. I believe that fractional forms, with their simple notation introduced here, not only can simplify the actual Cartan calculus needed for field theories, but they can make it also more transparent. Moreover fractional forms, like all Voronov-Zorich $r|s$-forms can be integrated, and their integral satisfy some nice properties, as I explain in section 5.5.

I will show in the third and last part of this thesis how all this can be useful in the definition of superfield theories.

Fractional coforms are special examples of what Voronov calls *twisted covariant dual Lagrangians satisfying the fundamental equations* or shortly *twisted dual forms* in [153] and [154], and they are the basis for the definition of his stable forms. Fractional mixed forms are examples of what Voronov called mixed forms, [153], [154]. I will introduce them in Chapter 6, together with the rules to perform with them a Cartan calculus and together with the definition and the proof of some properties of their integral.

Chapter 6 is indeed independent of the rest of this thesis and it is not necessary to read it in order to understand its third part. Therefore the material presented there is not treated in detail. It can be considered as a natural complement of chapter 5 and as a preliminary work for future studies, especially in the direction of Batalin-Vilkovisky and Bechi-Rouet-Stora-Tyutin superfield theories.

Chapter 5

Fractional forms and their integration on supermanifolds

Section 5.1 will be dedicated to fix some conventions and notations regarding supermanifolds.

In section 5.2 I will give the definition of Voronov-Zorich superforms and I will propose a natural extension of this definition.

In section 5.3 I will treat a subclass of superforms that I call Berezinian superforms.

In section 5.4 I will present the exterior multiplication of superforms by covectors and their contraction with vectors. Doing so, I fix an imprecision in the definition of the wedge product appearing in [154]. Moreover I point out and I resolve an ambiguity in the definition of the interior product by an odd vector, which until now had not been noticed. I will present then the commutation relations to which exterior and interior products obey, which were first shown by Voronov in [154].

I will then define fractional superforms and will consider a Cartan calculus for fractional superforms, which turns out to be quite intuitive.

In sections 5.3 and 5.4 I also introduce a simple system of notation which, I think, makes all the subject transparent and possibly makes it easier the use of superforms in field theory or other areas of mathematics. It is from the choice of this notation that arises the term "fractional".

In section 5.5 I will shortly explain how the objects previously defined can be used to built a theory of integration on supermanifolds. Indeed all the material presented in the following chapters of this thesis presupposes a good theory of integration of superforms on supermanifolds. I will make use of the theory first elaborated by Voronov and Zorich in [155, 156, 157, 158] and presented in a detailed exposition by Voronov in [152]. In fact I will only need the main definitions and the main theorems of that theory and I will explicitly recall them without giving detailed proofs. I'll propose some minor modifications, like the introduction of the integral over an immersed body, which suitable for the matter treated here.

5.1 Supermanifolds and their tangent and cotangent modules.

The use of anticommuting variables in mathematical physics can be traced back to the papers of 50's and 60's of J. Schwinger on the theory of quantized fields, see the book of Schwinger [143].

In his paper on the commutation laws of relativistic fields, Peierls, in 1952 [122], made use of anticommuting parameters, which he put in front of fermionic fields.

J. Martin in 1959, [109, 110], introduced a differential calculus for functions of anticommuting variables with the aim of extending Feynman's path-integral method of quantization to fermions, but his work remained largely unknown.

It was the work of F. A. Berezin on second quantization in 1965 [11] that really gave birth to supermathematics. It was followed by other works of Berezin and coauthors, for example Berezin with G. I. Kac on supergroups in 1970 [14] and Berezin with D. A. Leites in 1975 on supermanifolds [15].

During the 70's, the development of supersymmetric field theories in Physics, which began with the works of D. Volkov and V. Akulov [149] and J. Wess and B. Zumino [159], required a good mathematical foundation of the theory of supermanifolds and triggered a number of studies of different authors: for example again Berezin [12, 13], M. Batchelor [7] or Leites [103].

Two different, but essentially equivalent, approaches to supermanifolds can be undertaken, see [8]. Following A. Rogers [133], I call the first approach 'concrete' and the second one 'algebro-geometric'.

In the concrete approach a supermanifold is a set, more specifically it is a manifold modeled on some 'superspace' so that it has local coordinates, some of which take values in the even and some others in the odd part of a Grassmann algebra [133]. This approach has been developed mainly by Rogers, see for example [132], and B. DeWitt, [43].

In the algebro-geometric approach to supermanifolds, it is the sheaf of functions on a classical manifold which is extended, to become an anticommuting sheaf. This approach was initially developed by the Russian school following Berezin; for an introduction to the subject see Leites [103] and Y. Manin [105]; see also C. Bartocci, U. Bruzzo and D. Hernández Ruipérez [6].

For a shorter introduction to supermanifolds see also Hélein [69].

Recently a categorical point of view on supermanifolds has been proposed by some authors: see C. Sachse [138]. This approach strongly relies on the concept of functor of points, which, to my knowledge, was first introduced in supermathemathics by A. S. Schwarz in [141].

In this work, I will use the concrete approach. For an extended introduction to the subject and a more complete historical account, see the book of Rogers [133] from which I have also taken most of the notation.

I will use the remaining two parts of this section to fix some notations and conventions.

5.1.1 Supermanifolds.

For the definitions of superspaces, supermanifolds, tangent super-module, G^∞ superfunctions and G^∞ super-bundles in the concrete aprroach, I refer to [133].

In this thesis a superspace of dimension $n|m$ will be a real superspace:

$$\mathbb{R}_S^{n|m} := \underbrace{\mathbb{R}_{S,0} \times \cdots \times \mathbb{R}_{S,0}}_{n \text{ times}} \times \underbrace{\mathbb{R}_{S,1} \times \cdots \times \mathbb{R}_{S,1}}_{m \text{ times}}$$

where $\mathbb{R}_{S,0}$ and $\mathbb{R}_{S,1}$ are respectively the even and the odd part of the real Grasmann algebra with infinite number of generators; I will call sometime "even numbers" the elements of $\mathbb{R}_{S,0}$ and "odd numbers" the elements of $\mathbb{R}_{S,1}$. The presence of an infinite number of generators is important for certain proofs.

On $\mathbb{R}_S^{n|m}$ the DeWitt topology will be understood [43]. Supermanifolds M of dimension $n|m$ will be modeled on $\mathbb{R}_S^{n|m}$ with the DeWitt topology with the use of $n|m - G^\infty$ atlases of charts. The G^∞-maps are in superanalysis the natural generalization of C^∞-maps: see [133] for a proper definition. For a different approach to maps between supermanifolds see Hélein [68]. The subscript S will be often let to drop.

On supermanifolds I will use the following convention: indices of even coordinates will be indicated with Latin lowercase letters, indices of odd coordinates will be indicated with Greek lowercase letters; capital Latin letters will be used for indices running trough coordinates of both parities or when the parity of the coordinate is unknown or generic. The convention used for the coordinates on tangent spaces of supermanifolds will be introduced and explained below. I will follow the so called skew-commutative convention for the parities of forms and vectors of a supermanifold: it will also be explained below.

With the symbols $|\ |$ or deg( ), I'll denote the degree of a coordinate or of an index (according to the context), especially when they appear in a formula which is the exponent of (-1). The degree is 0 when the coordinate (or the index) is even and it is 1 when the coordinate (or the index) is odd. All operations undertaken with degrees will be understood as modulo $\mathbb{Z}_2$ if not otherwise stated.

On an open chart U of an atlas of the supermanifold X I'll use often the local coordinates (x^a, x^α), $a = 1, \cdots n$, $\alpha = 1, \cdots m$ (or the coordinates x^A with $A = 1, \cdots, n+m$), where $n|m$ is the superdimension of X. So we will have $|x^a| = 0$ and $|x^\alpha| = 1$ or $|a| = 0$ and $|\alpha| = 1$.

I'll often denote by ∂_a, ∂_α and ∂_A the operators $\frac{\partial}{\partial x^a}$, $\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial x^A}$.

5.1.2 Tangent and cotangent bundles of a supermanifold.

The first big difference which distinguishes the super framework from the classical one arises when one considers vectors on supermanifolds. The tangent space over a point x of a supermanifold X is not in fact a superspace, whereas a free supermodule over $\mathbb{R}_S$: I will call it T_xX.

It is possible to establish a natural one-to-one correspondence between the elements of the tangent supermodule T_xX and a superspace. If the supermanifold X has dimension $n|m$, the tangent supermodule over one of its point is a free supermodule of dimension $n|m$. If on a local chart U of X, x has coordinates (x^a, x^α), then a base for T_xX is provided by the derivations $(\partial_a; \partial_\alpha)$. In this paper I follow the so called skew-commutative convention for the parities of forms and vectors of a supermanifold: so I give to $\partial_A = \frac{\partial}{\partial x^A}$ the same parity of x^A: thus ∂_a will be even and ∂_α will be odd. Once fixed the degrees of the generators of the supermodule T_xX, we note that:

- an element of T_xX is even if it is the product of an even generator by an even element of the algebra $\mathbb{R}_S$ or if it is the product of an odd generator by an odd element of the algebra $\mathbb{R}_S$;
- an element of T_xX is odd if it is the product of an odd generator by an even element of the algebra $\mathbb{R}_S$ or if it is the product of an even generator by an odd element of the algebra $\mathbb{R}_S$.

For example: $\overline{v^a}\partial_a$ is even if all $\overline{v^a} \in \mathbb{R}_{S,0}$ are even numbers, whereas $\widetilde{v^a}\partial_a$is odd if all $\widetilde{v^a} \in \mathbb{R}_{S,1}$ are odd numbers; $\overline{v^\alpha}\partial_\alpha$ is even if all $\overline{v^\alpha} \in \mathbb{R}_{S,1}$ are odd numbers, whereas $\widetilde{v^\alpha}\partial_\alpha$ is odd if all $\widetilde{v^\alpha} \in \mathbb{R}_{S,0}$ are even numbers. A suitable version of Einstein convention on repeated superindexes is understood.

I call $T_{x,0}X$ the space of all even elements of T_xX and I call $T_{x,1}X$ the space of all odd elements of T_xX. We have then $T_xX = T_{x,0}X \oplus T_{x,1}X$. $T_{x,0}X$ has a natural structure of superspace of the same superdimension $n|m$ of X. Every element $\overline{v} \in T_{x,0}X$ can be written as $\overline{v} = \overline{v^a}\partial_a + \overline{v^\alpha}\partial_\alpha$ with $\forall a = 1, \cdots, n \quad \overline{v^a} \in \mathbb{R}_{S,0}$, $\forall \alpha = 1, \cdots, m \quad \overline{v^\alpha} \in \mathbb{R}_{S,1}$. So $(\overline{v^a}; \overline{v^\alpha})$ can be used as coordinates on $T_{x,0}X$. Also $T_{x,1}X$ has a natural structure of superspace, with superdimension $m|n$, endowed with the coordinates $(\widetilde{v^\alpha}; \widetilde{v^a})$. Lastly $T_xX = T_{x,0}X \oplus T_{x,1}X$ has itself a natural structure of superspaces of superdimension $n+m|m+n$, and with natural coordinates $(\overline{v^a}, \widetilde{v^\alpha}; \overline{v^\alpha}, \widetilde{v^a})$. So in the following I will often use the convention that coordinates on tangent spaces over a supermanifold are indicated with the same indices used for the coordinates in the supermanifolds and either with a bar over them, to indicate that they maintain the same parity of the corresponding coordinate on the base, or with a tilde over them, to indicate that they have opposite parity to the one of the corresponding coordinate on the base.

Note that $T_{x,1}X$ is naturally isomorphic to $\Pi T_{x,0}X$ in the category of superspaces, where Π is the parity-inversion functor. Very often in the literature (for example in the papers of Voronov) $T_{x,0}X$ is called the tangent space and it is indicated with T_xX, whereas $T_{x,1}X$ is indicated with ΠT_xX. I prefer to use the notation introduced above, where T_xX indicates the tangent supermodule or the tangent superspace built up from it. TX will be the tangent bundle on X, defined in the expected way. This notation can be useful in some situations.

Remark 22. *Each free supermodule A of superdimension $n|m$ over the algebra $\mathbb{R}_S$ can be put in a one to one correspondence with a superspace of superdimension $n+m|m+n$ with a technique analogue to the one used above, so that we can write $A \cong A_0 \oplus A_1$ where A_0 and A_1 are the even and odd part of the supermodule A.*

We will use this construction a couple of time in the following without coming back on its details.

In particular in the following sections we will meet the cotangent space T_x^*X of a $n|m$-supermanifold X over one of its points x; id est the space of left-$\mathbb{R}_S$-linear functions from T_xX to $\mathbb{R}_S$. I will call covectors its element. Note that T_x^*X is a right supermodule on $\mathbb{R}_S$ of dimension $n|m$. It can be seen as a superspace of dimension $n+m|m+n$ and can be split in an even component $T_{x,0}^*X$ of dimension $n|m$ and an odd component $T_{x,1}^*X$ of dimension $m|n$. Every even covector $\overline{p}$ can be written as $\overline{p} = dx^a\overline{p_a} + dx^\alpha\overline{p_\alpha}$ and every odd covector $\widetilde{p}$ can be written as $\widetilde{p} = dx^a\widetilde{p_a} + dx^\alpha\widetilde{p_\alpha}$. This is in agreement with the skew-commutative convention for which $\left|dx^A\right| = \left|x^A\right| = |A|$.

5.2 Superforms on supermanifolds

Starting from the concept of vector fields, it is not difficult to define tensor fields on a supermanifold: see [133] chapter 10. Then an almost obvious generalization of differential forms becomes readily available: they are the graded analogue of classical differential forms and they first appeared explicitly in the literature in a paper of B. Kostant in 1977 [101].

To build a Lagrangian theory, as seen in chapter 1, the main ingredient is the Lagrangian n-form on $J^1\pi$. The Lagrangian has then to be integrated on the image of a section of $J^1\pi$, to obtain the action A of the theory on a field configuration. If one want to extend the Lagrangian formalism to a superfield theory, the integration of the Lagrangian is the most delicate point to address. First of all it is necessary to use a suitable definition of integral on superspaces. At this scope, for physical applications, it is used the Berezin integral, first introduced in [11]. But, if one tries and uses differential n-forms à la Kostant to build an integration theory on supermanifolds, one soon falls in some troubles, and in fact it doesn't exist a satisfactory theory of integration of "naive" forms.

One can instead use other objects of integration, like Berezinian tensor densities, whose definition is natural when one disposes of tensors and of Berezin integral. The drawback is that densities do not share many of the features of forms which allow the use of a geometrical language in describing field theories. For example they do not naturally allow contraction with super vector fields which generate supersymmetries.

The quest becomes then necessary for objects which on one side share some of the tensor properties of forms and on the other side are suitable for building a consistent and possibly simple theory of integration.

At this scope I will use the forms introduced and studied by Voronov and Zorich in [155, 156, 157, 158] and in [152]. Voronov and Zorich worked on the basis of previous ideas of the Russian school. J.N. Bernstein and Leites in 1977 had defined 'integral forms' as tensor products of multivector fields with Berezin volume forms, [16]; then they defined 'pseudodifferentials' forms [17], which are beautiful objects, but which do not have any natural grading and do not solve in a satisfactory way the problem of integration on submanifolds of a supermanifold (moreover they cannot be integrated on supermanifolds whose odd part is unoriented). A.S. Schwarz, M.A. Baranov, A.V. Gajduk, O.M. Khudaverdian and A.A. Rosly, in the beginning of 80's, [55, 5, 136] studied Berezinian densities and introduced the concept of closed densities.

In subsection 5.2.1 I will give a very short introduction of the theory of Voronov and Zorich; in subsection 5.2.2 I will define extended forms, which are a natural extension of Voronov-Zorich forms.

5.2.1 Voronov and Zorich superforms

Voronov and Zorich in their papers often use the word "Lagrangian" in a way which is not exactly equivalent to the way it is used here: I will therefore use a vocabulary slightly different than the one used for example in [152]. I will present here a version of Voronov-Zorich theory which allows superforms with poles: see [9]. Let's see the main definitions.

Recall that $GL(n|m)$ is the supergroup of invertible $\mathbb{R}_S$-linear even maps between a supermodule A over $\mathbb{R}_S$ of dimensions $n|m$ and a supermodule B of the same superdimension (note that $n|m$ is the dimension of A and B as supermodules and not as superspaces, see Remark 22). When fixed bases of A and B are chosen, an element $g \in GL(n|m)$ can be written as a $(n, m) \times (n, m)$-dimensional supermatrix:

$$g = \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{1,0} & g_{1,1} \end{pmatrix}$$

where $g_{0,0}$ is a $n \times n$ invertible matrix with even entries; $g_{0,1}$ is a matrix with n rows and m columns with odd entries; $g_{1,0}$ is a matrix with m rows and n columns with odd entries and $g_{1,1}$ is a $m \times m$ invertible matrix with even entries.

An element $a \in A$ can be written as the column of its coordinates with-respect to the chosen basis; the supermatrix g acts from the left with the usual rule of multiplications of matrices and the image $b = g(a) \in B$ is the column of its coordinates in the base chosen on B. Note that, when $A = B$, one usually chooses the same base on the two copies of the supermodule.

Since $A = A_0 \oplus A_1$, see Remark 22, we can consider the restriction of g to A_0 and then the restriction to A_1. Since, by definition of $GL(n|m)$, g is even, we have that $g(A_0) \subset B_0$ and $g(A_1) \subset B_1$; so, when necessary, we can consider g as a $\mathbb{R}_{S,0}$-linear invertible map between the superspace A_0 of dimension $n|m$ and the superspace B_0 of the same dimension.

Recall that if $g \in GL(n|m)$ is written as a supermatrix, we have:

Definition 23. *The superdeterminant, also called Berezinian, of $g \in GL(n|m)$ is the element of $\mathbb{R}_S$ obtained with the following formula:*

$$\mathrm{sdet}_{n,m}(g) = \mathrm{Ber}_{n,m}(g) := \det\Big(g_{0,0} - g_{0,1}g_{1,1}^{-1}g_{1,0}\Big)\det\big(g_{1,1}^{-1}\big) \tag{5.1}$$

Where the determinant of the matrices involved is calculated with the usual rule. Note that $\forall g \in GL(n|m)$, $\mathrm{Ber}_{n,m}(g)$ is in fact an element of $\mathbb{R}_{S,0}$ and, to be more precise, an invertible one.

I extend this definition to any $(n, m) \times (n, m)$-dimensional supermatrix, regardless of the parities of its entries, provided that its block $g_{1,1}$ has even entries and it is invertible. In this more general case, the entries of the matrix in the first parenthesis in equation (5.1) may be odd, so it is important to fix a definition of det which fix the order of terms involved in multiplications, because the result of those multiplications may vary according to the order chosen to perform them.

Definition 24. *If g is a $(n+m)\times(n+m)$ matrix with entries belonging to $\mathbb{R}_S$ and so that its submatrix g_{11} is invertible, then we pose:*

$$\operatorname{sdet}_{n,m}(g) := \det\Big(g_{0,0} - g_{0,1}g_{1,1}^{-1}g_{1,0}\Big)\det\big(g_{1,1}^{-1}\big) \tag{5.2}$$

where the determinants appearing in the second member of the equality (5.2) are the polynomials calculated with the usual definition provided that in each of their monomial the entries of the matrix involved are multiplied following an order by column: so that elements of first column come first, then elements of second column and so on.

Note that in this more general case the superdeterminant may be an odd element of $\mathbb{R}_S$. Note that with this definition the superdeterminant keeps the property of additivity in the first n rows and in the first n columns.

Let $g \in GL(n|m)$ be written as a supermatrix (after having chosen arbitrarily the necessary basis) and let V be a superspace of dimension $o|p$, then there is a left action of $GL(n|m)$ onto $\underbrace{V\times\cdots\times V}_{n}\times\underbrace{\Pi V\times\cdots\times\Pi V}_{m}$, defined in this way:

$$\forall g\in GL(n|m),\ \forall\overline{v_1},\cdots,\overline{v_n}\in V,\ \widetilde{v_1},\cdots,\widetilde{v_m}\in\Pi V,\ g\cdot\begin{pmatrix}\overline{v_1}\\ \vdots\\ \overline{v_n}\\ \widetilde{v_1}\\ \vdots\\ \widetilde{v_m}\end{pmatrix} = \begin{pmatrix}g_{0,0} & g_{0,1}\\ g_{1,0} & g_{1,1}\end{pmatrix}\begin{pmatrix}\overline{v_1}\\ \vdots\\ \overline{v_n}\\ \widetilde{v_1}\\ \vdots\\ \widetilde{v_m}\end{pmatrix} \tag{5.3}$$

Where, in the right side of the last equation, the product is the usual matrix product with attention given to the order in products of entries and where an element $\overline{v}\in V$, multiplied on the left by an odd number, gives an element of ΠV in a natural way and vice-versa.

Note that in equation (5.3), as well as in most of formula in this thesis, I write each vector v as the line of its components. I use here the convention adopted in Voronov papers, which is indeed different from the convention usually adopted in papers on classical differential geometry, where vectors are usually written in column.

We are now in the position to understand the definition of a superform given by Voronov and Zorich.

Let's consider a supermanifold X of dimension $n|m$ and one of its point $x\in U\subset X$, where U is a local chart of X. On U we have local coordinates x^A. Let T_xX be the tangent module of X over x. A base for T_xX is given by $(\partial_A|_x)_{A=1\dots n|m}$. We can identify a point $u\in T_xX$ by its coordinates u^A with-respect to the chosen basis. On T_xX we can consider the topology inherited from TX. We have then the following:

Definition 25 (Voronov and Zorich)**.** *A form of degree $r|s$ over a point $x\in X$, supermanifold of dimension $n|m$, is a G^∞ map $\omega:\ O\subset\underbrace{T_{x,0}X\times\cdots T_{x,0}X}_{r}\times\underbrace{T_{x,1}X\times\cdots T_{x,1}X}_{s}\longrightarrow\mathbb{R}_S$, which satisfies the following: $\forall v\in O$, open subset of $\underbrace{T_{x,0}X\times\cdots T_{x,0}X}_{r}\times\underbrace{T_{x,1}X\times\cdots T_{x,1}X}_{s}$:*

$$\forall g\in GL(r|s),\ \omega(g\cdot v) = \omega(v)\operatorname{Ber}_{r,s}(g) \tag{5.4}$$

$$\frac{\partial^2 \omega}{\partial v_G^B \partial v_F^A} + (-1)^{|G||F|+(|G|+|F|)|A|} \frac{\partial^2 \omega}{\partial v_F^B \partial v_G^A} = 0 \tag{5.5}$$

where $A, B = 1, \cdots, n+m$ are the indices in the space T_xX and so also in both spaces $T_{x,0}X$ and $T_{x,1}X$ with their usual degree; v_F^A is the A-th coordinates of v_F in the local base $(\partial_A|_x)_A$; F runs from 1 to $r+s$ and we have $v_F \in T_{x,|F|}X$, where we set $|F| = 0$ when $F = 1, \cdots, r$ and $|F| = 1$ when $F = r+1, \cdots, r+s$.

It can be seen that the definition does not depend on the choice of the chart U and of the corresponding basis for T_xX.

An important consequence of (5.5), which in some sense justifies it, is noted below in remark 67 in the section treating the integration of superforms.

Note that, by definition, if $v \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$ is such that $v_{r+1}, \cdots, v_{r+s}$ are linear dependent, then there are only two possibility: either $v \notin O$ (so that ω is not defined on v), or $\omega(v) = 0$.

The space of $r|s$-forms over x has a natural structure of free right supermodule over $\mathbb{R}_S$; in fact, if ω is a $r|s$-superform, $\lambda \in \mathbb{R}_S$ and $v \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, then we can define $\omega\lambda$ with $\omega\lambda(v) := \omega(v)\lambda$.

I call the space of $r|s$-superform $\Lambda_x^{r|s}$, as a shortcut for $\Lambda^{r|s}T_x^*X$ and we have, as usual, $\Lambda_x^{r|s} = \Lambda_{x,0}^{r|s} \oplus \Lambda_{x,1}^{r|s}$, where $\Lambda_{x,0}^{r|s}$ and $\Lambda_{x,1}^{r|s}$ are respectively the even and the odd part of $\Lambda_x^{r|s}$ and they are superspaces. Note that $\omega \in \Lambda_{x,0}^{r|s}$ if $\forall v \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, $\omega(v) \in \mathbb{R}_{S,0}$ and $\omega \in \Lambda_{x,1}^{r|s}$ if $\forall v \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, $\omega(v) \in \mathbb{R}_{S,1}$.

We can give to $\Lambda_x^{r|s}$ the structure of left supermodule over over $\mathbb{R}_S$ using the following definition: $\forall \lambda \in \mathbb{R}_S$, $\forall \omega \in \Lambda_x^{r|s}$, $\forall v \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$: $\lambda\omega(v) := (-1)^{|\lambda||\omega|}\omega\lambda(v)$.

In the usual way we can build the fiber bundles $\Lambda_0^{r|s}X$, $\Lambda_1^{r|s}X$ and $\Lambda^{r|s}X$.

Definition 26. *A G^∞ section of the bundle $\Lambda^{r|s}X$ is called a differential $r|s$-form. The space of $r|s$-form over X is called $\Omega^{r|s}X := \Gamma\left(\Lambda^{r|s}X\right)$.*

Definition 27. *The operator of exterior derivation d is defined on forms ω of degree $r|s$ by the formula*

$$\begin{aligned} &\forall(\overline{v_1}, \cdots, \overline{v_r}, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r+1} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}, \\ &\forall \omega \in \Omega^{r|s}X : \\ &d\omega(\overline{v_1}, \cdots, \overline{v_r}, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) := (-1)^r \, \overline{v_{r+1}^A} \left(\frac{\partial \omega}{\partial x^A} - (-1)^{|A||F|} v_F^B \frac{\partial^2 \omega}{\partial x^B \partial v_F^A} \right) \end{aligned} \tag{5.6}$$

where the index F runs from 1 to $r+s$ and its parity is the parity of the corresponding vector so that $|F| = 0$ when $v_F = \overline{v_F} \in T_0X$ and $|F| = 1$ when $v_F = \widetilde{v_{F-r}} \in T_1X$; the indices A and B run from 1 to $n+m$ and their parities are defined in the obvious way following the parities of coordinates on X.

Proposition 28. *The operator of exterior derivation d is well defined, it sends $r|s$-forms to $r+1|s$-forms and it is $\mathbb{R}_S$-linear.*

Proof. See [152]. ∎

5.2.2 Preforms and extended forms

I define preforms in the following way:

Definition 29. *A preform of degree $r|s$ over a point $x \in X$, X of dimension $n|m$, is a G^∞ map $\omega : \underbrace{T_xX \times \cdots T_xX}_{r+s} \longrightarrow \mathbb{R}_S$, which, when restricted to $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, is a form of degree $r|s$.*

Note that, with these definitions, one can prove that there is a one-to-one correspondence between r-forms à la Kostant (see [133] chapter 10 for their definition in a G^∞ setting) and $r|0$-forms. In fact a r-form can be proven to be a $r|0$-preform.

One may asks himself if all $r|0$-preforms are also r-forms à la Kostant, and the answer is no. Not only: there exist $1|0$-preforms which are $\mathbb{R}_{S,0}$-linear on all T_xX (which implies that they are linear separately on $T_{x,0}X$ and $T_{x,1}X$), which on $T_{x,0}X$ coincide with a 1-form, but who are not 1-forms.

Consider for example the superspace $V^{0|m}$: it is a supermanifold and its tangent module $T_xX = T_{x,0}X \oplus T_{x,1}X$ over a point x is isomorphic to $V \oplus \Pi V$. If (x^α), $\alpha = 1, \cdots, m$ are the coordinates on V, then a vector $v \in T_xV$ can be written as $v = \overline{v^\alpha}\partial_\alpha|_x + \widetilde{v^\alpha}\partial_\alpha|_x$ with $\overline{v^\alpha} \in \mathbb{R}_{S,1}$, $\widetilde{v^\alpha} \in \mathbb{R}_{S,0}$, $\overline{v^\alpha}\partial_\alpha|_x \in T_{x,0}X$ and $\widetilde{v^\alpha}\partial_\alpha|_x \in T_{x,1}X$. The 1-form à la Kostant $dx^\alpha|_x$ acts on v in this way: $dx^\alpha|_x(v) = v^\alpha = \overline{v^\alpha} + \widetilde{v^\alpha}$. Now let's consider the function $\omega : T_xX \longrightarrow \mathbb{R}_S$ defined by $\omega(v) = \overline{v^\alpha} - \widetilde{v^\alpha}$. It is a $1|0$-preform, it is $\mathbb{R}_{S,0}$-linear on T_xX, when restricted to $T_{x,0}X$ it coincides with $dx^\alpha|_x$, but it is not a 1-form à la Kostant, and in fact it does not coincide with $dx^\alpha|_x$ on all T_xX.

Similar phenomena occur for r greater than 1.

Definition 30. *A form of degree $r|s$ over a point $x \in X$, X of dimension $n|m$, is said to be extended in the first argument if it is a G^∞ map $\widehat{\omega} : T_xX \times \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r-1} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$ which is $\mathbb{R}_S$-linear in the first argument, $\mathbb{R}_{S,0}$-linear and antisymmetric in the $r-1$ following arguments, and which, when restricted to $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, is a form of degree $r|s$.*

An extended $1|0$-form is a 1-form à la Kostant. An extended $1|0$-form α can be written on a local chart as $\alpha = dx^A\alpha_A$.

I chose the convention to fix the degree of dx^A as $\left|dx^A\right| := |A|$; this is far from being widely accepted, but it seems to me the most natural and most useful in the context of this work. This convention is sometime called the skew-commutative convention, while the other one is called commutative. Note that, when using the commutative convention, the wedge product is often defined as a graded commutative product of differentials. This definition differs from the one that I use here later by a sign when the wedge product of dx^a with dx^α is considered. Note also that the two different corresponding de Rham complexes are called by Manin [105] 'even' and 'odd' complexes: I stress the fact that this use of the terms 'even' and 'odd' is different than the one made here.

It is immediate to see by a direct calculation that the operator of exterior derivative defined above sends the coordinate local function x^A (which can be seen as a $0|0$-form) to the $1|0$-form dx^A. So the notation is consistent.

I call T_x^*X the space of extended $1|0$-form over a point x of a supermanifold X. It is a right free supermodule whose even generators are the forms dx^a and the odd generators are the forms dx^α. It can be given also the structure of left supermodule with the technique seen in the previous section, so that: $\forall\lambda \in \mathbb{R}_S$, $\forall\mu \in T_x^*X$ and $\forall v \in T_xX$, $(\lambda\mu)(v) = (-1)^{|\lambda||\mu|}(\mu\lambda)(v) = (-1)^{|\lambda||\mu|}\mu(v)\lambda$. Note that, with this definition, we have that in general: $(\lambda\mu)(v) \neq (-1)^{|\lambda||\mu(v)|}\mu(v)\lambda$ and $(\lambda\mu)(v) \neq \lambda[\mu(v)]$. A similar situation will occur for extended superforms of higher degree: I will not comment on it again.

Using the same technique described at the beginning of the chapter it is possible to build the super cotangent fiber bundle T^*X: on a local chart it has coordinates $(x^a, x^\alpha; \overline{\alpha_a}, \overline{\alpha_\alpha}, \widetilde{\alpha_a}, \widetilde{\alpha_\alpha})$. T_0^*X is the even part of T^*X and it has coordinates $(x^a, x^\alpha; \overline{\alpha_a}, \overline{\alpha_\alpha})$. T_1^*X is the odd part of T^*X and it has coordinates $(x^a, x^\alpha; \widetilde{\alpha_a}, \widetilde{\alpha_\alpha})$. T_0^*X and T_1^*X have fibers type which are superspaces of dimension $n|m$ and $m|n$ respectively, accordingly with the degree convention that I chose above.

Definition 31. *A form of degree $r|s$ over a point $x \in X$, X of dimension $n|m$, is said to be extended in the arguments $(v_{a_1}, \cdots, v_{a_k})$, with $k < r$, if it is a G^∞ map $\widehat{\omega}$: $T_{x,0}X \times \cdots \times T_xX \times \cdots \times T_{x,0}X \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$ (where T_xX substitutes $T_{x,0}X$ k times in the positions $a_1, \cdots, a_k$) which satisfies the following conditions:*

- *when restricted to $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, it is a form of degree $r|s$,*
- $\forall(\overline{v_1}, \cdots, \overline{v_{r-k}}, \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r-k} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$,
  $\widehat{\omega}(\overline{v_1}, \cdots, \cdot_1, \cdots, \cdot_k, \cdots, \overline{v_{r-k}}, \widetilde{v_1}, \cdots, \widetilde{v_s})$, *where the free arguments are in the positions $a_1, \cdots, a_k$, is a "naive" k-form à la Kostant,*
- $\forall(v_1, \cdots, v_k, \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_xX \times \cdots T_xX}_{k} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$,
  $\widehat{\omega}(v_1, \cdots, \cdot_1, \cdots, \cdot_{r-k}, \cdots, v_k, \widetilde{v_1}, \cdots, \widetilde{v_s})$, *where the free arguments are in the positions different than $a_1, \cdots, a_k$, is $\mathbb{R}_{S,0}$-linear and antisymmetric.*

We call $\widehat{\omega}$ an extended-form, when it is extended in the above sense in all its even arguments.

We say that $\widehat{\omega}$ extends the $r|s$-form ω if, when restricted to $\underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$, it coincides with ω.

Remark 32. *The space $T_{x,0}X \times \cdots \times T_xX \times \cdots \times T_{x,0}X \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$ (where T_xX substitutes $T_{x,0}X$ k times in the positions $a_1, \cdots, a_k$) could be more precisely denoted with the help of the* Ordered Cartesian Product*:*

$$O^r\left[A^{(a_1,\ldots,a_k)} \times B^{\overline{(a_1,\ldots,a_k)}}\right] := C_1 \times \cdots \times C_r$$

where:

$$C_i = A \text{ if } i \in (a_1, \ldots, a_k)$$
$$C_i = B \text{ if } i \notin (a_1, \ldots, a_k)$$

with $k < r$.

Then we could write:

$$\begin{aligned}&T_{x,0}X \times \cdots \times T_xX \times \cdots \times T_{x,0}X \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}\\ \equiv &O^r\left[T_xX^{(a_1,\ldots,a_k)} \times T_{x,0}X^{\overline{(a_1,\ldots,a_k)}}\right] \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}.\end{aligned}$$

I did not use this notation in Definition 31, neither I will use it in the following, because I judge that the notation used, although less precise, is lighter and more intuitive.

Proposition 33. *For every form ω of degree $r|s$ over a point $x \in X$, X of dimension $n|m$, there is one and only one form $\widehat{\omega}$ which extends ω in the arguments $(v_{a_1}, \cdots, v_{a_k})$, with $k < r$.*

Proof. Let me recall that a k-form à la Kostant over a point $x \in X$ is a G^∞ map $\widehat{o}$: $\underbrace{T_xX \times \cdots T_xX}_{k} \longrightarrow \mathbb{R}_S$ such that $\forall (v_1, \cdots, v_k) \in \underbrace{T_xX \times \cdots T_xX}_{k}$, $\forall H \in \mathbb{R}_S$, $\forall i \in \{1, \cdots, k-1\}$,

$$\widehat{o}(v_1, \cdots, Hv_i, \cdots, v_k) = (-1)^{|H| \sum_{l=1}^{i-1} |v_l|} H \widehat{o}(v_1, \cdots, v_i, \cdots, v_k) \tag{5.7}$$

and

$$\widehat{o}(v_1, \cdots, v_i, v_{i+1}, \cdots, v_k) = (-1)^{1+|v_i||v_{i+1}|} \widehat{o}(v_1, \cdots, v_{i+1}, v_i, \cdots, v_k) \tag{5.8}$$

To prove the existence of a map $\widehat{\omega}$ which satisfies the conditions of definition 30 we can argument by induction on the number k of extended entries of $\widehat{\omega}$.

When $k = 0$ we pose $\widehat{\omega}^0 = \omega$ and there is nothing to prove.

Suppose by induction that it is possible to find an extension of ω in the arguments $(v_{a_1}, \cdots, v_{a_l})$ with $l < k$, let's call it $\widehat{\omega}^l$, then we pose:

$$\begin{aligned}
&\widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_l}, \cdots, v_{a_{l+1}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) := \\
&\quad := \widehat{\omega}^l(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_l}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) + \\
&\quad + (-1)^{\sum_{i=1}^{l} |v_{\alpha_i}|} \frac{\partial}{\partial \varepsilon} \widehat{\omega}^l(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_l}, \cdots, \varepsilon \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)
\end{aligned} \tag{5.9}$$

where $\varepsilon \in \mathbb{R}_{S,1}$ is an odd parameter.

With some patience and calculations one can see that $\widehat{\omega}^{l+1}$ extends ω in the arguments $(v_{a_1}, \cdots, v_{a_{l+1}})$. Hence (5.9) together with $\widehat{\omega}^0 = \omega$ can be taken as the inductive definition of $\widehat{\omega}$.

To prove the uniqueness of $\widehat{\omega}$, we argue by contradiction and induction again on the number k of extended entries.

For $k = 0$ obviously, $\widehat{\omega}^0$ must be equal to ω and so the uniqueness is proved.

Suppose the uniqueness is proved for l extended entries, then suppose by contradiction that there is some $\widehat{\omega}'^{l+1} \neq \widehat{\omega}^{l+1}$ where $\widehat{\omega}^{l+1}$ is defined by (5.9); then there would be $(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_{l+1}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdot\cdot$ such that

$$\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_{l+1}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \neq \widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_{l+1}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s).$$

and since we must have

$$\begin{aligned}
&\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, v_{a_{l+1}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\
&= \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\
&\quad + \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)
\end{aligned}$$

by linearity of $\widehat{\omega}'^{l+1}$ in the argument $v_{a_{l+1}}$.

$$\begin{aligned}
&= \widehat{\omega}^{l}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\
&\quad + \ \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\
&= \widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\
&\quad + \ \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)
\end{aligned}$$

by the induction hypothesis of uniqueness of the extension of ω in the arguments $(v_{a_1}, \cdots, v_{a_l})$.

This implies that we would have:

$$\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \neq \widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)$$

But on the other hand we would have that:

$$\begin{aligned}\forall \eta \in \mathbb{R}_{S,1}: \quad & \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}} + \eta\widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) = \\ =&\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\ &+ \widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \eta\widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)\end{aligned}$$

by linearity of $\widehat{\omega}'^{l+1}$ in the argument $v_{a_{l+1}}$;

$$\begin{aligned}=&\widehat{\omega}^{l}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\ &+ \eta(-1)^{\sum_{i=1}^{l}|v_{a_i}|}\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)\end{aligned}$$

by the induction hypothesis of uniqueness of the extension of ω in the arguments $(v_{a_1}, \cdots, v_{a_l})$ and by the $\mathbb{R}_S$-linearity of $\widehat{\omega}'^{l+1}$ in the argument $v_{a_{l+1}}$;

$$\begin{aligned}=&\widehat{\omega}^{l}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \overline{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) \\ &+ \eta(-1)^{\sum_{i=1}^{l}|v_{a_i}|}\widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) =\end{aligned}$$

again by the induction hypothesis of uniqueness of the extension of ω in the arguments $(v_{a_1}, \cdots, v_{a_l})$ and by the inductive definition of $\widehat{\omega}^{l+1}$.

And so we would have

$$\forall \eta \in \mathbb{R}_{S,1}:$$
$$\eta\widehat{\omega}'^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s) = \eta\widehat{\omega}^{l+1}(\overline{v_1}, \cdots, v_{a_1}, \cdots, \widetilde{v_{a_{l+1}}}, \cdots, \overline{v_r}, \widetilde{v}_1, \cdots, \widetilde{v}_s)$$

and since R_S has infinite odd generators, this is a contradiction. ■

One can also give the definition of $r|s$-differential-preforms and $r|s$-differential-extended-forms in the expected way. Note although that the space of extended forms over a point is not a natural bilateral supermodule, but only a natural right free supermodule. The structure of left supermodule can be given with the sign rule as seen above.

The differential $r|s$-forms, preforms and extended forms can be pull-back by G^∞ maps in the usual way, see [152] pag.60.

5.3 Berezinian $r|s$-forms

In this section I want to study an important class of r|s-forms.

Voronov, in [154], example 1.1, claims that there exists a particularly simple class of $r|s$-forms, built from the superdeterminant of arrays of 1|0-forms. This kind of $r|s$-forms appears naturally in Physics and they are used by A. Belopolsky in [9, 10] and by P. A. Grassi and M. Marescotti in [65] in the context of string theory and were already used by O. M. Khudaverdian in [91] in the context of Batalin-Vilkovisky theory. Belopolsky called them Plücker forms and he gave an indirect proof that they indeed satisfy the conditions (5.4) and (5.5). The proof is based on the fact they can be built by Baranov-Schwarz transform followed by a sign twist, starting from a special kind of Bernstein-Leites pseudodifferential forms; since Voronov (see [152]) has proved that the Baranov-Schwarz transform sends pseudodifferential forms to twisted Voronov-Zorich forms, then the proof follows.

Those special Voronov-Zorich forms also reveal to be very important in the third part of this thesis: I call them Berezinian forms because they can be built from $1|0$-forms, with the help of a superdeterminant. However neither Voronov nor Belopolsky nor Khudaverdian do give in their papers a full direct proof of the fact that those forms satisfy indeed (5.4) and (5.5). Since I could not find any such proof in the published literature and since I couldn't find myself any trivial proof, I use this subsection to give a definition of Berezinian forms and to present such a proof (indeed an elaborated one).

To this purpose, I need first to prove three technical lemmas.

Lemma 34. *Let D be an invertible $s \times s$ matrix whose entries d^i_j (where i identifies the i-th column and j the j-th row) are even-numbers id est elements of $\mathbb{R}_{S,0}$, then the following identity holds:*

$$\frac{\partial \det D}{\partial d^i_j} \cdot \frac{\partial \det D}{\partial d^k_l} - \frac{\partial \det D}{\partial d^k_j} \cdot \frac{\partial \det D}{\partial d^i_l} = \frac{\partial^2 \det D}{\partial d^i_j \partial d^k_l} \cdot \det D \tag{5.10}$$

Proof. If all the entries of D belong to $\mathbb{R}$, we can proceed as follow. We fix a base e^i, $i = 1, \cdots, s$ of a real vector space V of dimension s and then we consider its dual basis θ_j, $j = 1, \cdots, s$ of covectors belonging to V^*. The matrix D is then associated to the linear operator from V to V sending each e^i to the vector $d^i_j e^j$. We then note that $\forall v^1, \cdots, v^s \in V \quad \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(Dv^1, \cdots, Dv_s) = (\det D) \cdot \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(v^1, \cdots, v_s)$ and we consider the real number $\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(De^1, \cdots, DD^{-1}e^j, \cdots, DD^{-1}e^l, \cdots, De^s)$, where the matrices DD^{-1} occur in the positions i and k respectively. Using the definition and the properties of the pullback we have then:

$$\begin{aligned}
&\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(De^1, \cdots, \underbrace{DD^{-1}e^j}_{i}, \cdots, \underbrace{DD^{-1}e^l}_{k}, \cdots, De^s) \\
&= (\det D) \cdot \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(e^1, \cdots, \underbrace{D^{-1}e^j}_{i}, \cdots, \underbrace{D^{-1}e^l}_{k}, \cdots, e^s) = \\
&= (\det D) \cdot \theta_i \wedge \theta_k(D^{-1}e^j, D^{-1}e^l) = \det D \left[\left(D^{-1}\right)^j_i \left(D^{-1}\right)^l_k - \left(D^{-1}\right)^l_i \left(D^{-1}\right)^j_k \right] = \\
&= \frac{1}{\det D} \left[\frac{\partial \det D}{\partial d^i_j} \cdot \frac{\partial \det D}{\partial d^k_l} - \frac{\partial \det D}{\partial d^k_j} \cdot \frac{\partial \det D}{\partial d^i_l} \right]
\end{aligned}$$

On the other hand, we have that:

$$\begin{aligned}
&\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(De^1, \cdots, \underbrace{DD^{-1}e^j}_{i}, \cdots, \underbrace{DD^{-1}e^l}_{k}, \cdots, De^s) \\
&= \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_s(De^1, \cdots, \underbrace{e^j}_{i}, \cdots, \underbrace{e^l}_{k}, \cdots, De^s) \\
&= (-1)^{i-1+k-2+j-1+l-2+b} \Theta^{jl}(De^1, \cdots, De^s)_{ik} \\
&= \frac{\partial^2 \det D}{\partial d^i_j \partial d^k_l}
\end{aligned}$$

where $b = 0$ when $j < l$ and $i < k$ or when $j > l$ and $i > k$, $b = 1$ when $j < l$ but $i > k$ or when $j > l$ and $i < k$; where $\Theta^{jl} = \theta_1 \wedge \cdots \wedge \theta_s$ without θ_j and θ_l in the wedge product and $\Theta^{jl}(De^1, \cdots, De^s)_{ik}$ means Θ^{jl} applied to the vectors $(De^1, \cdots, De^s)$ where in the list do not appear the vectors De^i and De^k.

The same argument can be adapted to matrices with entries in $\mathbb{R}_{s,0}$ keeping in mind the properties of G^∞ and their even derivatives. So the theorem is proved. ∎

Lemma 35. *Let* $M = \begin{pmatrix} v_0^E & v_0^0 & v_0^1 \\ v_f^E & A & B \\ v_\varphi^E & C & D \end{pmatrix}$ *be a supermatrix of size* $(r+1|s)\times(r+1|s)$ *where* v_0^0 *is a line vector with* r *components* v_0^B *(*B *runs from 1 to* r*),* v_0^1 *is a line vector with* s *components* v_0^γ *(*γ *runs form 1 to* s*),* v_f^E *is a column vector of* r *components (*f *runs from 1 to* r*),* v_φ^E *is a column vector of* s *components (*φ *runs from 1 to* s*),* A *is a* $r\times r$ *matrix with entries* a_f^B *,* B *is a* $r\times s$ *matrix with entries* a_f^γ *(*γ *runs form 1 to* s*),* C *is a* $s\times r$ *matrix with entries* a_φ^B *and* D *is a* $s\times s$ *matrix with entries* a_φ^γ*, with possibly some of the first* $r+1$ *columns of* M *with inverted parity with respect to the one usual in a supermatrix. Let's call* G *the supermatrix* $r|s\times r|s$*-dimensional* $G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *where possibly some of the* r *first columns of* G *have inverse parity with respect to the usual one. We have the following formula:*

$$\begin{aligned}\operatorname{sdet}_{r+1|s} M =& v_0^E \operatorname{sdet}_{r|s} G \\ & - v_f^E \operatorname{sdet}_{r|s} \partial_0^f G \\ & - (-1)^{|E|} v_\varphi^E \det \partial_f^\varphi D \frac{1}{\det D} \operatorname{sdet}_{r|s} \partial_0^f G \\ & + (-1)^{|E|} v_\varphi^E \det \partial_0^\varphi D \frac{1}{\det D} \operatorname{sdet}_{r|s} G\end{aligned} \tag{5.11}$$

where $|E|$ *denotes the parity of the column 0 of* M *which is equal to 0 if the entries in that column have the usual parity of supermatrices and it is equal to 1 if they have inverse parity with respect to the usual one;* $\partial_0^f G$ *is the matrix obtained by* G *substituting the* f*-th line with the corresponding entries of the line 0 of* M*;* $\partial_f^\varphi D$ *is the matrix obtained by* D *where substituting the* φ*-th line with the corresponding entries of the* f*-th line of* B*;* $\partial_0^\varphi D$ *is the matrix obtained by* D *substituting the* φ*-th line with the corresponding entries of the line 0 of* M *(id est the components of* v_0^1*); and where all the determinants and superdeterminants are calculated according with the formula introduced in definition 24.*

Proof. By direct calculation and developing for the first column:

$$\begin{aligned}\operatorname{sdet}_{r+1|s} M &= \operatorname{sdet}_{r+1|s} \begin{pmatrix} v_0^E & v_0^0 & v_0^1 \\ v_f^E & A & B \\ v_\varphi^E & C & D \end{pmatrix} = \det_{r+1} \left[\begin{pmatrix} v_0^E & v_0^0 \\ v_f^E & A \end{pmatrix} - \begin{pmatrix} v_0^1 \\ B \end{pmatrix} D^{-1} \begin{pmatrix} v_\varphi^E & C \end{pmatrix} \right] = \\ &= \det_{r+1} \left[\begin{pmatrix} v_0^E & v_0^0 \\ v_f^E & A \end{pmatrix} - \begin{pmatrix} v_0^1 D^{-1} v_\varphi^E & v_0^1 D^{-1} C \\ B D^{-1} v_\varphi^E & B D^{-1} C \end{pmatrix} \right] = \\ &= \det_{r+1} \left[\begin{pmatrix} v_0^E - v_0^1 D^{-1} v_\varphi^E & v_0^0 - v_0^1 D^{-1} C \\ v_f^E - B D^{-1} v_\varphi^E & A - B D^{-1} C \end{pmatrix} \right] = \\ &= v_0^E \operatorname{sdet}_{r|s} G - v_0^1 D^{-1} v_\varphi^E \operatorname{sdet}_{r|s} G - v_f^E \operatorname{sdet}_{r|s} \partial_0^f G + a_f^\gamma \left(D^{-1}\right)_\gamma^\varphi v_\varphi^E \operatorname{sdet}_{r|s} \partial_0^f G\end{aligned}$$

where $\left(D^{-1}\right)_\gamma^\varphi$ is the element of D^{-1} in the γ-th line and φ-th column.

But:

$$\left(D^{-1}\right)_\gamma^\varphi = \frac{\partial \det D}{\partial a_\varphi^\gamma} \cdot \frac{1}{\det D}$$

and

$$a_f^\gamma \left(D^{-1}\right)_\gamma^\varphi = \det \partial_f^\varphi D \cdot \frac{1}{\det D}$$

then, since the parity of elements in B is always 1, we have that:

$$a_f^\gamma \left(D^{-1}\right)_\gamma^\varphi v_\varphi^E = (-1)^{|E|+1} v_\varphi^E \det \partial_f^\varphi D \cdot \frac{1}{\det D}$$

Moreover $v_0^1 D^{-1} v_\varphi^E$ stands for:

$$\begin{aligned} v_0^\gamma \left(D^{-1}\right)_\gamma^\varphi v_\varphi^E &= v_0^\gamma \frac{\partial \det D}{\partial a_\varphi^\gamma} \cdot \frac{1}{\det D} v_\varphi^E = \det \partial_0^\varphi D \cdot \frac{1}{\det D} v_\varphi^E \\ &= (-1)^{|E|+1} v_\varphi^E \det \partial_0^\varphi D \cdot \frac{1}{\det D} \end{aligned}$$

again because the parity of all v_0^γ is 1.

Note that all the calculations depends on the parity of the column 0 of M, as it is clear from the result, but do not depend on the parity of the columns of M going from 1 to r. So the theorem is proved. ■

Lemma 36. *Let G be a $r|s \times r|s$-dimensional supermatrix, $G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where possibly some of the first r columns of G have inverted parity with respect to the usual one. Let's call g_F^B the elements of G, with $F, B = 1, \ldots, r|s$; let $\overline{g_0^B} \in \mathbb{R}_S$ be the $r|s$ components of an even vector. We have the following formulas:*

1. $g_0^B \frac{\partial}{\partial g_f^B} \operatorname{sdet} G = \operatorname{sdet} \partial_0^f G$
2. $g_0^B \frac{\partial}{\partial g_\varphi^B} \operatorname{sdet} G = -\frac{1}{\det_s D} \det_s \partial_0^\varphi D \operatorname{sdet}_{r|s} G + \frac{1}{\det_s D} \sum_{f=1}^r \det_s \partial_f^\varphi D \operatorname{sdet}_{r|s} \partial_0^f G$

where $\partial_0^f G$ is the matrix obtained by G substituting the line f with the elements g_0^B and $\partial_0^\varphi D$ is the matrix obtained by D substituting the line φ with the elements g_0^β and where $f = 1, \ldots, r$ and $\varphi = r+1, \ldots, r+s$.

Proof. The first claim is obvious for the linearity of the superdeterminant in his first r lines.

To calculate $\overline{g_0^B} \frac{\partial}{\partial g_\varphi^B} G$, it is useful to rewrite G as:

$$G := \operatorname{sdet}_{r|s} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det_r \left(A - BD^{-1}C\right) \frac{1}{\det_s D}$$

with:

$$A := \begin{pmatrix} \overline{g_1^1} & \cdots & \overline{g_1^r} \\ \vdots & \ddots & \vdots \\ \overline{g_r^1} & \cdots & \overline{g_r^r} \end{pmatrix}; \quad B := \begin{pmatrix} \overline{g_1^{r+1}} & \cdots & \overline{g_1^{r+s}} \\ \vdots & \ddots & \vdots \\ \overline{g_r^{r+1}} & \cdots & \overline{g_r^{r+s}} \end{pmatrix};$$

$$C := \begin{pmatrix} \widetilde{g_1^1} & \cdots & \widetilde{g_1^r} \\ \vdots & \ddots & \vdots \\ \widetilde{g_s^1} & \cdots & \widetilde{g_s^r} \end{pmatrix}; \quad D := \begin{pmatrix} \widetilde{g_1^{r+1}} & \cdots & \widetilde{g_1^{r+s}} \\ \vdots & \ddots & \vdots \\ \widetilde{g_s^{r+1}} & \cdots & \widetilde{g_s^{r+s}} \end{pmatrix}$$

We then have:

$$\begin{aligned} \overline{g_0^\beta} \frac{\partial}{\partial g_\varphi^\beta} G &= \overline{g_0^\beta} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_r \left(A - BD^{-1}C\right) \frac{1}{\det_s D}\right] \\ &= -\overline{g_0^\beta} \frac{1}{(\det_s D)^2} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_s D\right] \det_r \left(A - BD^{-1}C\right) + \overline{g_0^\beta} \frac{1}{\det_s D} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_r \left(A - BD^{-1}C\right)\right] \\ &= -\overline{g_0^\beta} \frac{1}{\det_s D} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_s D\right] G + \overline{g_0^\beta} \frac{1}{\det_s D} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_r \left(A - BD^{-1}C\right)\right] \\ &= -\frac{1}{\det_s D} \det_s \partial_0^\varphi D \operatorname{sdet}_{r|s} G + \overline{g_0^\beta} \frac{1}{\det_s D} \frac{\partial}{\partial g_\varphi^\beta} \left[\det_r \left(A - BD^{-1}C\right)\right] \end{aligned} \tag{5.12}$$

where $\partial_0^\varphi D$ is the matrix obtained by D substituting the entries in the φ-th line with the corresponding entries g_0^β. And:

$$\begin{aligned}
&\overline{g_0^\beta}\frac{1}{\det_s D}\frac{\partial}{\partial g_\varphi^\beta}\left[\det_r\left(A-BD^{-1}C\right)\right]+\overline{g_0^b}\frac{\partial}{\partial g_\varphi^b}G\\
&=\frac{1}{\det_s D}\left\{\overline{g_0^\beta}\frac{\partial}{\partial g_\varphi^\beta}\left[\det_r\left(A-BD^{-1}C\right)\right]+\overline{g_0^b}\frac{\partial}{\partial g_\varphi^b}\left[\det_r\left(A-BD^{-1}C\right)\right]\right\}
\end{aligned} \tag{5.13}$$

To proceed with the calculation we call $E:=\left(A-BD^{-1}C\right)$ and $E_f^l:=\left(A-BD^{-1}C\right)_f^l$. I will from now on omit the overbar on g_0.

We note that:

$$(BD^{-1})_f^\gamma=\frac{1}{\det_s D}\det_s \partial_f^\gamma D \tag{5.14}$$

and so:

$$\left(BD^{-1}C\right)_f^l=\frac{1}{\det_s D}\det_s \partial_f^\gamma D g_\gamma^l \tag{5.15}$$

and

$$E_f^l=g_f^l-\frac{1}{\det_s D}\det_s \partial_f^\gamma D g_\gamma^l \tag{5.16}$$

We have to calculate $g_0^B\frac{\partial}{\partial g_\varphi^B}\left[\det_r E\right]$. In the polynomial $\det_r E$, each monomial is a product of r factors each of which is an element of E.

The operator $g_0^b\frac{\partial}{\partial g_\varphi^b}$ acts on each monomial by transforming it in a polynomial: it acts as a derivation and it maps each factor E_f^l which has the column l even into the factors $k^l\frac{1}{\det_s D}\det_s \partial_f^\varphi D g_0^l$, where k^l is a sign depending only on l, leaving each time the other factors invariant.

The operator $g_0^\beta\frac{\partial}{\partial g_\varphi^\beta}$ acts on each monomial in a slightly more complicated way, transforming it in a polynomial as follows: it acts as a derivation and it maps each factor E_f^l with column l odd into the factors:

$$\left[k^l\frac{1}{\det_s D}\det_s \partial_f^\varphi D g_0^l-k^l g_0^\beta\frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D}\det_s \partial_f^\gamma D\right)g_\gamma^l\right]$$

leaving other factors invariant; it maps each factor E_f^l with column l even into the factors:

$$\left[-k^l g_0^\beta\frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D}\det_s \partial_f^\gamma D\right)g_\gamma^l\right]$$

leaving other factors invariant.

So we have:

$$
\begin{aligned}
&g_0^B \frac{\partial}{\partial g_\varphi^B}\left[\det_r E\right] = g_0^B \frac{\partial}{\partial g_\varphi^B}\left[\sum_p \sigma(p)\prod_{l=1}^r E^l_{p(l)}\right] = \sum_p \sigma(p) g_0^B \frac{\partial}{\partial g_\varphi^B}\prod_{l=1}^r E^l_{p(l)} \\
&= \sum_p \sigma(p)\sum_{i=1}^r\left\{\prod_{l<i} E^l_{p(l)}\left[k^i \frac{1}{\det_s D}\det_s \partial^\varphi_{p(i)} D g_0^i - k^i g_0^\beta \frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D}\det_s \partial^\gamma_{p(i)} D\right) g^i_\gamma\right]\prod_{l>i} E^l_{p(l)}\right\} \\
&= \sum_p \sigma(p)\sum_{i=1}^r\left\{\prod_{l<i} E^l_{p(l)}\left[k^i \frac{1}{\det_s D}\det_s \partial^\varphi_{p(i)} D g_0^i\right]\prod_{l>i} E^l_{p(l)}\right\} + \\
&\quad - \sum_p \sigma(p)\sum_{i=1}^r\left\{\prod_{l<i} E^l_{p(l)}\left[k^i g_0^\beta \frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D}\det_s \partial^\gamma_{p(i)} D\right) g^i_\gamma\right]\prod_{l>i} E^l_{p(l)}\right\}
\end{aligned}
\tag{5.17}
$$

On the right hand side we have two sums. The first one becomes:

$$
\begin{aligned}
&\sum_p \sigma(p)\sum_{i=1}^r\left\{\prod_{l<i} E^l_{p(l)}\left[k^i \frac{1}{\det_s D}\det_s \partial^\varphi_{p(i)} D g_0^i\right]\prod_{l>i} E^l_{p(l)}\right\} \\
&= \sum_p \sigma(p)\sum_{i=1}^r \frac{1}{\det_s D}\det_s \partial^\varphi_{p(i)} D\left\{\prod_{l<i} E^l_{p(l)}\left[g_0^i\right]\prod_{l>i} E^l_{p(l)}\right\} \\
&= \sum_{i=1}^r \frac{1}{\det_s D}\sum_p \sigma(p)\det_s \partial^\varphi_{p(i)} D\left\{\prod_{l<i} E^l_{p(l)}\left[g_0^i\right]\prod_{l>i} E^l_{p(l)}\right\}
\end{aligned}
\tag{5.18}
$$

where p is a permutation of r elements and $\sigma(p)$ is the sign of the permutation p. Note that the factors k^i disappear because the operator $g_0^B \frac{\partial}{\partial g_\varphi^B}$ and the operator $\frac{1}{\det_s D}\det_s \partial^\varphi_{p(i)} D$ have the same parity and they "jump" the same factors during the calculation. Moreover their parity is 1 and this fix all the signs in the second line of (5.17).

To calculate the second term in the last line of (5.17), let's first calculate

$$
\begin{aligned}
&- g_0^\beta \frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D}\det_s \partial^\gamma_{p(i)} D\right) g^i_\gamma \\
&= -g_0^\beta \frac{\partial}{\partial g_\varphi^\beta}\left(\frac{1}{\det_s D} v^\varepsilon_{p(i)} \frac{\partial}{\partial g^\varepsilon_\gamma}\det_s D\right) g^i_\gamma \\
&= g_0^\beta \frac{1}{(\det_s D)^2}\frac{\partial}{\partial g_\varphi^\beta}\left(\det_s D\right) g^\varepsilon_{p(i)} \frac{\partial}{\partial g^\varepsilon_\gamma}\left(\det_s D\right) g^i_\gamma - g_0^\beta \frac{1}{\det_s D} g^\varepsilon_{p(i)} \frac{\partial^2}{\partial g^\beta_\varphi \partial g^\varepsilon_\gamma}\left(\det_s D\right) g^i_\gamma \\
&= g_0^\beta \frac{1}{(\det_s D)^2}\frac{\partial}{\partial g_\varphi^\varepsilon}\left(\det_s D\right) g^\varepsilon_{p(i)} \frac{\partial}{\partial g^\beta_\gamma}\left(\det_s D\right) g^i_\gamma
\end{aligned}
$$

where the last equality holds because of Lemma 34 which can be applied because D is invertible in the domain of definition of G.

Then the second term in the last line of (5.17) becomes:

$$\begin{aligned}
&\sum_p \sigma(p) \sum_{i=1}^r \left\{ \prod_{l<i} E^l_{p(l)} \left[k^i g_0^\beta \frac{1}{(\det_s D)^2} \frac{\partial}{\partial g^\varepsilon_\varphi} \left(\det_s D \right) g^\varepsilon_{p(i)} \frac{\partial}{\partial g^\beta_\gamma} \left(\det_s D \right) g^i_\gamma \right] \prod_{l>i} E^l_{p(l)} \right\} \\
&= -\sum_p \sigma(p) \sum_{i=1}^r \left\{ \prod_{l<i} E^l_{p(l)} \left[k^i \frac{1}{(\det_s D)^2} \det_s \partial^\varphi_{p(i)} D \det_s \partial^\gamma_0 D v^{A_i}_\gamma \right] \prod_{l>i} E^l_{p(l)} \right\} \\
&= \sum_{i=1}^r \frac{1}{(\det_s D)} \sum_p \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[-\frac{1}{(\det_s D)} \det_s \partial^\gamma_0 D g^i_\gamma \right] \prod_{l>i} E^l_{p(l)} \right\}
\end{aligned} \tag{5.19}$$

where the k^i disappear from the calculation for the same reason explained above.

Remembering (5.15), we can write that:

$$-\frac{1}{(\det_s D)} \det_s \partial^\gamma_0 D g^i_\gamma = -\left[\partial_0^{p(i)} \left(BD^{-1}C \right) \right]^i_{p(i)}$$

so we can put together (5.18) and (5.19), we can substitute in (5.17) and we obtain:

$$\begin{aligned}
g_0^B \frac{\partial}{\partial g^B_\varphi} \left[\det_r E \right] =& \sum_{i=1}^r \frac{1}{\det_s D} \sum_p \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[g^i_0 \right] \prod_{l>i} E^l_{p(l)} \right\} \\
&+ \sum_{i=1}^r \frac{1}{(\det_s D)} \sum_p \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[-\frac{1}{(\det_s D)} \det_s \partial^\gamma_0 D g^i_\gamma \right] \prod_{l>i} E^l_{p(l)} \right\} \\
=& \sum_{i=1}^r \frac{1}{\det_s D} \sum_p \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[\partial_0^{p(i)} A \right]^i_{p(i)} \prod_{l>i} E^l_{p(l)} \right\} \\
&- \sum_{i=1}^r \frac{1}{(\det_s D)} \sum_p \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[\partial_0^{p(i)} \left(BD^{-1}C \right) \right]^i_{p(i)} \prod_{l>i} E^l_{p(l)} \right\} \\
=& \frac{1}{\det_s D} \sum_p \sum_{i=1}^r \sigma(p) \det_s \partial^\varphi_{p(i)} D \left\{ \prod_{l<i} E^l_{p(l)} \left[\partial_0^{p(i)} E \right]^i_{p(i)} \prod_{l>i} E^l_{p(l)} \right\} \\
=& \frac{1}{\det_s D} \sum_p \sum_{f=1}^r \sigma(p) \det_s \partial^\varphi_f D \left\{ \prod_{l<f} E^{p^{-1}(l)}_l \left[\partial_0^f E \right]^{p^{-1}(f)}_f \prod_{l>f} E^{p^{-1}(l)}_l \right\} \\
=& \frac{1}{\det_s D} \sum_{f=1}^r \det_s \partial^\varphi_f D \det_r \partial^f_0 E \\
=& \sum_{f=1}^r \det_s \partial^\varphi_f D \, \mathrm{sdet}_{r|s} \, \partial^f_0 G
\end{aligned} \tag{5.20}$$

where $\partial^f_0 E$ is obtained by the matrix E by substituting the f-th line with the corresponding entries of g^B_0.

Keeping into account the results obtained in (5.12),(5.13) and (5.20) we can write that:

$$\begin{aligned}
\overline{g_0^B}\frac{\partial}{\partial g_\varphi^B}G &= \overline{g_0^\beta}\frac{\partial}{\partial g_\varphi^\beta}G + \overline{g_0^b}\frac{\partial}{\partial g_\varphi^b}G \\
&= -\frac{1}{\det_s D}\det_s \partial_0^\varphi D\,\mathrm{sdet}_{r|s}\,G + \overline{g_0^\beta}\frac{1}{\det_s D}\frac{\partial}{\partial g_\varphi^\beta}\left[\det_r\left(A - BD^{-1}C\right)\right] + \overline{g_0^b}\frac{\partial}{\partial g_\varphi^b}G \qquad (5.21)\\
&= -\frac{1}{\det_s D}\det_s \partial_0^\varphi D\,\mathrm{sdet}_{r|s}\,G + \frac{1}{\det_s D}\sum_{f=1}^{r}\det_s \partial_f^\varphi D\,\mathrm{sdet}_{r|s}\,\partial_0^f G
\end{aligned}$$

And the theorem is proved. ■

I can now prove the following three theorems which allow to identify the Berezinian class of Voronov-Zorich superforms:

Theorem 37. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local adapted chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let v be any $(\widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$, let $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ be s different odd indices chosen in the the set $\{n+1, \cdots, n+m\}$; then the function*

$$\omega \;:\; O \subset \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} \;\longrightarrow\; \mathbb{R}_S$$

defined by:

$$\omega(v) := \mathrm{sdet}_{0,s}\begin{pmatrix} \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} = \frac{1}{\det_s\begin{pmatrix} \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}} \qquad (5.22)$$

is a $0|s$-form over $x \in U$.

Proof. Let's call D the matrix $\begin{pmatrix} \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$ and let's call Δ its determinant.

To prove that ω satisfy (5.4), we take a generic $g \in GL(0|s)$, then $g \cdot v$ is the matrix obtained multiplying, with the usual matrix product, the matrix g times the matrix obtained writing each $\widetilde{v_\beta}$ as a line made of its components, id est the matrix D; then we have:

$$\omega(g\cdot v) = \mathrm{sdet}_{0,s}\,(g\cdot D) = \frac{1}{\det_s\,(g\cdot D)} = \frac{1}{\det_s g\,\det_s\,(D)} = \mathrm{sdet}_{0,s}\,g\cdot\omega(v)$$

To prove that ω satisfies (5.5), we have to prove that $\forall i,j,k,l = 1\cdots,s$ $\frac{\partial^2\Delta^{-1}}{\partial v_j^{\alpha_l}\partial v_i^{\alpha_k}} = \frac{\partial^2\Delta^{-1}}{\partial v_j^{\alpha_k}\partial v_i^{\alpha_l}}$. We can perform a direct calculation:

$$\begin{aligned}
\frac{\partial^2\Delta^{-1}}{\partial v_j^{\alpha_l}\partial v_i^{\alpha_k}} &= -\frac{\partial}{\partial v_j^{\alpha_l}}\left(\frac{1}{\Delta^2}\frac{\partial\Delta}{\partial v_i^{\alpha_k}}\right) = \frac{2}{\Delta^3}\frac{\partial\Delta}{\partial v_j^{\alpha_l}}\frac{\partial\Delta}{\partial v_i^{\alpha_k}} - \frac{1}{\Delta^2}\frac{\partial^2\Delta}{\partial v_j^{\alpha_l}\partial v_i^{\alpha_k}} \\
&= \frac{2}{\Delta^3}\frac{\partial\Delta}{\partial v_j^{\alpha_k}}\frac{\partial\Delta}{\partial v_i^{\alpha_l}} + \frac{1}{\Delta^2}\frac{\partial^2\Delta}{\partial v_j^{\alpha_l}\partial v_i^{\alpha_k}} = \frac{2}{\Delta^3}\frac{\partial\Delta}{\partial v_j^{\alpha_k}}\frac{\partial\Delta}{\partial v_i^{\alpha_l}} - \frac{1}{\Delta^2}\frac{\partial^2\Delta}{\partial v_j^{\alpha_k}\partial v_i^{\alpha_l}} \\
&= \frac{\partial^2\Delta^{-1}}{\partial v_j^{\alpha_k}\partial v_i^{\alpha_l}}
\end{aligned}$$

where we made use of Lemma 34. The theorem is proved. ∎

Theorem 38. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let w be any $(\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ be s different odd indices chosen in the the set $\{n+1, \cdots, n+m\}$; let $A_1, A_2, \cdots, A_{r+1}$ be $r+1$ even or odd indices (possibly equal) chosen in the set $\{1, \cdots, n+m\}$; then the function*

$$\omega \ : \ O \subset \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} \ \longrightarrow \ \mathbb{R}_S$$

defined by:

$$\omega(w) := \mathrm{sdet}_{r+1,s} \begin{pmatrix} \overline{v_1^{A_{r+1}}} & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_{r+1}}} & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \overline{v_{r+1}^{A_{r+1}}} & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\alpha_s}} \\ \widetilde{v_1^{A_{r+1}}} & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_{r+1}}} & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} \tag{5.23}$$

is an $r+1|s$-form over $x \in U$, O being precisely the subset where the formula (5.23) *is well defined.*

Moreover we have that

$$\omega = d\left(x^{A^{r+1}}\theta\right) \tag{5.24}$$

where θ is the $r|s$-form defined by:

$$\forall x \in U,\ \forall v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} :$$

$$\theta(v) := \mathrm{sdet}_{r,s} \begin{pmatrix} \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_r}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} \tag{5.25}$$

Note that we don't assume as hypothesis that the indices A_i are different from the indices α_j.

Proof. I will prove the theorem by induction on $r+1$.

- If $r+1 = 0$ and $s \neq 0$ then the theorem reduces to theorem 37 which has already been proved.
- If $r+1 = 0$ and $s = 0$, then there is nothing to prove.
- If $r+1 = 1$ and $s = 0$, then the proof is trivial.

Suppose now that the theorem has already been proved for $r|s$. Let's consider θ defined by (5.25): it is an $r|s$-form by the inductive hypothesis; so also $x^{A_{r+1}}\theta$ is a $r|s$-form as it can be easyly seen. If I prove that $\omega = d\left(x^{A_{r+1}}\theta\right)$, because of proposition 28, our theorem is proved.

Let's calculate $d\left(x^{A_{r+1}}\theta\right)$ using definition 27. Using the definition of superdeterminant, we can easily prove that:

$$\omega(w) := \mathrm{sdet}_{r+1,s}\begin{pmatrix} \overline{v_1^{A_{r+1}}} & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_{r+1}}} & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \overline{v_{r+1}^{A_{r+1}}} & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\alpha_s}} \\ \widetilde{v_1^{A_{r+1}}} & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_{r+1}}} & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} =$$

$$= (-1)^r\,\mathrm{sdet}_{r+1,s}\begin{pmatrix} \overline{v_{r+1}^{A_{r+1}}} & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\alpha_s}} \\ \overline{v_1^{A_{r+1}}} & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^{A_{r+1}}} & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^{A_{r+1}}} & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^{A_{r+1}}} & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix}$$

In the following I call v_F^A which stands for v_F^a, v_F^α, the components of the multivector v and $\overline{v_{r+1}^a}$, $\overline{v_{r+1}^\alpha}$ the components of the vector $\overline{v_{r+1}}$, with $F \in \{f, \varphi\}$ and $f \in \{1, \cdots, r\}$, $\varphi \in \{1, \cdots, s\}$, $a \in \{1, \cdots, n\}$, $\alpha \in \{1, \cdots, m\}$. I will also need to indicate the components of v which explicitely appear in the definition of θ (5.25) and ω (5.23): I will call them $v_F^{A_l}$ and $v_F^{\alpha_\lambda}$, with $l \in \{1, \cdots, r\}$ and $\lambda \in \{1, \cdots, s\}$. I will call $\overline{v_{r+1}^{A_{r+1}}}$, $\overline{v_{r+1}^{A_l}}$ and $\overline{v_{r+1}^{\alpha_\lambda}}$ the components of $\overline{v_{r+1}}$ which appear in (5.23).

We have that:

$$\begin{aligned} (-1)^r d\left(x^{A_{r+1}}\theta\right)(w) =& \overline{v_{r+1}^B}\left(\delta_B^{A_{r+1}}\theta(v) - (-1)^{|B||F|}v_F^C\frac{\partial}{\partial v_F^B}(-1)^{|C|(|B|+|F|)}\delta_C^{A_{r+1}}\theta(v)\right) \\ =& \overline{v_{r+1}^{A_{r+1}}}\theta(v) - (-1)^{[|B||F|+|A_{r+1}|(|B|+|F|)]}\overline{v_{r+1}^B}v_F^{A_{r+1}}\frac{\partial}{\partial v_F^B}\theta(v) \\ =& \overline{v_{r+1}^{A_{r+1}}}\theta(v) - (-1)^{[|B||f|+|A_{r+1}|(|B|+|f|)+|A_{r+1}||B|]}\overline{v_f^{A_{r+1}}}\,\overline{v_{r+1}^B}\frac{\partial}{\partial v_f^B}\theta(v) \\ &- (-1)^{[|B||\varphi|+|A_{r+1}|(|B|+|\varphi|)+(|A_{r+1}|+1)|B|]}\widetilde{v_\varphi^{A_{r+1}}}\,\overline{v_{r+1}^B}\frac{\partial}{\partial v_\varphi^B}\theta(v) \\ =& v_{r+1}^{A_{r+1}}\theta(v) - \overline{v_f^{A_{r+1}}}\theta(\partial_{r+1}^f v) - (-1)^{|A_{r+1}|}\widetilde{v_\varphi^{A_{r+1}}}\,\overline{v_{r+1}^B}\frac{\partial}{\partial v_\varphi^B}\theta(v) \end{aligned} \tag{5.26}$$

where $\theta(\partial_{r+1}^f v)$ is the superdeterminant of the matrix obtained by the matrix in formula 5.25 by substituting the line f with the last $r|s$ columns of line $r+1$ in the matrix in formula 5.23; id est $\theta(\partial^f_{r+1}v)$ is the value obtained by letting θ act on the multivector obtained by substituting in v the f-th vector with $\overline{v_{r+1}}$.

To calculate $\widetilde{v^{A_{r+1}}_\varphi}\overline{v^B_{r+1}}\frac{\partial}{\partial v^B_\varphi}\theta(v)$, it is useful to rewrite $\theta(v)$ as:

$$\theta(v) := \mathrm{sdet}_{r|s}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det_r\left(A - BD^{-1}C\right)\frac{1}{\det_s D}$$

with:

$$A := \begin{pmatrix} \overline{v^{A_1}_1} & \cdots & \overline{v^{A_r}_1} \\ \vdots & \ddots & \vdots \\ \overline{v^{A_1}_r} & \cdots & \overline{v^{A_r}_r} \end{pmatrix}; \quad B := \begin{pmatrix} \overline{v^{\alpha_1}_1} & \cdots & \overline{v^{\alpha_s}_1} \\ \vdots & \ddots & \vdots \\ \overline{v^{\alpha_1}_r} & \cdots & \overline{v^{\alpha_s}_r} \end{pmatrix};$$
$$C := \begin{pmatrix} \widetilde{v^{A_1}_1} & \cdots & \widetilde{v^{A_r}_1} \\ \vdots & \ddots & \vdots \\ \widetilde{v^{A_1}_s} & \cdots & \widetilde{v^{A_r}_s} \end{pmatrix}; \quad D := \begin{pmatrix} \widetilde{v^{\alpha_1}_1} & \cdots & \widetilde{v^{\alpha_s}_1} \\ \vdots & \ddots & \vdots \\ \widetilde{v^{\alpha_1}_s} & \cdots & \widetilde{v^{\alpha_s}_s} \end{pmatrix}$$

By Lemma 36, we then have that:

$$\overline{v^B_{r+1}}\frac{\partial}{\partial v^B_\varphi}\theta(v) = -\frac{1}{\det_s D}\det_s \partial^\varphi_{r+1}D\theta(v) + \frac{1}{\det_s D}\det_s \partial^\varphi_f D\theta(\partial^f_{r+1}v) \tag{5.27}$$

Keeping into account (5.27), we can rewrite (5.26) as follows:

$$\begin{aligned}(-1)^r d\left(x^{A_{r+1}}\theta\right)(w) =& v^{A_{r+1}}_{r+1}\theta(v) - \overline{v^{A_{r+1}}_f}\theta(\partial^f_{r+1}v) - (-1)^{|A_{r+1}|}\widetilde{v^{A_{r+1}}_\varphi}\overline{v^B_{r+1}}\frac{\partial}{\partial v^B_\varphi}\theta(v) \\ =& v^{A_{r+1}}_{r+1}\theta(v) - \overline{v^{A_{r+1}}_f}\theta(\partial^f_{r+1}v) + (-1)^{|A_{r+1}|}\widetilde{v^{A_{r+1}}_\varphi}\frac{1}{\det_s D}\det_s \partial^\varphi_{r+1}D\theta(v) \\ &- (-1)^{|A_{r+1}|}\widetilde{v^{A_{r+1}}_\varphi}\frac{1}{\det_s D}\det_s \partial^\varphi_f D\theta(\partial^f_{r+1}v)\end{aligned} \tag{5.28}$$

Using (5.11), we can develop the matrix defining ω in (5.23) and then we can compare the obtained development with (5.28). We can see that the claim is proved. ■

Note that

Theorem 39. *Let X be an $n|m$-dimensional supermanifold and TX its tangent space. Let $x \in X$, let w be any $(\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$; let $\theta^1 \dots \theta^s$ be s linear independent odd $1|0$-forms; let $\Theta^1, \cdots, \Theta^r$ be r even or odd $1|0$-forms (possibly linear dependent); let*

$$\mathcal{O} \subset \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s}$$

be the subset where the function

$$\omega : \mathcal{O} \subset \underbrace{T_{x,0}X \times \cdots T_{x,0}X}_{r} \times \underbrace{T_{x,1}X \times \cdots T_{x,1}X}_{s} \longrightarrow \mathbb{R}_S$$

$$\omega(w) := \operatorname{sdet}_{r,s} \begin{pmatrix} \Theta^1(\overline{v_1}) & \cdots & \Theta^r(\overline{v_1}) & \theta^1(\overline{v_1}) & \cdots & \theta^s(\overline{v_1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Theta^1(\overline{v_r}) & \cdots & \Theta^r(\overline{v_r}) & \theta^1(\overline{v_r}) & \cdots & \theta^s(\overline{v_r}) \\ \Theta^1(\widetilde{v_1}) & \cdots & \Theta^r(\widetilde{v_1}) & \theta^1(\widetilde{v_1}) & \cdots & \theta^s(\widetilde{v_1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Theta^1(\widetilde{v_s}) & \cdots & \Theta^r(\widetilde{v_s}) & \theta^1(\widetilde{v_s}) & \cdots & \theta^s(\widetilde{v_s}) \end{pmatrix} \tag{5.29}$$

is well defined.

If $\mathcal{O}$ is not empty, then ω is an $r|s$-form over $x \in X$,

Proof. Since $\theta^1 \dots \theta^s$ are s linear independent odd $1|0$-forms, if $\mathcal{O}$ is not empty, there is a chart $U \subset X$ containing x and with local coordinates (x^a, ξ^α) such that over x: $\theta^1 = d\xi^1 \dots \theta^s = d\xi^s$. Moreover, since by definition, the superdeterminant in (5.29) is additive in the first r columns and satisfy a graded version of homogeneity in each of them when the entries in each column of the first block share the same parity as well as the entries in each column of the second block, then it follows that, in those coordinates, $\omega(w)$ is expressed as a linear combination, with coefficients in $\mathbb{R}_S$, of terms each of which is of the type of those appearing in theorem 38. By theorem 38, each of these term is a Voronov-Zorich form over the point x.

Since Voronov and Zorich have proved that the conditions (5.4) and (5.5), given pointwise, are well posed and do not depend from the choice of the local coordinates, the theorem is proved. ■

Definition 40. *I call Berezinian-superforms the superforms of the type defined with* (5.29) *and their $\mathbb{R}_S$-linear combinations.*

Particularly interesting are $n|m$-forms over a $n|m$-dimensional supermanifold X. We have the following:

Theorem 41. *Let ω and ω' be $n|m$-forms over the $n|m$-dimensional supermanifold X, so that $\forall x \in X$, $\omega\big|_x$ and $\omega'\big|_x$ do not vanish; if there exists a section $v \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{n} \times \underbrace{T_1X \times \cdots T_1X}_{m})$ such that, for every $x \in X$, $v\big|_x = \left(\overline{v_{x,1}}, \cdots, \overline{v_{x,n}}, \widetilde{v_{x,n+1}}, \cdots, \widetilde{v_{x,n+m}}\right)$ is a base for the free module T_xX and $\omega\big|_x(v\big|_x)$ is invertible , then there exist a G^∞ function $f \in G^\infty(X)$, so that $\omega' = f\omega$. Moreover f is everywhere non vanishing in X.*

Proof. Let $v \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{n} \times \underbrace{T_1X \times \cdots T_1X}_{m})$ be as in the hypothesis, so that $\forall x \in X$, $\omega\big|_x(v\big|_x)$ is invertible in $\mathbb{R}_S$, and fix $f(x) := \omega'\big|_x(v\big|_x)\left[\omega\big|_x(v\big|_x)\right]^{-1}$.

We have that, if $v'\big|_x = (\overline{v'_{x,1}} \cdots \overline{v'_{x,n}}, \widetilde{v'_{n+1}}, \cdots, \widetilde{v'_{n+m}})$ is so that $(\overline{v'_{x,1}} \cdots \overline{v'_{x,n}})$ and $(\widetilde{v'_{n+1}}, \cdots, \widetilde{v'_{n+m}})$ are linear independent, then a $g \in GL(n|m)$ exists so that $v' = gv$. We have then for every such v'

$$\begin{aligned} \omega'\big|_x(v') &= \omega'\big|_x(gv) = \operatorname{Ber}_{n,m}(g)\omega'\big|_x(v) = \operatorname{Ber}_{n,m}(g)f(x)\omega\big|_x(v) = \\ &= f(x)\omega\big|_x(gv) = f(x)\omega\big|_x(v') \end{aligned}$$

Finally $\omega'\big|_x$ does not vanish $\forall x \in X$, so $f(x) \neq 0$ $\forall x$, and we conclude that for any u in its domain of definition $\omega'\big|_x(u) = f(x)\omega\big|_x(u)$. ■

This proves that the dimension of the free supermodule $\Lambda_x^{n|m}$ is 1 for every $x \in X$ and that all $n|m$-forms on a $n|m$-supermanifold are Berezinian superforms.

For a study of $\Lambda_x^{r|m}$ on a $n|m$-manifold, see [152], where Voronov shows the connection between the $r|m$-forms and integral and pseudodifferential forms defined by Bernstein and Leites [16, 17].

We can see that $n|m$-forms over $n|m$-dimensional supermanifolds behave much alike n-forms on n-dimensional manifolds. For example:

Proposition 42. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; the function*

$$\beta \ : \ \mathcal{O} \subset \underbrace{T_0U \times \cdots T_0U}_{n} \times \underbrace{T_1U \times \cdots T_1U}_{m} \ \longrightarrow \ \mathbb{R}_S$$

defined by:

$$\beta := \mathrm{sdet}_{n,m} \begin{pmatrix} \overline{v_1^1} & \cdots & \overline{v_1^n} & \overline{v_1^{n+1}} & \cdots & \overline{v_1^{n+m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_n^1} & \cdots & \overline{v_n^n} & \overline{v_n^{n+1}} & \cdots & \overline{v_n^{n+m}} \\ \widetilde{v_1^1} & \cdots & \widetilde{v_1^n} & \widetilde{v_1^{n+1}} & \cdots & \widetilde{v_1^{n+m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_m^1} & \cdots & \widetilde{v_m^n} & \widetilde{v_m^{n+1}} & \cdots & \widetilde{v_m^{n+m}} \end{pmatrix} \tag{5.30}$$

is an $n|m$-form over U: we call it the canonical $n|m$-form of U.

Proof. Obvious after theorem 38 ■

So, combining theorem 41 and proposition 42, we can prove:

Theorem 43. *Every $n|m$-form $\mathcal{L} \in \Omega^{n|m}X$ over a $n|m$-dimensional manifold X can be written in local coordinates as:*

$$\mathcal{L} = L(x^A)\beta$$

and if $\mathcal{L}$ is everywhere non vanishing on the local chart, then $L(x^A)$ is everywhere non vanishing too.
Changing local chart, L transform as a $n|m$-density, that is

$$L'(x^{A'}) = \mathrm{Ber}_{n,m}\Big(\frac{\partial x^{A'}}{\partial x^A}\Big)^{-1} L(x^A)$$

where $(\frac{\partial x^{A'}}{\partial x^A})^{-1}$ is the inverse matrix of the tangent map of the transformation function of local coordinates.

Note that if one assigns conventionally the $\mathbb{Z}_2$ degree to an $n|m$-form, assigning the same degree to all its local representative, the degree is well defined. I choose the following convention: $|\mathcal{L}| = |L|$; where $|L| = 0$ if it takes only even values, and 1 if it takes only odd values: it is consistent because, by change of coordinates, $|L|$ doesn't change, being $\Big|\mathrm{Ber}_{n,m}\Big(\frac{\partial x^{A'}}{\partial x^A}\Big)\Big| = 0$. This convention agrees with the parity convention already adopted above in the previous two sections.

We will denote the Berezinian forms defined by (5.29) in this way:

$$\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{5.31}$$

Note the difference between the notation in (5.31) and similar notations like:

$$\omega = \Theta^1 \wedge \cdots \wedge \Theta^r \odot v_1 \odot \cdots \odot v_s \tag{5.32}$$

where $v_1 \dots v_s$ are odd vector fields; or

$$\omega = dx^1 \wedge \cdots \wedge dx^r \frac{\partial^s}{\partial x^{m+1} \dots \partial x^{m+s}} \tag{5.33}$$

where x^A are local coordinates on a $n|m$-manifold.

Notations like the one in (5.32) are usually restricted only to $n|m$-forms on $n|m$-manifolds, or to integral forms, see for example [114, 116]; notations like the one in (5.33) are usually restricted only to $n|m$-forms on $n|m$-manifolds and they already appeared in the literature in several variants, especially in the context of Berezinian densities, see for example [146].

I propose the use of (5.31) which has to be intended as a shortcut for (5.29).

Another important class of superforms are $r|0$-forms. It is easy to prove that every extended $r|0$-form is a Berezinian form. In particular every extended $1|0$-form μ is a Berezinian form and can be written as $dx^A \mu_A$.

Following the same convention used for elements of the cotangent bundle, id est for Kostant 1-form, we have that $\left|dx^A\right| = \left|x^A\right| = |A|$. The parity of μ then follows automatically.

Every extended $r|0$-form ω can be written as a $\mathbb{R}_S$-linear combination of forms of the type of $\theta = dx^{A_1} \wedge \cdots dx^{A_r}$. I use the following natural convention:

$$|\theta| = \left|dx^{A_1} \wedge \cdots \wedge dx^{A_r}\right| := \left|dx^{A_1}\right| + \cdots + \left|dx^{A_r}\right| \tag{5.34}$$

The degree of a generic $r|0$-form ω follows.

For a form of the type defined with 5.31 I set the following convention:

$$|\omega| = \left|\frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}\right| := \left|\Theta^1 \wedge \cdots \wedge \Theta^r\right| = |\Theta_1| + \cdots + |\Theta_r| \tag{5.35}$$

5.4 Fractional $r|s$-forms and Cartan calculus

In his work on 'stable forms' [154], Voronov introduced, for every $1|0$-form α, an operator $e(\alpha)$ acting on 'stable forms'. Voronov then says that this operator corresponds to an operator, indicated with e_α, which act on $r|s$-forms, sending them to $r+1|s$ forms and which behaves like the wedge product of α with a $r|s$-form. Voronov, [154] formula 17 pag. 9, gives the following formula in coordinates for e_α:

$$e_\alpha = (-1)^r \left(v^A_{r+1}\alpha_A - (-1)^{|\alpha||F|+|B|} v^A_F \alpha_A v^B_{r+1} \frac{\partial}{\partial v^B_F} \right) \tag{5.36}$$

Formula (5.36) is in fact imprecise. This can be seen if one considers on a supermanifold the very simple $0|1$-form ω defined by:

$$\omega(\widetilde{v_1}) = \frac{1}{\widetilde{v_1^\gamma}} \tag{5.37}$$

and the $1|0$-form $\alpha = dx^\beta$. Then, using (5.36), one would find that:

$$\begin{aligned}
e_\alpha\omega(\overline{v_0},\widetilde{v}_1) &= e_{dx^\beta}\omega(\overline{v_0},\widetilde{v}_1) \\
&= (-1)^0\left[\overline{v_0^A}\delta_A^\beta - (-1)^{1\cdot 1+1}\widetilde{v_1^A}\delta_A^\beta\overline{v_0^\gamma}\frac{\partial}{\partial\widetilde{v_1^\gamma}}\right]\frac{1}{\widetilde{v_1^\gamma}} \\
&= (-1)^0\left[\overline{v_0^\beta} - (-1)^{1\cdot 1+1}\widetilde{v_1^\beta}\overline{v_0^\gamma}\frac{\partial}{\partial\widetilde{v_1^\gamma}}\right]\frac{1}{\widetilde{v_1^\gamma}} \\
&= \frac{\overline{v_0^\beta}}{\widetilde{v_1^\gamma}} + \frac{\widetilde{v_1^\beta}\overline{v_0^\gamma}}{(\widetilde{v_1^\gamma})^2}
\end{aligned} \tag{5.38}$$

But the form defined by (5.38) does not satisfy (5.5).

The correct definition of a wedge product operator of a $1|0$-form α acting on generic $r|s$-forms is given by Belopolsky in [9], it is used also in [65] and it corresponds to the following formula:

$$e_\alpha = (-1)^r\left(v^A_{r+1}\alpha_A - (-1)^{|\alpha||F|}v_F^A\alpha_A v^B_{r+1}\frac{\partial}{\partial v_F^B}\right) \tag{5.39}$$

Belopolsky doesn't exhibit the proof that his definition is well posed. I will not give here a full proof, because it is not necessary for what follows. I will instead show that (5.39) is well posed for Berezinian forms and I will show that for Berezinian forms it reduces to an intuitive formula. We have in fact:

Proposition 44. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $w := (\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let θ be the $r|s$-superform defined with* (5.25)*, and let $\mu := dx^A\mu_A$ be a $1|0$-superform, then the function defined by:*

$$\mu\wedge\theta(w) := \operatorname{sdet}_{r+1,s}\begin{pmatrix}
\overline{v_1^A}\mu_A & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{v_r^A}\mu_A & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_{r-}}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\
\overline{v_{r+1}^A}\mu_A & \overline{v_{r+1}^{A_1}} & \cdots & \overline{v_{r+1}^{A_r}} & \overline{v_{r+1}^{\alpha_1}} & \cdots & \overline{v_{r+1}^{\mu_s}} \\
\widetilde{v_1^A}\mu_A & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{v_s^A}\mu_A & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}}
\end{pmatrix} \tag{5.40}$$

is a $r+1|s$ coordinates-superform over x.

Proof. It is obvious after theorem 38. ■

And:

Proposition 45. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $w := (\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$, let $v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let θ be the $r|s$-superform* *defined with* (5.25)*, and let* $\mu := dx^A\mu_A$ *be a* $1|0$*-superform; if* $e_\mu\theta$ *is defined by* (5.39) *and* $\mu \wedge \theta$ *is defined by* (5.40)*, then:*

$$e_\mu\theta(w) = \mu \wedge \theta(w) \tag{5.41}$$

Proof. First of all we note that equation (5.39) well defines an operator, indeed, if one considers the operator defined on an other local chart U', it is easy to see that the two operators coincide on $U \cap U'$. When (5.41) is proved, this will also prove that definition (5.40) does not depend on local coordinates.

Then, if we apply e_μ to θ, we obtain:

$$e_\mu\theta(w) = (-1)^r \left(v^B_{r+1}\mu_B\theta(v) - (-1)^{|\mu||F|} v^B_F \mu_B v^A_{r+1} \frac{\partial\theta(v)}{\partial v^A_F} \right) \tag{5.42}$$

where the index A takes the values $A_1, \dots, A_r, \alpha_1, \dots, \alpha_s$, whereas B runs from 1 to $n|m$ and F runs from 1 to $r|s$.

Using Lemma 36, we can rewrite (5.42) as:

$$\begin{aligned} e_\mu\theta(w) =& (-1)^r \left[v^B_{r+1}\mu_B\theta(v) - (-1)^{|\mu||f|} v^B_f \mu_B \left(\theta(\partial^f_{r+1} v) \right) \right] \\ & - (-1)^r(-1)^{|\mu||\varphi|} \left[v^B_\varphi \mu_B \left(-\frac{1}{\det_s D} \det_s \partial^\varphi_{r+1} D\theta(v) + \frac{1}{\det_s D} \det_s \partial^\varphi_f D\theta(\partial^f_{r+1} v) \right) \right] \end{aligned} \tag{5.43}$$

But, by Lemma 35, (5.43), is equivalent to (5.40), and the theorem is proved. ■

After proposition 44, if $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$, and μ is a $1|0$-form, then we can write:

$$\mu \wedge \omega = \mu \wedge \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} = \frac{\mu \wedge \Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{5.44}$$

It is easy to see that:

$$\forall \lambda \in \mathbb{R}_S,\ \mu \wedge \lambda\theta = (-1)^{|\mu||\lambda|}\lambda\mu \wedge \theta$$

So the wedge product can be extended by $\mathbb{R}_S$-linearity to linear combinations of $r|s$-superforms. Then we have the following:

Proposition 46. $\forall \lambda \in \mathbb{R}_S,\ \forall \mu, \nu \in \Omega^{1|0}X,\ \forall \theta,\ \forall \tau$ *$r|s$-Berezinian-superforms we have that:*

$$\mu \wedge (\theta + \tau) = \mu \wedge \theta + \mu \wedge \tau \tag{5.45}$$

$$(\mu + \nu) \wedge \theta = \mu \wedge \theta + \nu \wedge \theta \tag{5.46}$$

$$\mu \wedge \lambda\theta = \mu\lambda \wedge \theta = (-1)^{|\mu||\lambda|}\lambda\mu \wedge \theta \tag{5.47}$$

Proof. (5.45) is true by definition; (5.46) and (5.47) can be verified by direct calculations using the definition (5.40) and the properties of the superdeterminant defined with 24. ■

Moreover it is immediate to see that relation (5.24) becomes:

$$\omega = d(x^{A_{r+1}}\theta) = dx^{A_{r+1}} \wedge \theta$$

and more in general:

Proposition 47. *For the $r|s$-Berezinian-form θ defined by* (5.25) *on the supermanifold X and for every G^∞-function f on X, we have*

$$d(f\theta) = df \wedge \theta \tag{5.48}$$

To build up the Cartan calculus, we still need to define an inner product between a vector field and a $r|s$-form. We will see in a while that my definition of interior product by an even vector field coincides with the ones given by Voronov and Belopolsky and used also by Grassi and Marescotti. The interior product by an odd vector field is instead defined in the literature with a certain ambiguity, because it is not given enough attention to the necessity of extending a superforms before to contract it with an odd vector in one of its first arguments. Since the extension may not be unique without the assumption that I made in definition 31, then the lack of this last definition introduces an ambiguity. We have the following:

Lemma 48. *Let X be a supermanifold of dimension $n|m$ and let ω be any $r|s$-form on it, extended in its first argument, if u is a tangent vector field to X, then $\omega_u(\cdot) := \omega(u, \cdot)$ is a $r-1|s$ form on X.*

Proof. We have to prove that the conditions (5.4) and (5.5) hold for ω_u. When u is even, the second condition is automatically satisfied because it is satisfied by ω. To prove the first condition, let's take $v = (\overline{v_1}, \cdots, \overline{v_{r-1}}, \widetilde{v_1}, \cdots, \widetilde{v_s}) \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{r-1} \times \underbrace{T_1X \times \cdots T_1X}_{s})$ and let's consider a $g \in GL(r-1|s)$ acting on v according to 5.3 so that $gv = v'$; we can then build $g' \in GL(r|s)$ as the $r|s$-supermatrix $g' := \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$. We can make g' act on the couple $(u, v) \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{r} \times \underbrace{T_1X \times \cdots T_1X}_{s})$ in this way $g'(u, v) := (u, gv)$. A straightforward calculation gives $\operatorname{Ber}_{r,s}(g') = \operatorname{Ber}_{r-1,s}(g)$. So we have that $\omega_u(v') = \omega_u(gv) = \omega(u, gv) = \omega(g'(u, v)) = \operatorname{Ber}_{r,s}(g')\omega(u, v) = \operatorname{Ber}_{r-1,s}(g)\omega_u(v)$.

If u is odd, let's take any odd generator η of $\mathbb{R}_S$, then ηu is even and we can apply to it the same argument seen above. Moreover $\forall \eta,\ \omega_{\eta u}(v') = \eta\omega_u(v')$ and $\omega_{\eta u}(v) = \eta\omega_u(v)$ for the $\mathbb{R}_S$-linearity in the first argument of the extended form ω. So we have that: $\forall \eta,\ \eta\omega_u(v') = \eta \operatorname{Ber}_{r-1,s}(g)\omega_u(v)$ and since $\mathbb{R}_S$ has infinite odd generator, it must be $\omega_u(v') = \operatorname{Ber}_{r-1,s}(g)\omega_u(v)$.

To prove that (5.5) is satisfied by ω_u also when u is odd, we note that ω is the extension in the first argument of its restriction to $\Gamma(\underbrace{T_0X \times \cdots T_0X}_{r} \times \underbrace{T_1X \times \cdots T_1X}_{s})$. So by 33 we have that:

$$\forall u \in \Gamma(T_1X),\ \forall v \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{r-1} \times \underbrace{T_1X \times \cdots T_1X}_{s}),\ \forall \varepsilon \in \mathbb{R}_{S,1}$$

$$\omega_u(v) = \omega(u, v) = \frac{\partial}{\partial \varepsilon}\omega(\varepsilon u, v)$$

and consequently:

$$\begin{aligned}
\frac{\partial^2}{\partial v_G^B \partial v_F^A}\omega_u(v) =& \frac{\partial^2}{\partial v_G^B \partial v_F^A}\frac{\partial}{\partial \varepsilon}\omega(\varepsilon u, v) = (-1)^{|G|+|B|+|F|+|A|}\frac{\partial}{\partial \varepsilon}\frac{\partial^2}{\partial v_G^B \partial v_F^A}\omega(\varepsilon u, v) \\
=& -(-1)^{|G|+|B|+|F|+|A|}(-1)^{|G||F|+(|G|+|F|)|A|}\frac{\partial}{\partial \varepsilon}\frac{\partial^2}{\partial v_F^B \partial v_G^A}\omega(\varepsilon u, v) \\
=& -(-1)^{|G||F|+(|G|+|F|)|A|}\frac{\partial^2}{\partial v_F^B \partial v_G^A}\frac{\partial}{\partial \varepsilon}\omega(\varepsilon u, v) \\
=& -(-1)^{|G||F|+(|G|+|F|)|A|}\frac{\partial^2}{\partial v_F^B \partial v_G^A}\omega_u(v)
\end{aligned}$$

which is (5.5). ∎

I will call the form ω_u also $u \lrcorner \omega$ or $i_u\omega$.

When ω is a Berezinian-form, then I call $i_u\omega$ a contracted Berezinian-form: note that in general it may not be a Berezinian form.

If $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$ and $v = v^A \partial_A$, we can write:

$$u \lrcorner \omega = v^A \partial_A \lrcorner \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{5.49}$$

where formula (5.49) has to be considered a shortcut for the formulas described in Lemma 48.

I set the following convention:

$$|u \lrcorner \omega| := |u| + |\omega| \tag{5.50}$$

The same convention will be used in the following whenever interior products are involved.

Voronov in [152, 154] gives a definition of the interior product by the following:

Proposition 49. *Let X be a supermanifold of dimension $n|m$ and let ω be any $r|s$-form on it. If u is a tangent vector field to X locally defined by $u = u^A \partial_A$, then then the operator i_u defined in local coordinates by*

$$i_u = (-1)^{r-1} u^A \frac{\partial}{\partial v_r^A} \tag{5.51}$$

sends ω to $i_u\omega$, which is a $r-1|s$-form on X.

If ω is a Berezinian-form, then it is easy by a straightforward calculation to prove that my definition agrees with the one given by Voronov.

Since every $r|s$-forms is $\mathbb{R}_{S,0}$-linear in its first r arguments (see [152]), then it is equally easy to prove that my definition and Voronov definition are equivalent for every $r|s$-form.

What is incorrect is to deduce from those definitions that making an interior product by an odd vector is equivalent to substitute the odd vector as the first argument of the original form. This may be without sense or could lead to ambiguities. Indeed before contracting a form with an odd vector, it always necessary to extend it (for example with (5.9)).

New forms can obviously be obtained by repeated contractions with different vector fields. Note that: to extend a form in its first argument, to perform a contraction with an odd vector, followed by an other extension in the new first argument of the new form and another contraction by an odd vector is not equivalent to extend the original form in its two first arguments with (5.9) and then contract it in its two first arguments with two odd vectors. For example, if ω is a form and $\widehat{\omega}$ is its extension in the first two arguments, if $\widetilde{v}$ and $\widetilde{u}$ are odd vectors, then:

$$\widetilde{v} \lrcorner \left(\widetilde{u} \lrcorner \omega \right) \neq \widehat{\omega} \left(\widetilde{u}, \widetilde{v} \right)$$

In other words, if $\widehat{\widetilde{u} \lrcorner \omega}$ is the extension of $\widetilde{u} \lrcorner \omega$ in its first argument, we have that:

$$\widehat{\widetilde{u} \lrcorner \omega} \left(\cdot \right) \neq \widehat{\omega} \left(\widetilde{u}, \cdot \right)$$

This is consistent, if one thinks that $\widehat{\widetilde{u} \lrcorner \omega}$ is $\mathbb{R}_S$-linear in its first argument, while $\widehat{\omega}(\cdot, \cdot)$, being a Kostant-form, must obey to (5.8) and therefore it is not $\mathbb{R}_S$-linear in its second argument.

It is useful to define a wedge product between a $1|0$-form and a contracted Berezinian-form. We can make so with the help of the following:

Proposition 50. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let θ be the $r|s$-Berezinian-superform defined with (5.25); let η be a odd generator of $\mathbb{R}_{S,1}$; let $\theta_{\bar{u}}$ and $\theta_{\tilde{u}}$ be its contractions with the even vector field $\bar{u}$ and $\tilde{u}$ and let $\mu := dx^A\mu_A$ be a $1|0$-superform, then the functions defined by:*

$$\mu \wedge \theta_{\bar{u}}(v) := \mathrm{sdet}_{r+1,s} \begin{pmatrix} \overline{v_1^A}\mu_A & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ 0 & \overline{u^{A_1}} & \cdots & \overline{u^{A_r}} & \overline{u^{\alpha_1}} & \cdots & \overline{u^{\alpha_s}} \\ \overline{v_2^A}\mu_A & \overline{v_2^{A_1}} & \cdots & \overline{v_2^{A_r}} & \overline{v_2^{\alpha_1}} & \cdots & \overline{v_2^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^A}\mu_A & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_r-}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^A}\mu_A & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^A}\mu_A & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} \tag{5.52}$$

and

$$\mu \wedge \theta_{\tilde{u}}(v) := (-1)^{|\mu|} \frac{\partial}{\partial \eta} \mathrm{sdet}_{r+1,s} \begin{pmatrix} \overline{v_1^A}\mu_A & \overline{v_1^{A_1}} & \cdots & \overline{v_1^{A_r}} & \overline{v_1^{\alpha_1}} & \cdots & \overline{v_1^{\alpha_s}} \\ 0 & \eta\widetilde{u^{A_1}} & \cdots & \eta\widetilde{u^{A_r}} & \eta\widetilde{u^{\alpha_1}} & \cdots & \eta\widetilde{u^{\alpha_s}} \\ \overline{v_2^A}\mu_A & \overline{v_2^{A_1}} & \cdots & \overline{v_2^{A_r}} & \overline{v_2^{\alpha_1}} & \cdots & \overline{v_2^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{v_r^A}\mu_A & \overline{v_r^{A_1}} & \cdots & \overline{v_r^{A_r-}} & \overline{v_r^{\alpha_1}} & \cdots & \overline{v_r^{\alpha_s}} \\ \widetilde{v_1^A}\mu_A & \widetilde{v_1^{A_1}} & \cdots & \widetilde{v_1^{A_r}} & \widetilde{v_1^{\alpha_1}} & \cdots & \widetilde{v_1^{\alpha_s}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{v_s^A}\mu_A & \widetilde{v_s^{A_1}} & \cdots & \widetilde{v_s^{A_r}} & \widetilde{v_s^{\alpha_1}} & \cdots & \widetilde{v_s^{\alpha_s}} \end{pmatrix} \tag{5.53}$$

are $r|s$-forms.

Moreover we have again that $\forall \lambda \in \mathbb{R}_S$, $\forall \mu, \nu \in \Omega^{1|0}X$ which have the same parity:

$$(\mu + \nu) \wedge \theta_u = \mu \wedge \theta_u + \nu \wedge \theta_u \tag{5.54}$$

$$\mu \wedge \lambda\theta_u = \mu\lambda \wedge \theta_u = (-1)^{|\mu||\lambda|}\lambda\mu \wedge \theta_u \tag{5.55}$$

Proof. The proof that both (5.52) and (5.53) define a $r|s$-form relies on the fact that

$$\mu \wedge (i_u)\theta + (-1)^{|\mu||u|}\, i_u(\mu \wedge \theta) = (-1)^{|\mu||u|}\, \mu(u)\theta \tag{5.56}$$

which can be checked by direct calculation starting from (5.40), (5.52) and (5.53).

The proof of (5.54) and (5.55) can be done by direct calculation. ■

We have then the following three propositions, the second of which is analogous to proposition 45:

Proposition 51. *The wedge product defined for fractional forms with (5.40), is a special case* *of the wedge product defined locally for all superforms with the formula:*

$$\begin{aligned}
&\forall \text{ local chart } U,\ \forall \mu \in \Omega^{1|0}U,\ \forall \theta \in \Omega^{r|s}U,\\
&\forall v = (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s},\\
&\forall w := (v, \overline{v_{r+1}}) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s} : \\
&\mu \wedge \theta(w) := (-1)^r \left[\mu(\overline{v_{r+1}}) - (-1)^{|\mu||F|} \mu(v_F) \overline{v^A_{r+1}} \frac{\partial}{\partial v^A_F} \right] \theta(v)
\end{aligned} \tag{5.57}$$

Id est: on Berezinian superform, $\mu\wedge$ acts as e_μ.

Proof. The theorem is just a corollary of theorems 44 and 45 which have been proved with the same calculation techniques used for proving theorem 38. ■

Proposition 52. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $w := (\overline{v_1}, \cdots, \overline{v_{r+1}}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r+1} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let θ be the $r|s$-Berezinian-superform defined with* (5.25)*; let $\theta_{\bar{u}}$ and $\theta_{\tilde{u}}$ be its contractions with respectively the even vector field $\bar{u}$ and the odd vector field $\tilde{u}$ and let $\mu := dx^A\mu_A$ be a $1|0$-superform: if $e_\mu\theta_{\bar{u}}$ and $e_\mu\theta_{\tilde{u}}$ are defined by* (5.39) *and $\mu \wedge \theta_{\bar{u}}$ and $\mu \wedge \theta_{\tilde{u}}$ are defined by* (5.52) *and* (5.53)*, then:*

$$\begin{aligned}
e_\mu\theta_{\bar{u}}(w) &= \mu \wedge \theta_{\bar{u}}(w)\\
e_\mu\theta_{\tilde{u}}(w) &= \mu \wedge \theta_{\tilde{u}}(w)
\end{aligned} \tag{5.58}$$

Proof. First of all we note that, since (5.39) well define an operator, the proof of (5.58) is also a proof that definitions (5.52) and (5.53) are well given and don't depend on coordinates.

Then, for what we have seen above, we note that $\theta_u = i_u\theta$, where i_u is defined by (5.51). Using (5.39) and (5.51), we can therefore easily see that:

$$e_\mu\theta_u = -(-1)^{|\mu||u|} i_u(e_\mu\theta) + (-1)^{|\mu||u|} \mu(u)\theta \tag{5.59}$$

Remembering that $e_\mu\theta = \mu \wedge \theta$ and comparing (5.59) with (5.56), we have that:

$$e_\mu\theta_u = \mu \wedge \theta_u$$

■

We can therefore set the following definition:

Definition 53. *Let $(U, x^A, \overline{v^A}, \widetilde{v^A})$ be a local chart of TX, tangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $v := (\overline{v_1}, \cdots, \overline{v_r}; \widetilde{v}_1, \cdots, \widetilde{v}_s) \in \underbrace{T_{x,0}U \times \cdots T_{x,0}U}_{r} \times \underbrace{T_{x,1}U \times \cdots T_{x,1}U}_{s}$; let θ be any $r|s$-superform; let $\mu := dx^A\mu_A$ be a $1|0$-superform, then:*

$$\mu \wedge \theta := e_\mu\theta$$

where e_μ is defined by (5.39).

And we have then:

Proposition 54.

$$\begin{aligned}&\forall \mu,\ \nu \in \Omega^{1|0}X,\ \forall \omega \in \Omega^{r|s}X,\\ &\mu \wedge \nu \wedge \omega = (-1)^{|\mu||\nu|+1}\nu \wedge \mu \wedge \omega\end{aligned} \tag{5.60}$$

Proof. By direct calculation using (5.57) or, with easier calculations, for the class of Berezinian-forms and contracted Berezinian-forms, using (5.40), (5.52) and (5.53). ■

A short comment on the formula (5.56) for the commutator between interior and exterior products: in [65] and [9] a different formula is given. Note that, formula 3.6 in [65], defining the interior product, does not make sense without a definition of an extension of superforms. If one assumes that for a generic Berezinian superform its extension were obtained by simply using the superdeterminant formula (5.25), then formula 3.6 in [65] would not coincide with my definition (5.53) and moreover formula 3.6 would not lead to a superform satisfying (5.4) and (5.5) (as can be seen with simple counterexamples).This explain also the difference between (5.56), which involves a superanticommutator, and the corresponding formula in [65] (the third equality of 3.8) which instead involves a supercommutator.

Using (5.54) and (5.55), the wedge product can be extended to the space of all contracted Berezinian-forms by $\mathbb{R}_S$ linearity.

Using 5.60, if α is a $t|0$-form and ω is a $r|s$-form, it is easy to define the wedge product $\alpha \wedge \omega$.

Using (5.59) and (5.56), we can repeatedly apply the operator i_u to a contracted Berezinian-form obtaining a class of new forms, which we may call repeatedly contracted Berezinian forms, for which the interior products with vector fields and the exterior product with 1|0-forms are well defined and enjoy the properties described by (5.56), (5.54) and (5.55). Moreover, for this class of forms, it is possible to explicitly calculate the wedge and interior products in term of superdeterminants of the components of the 1-forms and vectors involved obtaining, results analogous to the ones of theorem 50. In the following chapters of this thesis, I will use only Berezinian and contracted Berezinian forms, so formula (5.40), (5.52) and (5.53) will be enough to explicitly calculate their values.

If $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$ and $v_1 = v_1^A \partial_A, \ldots, v_p = v_p^B \partial_B$, using the properties seen above, we can write:

$$v_p \lrcorner \ldots \lrcorner v_1 \lrcorner \omega = v_1 \wedge \cdots \wedge v_p \lrcorner \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s} \tag{5.61}$$

We can therefore give the following:

Definition 55. *The class of superforms obtained by Berezinian superforms repeatedly performing wedge products by* 1|0*-superforms and contraction by vector fields and their* $\mathbb{R}_S$*-linear combinations is called the class of fractional superform.*

The name is justified by the notation proposed with (5.44), (5.49) and(5.61).

It is important to know how the operator d act on fractional forms and how its action is correlated with the action of the wedge product and of the interior product. We have:

Proposition 56.

$$\begin{aligned}&\forall f \in \Omega^{0|0}X = \mathcal{F}(X),\ \forall \omega \in \Omega^{r|s}X,\\ &d(f\omega) = df \wedge \omega + f d\omega\end{aligned} \tag{5.62}$$

Proof. By direct calculation using (5.6) and the definition of $e_\mu\theta$. ∎

Since we define the class of fractional superforms as the smallest subclass of general $r|s$-forms containing the Berezinian forms and closed under contraction and wedge product, the natural question arises whether this subclass is a proper subclass of Voronov-Zorich forms or it is coincident with the class of all Voronov-Zorich forms.

I don't have now an answer to this question.

For example: if

$$\omega = \frac{1}{\widetilde{\theta}}$$

with $\widetilde{\theta} = dx^A\widetilde{\theta_A}$, $|\theta| = 1$ and with $d\widetilde{\theta} \neq 0$, then I don't know if $d\omega$ is a fractional form or not, although I guess it is not.

Then it may be that the class of fractional superforms is not closed under exterior derivation.

The example above, by the way, shows that the action of the operator d on a fractional form, and even on a Berezinian form, is not always easy to compute.

We have the following:

Proposition 57. *If μ is a $1|0$-form and ω is and $r|s$-form:*

$$d(\mu \wedge \omega) = d\mu \wedge \omega - \mu \wedge d\omega$$

Proof. By direct calculation remembering definition 53 and using 5.39 and 5.6 . ∎

One other property of $n|m$-forms, easy to demonstrate, is that they can be pullback by G^∞ maps in the expected way.

Using (5.6) and (5.57), it is easy to prove that the pullback of forms trough a G^∞ map between supermanifold commutes with the exterior derivative and that the pullback of a wedge product is the wedge product of the pullbacks.

Voronov and Zorich have shown that, with their definitions of exterior derivative, interior product and pull back, the following Cartan formula holds for the Lie derivative along an even or an odd vector field u:

$$\begin{aligned} &\forall \omega \in \Omega^{r|s}X,\ \forall u \in \Gamma(TX): \\ &\mathrm{Lie}_u\,\omega = d(i_u\omega) + i_u(d\omega) \end{aligned} \tag{5.63}$$

keeping in mind that (5.63) makes sense whenever $i_u\omega$ is well defined.

Since my definitions of exterior derivative and pull back for fractional superforms and my definition of interior product agree with the ones of Voronov and Zorich, then (5.63) also holds for Berezinian, for contracted Berezinian and for fractional superforms.

I can finally define a new kind of product which is an extension of the exterior product and which had not been taken in consideration by Voronov.

Definition 58. *The exterior product between the two Berezinian forms $\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}$ and $\frac{\Gamma^1\wedge\cdots\wedge\Gamma^r}{\gamma^1\odot\cdots\odot\gamma^s}$ is the $l+r|d+s$ Berezinian form defined by:*

$$\left(\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\right)\wedge\left(\frac{\Gamma^1\wedge\cdots\wedge\Gamma^r}{\gamma^1\odot\cdots\odot\gamma^s}\right) := \frac{\Theta^1\wedge\cdots\wedge\Theta^l\wedge\Gamma^1\wedge\cdots\wedge\Gamma^r}{\theta^1\odot\cdots\odot\theta^d\odot\gamma^1\odot\cdots\odot\gamma^s} \tag{5.64}$$

This definition will reveal to be useful in section 6.3. One could try to extend it to non Berezinian forms. This could be achieved by defining, for each $0|1$-form α, the operator e_α sending every $r|s$-form ω to a $r|s+1$ form in a way analogous to the one of the operator defined by (5.39). This could be useful in the context of string theory for defining picture changing operators, see [9, 10] but, since it is not necessary for what it follows, I will not undertake this path.

5.5 Integral of superforms on supermanifolds

The main interesting feature of $r|s$-forms is that they can be integrated on $r|s$-submanifolds. Voronov and Zorich in [155, 156, 157, 158] have in fact shown that $r|s$-forms can be integrated over $r|s$-supermanifold with boundary, whence a suitable definition of boundary is given and that a super version of Stokes theorem then holds. For details on the theory of integration see Voronov [152].

For a short and good account on the history of the theory of integration on supermanifolds, with all the main references quoted, see the introduction of [154]; for an other list of references see the bibliographical notes of [152]. A recent review on integration of integral forms oriented to physical applications is Witten's paper [163].

Here I will give only the main definitions and results without many comments. I will introduce only the material which is necessary to develop a super field theory in the next two parts of this thesis.

Remember that for every G^∞-supermanifold X of dimension $r|s$ there is a well defined map, called body, and usually denoted by ϵ, from X to $\underline{X}$, being $\underline{X}$ a r-dimension real C^∞-manifold. For details see [133].

Definition 59. *I call body immersion every injective C^∞ map b from $\underline{X}$ to X such that:*

$$\forall \underline{x} \in \underline{X},\ \epsilon b(\underline{x}) = \underline{x} \tag{5.65}$$

The body-immersion map b fix a "real" slice $b(\underline{X})$ in X: with a little abuse of language I will call this image the immersed body of X even if it is not obviously uniquely defined, neither can be in general canonically defined.

Remark 60. *Batchelor with her theorem, in [7] (see also [133], Chapter 8), shows that to every vector fiber bundle E with an n-dimensional C^∞ base $\underline{X}$ and an m-dimensional fiber, corresponds a $n|m$-dimensional G^∞ supermanifold (with DeWitt topology), which, using Rogers notation, we can call $S(\underline{X}, E)$, whose body is $\underline{X}$. Conversely: every G^∞ $n|m$-dimensional supermanifold with DeWitt topology is superdiffeomorphic to a supermanifold of the type $S(\underline{X}, E)$, with E uniquely determined by X.*

We could show that every G^∞ $n|m$-dimensional supermanifold X with a fixed immersed body corresponds to a vector fiber bundle E with a fixed global section, but this correspondence is not 1 to 1: two different immersions may correspond to the same global section. This means that the immersion of the body contains more information than the global section of the corresponding Batchelor vector bundle. We will see in the following how this information may be useful to define an integral over a supermanifold.

To avoid any ambiguity, we could from now on consider only supermanifolds X with a fixed immersed body: in this case we could call it, with a little abuse of notation, $b(X)$, as a shortcut for $b(\underline{X}) = b(\epsilon(X))$. Note that, once an immersed body is fixed, if $U \subset X$, then $b(U) := U \cap b(X)$ is well defined. Note moreover that, if on a chart U the local coordinates are $(x^a, x^{n+\alpha})$ and if, $\forall \underline{x} \in \underline{X}$ such that $b(\underline{x}) \in U$, we have that $x^{n+\alpha}(b(\underline{x})) = 0$ and $x^a(b(\underline{x})) \in \mathbb{R}$, then the same conditions are not necessarily satisfied on another local chart U'. However it is possible to demonstrate that, once fixed an immersed body, there always exists a sub-atlas of X such that, for every local chart $[U, (x^a, x^{n+\alpha})]$, $x^{n+\alpha}(b(\underline{x})) = 0$ and $x^a(b(\underline{x})) \in \mathbb{R}$. In the following I will always use such an atlas, unless I explicitly mention it.

If the supermanifold X is an open domain of $\mathbb{R}_{n|m}$, then an atlas can be chosen with a single chart with coordinates (x^A). The body of X is an open domain $\underline{X} \subset \mathbb{R}^n$ with coordinates $(\underline{x}^a), \underline{x}^a \in \mathbb{R}$, and there is a canonical immersed body $b(X)$ so that the body immersion map b sends $(\underline{x}^a)$ to $(x^a = \underline{x}^a; x^{n+\alpha} = 0)$.

If U is an open domain in $\mathbb{R}^{n|m}$ with coordinates $(x^a, x^{n+\alpha})$, and if f is a G^∞-function on U, we can write in coordinates:

$$f(x^a, x^{n+\alpha}) = x^\Lambda \hat{f}_\Lambda(x^a) \tag{5.66}$$

where the Greek capital letter Λ stands for a multiindex which can be 0 or can be a sequence of ordered integer numbers α_j, not mutually equal, chosen in the set going from $n+1$ to $n+m$. Here and in the following we will use the convention that the order in the multiindex must go from the smaller α_j to the bigger α_j when the multiindex is written as an apex and the opposite when it is written as a subscript. If the same letter is used for a apex multiindex and a subscript multiindex, then it is understood that they are obtained by inverses sequences of indexes. By definition: if $\Lambda = 0$, then $x^\Lambda = 1$; if $\Lambda = \alpha_1\alpha_2\cdots\alpha_k$, with $n < \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n+m$, then $x^\Lambda = x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_k}$. For example, if $m = 7$, $\alpha_1 = n+3, \alpha_2 = n+5, \alpha_3 = n+7$ and $\Lambda = \alpha_1\alpha_2\alpha_3$, then $x^\Lambda = x^{n+3}x^{n+5}x^{n+7}$, but $f_\Lambda = f_{\alpha_3\alpha_2\alpha_1} = f_{n+7,n+5,n+3}$. The function $\hat{f}_\Lambda$ is the Grassmann analytic continuation of a function f_Λ defined on $\underline{U}$ and with values in $\mathbb{R}_S$; for the definition of Grassmann analytic continuation see [133].

I then define the Berezin integral of f over U as:

$$\int_U Dx^a Dx^\alpha f = \int_U Dx^a Dx^\alpha x^\Lambda \hat{f}_\Lambda(x^a) := \int_{\underline{U}} D\underline{x}^a f_{\text{TOP}}(\underline{x}^a) \tag{5.67}$$

where $\hat{f}_{\text{TOP}} = \hat{f}_{n+m,n+m-1,\cdots,n+1}$ is the top component in the expansion (5.66) and $f_{\text{TOP}} = \hat{f}_{\text{TOP}}|_{\underline{U}}$ is the corresponding function on $\underline{U}$.

Note that, with this definition, $\int_U Dx^a Dx^\alpha$ is an operator from $G^\infty(U)$ to $\mathbb{R}_S$ which is $\mathbb{R}_S$-linear from the right.

Note also that this definition agrees with the definition given by Voronov in [152] and differs from the definition given by Rogers in [133], which gives rise to an operator $\mathbb{R}_S$-linear on the left: this depends on the choice of putting the coefficients $\hat{f}_\Lambda$ in the expansion (5.66) after the product of odd variables x^Λ; the two definitions give the same results when f_{TOP} is real or even and they differ when f_{TOP} is odd.

If we consider a non canonical immersed body $b(U)$, we can give the following definition of an integral of the function f over the pair $[U, b(U)]$:

$$\int_{[U,b(U)]} Dx^a Dx^\alpha f = \int_{[U,b(U)]} Dx^a Dx^\alpha x^\Lambda \hat{f}_\Lambda(x^a) := \int_{\underline{U}} D\underline{x}^a \hat{f}_{\text{TOP}}(b(\underline{x}^a)) \tag{5.68}$$

Unfortunately both definitions (5.67) and (5.68) pose some problems when the function f is not null at the boundary of U. If one try and use them directly to define an integral over a non compact supermanifold, then he falls quickly in some troubles. To avoid these troubles Voronov and Zorich proposed to define the integral over supermanifolds with boundaries.

We can define a $n|m$-dimensional domain of $\mathbb{R}^{n|m}$ with boundary in the following way: we consider a G^∞ function v from $\mathbb{R}^{n|m}$ to $\mathbb{R}^{1|0}$ and we define a domain with boundary as the pair $(U, \partial U)$, where ∂U, called the boundary, is the set of points x of $\mathbb{R}^{n|m}$ such that $v(x) = 0$ and

U, called the interior part, is the set of points x of $\mathbb{R}^{n|m}$ such that $\epsilon\,[v(x)] > 0$ (remember that $\epsilon\,[v(x)]$ is the body of $v(x)$ *id est* its real part). Note that it could be written $v(x) > 0$ without ambiguity.

It is possible to prove that $\epsilon\,(\partial U) = \partial\epsilon\,(U)$, where $\partial\epsilon\,(U) = \partial\underline{\mathrm{U}}$ is the natural boundary of $\epsilon\,(U) = \underline{\mathrm{U}}$. Note that $\partial\underline{\mathrm{U}}$ could be defined by the equation $\underline{v} = 0$, where $\underline{v} = \epsilon\left(v|_{\underline{\mathbb{R}^{n|m}}}\right)$ is a real function defined on $\mathbb{R}^n$.

Note that two different domains with boundary could have the same interior part (and different boundaries). Let's consider for example on $\mathbb{R}^{1|0}$, with variable x, the two functions $v = x$ and $u = x - \eta_1\eta_2$, where $\eta_1, \eta_2 \in \mathbb{R}_{S,1}$ are odd constants. The two functions define the two domains with boundary $(U_v, \partial U_v)$ and $(U_u, \partial U_u)$ with $U_v = U_u$ and with ∂U_v consisting in the point $x = 0$ and ∂U_v consisting in the point $x = \eta_1\eta_2$.

However it is possible to prove that if $(U, \partial U_1)$ and $(U, \partial U_2)$ are two domains with boundary which have the same interior part and which are defined by the two functions v_1 and v_2, then $\epsilon\,(\partial U_1) = \epsilon\,(\partial U_2)$; moreover $\epsilon\,(\partial U_1) = \epsilon\,(\partial U_2) = \partial\underline{\mathrm{U}}$ and $\underline{v_1} = \underline{v_2}$. In the example above: $\underline{v} = \underline{v} = \underline{x}$.

We can now give the following:

Definition 61 (Voronov and Zorich)**.** *The Berezin integral of the function f over a domain with boundary $(U, \partial U)$ defined by the function u is:*

$$\int_{(U,\partial U)} Dx^a Dx^\alpha f := \int_U Dx^a Dx^\alpha x^\Lambda \hat{f}_\Lambda(x^a)\hat{\theta}\,(u(x^a, x^\alpha)) \tag{5.69}$$

where $\int_U Dx^b Dx^{r+\beta}$ is the Berezin integral defined above, and where $\hat{\theta}$ is the Grassmann analytic continuation to $\mathbb{R}^{1|0}$ of the classical Heaviside θ function defined on $\mathbb{R}^1$.

Note that, since the Grassmann analytic continuation of the classical θ involves its derivatives and since the first derivative of θ is the Dirac δ distribution, it is clear that the integral defined above automatically include boundary terms, id est a Berezinian integral over the boundary ∂U and consequently a real integral over $\partial\underline{U}$. The definition of δ distribution on $\mathbb{R}^{n|m}$ is straightforward.

I want to present a small extension of definition 61:

Definition 62. *The Berezin integral of the function f over over the triplet $[(U, \partial U)\,, b(U)]$, where $(U, \partial U)$ is a domain with boundary defined by the function u ($u > 0$ on U) and $b(U)$ is an immersed body (possibly non canonical) of U, such that $\partial b(U) = b\,(\partial\underline{U}) \subset \partial U$, is defined by:*

$$\int_{[(U,\partial U),b(U)]} Dx^a Dx^\alpha f := \int_{[U,b(U)]} Dx^a Dx^\alpha x^\Lambda \hat{f}_\Lambda(x^a)\hat{\theta}\,(u(x^a, x^\alpha)) \tag{5.70}$$

where $\int_{[U,b(U)]} Dx^a Dx^\alpha$ is the Berezin integral over immersed body defined above with (5.68)*, and where $\hat{\theta}$ is the Grassmann analytic continuation to $\mathbb{R}^{1|0}$ of the classical θ function defined on $\mathbb{R}^1$.*

Remark 63. *Definitions 61 and 62 are always well posed and give a finite result when $U \cup \partial U$ is compact. If $(U, \partial U_1)$ and $(U, \partial U_2)$ are two domains with boundary which have the same interior part U and different boundaries, which are defined by the two functions u_1 and u_2, then we have in general that $\int_{(U,\partial U_1)} \neq \int_{(U,\partial U_2)}$. This fact will be crucial in allowing a good definition of integration over supermanifolds with boundaries.*

Using the notion of domain with boundary it is easy to give a definition of supermanifold with boundary. I will not give it here explicitly. Sometime a $r|s$-supermanifold with boundary

X contained in a $r|s$-supermanifold Y can be defined starting directly from a function v defined on Y.

We can now define the integral of a super-form.

Definition 64. *Let Y be an $n|m$-supermanifold. Let X be a $r|s$-supermanifold with boundary ∂X. Let $\{(U_\alpha, \partial U_\alpha), \rho_\alpha\}$ be a partition of unity of X, with $(U_\alpha, \partial U_\alpha)$ open domains with boundary of $\mathbb{R}^{r|s}$ defined by the functions u_α ($u_\alpha > 0$ on U_α), let every U_α be included in a local chart with coordinates x^B; let $b(X)$ be an immersed body of X and $b(U_\alpha)$ the corresponding immersed bodies (possible non canonical) on U_α; let $i : X \longrightarrow Y$ be an immersion of X in Y which is a G^∞-map; let ω be a $r|s$-form defined on Y; then:*

$$\begin{aligned}\int_{i[(X,\partial X),b(X)]} \omega &:= \sum_\alpha \rho_\alpha \int_{[(U_\alpha,\partial U_\alpha),b(U_\alpha)]} Dx^b Dx^{r+\beta} i^* \omega (\partial_1, \cdots, \partial_{r+s}) \\ &= \sum_\alpha \rho_\alpha \int_{[U_\alpha, b(U_\alpha)]} Dx^b Dx^{r+\beta} i^* \omega (\partial_1, \cdots, \partial_{r+s}) \hat{\theta}(u_\alpha)\end{aligned} \tag{5.71}$$

Note that, in order to avoid to weight to much the notation, in Formula (5.71) I did not write explicitly the coordinates functions from X to the charts U_α.

Remark 65. *The same definition can be used when X has no boundary. In this case the boundary terms over each ∂U_α do not appear in the sum* (5.71)*.*

Note that we can take a simpler version of definition 64, if we fix an immersed body and we use adapted (existing) atlases for which, in each chart: $b(U_\alpha) = \underline{U_\alpha}$ is the canonical body of $b(U_\alpha)$. For this case, the proof that the definition is well posed and that it doesn't depend on the choice of the charts can be found in [152]. The proof is based on the property (5.4) of superforms. The proof is also based on the fact that, changing coordinates, it changes also the local function defining the boundary and consequently it changes the contribution of the $\hat{\theta}$ functions in the integral.

I decided to give this slightly more complicated definition because it allows more general changes of coordinates, namely those changes of coordinates for which the immersed body does not correspond to the canonical bodies of the charts involved in the change. In the following I give a couple of super simple examples to explain how the definition works.

Let $Y = \mathbb{R}^{1|2}$. Let's take on Y the coordinates (x, ξ^1, ξ^2). Let's consider the superform $\omega = \frac{dx}{d\xi^1 \odot d\xi^2}\left[\hat{f}_0(x) + \xi^1\xi^2\widehat{f_{21}}(x)\right]$. Let's consider the submanifold with boundary X of Y defined by $v = x$ when $x < 1$ and by $u = 1 - x$ when $x > 0$. Let's use on X the coordinates of Y. The immersion of X in Y is then given locally by the identity. Let's consider the immersed body of X defined by $x(b(X)) = \underline{x}$, $\xi^1(b(X)) = 0$ and $\xi^2(b(X)) = 0$, with understandable notation. Then:

$$\begin{aligned}\int_{i[(X,\partial X),b(X)]} \omega &= \int_{b(X)} dx \int Dx^{1+\beta} \left[\hat{f}_0(x) + \xi^1\xi^2\widehat{f_{21}}(x)\right]\left[\hat{\theta}(x) - \hat{\theta}(x-1)\right] \\ &= \int d\underline{x} f_{21}(\underline{x}) \left[\theta(\underline{x}) - \theta(\underline{x} - 1)\right] \\ &= \int_0^1 d\underline{x} f_{21}(\underline{x})\end{aligned}$$

If we perform a change of coordinates which leaves ξ unchanged and such that $y = x + \xi^1\xi^2$, then in the new coordinates the body is given by $y(b(X)) = \underline{x}$, $\xi^1(b(X)) = 0$ and $\xi^2(b(X)) = 0$ and in the new local coordinates we have that $\omega = \frac{dy}{d\xi^1 \odot d\xi^2}\left[\widehat{f_0}(y-\xi^1\xi^2)+\xi^1\xi^2\widehat{f_{21}}(y-\xi^1\xi^2)\right]$. Then we have:

$$
\begin{aligned}
&\int_{i[(X,\partial X),b(X)]}\omega \\
&= \int_{b(X)} dy \int Dx^{1+\beta}\left[\widehat{f_0}(y-\xi^1\xi^2)+\xi^1\xi^2\widehat{f_{21}}(y-\xi^1\xi^2)\right]\left[\hat\theta\left(y-\xi^1\xi^2\right)-\hat\theta\left(y-\xi^1\xi^2-1\right)\right] \\
&= \int_{b(X)} dy \int Dx^{1+\beta}\left[\widehat{f_0}(y)-\xi^1\xi^2\frac{\partial\widehat{f_0}}{\partial y}(y)+\xi^1\xi^2\widehat{f_{21}}(y)\right]\left[\hat\theta\left(y\right)-\xi^1\xi^2\delta(y)-\hat\theta\left(y-1\right)+\xi^1\xi^2\delta(y-1)\right] \\
&= \int d\underline{x} f_{21}(\underline{x})\left[\theta(\underline{x})-\theta(\underline{x}-1)\right] = \int_0^1 d\underline{x} f_{21}(\underline{x})
\end{aligned}
$$

And we recover the previous result.

On the other hand, if we perform a change of coordinates which leaves the ξ unchanged and such that $y = x+\eta^1\eta^2$, with η^1 and η^2 odd constant belonging to $\mathbb{R}_{S,1}$, then in the new coordinates the body is given by $y\left(b(X)\right)=\underline{x}+\eta^1\eta^2$, $\xi^1\left(b(X)\right)=0$ and $\xi^2\left(b(X)\right)=0$ and in the new local coordinates we have that $\omega = \frac{dy}{d\xi^1 \odot d\xi^2}\left[\widehat{f_0}(y-\eta^1\eta^2)+\xi^1\xi^2\widehat{f_{21}}(y-\eta^1\eta^2)\right]$. Then we have:

$$
\begin{aligned}
&\int_{i[(X,\partial X),b(X)]}\omega \\
&= \int_{b(X)} dy \int Dx^{1+\beta}\left[\widehat{f_0}(y-\eta^1\eta^2)+\xi^1\xi^2\widehat{f_{21}}(y-\eta^1\eta^2)\right]\left[\hat\theta\left(y-\eta^1\eta^2\right)-\hat\theta\left(y-\eta^1\eta^2-1\right)\right] \\
&= \int_{b(X)} dy \int Dx^{1+\beta}\left[\widehat{f_0}(y)-\eta^1\eta^2\frac{\partial\widehat{f_0}}{\partial y}(y)+\xi^1\xi^2\widehat{f_{21}}(y-\eta^1\eta^2)\right]\left[\hat\theta\left(y\right)-\eta^1\eta^2\delta(y)-\hat\theta\left(y-1\right)+\eta^1\eta^2\delta(y-1)\right] \\
&= \int d\underline{x} f_{21}(\underline{x})\left[\theta(\underline{x})-\theta(\underline{x}-1)\right] = \int_0^1 d\underline{x} f_{21}(\underline{x})
\end{aligned}
$$

which gives us again the same value.

If we hadn't used the definition including the concept of immersed body, we could have not obtained this last result. Obviously we could have decided to not allow changes of coordinates which send a local canonical body to a local non canonical one. This is always possible because we can always cover a supermanifold with atlases for which the changes of coordinates are never of that kind.

I don't know if these kind of considerations were implicitly present in the original work of Voronov and Zorich. It is true that they become almost necessary if one works in the concrete framework for supermanifolds, which allows the use of odd constant.

In any case I think that the concept of immersed body clarifies better from a geometric point of view the notion of integration on a supermanifold. Moreover I will utilize that concept again in chapter 9 on Comparison Theorems, again with the idea that it can help in understanding the geometrical meaning of those theorems.

Remark 66. *Definition 64 is always well posed when $X\cup\partial X$ is compact, also if X alone is not compact.*

We could try and define an integral over X, when X is not compact, but $X\cup\partial X$ is compact, and this can be done if suitable conditions on ω are imposed. We have however to be careful: if

X is an open supermanifold immersed in an other supermanifold Y and X can be comptactified by adding a suitable boundary, then speaking of the integral on X may not have much sense. As we have seen, the same supermanifold X may have different boundaries, and the natural notion of integral of superforms is a notion defined on supermanifolds with boundary: changing the boundary in general changes the value of the integral, so there is not much point in looking for a value depending only on the interior part. If we look for a different definition of integral on supermanifolds with boundary, we may lose the advantages of the one here presented, the main of which is the validity of a super version of the Stokes theorem, as we see below.

For some comments on integrations on non compact manifolds and for a short accounts on approaches different from the one undertaken here, one can see Rogers [133]. One can also read the recent paper of A. Alldridge, J. Hilgert and W. Palzer, [2], who worked on older ideas of M. J. Rothstein, [137]. I am not going to compare their results to the one of Voronov and Zorich here presented, althought it would be interesting to do so.

In fact, for what follows in the third and fourth part of this thesis, dealing with a geometrically well defined variational foundation of super field theories, it is enough to have a good definition of integral on compact supermanifolds with boundary. What will be important are indeed variations of some integral for compactly supported variations of the fields.

Remark 67. *The integral defined with 64 satisfies an interesting property which is a consequence of* (5.5)*: if we take a compactly supported small variation of the immersion i, then the functional variation of $\int_{i(X)} \omega$ doesn't depend on derivatives of the immersion with respect to the coordinates on X which are of order higher than 1.*

Condition (5.5) *is imposed by Voronov and Zorich precisely because they wanted this property to be true.*

The functional variation of $\int_{i(X)} \omega$ can be then expressed as the integral: $\int_{i(X)} w \lrcorner d\omega$, where w is the vector field generating the infinitesimal compactly supported variation of i and where d is the operator defined by Voronov and Zorich.

The reader can see the original works of Voronov and Zorich and [152] for details. In section 6.3 I will give a short proof of an analogous property satisfied by the integral of mixed forms on a supermanifold.

We conclude with the following version of Stokes theorem for superforms:

Theorem 68 (Voronov and Zorich)**.** *If α is a superform, then:*

$$\int_U d\alpha = \int_{\partial U} \alpha \tag{5.72}$$

For the proof see Voronov [152].

Chapter 6

Fractional coforms, mixed forms and their integration on supermanifolds

In 1995 Khudaverdian, [91], had the idea to define *codensities* on supermanifolds.

Independently Voronov, in [153] and then in [154], gave the definition of *dual forms* on a supermanifold, which turned out to be special kind of codensities.

Voronov, in [153] and in [154] gave also the definition of *mixed forms* and showed how to perform a Cartan calculus with those new objects.

In section 6.1 I give a version of the definition of *dual forms* adapted to the notation used until now. I prefer to call coforms, what Voronov calls dual forms. Then I explain what I mean by fractional coforms.

In section 6.2 I introduce *mixed forms* and I define mixed fractional forms.

In section 6.3 I shortly explain how coforms and mixed forms can be integrated on submanifolds of a supermanifold.

All the material presented in this chapter is not necessary to build a multisymplectic superfield theory (which is the main scope of this thesis) and it is presented without many details. It can be consider as the natural companion of the material presented in the previous chapter and as an introduction for further and more complete studies.

The use of fractional coforms and fractional mixed forms, together with the theory of their integration, may be useful in the context of BRST (Bechi, Rouet, Stora, Tyutin) and BV (Batalin, Vilkovisky) superfield theories. I believe indeed that their use can make more transparent some aspects of BRST and BV approaches. So the results presented in this chapter can be considered as a preliminary work for a future study in that direction.

6.1 Berezinian and fractional $t|q$-coforms on supermanifolds.

I follow the path already undertaken in sections 5.2.1 and 5.3.

Let $g \in GL(t|q)$ be written as a supermatrix (after having chosen arbitrarily the necessary basis) and let V be a superspace of dimension $n|m$, then there is a right action of $GL(t|q)$ onto

$\underbrace{V^* \times \cdots \times V^*}_{t} \times \underbrace{\Pi V^* \times \cdots \times \Pi V^*}_{q}$, defined in this way:

$$\begin{aligned} &\forall g \in GL(t|q),\ \forall \overline{p^1}, \cdots, \overline{p^t} \in V^*,\ \widetilde{p^1}, \cdots, \widetilde{p^q} \in \Pi V^*, \\ &\left(\overline{p^1}, \cdots, \overline{p^t}; \widetilde{p^1}, \cdots, \widetilde{p^q}\right) \cdot g = \\ &= \left(\overline{p^1}, \cdots, \overline{p^t}; \widetilde{p^1}, \cdots, \widetilde{p^q}\right) \cdot \begin{pmatrix} g_{0,0} & g_{0,1} \\ g_{1,0} & g_{1,1} \end{pmatrix} \end{aligned} \tag{6.1}$$

Where, in the right side of the last equation, the product is the usual matrix product with attention given to the order in products of entries and where an element $\overline{p} \in V^*$, multiplied on the right by an odd number, gives an element of ΠV^* in a natural way and so on.

Let's consider a supermanifold X of dimension $n|m$ and one of its point $x \in U \subset X$, where U is a local chart of X. On U we have local coordinates x^F. Let $T^*_x X$ be the tangent module of X over x. A base for $T^*_x X$ is given by $\left(dx^F|_x\right)_{F=1\dots n|m}$. We can identify a point $p \in T^*_x X$ by its coordinates p_F with-respect to the chosen basis. On $T^*_x X$ we can consider the topology inherited from T^*X. We have then the following:

Definition 69 (Voronov)**.** *A coform of codegree $t|q$ over a point $x \in X$, supermanifold of dimension $n|m$, is a G^∞ map $w : \mathcal{O} \subset \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q} \longrightarrow \mathbb{R}_S$, which, $\forall p \in \mathcal{O}$, open subset of $\underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, satisfies the following two conditions:*

$$\forall g \in GL(t|q),\ w(p \cdot g) = w(p)\,\mathrm{Ber}_{t,q}(g) \tag{6.2}$$

$$\frac{\partial^2 w}{\partial p^B_G \partial p^A_F} + (-1)^{|G||F|+(|G|+|F|)|A|} \frac{\partial^2 w}{\partial p^B_F \partial p^A_G} = 0 \tag{6.3}$$

where $F, G = 1, \cdots, n+m$ are the indices in the space $T^*_x X$ and so also in both spaces $T^*_{x,0}X$ and $T^*_{x,1}X$ with their usual degree; p^A_F is the F-th coordinates of p^A in the local base $\left(dx^F|_x\right)_F$; A runs from 1 to $t+q$ and we have $p^A \in T^*_{x,|A|}X$, where we set $|A| = 0$ when $A = 1, \cdots, t$ and $|A| = 1$ when $A = t+1, \cdots, t+q$.

It can be seen that the definition does not depend on the choice of the chart U and of the corresponding basis for $T^*_x X$.

Note that, by definition, if $p \in \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$ is so that $p^{t+1}, \cdots, p^{t+q}$ are linearly dependent, then there are only two possibility: either $p \notin \mathcal{O}$ (so that w is not defined on p), or $w(p) = 0$. Moreover, in this second case, w being a G^∞ map on $\mathcal{O}$, we must have $w(p) = 0\ \forall p \in \mathcal{O}$.

It is easy to prove that the space of $t|q$-coforms over x is naturally a free left supermodule over $\mathbb{R}_S$; I call it $\Lambda_{t|q;x}$, as a shortcut for $\Lambda_{t|q}T_x X$ and we have, as usual, $\Lambda_{t|q;x} = \Lambda_{t|q;x,0} \oplus \Lambda_{t|q;x,1}$, where $\Lambda_{t|q;x,0}$ and $\Lambda_{t|q;x,1}$ are respectively the even and the odd part of $\Lambda_{t|q;x}$ and they are superspaces. The space $\Lambda_{t|q;x}$ can be given the structure of right supermodule by a sign rule analougous to the one used in sections 5.2 and 5.3.

In the usual way we can build the fiber bundles $\Lambda_{t|q;0}X$, $\Lambda_{t|q;1}X$ and $\Lambda_{t|q}X$.

Definition 70. *A G^∞ section of the bundle $\Lambda_{t|q}X$ is called a differential $t|q$-coform. The space of $t|q$-coforms over X is called $\Omega_{t|q}X := \Gamma\left(\Lambda_{t|q}X\right)$.*

The $t|q$-coforms can be extended in their first t arguments with techniques analogous the one used to extend $r|s$-forms. It is although more natural to extend them so that they have "right $\mathbb{R}_s$-linearity":

Definition 71. *A coform of degree $t|q$ over a point $x \in X$ of dimension (n, m) is said to be extended in the arguments $(p_{a_1}, \cdots, p_{a_k})$, with $k < t$, if it is a G^∞ map $\widehat{w}$: $T^*_{x,0}X \times \cdots \times T^*_xX \times \cdots \times T^*_{x,0}X \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$ (where T^*_xX substitutes $T_{x,0}X$ k times in the positions $a_1, \cdots, a_k$) which satisfies the following conditions:*

- *when restricted to $\underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, it is a coform of degree $p|q$,*
- $\forall(\overline{p^1}, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^q}) \in \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t-k} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, $\forall p^1, \ldots, p^k \in \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{k}$
  $\widehat{w}(\overline{p^1}, \cdots, p^1, \cdots, p^iH, \cdots, p^k, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^q})$
  $= (-1)^{|H|\sum_{l=i+1}^k |p^l|}\widehat{w}(\overline{p^1}, \cdots, p^1, \cdots, p^i, \cdots, p^k, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^q})H$
  and
  $\widehat{w}(\overline{p^1}, \cdots, p^1, \cdots, p^i, \cdots, p^{i+1}, \cdots, p^k, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^q})$
  $= (-1)^{1+|p^i||p^{i+1}|}\widehat{w}(\overline{p^1}, \cdots, p^1, \cdots, p^{i+1}, \cdots, p^i, \cdots, p^k, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^q})$
  where the p^l are in the positions $a_1, \cdots, a_k$,
- $\forall(\overline{p^1}, \cdots, \overline{p^{t-k}}, \widetilde{p^1}, \cdots, \widetilde{p^s}) \in \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t-k} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, $\forall p^1, \ldots, p^k \in \underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{k}$
  $\widehat{w}(p^1, \cdots, \cdot_1, \cdots, \cdot_{t-k}, \cdots, p^k, \widetilde{p^1}, \cdots, \widetilde{p^q})$, *where the free arguments are in the positions different than $a_1, \cdots, a_k$, is $\mathbb{R}_{S,0}$-linear and antisymmetric.*

We call $\widehat{w}$ an extended-coform, when it is extended in the above sense in all its even arguments.

We say that $\widehat{w}$ extends the $t|q$-coform w if, when restricted to $\underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, it coincides with w.

Proposition 72. *For every coform w of degree $t|q$ over a point $x \in X$ of dimension (n, m), there is one and only one coform $\widehat{w}$ which extends w in the arguments $(p^{a_k}, \cdots, p^{a_1})$, with $k < t$. The coform $\widehat{w}$ can be inductively defined by:*

$$\widehat{w}^0 = w \tag{6.4}$$

$$\begin{aligned}
&\widehat{w}^{l+1}(\overline{p^1}, \cdots, p^{a_{l+1}}, \cdots, p^{a_l}, \cdots, p^{a_1}, \cdots, \overline{p^t}, \widetilde{p^1}, \cdots, \widetilde{p^q}) \\
&\quad := \widehat{w}^l(\overline{p^1}, \cdots, \overline{p^{a_{l+1}}}, \cdots, p^{a_l}, \cdots, p^{a_1}, \cdots, \overline{p^t}, \widetilde{p^1}, \cdots, \widetilde{p^q}) \\
&\qquad + (-1)^{\sum_{i=1}^l |p^{\alpha_i}|}\widehat{w}^l(\overline{p^1}, \cdots, \widetilde{p^{a_{l+1}}}\varepsilon, \cdots, p^{a_l}, \cdots, p^{a_1}, \cdots, \overline{p^t}, \widetilde{p^1}, \cdots, \widetilde{p^q})\frac{\overleftarrow{\partial}}{\partial \varepsilon}
\end{aligned} \tag{6.5}$$

where $\varepsilon \in \mathbb{R}_{S,1}$ is an odd parameter.

An extended $1|0$-coform is clearly an element of the bidual of T_xX.

When V is a left supermodule over the superalgebra $\mathbb{R}_S$, its dual V^* is defined as the set of left $\mathbb{R}_S$-linear function from V to $\mathbb{R}_S$. V^* is then naturally a right free supermodule over $\mathbb{R}_S$. The bidual V^{**} is defined as the set of right $\mathbb{R}_S$-linear function between V^* and $\mathbb{R}_S$ and it is naturally a left supermodule over $\mathbb{R}_S$. Between V and its bidual V^{**} there exists a canonical one-to-one left-$\mathbb{R}_S$-linear correspondence such that, if $v \in V$, $\underline{v} \in V^{**}$ and $a \in V^*$:

$$\begin{aligned}
v &\longleftrightarrow \underline{v} \\
\underline{v}(a) &= a(v)
\end{aligned}$$

Note that this correspondence is not the only legitimate one. One could also think of something like $\underline{v}(a) = (-1)^{|a||\underline{v}|}a(v)$ although the left-linearity would be affected. I will not follow this path.

So to each extended $1|0$-coform, we can associate a corresponding vector. In particular, if w is the extended $1|0$-coform which sends $p = dx^A p_A$ to p_B, we can associate to w the vector ∂_B hence, with a little abuse of notation, we can write $w = \partial_B$. We will see in the following that a similar notation can be used also for higher degree coforms.

With techniques analogous to the one used in section 5.3 it is then possible to build Berezinian coforms:

Theorem 73. *Let $(U, x^A, \overline{p_A}, \widetilde{p_A})$ be a local chart of T^*X, cotangent space of a $n|m$-dimensional manifold X; let $x \in U$, let $\pi := (\overline{p^1}, \cdots, \overline{p^{t+1}}; \widetilde{p^1}, \cdots, \widetilde{p^q}) \in \underbrace{T^*_{x,0}U \times \cdots T^*_{x,0}U}_{t+1} \times \underbrace{T^*_{x,1}U \times \cdots T^*_{x,1}U}_{q}$; let $\alpha_1 < \alpha_2 < \cdots < \alpha_q$ be q different odd indices chosen in the the set $\{n+1, \cdots, n+m\}$; let $A_1, A_2, \cdots, A_{t+1}$ be $t+1$ even or odd indices (possibly equal) chosen in the set $\{1, \cdots, n+m\}$; then the function*

$$w \ : \ O \subset \underbrace{T^*_{x,0}U \times \cdots T^*_{x,0}U}_{t+1} \times \underbrace{T^*_{x,1}U \times \cdots T^*_{x,1}U}_{q} \longrightarrow \mathbb{R}_S$$

defined by:

$$w(\pi) := \mathrm{rsdet}_{t+1,q} \begin{pmatrix} \overline{p^1_{A_{t+1}}} & \cdots & \overline{p^t_{A_{t+1}}} & \overline{p^{t+1}_{A_{t+1}}} & \widetilde{p^1_{A_{t+1}}} & \cdots & \widetilde{p^q_{A_{t+1}}} \\ \overline{p^1_{A_1}} & \cdots & \overline{p^t_{A_1}} & \overline{p^{t+1}_{A_1}} & \widetilde{p^1_{A_1}} & \cdots & \widetilde{p^q_{A_1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{p^1_{A_t}} & \cdots & \overline{p^t_{A_t}} & \overline{p^{t+1}_{A_t}} & \widetilde{p^1_{A_t}} & \cdots & \widetilde{p^q_{A_t}} \\ \overline{p^1_{\alpha_1}} & \cdots & \overline{p^t_{\alpha_1}} & \overline{p^{t+1}_{\alpha_1}} & \widetilde{p^1_{\alpha_1}} & \cdots & \widetilde{p^q_{\alpha_1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overline{p^1_{\alpha_q}} & \cdots & \overline{p^t_{\alpha_q}} & \overline{p^{t+1}_{\alpha_q}} & \widetilde{p^1_{\alpha_q}} & \cdots & \widetilde{p^q_{\alpha_q}} \end{pmatrix} \tag{6.6}$$

is a $t+1|q$-coform over $x \in U$, O being precisely the subset where the formula (6.6) *is well defined.*

The operator rsdet appearing in (6.6) is nothing else than the superdeterminant of the matrix defined with (5.1) but with a different definition of determinant. Precisely, instead of using the definition of determinant given with (5.2), when calculating a rsdet, the determinants are computed by respecting the order by raw instead of the order by column. We have then:

$$\mathrm{rightdet}\, G = \det G^T$$

where G is a matrix with entries, belonging to $\mathbb{R}_S$, of both parities and G^T is its transposed.

This obviously introduce a difference between sdet and rsdet when the entries of the supermatrix are not all even in the first block, which may happen with the "extended" supermatrices used here. The order used in (6.6) is necessary so that w satisfies (6.2).

We can call *Berezinian* the coform defined by (6.6), and we can write it in this way:

$$w = \frac{\partial_{A_{t+1}} \wedge \partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \tag{6.7}$$

More generally, by an equation analogous the equation (5.29), we can define a Berezinian $t|q$-coform noted as:

$$w = \frac{v_1 \wedge \cdots \wedge v_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}} \tag{6.8}$$

where $v_1, \ldots, v_t$ are vectors, even or odd, and $\widetilde{v_1}, \ldots, \widetilde{v_q}$ are odd vectors.

Proceeding on the way followed in section 5.3, we can extend a Berezinian coform in the first argument with (6.5) and we can contract a $t+1|q$-coform by an even or an odd covector obtaining a $t|q$-coform. We can therefore define *contracted Berezinian coforms* and *repeatedly contracted Berezinian coforms.*

The contractions are made with the help of proposition 72. One can easily see that the contraction of a $t+1|q$-coform by a covector $p = dx^F p_F$ corresponds to the application of the operator acting from the right:

$$i_p = (-1)^t \frac{\overleftarrow{\partial}}{\partial p_F^{t+1}} p_F \tag{6.9}$$

.

This operator can be written also as:

$$\begin{aligned} i_p w =& (-1)^t (-1)^{|F|(|w|+|F|)} (-1)^{(|F|+|p|)(|w|+|F|)} p_F \frac{\partial}{\partial p_F^{t+1}} w \\ =& (-1)^t (-1)^{|p|(|w|+|F|)} p_F \frac{\partial}{\partial p_F^{t+1}} w \end{aligned} \tag{6.10}$$

.

Note that the operator i_p is not equivalent to the operator $e(p)$ defined by Voronov in [154] (which acts between different spaces), although they are related one to the other one.

We can then define a wedge product, or exterior product, of a Berezinian, or a Berezinian contracted, or a generic coform by a vector; the wedge product increases by one the even codegree of the coform. We have the following:

Proposition 74. *Let X be a supermanifold, for every $v \in TX$, the operator*

$$e_v \ : \ \Omega_{t|q} X \longrightarrow \Omega_{t+1|q} X$$

defined on every $\theta \in \Omega_{t|q} X$ by:

$$e_v \theta := \theta \cdot \left[(-1)^{|v||\theta|} v^A \overline{p_A^{t+1}} - (-1)^{|v||\theta|} (-1)^{|v||F|} \frac{\overleftarrow{\partial}}{\partial p_B^F} \overline{p_B^{t+1}} v^A p_A^F \right] (-1)^t \tag{6.11}$$

is well defined.

The following formula holds:

$$i_p e_v (\theta) = -e_v i_p (\theta) + (-1)^{|v||\theta|} \theta v(p) \tag{6.12}$$

and

$$i_p e_v (\theta) = -e_v i_p (\theta) + (-1)^{|p||\theta|} v(p) \theta \tag{6.13}$$

Proof. I will prove the proposition only for Berezinian and repeatedly contracted Berezinian coforms.

I omit the proof that (6.11) does not depend on the choice of the coordinates.

The fact that e_v sends Berezinian $t|q$-coforms to Berezinian $t+1|q$-coforms can be checked by direct calculation with the use of a formula analogous to (5.11) and valid for the developing of right-superdeterminants.

Formula (6.12) and (6.13) are clearly equivalent and can be proved to be true by direct calculations for a generic $t|q$-coform θ. In fact, let $\pi = \left(\overline{p^1}, \ldots, \overline{p^t}; \widetilde{p^1}, \ldots, \widetilde{p^q}\right)$, then we have:

$$\begin{aligned} i_p e_v \theta(\pi) &= i_p (-1)^{|v||\theta|} (-1)^t \left[\theta(\pi) v^A \overline{p_A^{t+1}} - (-1)^{|v||F|} \theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_B^F} \overline{p_B^{t+1}} v^A p_A^F\right] \\ &= (-1)^{|v||\theta|} \left[\theta(\pi) v^A \overline{p_A^{t+1}} - (-1)^{|v||F|} \theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_B^F} \overline{p_B^{t+1}} v^A p_A^F\right] \frac{\overleftarrow{\partial}}{\partial p_C^{t+1}} p_C \\ &= (-1)^{|v||\theta|} \left[\theta(\pi) v^A p_A - (-1)^{|v||F|} \theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_B^F} p_B v^A p_A^F (-1)^{|p|(|v|+|F|)}\right] \end{aligned}$$

and

$$\begin{aligned} e_v i_p \theta(\pi) =& (-1)^{(|p|+|\theta|)|v|} \left[\left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t} p_C\right) v^A \overline{p_A^t} - (-1)^{|v||Z|} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t} p_C\right) \frac{\overleftarrow{\partial}}{\partial p_B^Z} \overline{p_B^t} v^A p_A^Z\right] \\ =& (-1)^{|p||v|} (-1)^{|v||\theta|} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t} p_C\right) v^A \overline{p_A^t} \\ &- (-1)^{|p||v|} (-1)^{|v||\theta|} (-1)^{|v||Z|} (-1)^{(|p|+|C|)(|Z|+|B|)} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t}\right) \frac{\overleftarrow{\partial}}{\partial p_B^Z} p_C \overline{p_B^t} v^A p_A^Z \\ =& (-1)^{|p||v|} (-1)^{|v||\theta|} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t} p_C\right) v^A \overline{p_A^t} \\ &+ (-1)^{|p||v|} (-1)^{|v||\theta|} (-1)^{|v||Z|} (-1)^{(|p|+|C|)(|Z|+|B|)} (-1)^{|B||C|+(|B|+|C|)|Z|} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_B^t}\right) \frac{\overleftarrow{\partial}}{\partial p_C^Z} p_C \overline{p_B^t} v^A p_A^Z \\ =& (-1)^{|p||v|} (-1)^{|v||\theta|} \left(\theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^t} p_C\right) v^A \overline{p_A^t} \\ &+ (-1)^{|p||v|} (-1)^{|v||\theta|} (-1)^{|v||Z|+|p||Z|} \theta(\pi) \frac{\overleftarrow{\partial}}{\partial p_C^Z} p_C v^A p_A^Z \end{aligned}$$

where Z runs from 1 to $t-1|q$ and where (6.3) has been used. So (6.12) is proved.

Then, by iterated application of contractions and exterior products, the proposition is proved for Berezinian and repeatedly contracted Berezinian coforms. ■

Formula (6.13) is the analogoues for the coforms of (5.59) for the forms and it allows to perform part of Cartan calculus for coforms.

The operator e_v can be written also as:

$$e_v \cdot w := (-1)^t \left[v^F \overline{p_F^{t+1}} - (-1)^{|F||A|} v^G p_G^A \overline{p_F^{t+1}} \frac{\partial}{\partial p_F^A}\right] \cdot w \tag{6.14}$$

where w is a coform of codegree $t|q$, and it therefore coincides with the operator defined in [154] by formula (16) (note that in that formula there is an imprecision since the exponent r of (-1) should be p).

It can be proved that, if w is defined by (6.8) and u is a $1|0$ coform corresponding to the vector u, then

$$e_u w = \frac{u \wedge v_1 \wedge \cdots \wedge v_p}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}}$$

So we can write:

$$u \wedge (\cdot) := e_u(\cdot)$$

However, Formula (6.13) is not as intuitive as (5.59) is. For example, if w is defined by (6.8), μ is a $1|0$-form and u is a vector, we can compute:

$$\begin{aligned} u \wedge (\mu \lrcorner w) &= -\mu \lrcorner (u \wedge w) + (-1)^{|w||u|} \mu(u)\, w \\ &= -\mu \lrcorner \frac{u \wedge v_1 \wedge \cdots \wedge v_p}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}} + (-1)^{|w||u|} \mu(u) \frac{v_1 \wedge \cdots \wedge v_p}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}} \end{aligned} \tag{6.15}$$

where $|w| = |v_1| + \cdots + |v_p|$.

Formula (6.15) would become a bit more transparent if we wrote the interior product in this way:

$$w \lrcorner \mu := \mu \lrcorner w$$

assuming the convention to put forms always on the right and coforms (or vectors) always on the left when performing interior products. Then (6.15) becomes:

$$(u \wedge w) \lrcorner \mu = (-1)^{|w||u|} \mu(u)\, w - u \wedge (w \lrcorner \mu)$$

We can therefore call fractional coforms the coforms belonging to the class built up starting from Berezinian coforms and applying repeatedly the wedge product by vectors and the interior product with covectors. Their degree can be defined with conventions analogous to the one used to fix the degree of fractional form in sections 5.3 and 5.4.

To define a full Cartan calculus on fractional coforms we still have to define an exterior derivative. To do so, we first have to define fractional mixed forms and then carry on the path drawn by Voronov in [153] and [154]. We will do so at the end of the next section.

6.2 Mixed fractional forms on supermanifolds

We can define in a natural way the contraction of a Berezinian form by a Berezinian coform and of a Berezinian coform by a Berezinian form, so that the contraction is a symmetric pairing between forms and coforms.

Let's first consider the case when the degree of the Berezinian form is equal to the codegree of the Berezinian coform.

Let's see an example. Let be $\omega = \frac{\Theta}{\theta^1 \odot \theta^2}$ and $w = \frac{u}{\widetilde{v_1} \odot \widetilde{v_2}}$, with $|\Theta| = 1$ and $|u| = 0$; then we pose:

$$w \lrcorner \omega := \operatorname{sdet}_{1,2} \begin{pmatrix} \Theta(u) & \theta^1(u) & \theta^2(u) \\ \Theta(\widetilde{v_1}) & \theta^1(\widetilde{v_1}) & \theta^2(\widetilde{v_1}) \\ \Theta(\widetilde{v_2}) & \theta^1(\widetilde{v_2}) & \theta^2(\widetilde{v_2}) \end{pmatrix} \tag{6.16}$$

$$\omega \lrcorner w = \operatorname{rsdet}_{1,2} \begin{pmatrix} u(\Theta\eta) & u(\theta^1) & u(\theta^2) \\ \widetilde{v_1}(\Theta\eta) & \widetilde{v_1}(\theta^1) & \widetilde{v_1}(\theta^2) \\ \widetilde{v_2}(\Theta\eta) & \widetilde{v_2}(\theta^1) & \widetilde{v_2}(\theta^2) \end{pmatrix} \frac{\overleftarrow{\partial}}{\partial \eta} \tag{6.17}$$

where for every $1|0$-form μ and for every vector v we pose $v(\mu) := \mu(v)$, which is coherent with the considerations made at the beginning of the previous section. Then we can easily see that $w \lrcorner \omega = \omega \lrcorner w$.

In general, if $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$, if τ is an other $r|s$-form and $w = \frac{u_1 \wedge \cdots \wedge u_r}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_s}}$ and x is an other $r|s$-coform, we pose:

$$
\begin{aligned}
w \lrcorner \tau &:= \widehat{\tau}\left(u_1, \cdots, u_r; \widetilde{v_1}, \cdots, \widetilde{v_s}\right) \\
\omega \lrcorner x &:= \widehat{x}\left(\Theta^1, \cdots, \Theta^r; \theta^1, \cdots, \theta^s\right) \\
w \lrcorner \omega &:= \widehat{\omega}\left(u_1, \cdots, u_r; \widetilde{v_1}, \cdots, \widetilde{v_s}\right) \\
\omega \lrcorner w &:= \widehat{w}\left(\Theta^1, \cdots, \Theta^r; \theta^1, \cdots, \theta^s\right)
\end{aligned}
$$

where the symbol $\widehat{\ }$ indicates the extension of the form or of the coform in the arguments for which it needs to be extended.

Then it is possible to prove that $w \lrcorner \omega = \omega \lrcorner w$.

What happens when we try to contract a form with a coform and the degree of the form and the codegree of the coform do not coincide? If $\omega \in \Omega^{r|s}$ and $w = \frac{u_1 \wedge \cdots \wedge u_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}}$, with $t \leq r$ and $q \leq s$, then we can define:

$$
\omega \lrcorner w\left(\cdot\right) := w \lrcorner \omega\left(\cdot\right) := \widehat{\omega}\left(u_1, \cdots, u_t, \cdot; \widetilde{v_1}, \cdots, \widetilde{v_q}, \cdot\right)
$$

and, with considerations similar to the one used in the proof of Lemma (48), we can prove that $w \lrcorner \omega\left(\cdot\right)$ is a $r-t|s-q$-form.

Analogously: if $w \in \Omega_{t|q}$ and $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$, with $r \leq t$ and $s \leq q$, then we can define:

$$
w \lrcorner \omega\left(\cdot\right) := \omega \lrcorner w\left(\cdot\right) := \widehat{w}\left(\Theta^1, \cdots, \Theta^r, \cdot; \theta^1, \cdots, \theta^s, \cdot\right)
$$

and, with similar considerations, we can prove that $\omega \lrcorner w\left(\cdot\right)$ is a $t-r|q-s$-coform.

Note that, if μ is a $1|0$-form, then the operator $\mu \wedge (\cdot)$ coincides with the operator i_μ defined by (6.9).

Other questions are: what if $\omega \in \Omega^{r|s}$ and $w = \frac{u_1 \wedge \cdots \wedge u_t}{\widetilde{v_1} \odot \cdots \odot \widetilde{v_q}}$, with $q \leq s$ but $r < t$? What if $w \in \Omega_{t|q}$ and $\omega = \frac{\Theta^1 \wedge \cdots \wedge \Theta^r}{\theta^1 \odot \cdots \odot \theta^s}$, with $s \leq q$ but $t < r$?

We may extend the interior product to these new cases by defining new kind of objects. In particular we will see, in the following part of this section, that the last case will give rise to mixed fractional forms, and I will show that they are a special subclass of the mixed forms defined by Voronov in [153] and [154].

Let's first see what Voronov mixed forms are.

Let's consider a supermanifold X of dimension $n|m$ and a superspace $\mathbb{R}^{r|s}$. The space $M = X \times \mathbb{R}^{r|s}$ has a natural structure of supermanifold of dimension $n+r|m+s$. Its tangent space TM is isomorphic to $TX \oplus T\mathbb{R}^{r|s}$. We have as well that $T^*M \cong T^*X \oplus T\mathbb{R}^{r|s\,*}$.

Voronov in [153] and [154] gives the definition of mixed superforms on a supermanifold X making implicitly use of the auxiliary supermanifold M. Here in the following I give a definition of mixed superforms which is a variant to the one given in [154], adapted to my notation.

Definition 75. *Let X be a supermanifold of dimension $n|m$ and M be the supermanifold $M = X \times \mathbb{R}^{r|s}$. Let $x \in X$ and (with a little abuse of notation) let's call x also the point of M defined by $x := (x, 0)$.*

A mixed form of codegree $t|q$ and additional degree $r|s$ over a point $x \in X$ is a G^∞ map $w: \ O \subset \underbrace{T^*_{x,0}M \times \cdots T^*_{x,0}M}_{t} \times \underbrace{T^*_{x,1}M \times \cdots T^*_{x,1}M}_{q} \longrightarrow \mathbb{R}_S$, which, $\forall p \in O_X$ open subset of $\underbrace{T^*_{x,0}X \times \cdots T^*_{x,0}X}_{t} \times \underbrace{T^*_{x,1}X \times \cdots T^*_{x,1}X}_{q}$, $\forall o \in \underbrace{\left(\mathbb{R}^{r|s\,*}\right)_0 \times \cdots \left(\mathbb{R}^{r|s\,*}\right)_0}_{t} \times \underbrace{\left(\mathbb{R}^{r|s\,*}\right)_1 \times \cdots \left(\mathbb{R}^{r|s\,*}\right)_1}_{q}$, satisfies the following conditions:

–
$$\forall g \in GL(t|q),\ w\left[\begin{pmatrix} p \\ o \end{pmatrix} \cdot g\right] = w\left[\begin{pmatrix} p \\ o \end{pmatrix}\right] \mathrm{Ber}_{t,q}(g) \tag{6.18}$$

–
$$\forall h \in GL(r|s),\ \forall a \in Mat(r|s \times n|m) \text{ with entries } a_F^K,\ \text{with parities } \left|a_F^K\right| = |F| + |K|$$
$$w\left[\begin{pmatrix} p + a \cdot o \\ h \cdot o \end{pmatrix}\right] = \mathrm{Ber}_{t,q}(h) w\left[\begin{pmatrix} p \\ o \end{pmatrix}\right] \tag{6.19}$$

–
$$\frac{\partial^2 w}{\partial p_G^B \partial p_F^A} + (-1)^{|G||F| + (|G|+|F|)|A|} \frac{\partial^2 w}{\partial p_F^B \partial p_G^A} = 0 \tag{6.20}$$

–
$$\frac{\partial^2 w}{\partial p_G^B \partial o_K^A} + (-1)^{|G||K| + (|G|+|K|)|A|} \frac{\partial^2 w}{\partial o_K^B \partial p_G^A} = 0 \tag{6.21}$$

–
$$\frac{\partial^2 w}{\partial o_L^B \partial o_K^A} + (-1)^{|L||K| + (|L|+|K|)|A|} \frac{\partial^2 w}{\partial o_K^B \partial o_L^A} = 0 \tag{6.22}$$

where $\begin{pmatrix} p \\ o \end{pmatrix} \in \underbrace{T^*_{x,0}M \times \cdots T^*_{x,0}M}_{t} \times \underbrace{T^*_{x,1}M \times \cdots T^*_{x,1}M}_{q}$*; where* x^F *are the coordinates on* X *and* y^K *are the coordinates on* $\mathbb{R}^{r|s}$*; where* $F, G = 1, \cdots, n+m$ *are also the indices in the space* $T^*_x X$ *and so also in both spaces* $T^*_{x,0}X$ *and* $T^*_{x,1}X$ *with their usual degree;* p_F^A *is the* F*-th coordinates of* p^A *in the local base* $\left(dx^F|_x\right)_F$*;* $K, L = 1, \cdots, r+s$ *are the indices in the spaces* $\mathbb{R}^{r|s\,*}$, $\mathbb{R}^{r|s\,*}$ *and so also in both spaces* $\mathbb{R}_0^{r|s\,*}$ *and* $\mathbb{R}_1^{r|s\,*}$ *with their usual degree;* o_K^A *is the* K*-th coordinates of* o^A *in the local base* $\left(dy^K\right)_K$*;* A *and* B *run from 1 to* $t+q$ *and we have* $p^A \in T^*_{x,|A|}X$ *and* $o^A \in \mathbb{R}^{r|s\,*}_{|A|}$*, where* $|A| = 0$ *when* $A = 1, \cdots, t$ *and* $|A| = 1$ *when* $A = t+1, \cdots, t+q$.

The space of mixed forms of codegree $t|q$ and additional degree $r|s$ over a point $x \in X$ is indicated with $\Lambda^{r|s}_{t|q}$ as a shortcut for $\Lambda^{r|s}_{t|q} T_x X$.

We can easily define the vector bundle $\Lambda^{r|s}_{t|q} TX$ over the base X. Its G^∞ sections are the differential mixed forms of codegree $t|q$ and additional degree $r|s$. Note that they are sections of a bundle whose base is X and not M. This will be important for the definition of their integral in section 6.3.

The space of differential mixed forms of codegree $t|q$ and additional degree $r|s$ of a manifold X is indicated by $\Omega^{r|s}_{t|q} X$.

Conditions (6.18), (6.20), (6.21) and (6.22) tell us that a mixed form of codegree $t|q$ and additional degree $r|s$ over the point x of the manifold X is a coform of codegree $t|q$ over the point $(x, 0)$ of the manifold $M = X \times \mathbb{R}^{r|s}$ which also has to satisfy (6.19). To every differential mixed form w of codegree $t|q$ and additional degree $r|s$ over the manifold X, we can easily associates a differential coform $\breve{w}$ of codegree $t|q$ over the manifold $M = X \times \mathbb{R}^{r|s}$ which do not depends on the coordinates y^K of $\mathbb{R}^{r|s}$ and which also satisfies (6.19).

If one looks for fractional mixed forms on X, then one has to look for fractional coform on M which satisfy (6.19). Indeed we have that:

Proposition 76. *Let* (U, x^A, p_F) *be a local chart of* T^*X*, cotangent space of a* $n|m$*-dimensional manifold* X*; let* $\left(y^K\right)$ *be coordinates on* $\mathbb{R}^{r|s}$ *and* (o_K) *be coordinates on* $\mathbb{R}^{r|s\,*}$*; let* $x \in U$*;*

let $\pi := (\overline{p^1}, \cdots, \overline{p^{t+r}}; \widetilde{p^1}, \cdots, \widetilde{p^{q+s}}) \in \underbrace{T^*_{x,0}U \times \cdots T^*_{x,0}U}_{t+r} \times \underbrace{T^*_{x,1}U \times \cdots T^*_{x,1}U}_{q+s}$*;*
let $o := (\overline{o^1}, \cdots, \overline{o^{t+r}}; \widetilde{o^1}, \cdots, \widetilde{o^{q+s}}) \in \underbrace{\mathbb{R}_0^{r|s\,*} \times \cdots \times \mathbb{R}_0^{r|s\,*}}_{t+r} \times \underbrace{\mathbb{R}_1^{r|s\,*} \times \cdots \mathbb{R}_1^{r|s\,*}}_{q+s}$*;*
let $\alpha_1 < \alpha_2 < \cdots < \alpha_q$ *be* q *different odd indices chosen in the the set* $\{n+1, \cdots, n+m\}$*; let* $A_1, A_2, \cdots, A_t$ *be* t *even or odd indices (possibly equal) chosen in the set* $\{1, \cdots, n+m\}$*;*
then the function

$$w \;:\; O \subset \underbrace{T^*_{(x,0),0}M \times \cdots T^*_{(x,0),0}M}_{t+r} \times \underbrace{T^*_{(x,0),1}M \times \cdots T^*_{(x,0),1}M}_{q+s} \longrightarrow \mathbb{R}_S$$

$$w = \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}}$$

defined by:

$$w\begin{pmatrix}\pi \\ o\end{pmatrix} := \mathrm{rsdet}_{t+r,q+s}\begin{pmatrix}
\overline{p^1_{A_1}} & \cdots & \overline{p^{t+r}_{A_1}} & \widetilde{p^1_{A_1}} & \cdots & \widetilde{p^{q+s}_{A_1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{p^1_{A_t}} & \cdots & \overline{p^{t+r}_{A_t}} & \widetilde{p^1_{A_t}} & \cdots & \widetilde{p^{q+s}_{A_t}} \\
\overline{o^1_1} & \cdots & \overline{o^{t+r}_1} & \widetilde{o^1_1} & \cdots & \widetilde{o^{q+s}_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{o^1_r} & \cdots & \overline{o^{t+r}_r} & \widetilde{o^1_r} & \cdots & \widetilde{o^{q+s}_r} \\
\overline{p^1_{\alpha_1}} & \cdots & \overline{p^{t+r}_{\alpha_1}} & \widetilde{p^1_{\alpha_1}} & \cdots & \widetilde{p^{q+s}_{\alpha_1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{p^1_{\alpha_q}} & \cdots & \overline{p^{t+r}_{\alpha_q}} & \widetilde{p^1_{\alpha_q}} & \cdots & \widetilde{p^{q+s}_{\alpha_q}} \\
\overline{o^1_{r+1}} & \cdots & \overline{o^{t+r}_{r+1}} & \widetilde{o^1_{r+1}} & \cdots & \widetilde{o^{q+s}_{r+1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{o^1_{r+s}} & \cdots & \overline{o^{t+r}_{r+s}} & \widetilde{o^1_{r+s}} & \cdots & \widetilde{o^{q+s}_{r+s}}
\end{pmatrix} \tag{6.23}$$

is a mixed form of codegree $t+r|q+s$ *and additional degree* $r|s$ *over* $x \in U$*,* O *being precisely the subset where the formula* (6.23) *is well defined.*

Proof. By theorem 73, we are sure that w, defined by (6.23), satisfies (6.18), (6.20), (6.21) and (6.22). We still have to prove that it satisfies (6.19).

By the definition of w, we have that

$$w\left[\begin{pmatrix}\pi + a \cdot o \\ h \cdot o\end{pmatrix}\right] = w\left[\begin{pmatrix}id & a \\ 0 & h\end{pmatrix} \cdot \begin{pmatrix}\pi \\ o\end{pmatrix}\right]$$

Then, by direct calculation, using the properties of the rsdet and keeping into account of the specific parities of the elements of the matrices involved, one can see that

$$w\left[\begin{pmatrix}id & a \\ 0 & h\end{pmatrix} \cdot \begin{pmatrix}\pi \\ o\end{pmatrix}\right] = \mathrm{Ber}_{r,s}(h) \cdot w\left[\begin{pmatrix}\pi \\ o\end{pmatrix}\right]$$

and the theorem is proved. ∎

Note that the condition that all ∂_y do appear in the definition of ω, without repetitions and with all odd ∂_y at denominator, is precisely the condition which ensures that (6.19) is satisfied for a Berezinian coforms defined on M.

For a general mixed forms w of codegree $t|q$ and additional degree $r|s$, we can define an exterior product by a $1|0$-coform v, sending w to the mixed form $e_v w$ of codegree $t+1|q$ and additional degree $r|s$:

$$e_v \cdot w := (-1)^t \left[v^F \overline{p_F^{t+1}} - (-1)^{|F||A|} v^G p_G^A \overline{p_F^{t+1}} \frac{\partial}{\partial p_F^A} - (-1)^{|K||A|} v^G p_G^A \overline{o_K^{t+1}} \frac{\partial}{\partial o_K^A} \right] \cdot w \tag{6.24}$$

See [154] for the proof that e_v is well defined.

We can prove that, if w is defined by (6.23), so that $w = \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}}$, then:

$$\begin{aligned} e_v \cdot w =& e_v \cdot \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}} \\ =& \frac{v \wedge \partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}} \end{aligned}$$

So in general we can define, for a generic $w \in \Omega_{t|q}^{r|s}$:

$$v \wedge w := e_v \cdot w \tag{6.25}$$

Voronov, [154] theorem 1.4, has proved that:

$$e_v e_u = -(-1)^{|v||u|} e_u e_v \tag{6.26}$$

This is consistent with the definition given with (6.25).

For a general mixed forms w of codegree $t+1|q$ and additional degree $r|s$, we can define an interior product by a $1|0$-form μ:

$$\mu \lrcorner w := i_\mu w := (-1)^t w \frac{\overleftarrow{\partial}}{\partial p_F^{t+1}} \mu_F \tag{6.27}$$

.

This operator is the analogue of the operator defined with (6.9) and (6.10) and it can be written also as:

$$\begin{aligned} i_\mu w =& (-1)^t (-1)^{|F|(|w|+|F|)} (-1)^{(|F|+|\mu|)(|w|+|F|)} \mu_F \frac{\partial}{\partial p_F^{t+1}} w \\ =& (-1)^t (-1)^{|\mu|(|w|+|F|)} \mu_F \frac{\partial}{\partial p_F^{t+1}} w \end{aligned} \tag{6.28}$$

.

The operator i_α is related to the operator $e(\alpha)$ defined in [154], formula (14), but it is not equivalent to it. In fact, if w is a mixed form of codegree $t|q$ and additional degree $r|s$, we have:

$$i_\alpha w = (-1)^{t-1} (-1)^r (-1)^{|\alpha||w|} \sigma_{1|0}^{-1} e(\alpha) w$$

where $\sigma_{1|0}^{-1}$ is the homomorphism defined below.

It is immediate to prove that:

$$i_\alpha i_\beta = -(-1)^{|\alpha||\beta|} i_\beta i_\alpha \tag{6.29}$$

It is also possible to prove that:

$$i_\alpha e_v\left(w\right)=-e_v i_\alpha\left(w\right)+\left(-1\right)^{|v||w|}wv\left(\alpha\right)=-e_v i_\alpha\left(w\right)+\left(-1\right)^{|\alpha||w|}v\left(\alpha\right)w \tag{6.30}$$

which is the generalization to mixed forms of (6.12) and (6.13).

Remembering the considerations made at the beginning of this section, we also find out that:

Proposition 77. *Let (U,x^A,p_F) be a local chart of T^*X, cotangent space of a $n|m$-dimensional manifold X; let $\left(y^K\right)$ be coordinates on $\mathbb{R}^{r|s}$ and (o_K) be coordinates on $\mathbb{R}^{r|s\,*}$; let $x\in U$;*
let $\pi:=(\overline{p^1},\cdots,\overline{p^{t+r-l}};\widetilde{p^1},\cdots,\widetilde{p^{q+s-d}})\in\underbrace{T^*_{x,0}U\times\cdots T^*_{x,0}U}_{t+r-l}\times\underbrace{T^*_{x,1}U\times\cdots T^*_{x,1}U}_{q+s-d}$;
let $o:=(\overline{o^1},\cdots,\overline{o^{t+r-l}};\widetilde{o^1},\cdots,\widetilde{o^{q+s-d}})\in\underbrace{\mathbb{R}_0^{r|s\,*}\times\cdots\times\mathbb{R}_0^{r|s\,*}}_{t+r-l}\times\underbrace{\mathbb{R}_1^{r|s\,*}\times\cdots\mathbb{R}_1^{r|s\,*}}_{q+s-d}$;
let $\alpha_1<\alpha_2<\cdots<\alpha_q$ be q different odd indices chosen in the the set $\{n+1,\cdots,n+m\}$; let $A_1,A_2,\cdots,A_t$ be t even or odd indices (possibly equal) chosen in the set $\{1,\cdots,n+m\}$;
let $\Theta^1,\ldots,\Theta^l$ be $1|0$-forms on X of any parity and let $\theta^1,\ldots,\theta^d$ be odd $1|0$-forms on X, with $l\le r$ and $d\le s$; then the function

$$\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w\ :\ O\subset\underbrace{T^*_{(x,0),0}M\times\cdots T^*_{(x,0),0}M}_{t+r-l}\times\underbrace{T^*_{(x,0),1}M\times\cdots T^*_{(x,0),1}M}_{q+s-d}\longrightarrow\mathbb{R}_S$$

$$\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w=\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}\wedge\partial_{y^1}\wedge\cdots\wedge\partial_{y^r}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}\odot\partial_{y^{r+1}}\odot\cdots\odot\partial_{y^{r+s}}}$$

defined by:

$$\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w\begin{pmatrix}\pi\\ o\end{pmatrix}:=(-1)^{\frac{1}{2}\left[\left(\sum_{i=1}^{l}|\Theta^i|\right)^2+\sum_{i=1}^{l}|\Theta^i|\right]}\mathrm{rsdet}_{r+t,s+q}$$

$$\begin{pmatrix}
\Theta^1_{A_1}(\eta_l)^{|\Theta^1|} & \cdots & \Theta^l_{A_1}(\eta_1)^{|\Theta^l|} & \overline{p^1_{A_1}} & \cdots & \overline{p^{t+r-l}_{A_1}} & \theta^1_{A_1} & \cdots & \theta^d_{A_1} & \widetilde{p^1_{A_1}} & \cdots & \widetilde{p^{q+s-d}_{A_1}}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\Theta^1_{A_t}(\eta_l)^{|\Theta^1|} & \cdots & \Theta^l_{A_t}(\eta_1)^{|\Theta^l|} & \overline{p^1_{A_t}} & \cdots & \overline{p^{t+r-l}_{A_t}} & \theta^1_{A_t} & \cdots & \theta^d_{A_t} & \widetilde{p^1_{A_t}} & \cdots & \widetilde{p^{q+s-d}_{A_t}}\\
0 & \cdots & 0 & \overline{o^1_1} & \cdots & \overline{o^{t+r-l}_1} & 0 & \cdots & 0 & \widetilde{o^1_1} & \cdots & \widetilde{o^{q+s-d}_1}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \overline{o^1_r} & \cdots & \overline{o^{t+r-l}_r} & 0 & \cdots & 0 & \widetilde{o^1_r} & \cdots & \widetilde{o^{q+s-d}_r}\\
\Theta^1_{\alpha_1}(\eta_l)^{|\Theta^1|} & \cdots & \Theta^l_{\alpha_1}(\eta_1)^{|\Theta^l|} & \overline{p^1_{\alpha_1}} & \cdots & \overline{p^{t+r-l}_{\alpha_1}} & \theta^1_{\alpha_1} & \cdots & \theta^d_{\alpha_1} & \widetilde{p^1_{\alpha_1}} & \cdots & \widetilde{p^{q+s-d}_{\alpha_1}}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\Theta^1_{\alpha_q}(\eta_l)^{|\Theta^1|} & \cdots & \Theta^l_{\alpha_q}(\eta_1)^{|\Theta^l|} & \overline{p^1_{\alpha_q}} & \cdots & \overline{p^{t+r-l}_{\alpha_q}} & \theta^1_{\alpha_q} & \cdots & \theta^d_{\alpha_q} & \widetilde{p^1_{\alpha_q}} & \cdots & \widetilde{p^{q+s-d}_{\alpha_q}}\\
0 & \cdots & 0 & \overline{o^1_{r+1}} & \cdots & \overline{o^{t+r-l}_{r+1}} & 0 & \cdots & 0 & \widetilde{o^1_{r+1}} & \cdots & \widetilde{o^{q+s-d}_{r+1}}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \overline{o^1_{r+s}} & \cdots & \overline{o^{t+r-l}_{r+s}} & 0 & \cdots & 0 & \widetilde{o^1_{r+s}} & \cdots & \widetilde{o^{q+s-d}_{r+s}}
\end{pmatrix}\left(\frac{\overleftarrow{\partial}}{\partial\eta_1}\right)^{|\Theta^l|}\cdots \tag{6.31}$$

is a mixed form of codegree $t+r-l|q+s-d$ and additional degree $r|s$ over $x\in U$, O being precisely the subset where the formula (6.23) is well defined.

Proof. By theorem 73 and by what was said at the beginning of this section about the contraction of extended coforms by forms, we are sure that w, defined by (6.31), satisfies (6.18), (6.20), (6.21) and (6.22). We still have to prove that it satisfies (6.19).

By the definition of w, we have that

$$w\left[\begin{pmatrix}\pi + a\cdot o\\ h\cdot o\end{pmatrix}\right] = w\left[\begin{pmatrix}id & a\\ 0 & h\end{pmatrix}\cdot\begin{pmatrix}\pi\\ o\end{pmatrix}\right]$$

Then, again, by direct calculation, using the properties of the rsdet and keeping into account of the specific parities of the elements of the matrices involved, one can see that

$$w\left[\begin{pmatrix}id & a\\ 0 & h\end{pmatrix}\cdot\begin{pmatrix}\pi\\ o\end{pmatrix}\right] = \mathrm{Ber}_{r,s}(h)\cdot w\left[\begin{pmatrix}\pi\\ o\end{pmatrix}\right]$$

and the theorem is proved. ∎

Note that in this last case the conditions which ensure that (6.19) is satisfied are two: first it is necessary that all ∂_y do appear in the definition of ω, without repetitions and with all odd ∂_y at denominator, and second it is necessary that the forms Θ^i and θ^j are defined on X and not on all M. In other words, if we consider the forms Θ^i and θ^j as forms on M, then their projections on the space $\mathbb{R}^{r|s\,*}$ must be null.

Voronov in [154] define two series of homomorphisms:

Definition 78. *Let (U, x^A, p_F) be a local chart of T^*X, cotangent space of a $n|m$-dimensional manifold X; let $x \in U$;*
let $M = X\times\mathbb{R}^{r|s}$, $N = X\times\mathbb{R}^{r|s}\times\mathbb{R}^{a|b}$.
With a little abuse of notation, let's also set $x = (x, 0) \in M$ and $x = (x, 0, 0) \in N$;

let $\pi^1 := (\overline{p^1}, \cdots, \overline{p^t}; \widetilde{p^1}, \cdots, \widetilde{p^q}) \in \underbrace{T^*_{x,0}U\times\cdots T^*_{x,0}U}_{t}\times\underbrace{T^*_{x,1}U\times\cdots T^*_{x,1}U}_{q}$;

let $\rho^1_1 := (\overline{r^1}, \cdots, \overline{r^t}; \widetilde{r^1}, \cdots, \widetilde{r^q}) \in \underbrace{\mathbb{R}^{r|s\,*}_0\times\cdots\times\mathbb{R}^{r|s\,*}_0}_{t}\times\underbrace{\mathbb{R}^{r|s\,*}_1\times\cdots\mathbb{R}^{r|s\,*}_1}_{q}$;

let $\pi^2 := (\overline{p^{t+1}}, \cdots, \overline{p^{t+a}}; \widetilde{p^{q+1}}, \cdots, \widetilde{p^{q+b}}) \in \underbrace{T^*_{x,0}U\times\cdots T^*_{x,0}U}_{a}\times\underbrace{T^*_{x,1}U\times\cdots T^*_{x,1}U}_{b}$;

let $\rho^2_1 := (\overline{r^{t+1}}, \cdots, \overline{r^{t+a}}; \widetilde{r^{q+1}}, \cdots, \widetilde{r^{q+b}}) \in \underbrace{\mathbb{R}^{r|s\,*}_0\times\cdots\times\mathbb{R}^{r|s\,*}_0}_{a}\times\underbrace{\mathbb{R}^{r|s\,*}_1\times\cdots\mathbb{R}^{r|s\,*}_1}_{b}$;

let $\rho^1_2 := (\overline{o^1}, \cdots, \overline{o^t}; \widetilde{o^1}, \cdots, \widetilde{o^q}) \in \underbrace{\mathbb{R}^{a|b\,*}_0\times\cdots\times\mathbb{R}^{a|b\,*}_0}_{t}\times\underbrace{\mathbb{R}^{a|b\,*}_1\times\cdots\mathbb{R}^{a|b\,*}_1}_{q}$;

let $\rho^2_2 := (\overline{o^{t+1}}, \cdots, \overline{o^{t+a}}; \widetilde{o^{q+1}}, \cdots, \widetilde{o^{q+b}}) \in \underbrace{\mathbb{R}^{a|b\,*}_0\times\cdots\times\mathbb{R}^{a|b\,*}_0}_{a}\times\underbrace{\mathbb{R}^{a|b\,*}_1\times\cdots\mathbb{R}^{a|b\,*}_1}_{b}$;

let $\begin{pmatrix}\pi^1\\ \rho^1_1\end{pmatrix} \in \underbrace{T^*_{x,0}M\times\cdots T^*_{x,0}M}_{t}\times\underbrace{T^*_{x,1}M\times\cdots T^*_{x,1}M}_{q}$;

let $\begin{pmatrix}\pi^1\\ \rho^1_1\\ \rho^1_2\end{pmatrix} \in \underbrace{T^*_{x,0}N\times\cdots T^*_{x,0}N}_{t}\times\underbrace{T^*_{x,1}N\times\cdots T^*_{x,1}N}_{q}$;

let $\begin{pmatrix}\pi^2\\ \rho^2_1\\ \rho^2_2\end{pmatrix} \in \underbrace{T^*_{x,0}N\times\cdots T^*_{x,0}N}_{a}\times\underbrace{T^*_{x,1}N\times\cdots T^*_{x,1}N}_{b}$;

then:

$$\begin{aligned} \sigma_{a|b} :\ & \Lambda^{r|s}_{t|q} \longrightarrow \Lambda^{r+a|s+b}_{t+a|q+b} \\ \sigma^{-1}_{a|b} :\ & \Lambda^{r+a|s+b}_{t+a|q+b} \longrightarrow \Lambda^{r|s}_{t|q} \end{aligned} \tag{6.32}$$

are defined by:

$$\sigma_{a|b}\zeta \begin{pmatrix} \pi^1 & \pi^2 \\ \rho^1_1 & \rho^2_1 \\ \rho^1_2 & \rho^2_2 \end{pmatrix} := \zeta \begin{pmatrix} \pi^1 - \pi^2 \left(\rho^2_2\right)^{-1} \rho^1_2 \\ \rho^1_1 - \rho^2_1 \left(\rho^2_2\right)^{-1} \rho^1_2 \end{pmatrix} \mathrm{Ber}(\rho^2_2) \tag{6.33}$$

$$\sigma^{-1}_{a|b}\lambda \begin{pmatrix} \pi^1 \\ \rho^1_1 \end{pmatrix} := \lambda \begin{pmatrix} \pi^1 & 0 \\ \rho^1_1 & 0 \\ 0 & id \end{pmatrix} \tag{6.34}$$

where $\zeta \in \Lambda^{r|s}_{t|q}$ *and* $\lambda \in \Lambda^{r+a|s+b}_{t+a|q+b}$.

We have that:

Theorem 79 (Voronov 1999)**.** *The maps* $\sigma_{a|b}$ *and* $\sigma^{-1}_{a|b}$ *defined bye* (6.33) *and* (6.34) *are indeed well defined homomorphisms between* $\Lambda^{r|s}_{t|q}$ *and* $\Lambda^{r+a|s+b}_{t+a|q+b}$*; moreover we have that* $\sigma^{-1}_{a|b}$ *is the inverse of* $\sigma_{a|b}$.

Proof. See the proof of theorem 1.1 in [154]. ■

The morphism $\sigma^{-1}_{a|b}$ acts in a very simple way on the fractional mixed forms defined by (6.23) and (6.31). We have in fact the following two:

Proposition 80. *If* w *is the mixed fractional form of codegree* $t+r|q+s$ *and additional degree* $r|s$ *defined by* (6.23)*; if* $a < r$, $b < q$ *and if* $\sigma^{-1}_{a|b}$ *is the morphism between* $\Lambda^{r|s}_{t+r|q+s}$ *and* $\Lambda^{t+r-a|q+s-b}_{t+a|q+b}$ *defined by* (6.34)*, then we have that:*

$$\sigma^{-1}_{a|b}w = \sigma^{-1}_{a|b}\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}\wedge\partial_{y^1}\wedge\cdots\wedge\partial_{y^r}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}\odot\partial_{y^{r+1}}\odot\cdots\odot\partial_{y^{r+s}}} = \frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}\wedge\partial_{y^1}\wedge\cdots\wedge\partial_{y^{r-a}}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}\odot\partial_{y^{r+1}}\odot\cdots\odot\partial_{y^{r+s-b}}} \tag{6.35}$$

If $a = r$ *and* $b = s$*, then*

$$\sigma^{-1}_{r|s}w = \sigma^{-1}_{r|s}\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}\wedge\partial_{y^1}\wedge\cdots\wedge\partial_{y^r}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}\odot\partial_{y^{r+1}}\odot\cdots\odot\partial_{y^{r+s}}} = \frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}} \tag{6.36}$$

Proof. It is easy by direct calculation. ■

Proposition 81. *If* $\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w$ *is the mixed fractional form of codegree* $t+r-l|q+s-d$ *and additional degree* $r|s$ *defined by* (6.31)*; if* $a < r$, $a \le t+r-l$, $b < s$, $b \le q+s-d$ *and if* $\sigma^{-1}_{a|b}$ *is the morphism defined by* (6.34)*, then:*

$$\sigma^{-1}_{a|b}\left(\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w\right) = \frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}\wedge\partial_{y^1}\wedge\cdots\wedge\partial_{y^{r-a}}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}\odot\partial_{y^{r+1}}\odot\cdots\odot\partial_{y^{r+s-b}}} \tag{6.37}$$

if $a = r$, $a \le t+r-l$, $b = s$, $b \le q+s-d$, *id est if* $a = r$, $t-l \ge 0$, $b = s$, $q-d \ge 0$, *then:*

$$\sigma^{-1}_{a|b}\left(\frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner w\right) = \frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}\lrcorner\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}} \tag{6.38}$$

Proof. Again it can be easily done by direct calculation. ∎

Following an idea of Voronov, [154], I want to define the space of stable mixed forms of codegree $t|q$ as the direct limit of the spaces of mixed forms with codegree $t+r|q+s$ and additional degree $r|s$ for $r \to \infty$ and $s \to \infty$. In this way the stable mixed form can have negative even codegree.

Note that I use the name "stable mixed forms" and not stable forms. I will briefly explain at the end of this section what are the objects called by Voronov "stable forms".

For example the fractional mixed form:

$$\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner w = \frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}}$$

has a representative in the space of stable mixed forms and it is:

$$\lim_{a\to\infty,\, b\to\infty} \sigma_{a|b} \left(\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}} \right)$$

After what we have seen with propositions 80 and 81, I suggest to denote that representative in this way:

$$\begin{aligned} &\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \\ &:= \lim_{a\to\infty,\, b\to\infty} \sigma_{a|b} \left(\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^r}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q} \odot \partial_{y^{r+1}} \odot \cdots \odot \partial_{y^{r+s}}} \right) \end{aligned} \tag{6.39}$$

Formula (6.39) is a definition of a notation and it is valid even when $l > t$.

This notation suggests however a way to extend the interior product between forms and coforms also to the case when the even degree of the form exceed the even codegree of the coform. In fact, if when $l > t$ and $d < q$, we can pose:

Definition 82. *The interior product between the Berezinian form $\omega = \frac{\Theta^1\wedge\cdots\wedge\Theta^l}{\theta^1\odot\cdots\odot\theta^d}$ and the Berezinian coform $\frac{\partial_{A_1}\wedge\cdots\wedge\partial_{A_t}}{\partial_{\alpha_1}\odot\cdots\odot\partial_{\alpha_q}}$ is defined by:*

$$\begin{aligned} &\omega \lrcorner \left(\frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \right) \\ &:= \lim_{a\to\infty,\, b\to\infty} \sigma_{a|b} \left(\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t} \wedge \partial_{y^1} \wedge \cdots \wedge \partial_{y^{l-t}}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \right) \\ &\equiv \frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}} \end{aligned} \tag{6.40}$$

In this way we have extended the interior product so to answer to one of the questions posed at the beginning of the section. We can for example give sense to operations like:

$$d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

and by symmetry we can pose:

$$\frac{1}{\frac{\partial}{\partial \xi^1}} \lrcorner d\xi^1 := d\xi^1 \lrcorner \frac{1}{\frac{\partial}{\partial \xi^1}}$$

Note that definition 82 can be adopted also when the coform w is not Berezinian.
For example, if $l > t$ and $d < q$, and w is a generic $t|q$-coform, we can pose:

$$\left(\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d}\right) \lrcorner w := \lim_{a\to\infty,\, b\to\infty} \sigma_{a|b} \left(\frac{\Theta^1 \wedge \cdots \wedge \Theta^l}{\theta^1 \odot \cdots \odot \theta^d} \lrcorner \sigma_{l-t|0} w\right) \tag{6.41}$$

To give sense to operations like

$$\frac{\partial}{\partial \xi^1} \lrcorner \frac{1}{d\xi^1}$$

we have to define new kind of objects that we can call mixed forms of the second type. They can be defined following the same path undertaken here, starting from forms instead of starting from coforms. Then we would have that a mixed forms of second type of degree $r|s$ and additional codegree $t|q$ on a $n|m$-manifold X is a form of degree $r+t|s+q$ on the extended manifold $M = X \times \mathbb{R}^{t|q}$, which satisfies an additional condition of the type of (6.19) and which can be projected on the manifold X (so that its "coefficients" do not depend on the coordinates on $\mathbb{R}^{t|q}$). On mixed forms of the second types it is possible to define appropriate interior and exterior products so that they have properties analogous to the ones expounded here for the mixed forms of the first type. It is possible to define stable mixed forms of the second type.

Once this is done, one will have that, whenever they are defined, the interior product and the exterior product by $1|0$-forms and $1|0$-coforms, satisfy the commutation relations already seen in the sections 5.4 and 6.1.

Then it is possible to define fractional mixed forms of the second type and stable fractional mixed forms of the second type and to verify that the interior and exterior products have a simple action on them.

I will not show here the definitions of the mixed forms and fractional mixed forms of the second type and of the exterior and interior products acting on them, bu the reader can imagine them.

For fractional forms, fractional coforms and stable fractional mixed forms of the first type, we have defined a good amount of what one would call Cartan calculus, id est interior and exterior products, plus some extensions.

We still have to see how an exterior derivative can be defined on mixed forms of first type and on coforms (which can be considered as a special case of mixed forms of first type). We will see that the behavior of the exterior derivative will be quite nice on most of fractional coforms and fractional mixed forms (although, as already seen for fractional forms, on some of them this will not be the case).

In the following I will continue calling mixed forms the mixed forms of the first type.

Definition 83. *Let (U, x^A, p_F) be a local chart of T^*X, cotangent space of a $n|m$-dimensional manifold X; let $x \in U$; let $M = X \times \mathbb{R}^{r|s}$, $N = X \times \mathbb{R}^{r|s} \times \mathbb{R}^{1|0}$;*
with a little abuse of notation, let's also set $x = (x, 0) \in M$ and $x = (x, 0, 0) \in N$;
let $\pi := (\overline{p^1}, \cdots, \overline{p^t}; \widetilde{p^1}, \cdots, \widetilde{p^q}) \in \underbrace{T^*_{x,0}U \times \cdots T^*_{x,0}U}_{t} \times \underbrace{T^*_{x,1}U \times \cdots T^*_{x,1}U}_{q}$;

let $\rho := (\overline{r^1}, \cdots, \overline{r^t}; \widetilde{r^1}, \cdots, \widetilde{r^q}) \in \underbrace{\mathbb{R}_0^{r|s\,*} \times \cdots \times \mathbb{R}_0^{r|s\,*}}_{t} \times \underbrace{\mathbb{R}_1^{r|s\,*} \times \cdots \mathbb{R}_1^{r|s\,*}}_{q}$;

let $o := (\overline{o^1}, \cdots, \overline{o^t}; \widetilde{o^1}, \cdots, \widetilde{o^q}) \in \underbrace{\mathbb{R}_0^{1|0\,*} \times \cdots \times \mathbb{R}_0^{1|0\,*}}_{t} \times \underbrace{\mathbb{R}_1^{1|0\,*} \times \cdots \mathbb{R}_1^{1|0\,*}}_{q}$;

let $\begin{pmatrix} \pi \\ \rho \end{pmatrix} \in \underbrace{T^*_{x,0}M \times \cdots T^*_{x,0}M}_{t} \times \underbrace{T^*_{x,1}M \times \cdots T^*_{x,1}M}_{q}$;

let $\begin{pmatrix}\pi\\ \rho\\ o\end{pmatrix} \in \underbrace{T^*_{x,0}N \times \cdots T^*_{x,0}N}_{t} \times \underbrace{T^*_{x,1}N \times \cdots T^*_{x,1}N}_{q}$*;*
let w *be a mixed form of codegree* $t|q$ *and additional degree* $r|s$*, then the exterior derivative of* w *is the mixed form of codegree* $t|q$ *and additional degree* $r+1|s$ *defined by:*

$$dw\begin{pmatrix}\pi\\ \rho\\ o\end{pmatrix} := (-1)^r(-1)^{|A||F|}o^A\frac{\partial}{\partial x^F}\frac{\partial}{\partial p^A_F}w\begin{pmatrix}\pi\\ \rho\end{pmatrix} \tag{6.42}$$

Theorem 84. *If* w *is a coform of codegree* $t|q$*, and* f *a* G^∞ *function on a supermanifold* X*, then we have:*

$$d(fw) = fdw - (-1)^t(-1)^{|f||w|}df \lrcorner \sigma_{1,0}w \tag{6.43}$$

where $\sigma_{1,0}$ *is the homomorphism defined by* (6.33)*.*

Proof. By definition (6.42), we have that:

$$\begin{aligned}
d(fw)\begin{pmatrix}\pi\\ o\end{pmatrix} =& d(fw)\begin{pmatrix}p^A_F\\ o^A\end{pmatrix}\\
=&(-1)^{|A||F|}o^A\frac{\partial}{\partial x^F}\frac{\partial}{\partial p^A_F}\left[fw\left(\pi\right)\right]\\
=&fdw\begin{pmatrix}\pi\\ o\end{pmatrix} + (-1)^{|A||F|}(-1)^{|f|(|A|+|F|)}o^A\frac{\partial f}{\partial x^F}\frac{\partial}{\partial p^A_F}w(\pi)\\
=&fdw\begin{pmatrix}\pi\\ o\end{pmatrix} + (-1)^{|f||F|}\frac{\partial f}{\partial x^F}o^A\frac{\partial}{\partial p^A_F}w(\pi)
\end{aligned}$$

On the other hand, from (6.33), we have:

$$\sigma_{1|0}w\begin{pmatrix}\pi & \check{p}\\ o & \check{o}\end{pmatrix} = \sigma_{1|0}w\begin{pmatrix}p^A_F & \check{p_F}\\ o^A & \check{o}\end{pmatrix} = w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]\check{o}$$

and from definition (6.28) we obtain:

$$df \lrcorner \sigma_{1|0}w\begin{pmatrix}\pi & \check{p}\\ o & \check{o}\end{pmatrix} = \sigma_{1|0}w\begin{pmatrix}\pi^A_F & \check{p_F}\\ o^A & \check{o}\end{pmatrix} = (-1)^t(-1)^{|f|(|w|+|F|)}\frac{\partial f}{\partial x^F}\frac{\partial}{\partial \check{p_F}}\left\{w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]\check{o}\right\}$$

But:

$$\frac{\partial}{\partial \check{p_F}}\left\{w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]\check{o}\right\} = \check{o}\left(-\frac{o^A}{\check{o}}\right)\frac{\partial}{\partial p^A_F}w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]$$

and since

$$\frac{\partial}{\partial \check{p_F}}\left\{w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]\check{o}\right\} = \check{o}\frac{\partial}{\partial \check{p_F}}\left\{w\left[p^A_F\right]\right\}$$

because $\sigma_{1|0}w$ is linear in $\check{p}$, we have that:

$$\frac{\partial}{\partial \check{p_F}}\left\{w\left[p^A_F - \check{p_F}\frac{o^A}{\check{o}}\right]\check{o}\right\} = \check{o}\left(-\frac{o^A}{\check{o}}\right)\frac{\partial}{\partial p^A_F}w\left[p^A_F\right] = -o^A\frac{\partial}{\partial p^A_F}w\left[p^A_F\right]$$

and:

$$df \lrcorner \sigma_{1|0}w\begin{pmatrix}\pi & \check{p}\\ o & \check{o}\end{pmatrix} = -(-1)^t(-1)^{|f|(|w|+|F|)}\frac{\partial f}{\partial x^F}o^A\frac{\partial}{\partial p^A_F}w\left[p^A_F\right]$$

and the theorem is proved. ■

Note that formula (6.43) is more transparent when it is written in this way:

$$d(wf) = dw \cdot f + (-1)^{t+1}\sigma_{1,0} w \lrcorner df$$

A similar theorem can be proved with a little bit more calculations for the case when w is a mixed form of codegree $t|q$ and additional degree $r|s$.

If we work on stable mixed forms, remembering definition 82 and its generalisation (6.41), the equation (6.43) becomes:

$$d(fw) = fdw - (-1)^t(-1)^{|f||w|} df \lrcorner w \tag{6.44}$$

or equivalently:

$$d(wf) = dw \cdot f - (-1)^t w \lrcorner df \tag{6.45}$$

From the definition of the operator d, (6.42), a straightforward calculation gives:

$$d\left(\partial_F\right) = d\left(\frac{1}{\partial_\phi}\right) = 0 \tag{6.46}$$

Then (6.46) and (6.44) give:

Proposition 85. *If v is a Berezinian $1|0$-coform (id est a vector field), with components v^F in the local coordinates x^F of the $n|m$ dimensional supermanifold X, then:*

$$d(v) = d\left(v^F \partial_F\right) = (-1)^{(|v|+|F|)|F|} dv^F \lrcorner \partial_F = (-1)^{(|v|+|F|)|F|} \frac{\partial v^F}{dx^F} \tag{6.47}$$

and:

$$d\left(f^\phi \frac{1}{\partial_\phi}\right) = -df^\phi \lrcorner \frac{1}{\partial_\phi} \tag{6.48}$$

Voronov has shown how to calculate the commutator between the operator d and the operator e_v for every vector field v, see the homotopy identity for mixed forms: formula (40) in [154]. The homotopy identity involves the Lie derivative of a mixed form along the vector field v, which I do not introduce here. It is enough to note that it yields the following expected property:

$$d\left(\frac{\partial_{A_1} \wedge \cdots \wedge \partial_{A_t}}{\partial_{\alpha_1} \odot \cdots \odot \partial_{\alpha_q}}\right) = 0 \tag{6.49}$$

Formula (6.46), (6.45), (6.25), (6.24), (6.27), (6.26), (6.29), (6.30) and (6.49) allow a full and simple Cartan calculus on stable fractional mixed forms of the first type. The only exception would be the coforms of the type

$$\frac{1}{\widetilde{v}}$$

with $d\widetilde{v} \neq 0$, and the mixed forms obtained from them.

In fact we have that $d\left(\frac{1}{\widetilde{v}}\right)$ is not in general a fractional mixed form.

I propose to use the collective name of fractional forms to indicate the union of the sets of fractional forms, fractional coforms and fractional stable mixed forms of the first and the second type. What has been done in this and the previous chapter is to establish the rules for the Cartan calculus for fractional forms intended in this broader sense.

I conclude this section recalling that Voronov in [153] and [154] calls space of stable $r|s$-forms on the $n|m$-dimensional supermanifold X the space defined as

$$\mathbf{\Lambda}^{r|s}TX := \lim_{\substack{a,b \\ \longrightarrow}} \Lambda^{r+a|s+b}_{n+a|m+b}TX$$

With this definition Voronov can represent the space of forms in the space of mixed forms and can then extend $r|s$-forms to negative r. This reveals to be useful when studying the cohomology of supermanifolds.

For the purpose of this work, which was to establish an extended Cartan calculus for the special case of fractional forms, it seemed to me more useful to define the spaces of stable mixed fractional forms of the first and of the second type and to represent the space of proper fractional forms inside the space of stable mixed fractional forms of the second type.

6.3 Integral of coforms and mixed forms on supermanifolds

The integration of dual densities (codensities) on supermanifolds has been studied by Gajduk, Khudaverdian and Schwarz in [55] and by Khudaverdian in [91]. Khudaverdian shows the link between dual densities and integral forms and explains how their integration is a crucial ingredient for the developing of the Batalin-Vilkovisky (BV) theory.

The theory of integration of coforms and mixed forms on a supermanifold has been developed by Voronov in [153]. Voronov shows that the coforms (which he call dual forms) are special class of dual densities (or codensities) and they can be integrated on dual paths (or copaths) on a supermanifolds.

I will present very briefly in this section the main aspects of this theory. It will then be clear that the integration of coforms and mixed forms has a connection with the BRST theory and the BV theory, which is one of the reasons why it could be interesting also for Physics.

As we will see the the definition of integral of coforms and mixed forms is the main reason for the definitions of those objects, seen in the previous two sections.

Voronov in [153] gives a definition of copath. I give here an adapted version:

Definition 86. *Consider a coordinate open domain U in a supermanifold X of dimension $n|m$; U has dimension $n|m$ as well. A copath in U is a function $f \in G^\infty(U, \mathbb{R}^{t|q})$ whose components constitute an array of independents functions $f^A \in G^\infty(U)$ enumerated by indices A, which run over even and odd values from 1 to $t|q$, so that f^A can be formally treated as coordinates on $\mathbb{R}^{t|q}$. The dimension $t|q$ is called the codimension of the copath.*

The subset $Y \subset U$ defined by $y \in Y \Leftrightarrow f(y) = 0$, id est by $f^A(y) = 0$, is a closed submanifold of X and, since the functions f^A are independents, it has codimension $t|q$. The multidimensional differential df is constituted by the array df^A which is an array of $1|0$-forms on U. In fact df can be considered a shortcut for the $t|q$ Berezinian superform defined by $df := \frac{\overline{df^1}\wedge\cdots\wedge\overline{df^t}}{\widetilde{df^1}\odot\cdots\odot\widetilde{df^q}}$; so that, with the notation used in the previous section, we have: $w(df) = df \lrcorner w$.

If w is a coform of codegree $t|q$, we can define its integral on Y with:

$$I_{\beta,Y}[w] := \int_Y w := \int_U \beta\ \delta\left(f\right) w(df) \tag{6.50}$$

where β is a fixed $n|m$-form on U; the last integral in (6.50) is the integral of superforms defined in section 5.5; δ is the Dirac distribution on the superspace $\mathbb{R}^{t|q}$ and we can define the operator

$\int_U \beta \; \delta\,(f)$, by:

$$\int_U \beta \; \delta\,(f) := \int_Y \alpha \tag{6.51}$$

where α is any superform such that $df \wedge \alpha = \beta$, where the operator $\wedge$ is defined by (5.64). (Note that if $df \wedge \alpha = \beta$ and $df \wedge \alpha' = \beta$, then $\alpha|_Y = \alpha'|_Y$).

Equation (6.51) can be considered a definition of the Dirac δ. For an introduction to distributions on manifolds see [] and for distributions on supermanifolds see [].

Note that in (6.50) we could use the canonical $n|m$-form on U, so that, if $(x^g, x^\gamma) = (x^G)$, $G = 1, \ldots, n|m$, are the coordinates on U, $\beta = \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$. In this case we should remember however that, if on U we use new coordinates $(x^{G'})$, then in (6.50) we must keep using the old form β and not the new canonical form β' for the new coordinate system; otherwise the integral in (6.50) would not be well defined. This is what I mean when I write that β is a fixed chosen $n|m$-form on U. The integral $I_{\beta,\,Y}[w]$ depends on the choice of the $n|m$ form β.

For (6.50) to be well posed we have also to control what happens if the same submanifold Y is defined by a different array of functions g^B. This in fact amounts to treat $\mathbb{R}^{t|q}$ as a supermanifold, perform a change of coordinates on it, and then express the function f in the new coordinates. If h^B constitute an array of change of coordinates functions, then Y is defined by $y \in Y \Leftrightarrow h^B\,[f(y)] = h^B(0)$. If we set $g^B := h^B \circ f - h^B(0)$, then Y is defined by $y \in Y \Leftrightarrow g^B(y) = 0$.

We have to study what happens to $I_Y[w]$ when coordinates are changed on the target space $\mathbb{R}^{t|q}$. In fact we have that:

$$\begin{aligned}
\int_U \beta \; \delta\,(g)\, w(dg) &= \pm \int_U \beta \; \delta\,(h \circ f - h(0))\, w(dg) = \pm \int_U \beta \; \frac{\delta\,(f)}{\operatorname{Ber}\left(\frac{\partial g^B}{\partial f^A}\right)} w(dg) \\
&= \pm \int_U \beta \; \frac{\delta\,(f)}{\operatorname{Ber}\left(\frac{\partial g^B}{\partial f^A}\right)} w(df^A \frac{dg^B}{df^A}) = \pm \int_U \beta \; \frac{\delta\,(f)}{\operatorname{Ber}\left(\frac{\partial g^B}{\partial f^A}\right)} w(df) \operatorname{Ber}\left(\frac{\partial g^B}{\partial f^A}\right) \\
&= \pm \int_U \beta \; \delta\,(f)\, w(df)
\end{aligned} \tag{6.52}$$

To obtain (6.52) I made use of a property of the Dirac δ on supermanifolds (see **??**). The $\pm$ sign appearing in the formula depends on the sign of $\operatorname{Ber}\left(\frac{\partial g^B}{\partial f^A}\right)$ and can be set to 1 if one choose to integrate only on positively orientated submanifolds Y and if one chooses h so that it preserves the orientation. One can then change the sign in the definition of the integral when one wants to perform an integration on negatively orientated manifolds. All this leads to a well posed definition of integral if the orientations of submanifolds Y are defined with the so called $(+,-)$-coorientation convention: see [152, 154].

Note that, as pointed out by Voronov, [153], the integral (6.50) would not make sense if the f^a or if the f^α were not independent. In the first case the Dirac δ would not be well defined, in the second case $w(df)$ would not be defined.

Note that (6.52) holds precisely because w satisfies (6.2).

Since β is fixed, by definition the integral in (6.52) does not depend on the choice of the coordinate system on U as well.

With a partition of unity argument, we can extend the definition (6.52) to the case when Y is not covered by a single coordinate chart U, but this is possible only if we have picked up a form β defined on all Y. In particular we could use a form β defined on all X, when this is possible.

The integral of coforms defined with (6.50) has a nice property. Suppose that δf^A are compactly supported small variations of the functions f^A and suppose that $Y+\delta Y$ is a submanifold defined by $y \in Y+\delta Y \Leftrightarrow (f+\delta f)(y)=0$. Then δY is a compactly supported small deformation of Y and we have:

Proposition 87. *If $\delta I[w] := \int_{Y+\delta Y} w - \int_Y w$, and $\frac{\delta I}{\delta f}$ is the functional variation of I with-respect of the variation of f, then $\frac{\delta I}{\delta f}$ depends only on the first order derivatives of the f^A with respect to the x^F.*

Proof. In a short-while I will give a more general proof for an analogous theorem, valid for the integral of mixed forms. Since coforms can be considered as mixed forms of additional degree $0|0$, then this proposition will be a corollary. ■

Let's now see the definition of the integral of a mixed form.

Definition 88. *Let X be a supermanifold of dimension $n|m$, let $U \subset X$ be an open domain contained in a single chart of X with coordinates (x^F). Let y^K be the coordinates on the superspace $\mathbb{R}^{r|s}$. Let $M := X \times \mathbb{R}^{r|s}$; let $Y \subset U \times \mathbb{R}^{r|s}$ be a closed submanifold of M defined by the copath $f \in G^\infty\left(U \times \mathbb{R}^{r|s}, \mathbb{R}^{t|q}\right)$. Let β be a fixed $n|m$-form on U and α be a fixed $r|s$-form on $\mathbb{R}^{r|s}$;*
let w be a mixed form on X of codegree $t|q$ and additional degree $r|s$ and let be $\check{w}$ its associated coform on M, which does not depends on the coordinates y^K;
then we define the integral of w over Y by:

$$I_{\beta,\alpha,Y}[w] := \int_Y w := \int_Y \check{w} = \int_{U\times\mathbb{R}^{r|s}} \beta \wedge \alpha \ \delta(f)\, \check{w}(df) \tag{6.53}$$

Note that both β and α are fixed: α can be the canonical form on $\mathbb{R}^{r|s}$ for the coordinates (y^K): $\alpha = \frac{dy^1\wedge\cdots\wedge dy^r}{dy^{r+1}\odot\cdots\odot dy^{r+s}}$; but in this case, if we perform a change of coordinate, α doesn't change and it isn't therefore anymore the canonical form for the new coordinates. In fact, in (6.53), we could even choose to put any fixed $t+r|q+s$-form on M. The value of the integral $I_Y[w]$ doesn't change if we perform a change of coordinates on M, even if the change of coordinates on $\mathbb{R}^{r|s}$ depends on the coordinates on X. Moreover, using an argument analogous to the one used in (6.52), and remembering that the w and $\check{w}$ satisfy (6.18), one can see that the value of the integral $I_{\beta,\alpha,Y}[w]$ doesn't change when coordinates are changed on the target space $\mathbb{R}^{t|q}$. So, whence β and α are chosen, the integral $I_{\beta,\alpha,Y}[w]$ is well defined.

The next result, which I'm going to prove, is a generalization of proposition 87:

Proposition 89. *Suppose that δf^A are compactly supported small variations of the functions f^A and suppose that $Y+\delta Y$ is a submanifold defined by $y \in Y+\delta Y \Leftrightarrow (f+\delta f)(y)=0$, so that δY is a compactly supported small deformation of Y. If $\delta I[w] := \int_{Y+\delta Y} w - \int_Y w$, and $\frac{\delta I}{\delta f}$ is the functional variation of I with-respect of the variation of f, then $\frac{\delta I}{\delta f}$ depends only on the first order derivatives of the f^A with respect to the x^F.*

Proof. We can treat $I_{\beta,\alpha,Y}[w]$ as if it were the action of a singular Lagrangian defined with the use of a Dirac δ distribution. Since δf^A are compactly supported, to calculate the functional variation $\frac{\delta I}{\delta f}$, it suffices to use standard techniques, but paying attention to the parities of the object involved and remembering the properties of the derivative of the δ distribution. Identifying

w with $\check{w}$, we get the following:

$$\begin{aligned}\frac{\delta I}{\delta f^A} =&\frac{\partial\delta(f)}{\partial f^B}\delta^A_B w(df) - \frac{\partial\delta(f)}{\partial f^B}(-1)^{|A||G|+|A||\delta|+|B||G|+|B||\delta|+|B|}\frac{\partial f^B}{\partial x^G}\frac{\partial w}{\partial p^A_G}(df)\\ &- \frac{\partial\delta(f)}{\partial f^B}(-1)^{|A||K|+|A||\delta|+|B||K|+|B||\delta|+|B|}\frac{\partial f^B}{\partial y^K}\frac{\partial w}{\partial o^A_K}(df)\\ &- (-1)^{|A||F|+|A||\delta|}\delta(f)\frac{\partial^2 w}{\partial x^F\partial p^A_F}(df)\\ &- (-1)^{|A||F|+|A||\delta|}\delta(f)\left[\frac{\partial^2 f^C}{\partial x^F\partial x^G}\frac{\partial^2 w}{\partial p^C_G\partial p^A_F}(df) + \frac{\partial^2 f^C}{\partial x^F\partial y^K}\frac{\partial^2 w}{\partial o^C_K\partial p^A_F}(df)\right]\\ &- (-1)^{|A||K|+|A||\delta|}\delta(f)\left[\frac{\partial^2 f^C}{\partial y^K\partial y^I}\frac{\partial^2 w}{\partial o^C_I\partial o^A_K}(df) + \frac{\partial^2 f^C}{\partial y^K\partial x^F}\frac{\partial^2 w}{\partial p^C_F\partial o^A_K}(df)\right]\end{aligned} \tag{6.54}$$

but the last two lines disappear because w and $\check{w}$ satisfy (6.20), (6.21) and (6.22), and the proposition is proved. ■

Note that in fact it can be proved that also the second line in (6.54) vanishes, because w satisfies (6.19), see [153].

Remembering (6.42), the third line in (6.54), can be rewritten in terms of dw and in fact it constitutes the justification for the definition (6.42).

Proposition 87 is a corollary of proposition 6.54 and it holds precisely because w in 87 satisfies (6.3).

The integral $I_{\beta,\alpha,Y}[w]$ satisfies an other interesting property, consequence of (6.19):

Proposition 90. *Let f be a copath defining Y. If some components $\breve{f}^Z$ of f are of the type $\breve{f}^Z = g^Z(y^K) - h^Z(x^F)$, where $g^Z \in G^\infty(\mathbb{R}^{r|s})$ and $h^Z \in G^\infty(X)$ for some Z, then the integral $I_Y[w]$ does not depend on the functions h^Z.*

Proof. The property is a consequence of (6.19) because $d\breve{f}^Z = dy^K\frac{\partial g^Z}{\partial y^K} - dx^F\frac{\partial h^Z}{\partial x^F}$. Since (6.19) holds, $w(df)$ does not depend on $\frac{\partial h^Z}{\partial x^F}$ and so it does not depend on h^Z.

See proposition 3.1 of [153]. ■

This property can be easily checked on some examples if one uses fractional mixed forms.

I finish this chapter with the following:

Theorem 91. *Let γ be a $n|m$-form defined on the $n|m$-manifold X. Let $f \in G^\infty(U,\mathbb{R}^{t|q})$. Let $Z \subset X$ be the submanifold of X of codimension $t|q$, defined by $z \in Z \Leftrightarrow f(z) = 0$, id est by $f^A(z) = 0$, with $A = 1,\dots,t|q$ and $t \le n$, $q \le m$. Let w be a Berezinian coform on X of codegree $t|q$; let δ be the Dirac distribution defined with (6.51); then we have that:*

$$I_{\gamma,Z}[w] = \int_Z w = \int_X \gamma\ \delta(f)w(df) = \int_Z w \lrcorner \gamma = I_Z[w\lrcorner\gamma] \tag{6.55}$$

Proof. From the definition (6.51), we have that:

$$\int_X \gamma\ \delta(f)w(df) = \int_Z \beta\, w(df)$$

where β is any $n-t|m-q$ Berezinian form such that:

$$df\wedge\beta = \frac{d\overline{f^1}\wedge\cdots\wedge d\overline{f^t}}{d\widetilde{f^1}\odot\cdots\odot d\widetilde{f^q}}\wedge\beta = \gamma$$

But, since $\forall v \in TZ$, $v \lrcorner df = 0$, then:

$$\beta\, w(df)|_Z = [w \lrcorner\, (df \wedge \beta)]\,|_Z = (w \lrcorner \gamma)|_Z$$

and the theorem is proved. ■

Note that $|\beta| = |\gamma| = |df| = 0$ and so $\beta\, w(df) = w(df)\, \beta$.

Part III

Super multisymplectic field theories

Introduction to Part III

I repeat in this introduction to the third part of my thesis some concepts already presented in the main Introduction to this thesis, adding some details and some references.

The theories of superfields has been studied extensively since the 70's, when supersymmetry began to play an important role in Physics.

A supersymmetric field theory can be usually presented in two different ways: as a field theory on a classical manifold, with fermionic and bosonic components of the field (the components approach), or as a theory defined on a supermanifold with even and odd coordinates (the superfield approach).

When one uses the components approach, the field equations can be derived by a variational principle with an action defined as the integral on the classical base manifold of a Lagrangian density. The action obviously involves both the bosonic (commuting) and the fermionic (anticommuting) components of the fields, treating them accordingly to the respective parities.

To express the action principle in a geometric language, we use Lagrangian densities defined in terms of differential forms. Since the Lagrangian density typically depends on the derivative of the components of the fields (which can be bosonic or fermionic), even if the theory is defined on a classical (bosonic) manifold, it is clear that it is necessary to develop a calculus for differential forms valid also for the fermionic sector. This is not simple at all and it has been done by D. Hernández Ruipérez and J. Muñoz Masqué during 80's, exactly for the case when the base manifold is classical. In [76, 77, 78, 118, 119] they have indeed developed a graded variational calculus for Lagrangian densities defined in terms of graded Kostant differential forms and they have obtained the corresponding Lagrangian formalism (Euler-Lagrange equations, Poincaré-Cartan forms, Noether invariants, etc.).

When the base manifold is not classical and it is a supermanifold, like in the superfield approach to supersymmetric theories, then the task is even more difficult. The theory can still be derived by a variational principle, but the action in this case is defined as the Berezinian integral of a Lagrangian density, which must be therefore a Berezinian volume density.

In 1987 in [80] Hernández Ruipérez and Muñoz Masqué recognize: "the lack of an intrinsic definition of a suitable notion of intermediate Berezinian densities with its Cartan exterior calculus, prevents us from developing a Lagrangian formalism...", meaning a full Lagrangian formalism valid also for the case when the base manifold is a supermanifold. Nonetheless, in [79] and [80] they arrived to an intrinsic formulation of the notion of Berezinian Lagrangian density and Berezinian critical sections. Moreover, when a theory can be expressed both with the components and with the superfield approach, they showed the way to relate the critical sections of the Berezinian Lagrangian density, defined on the supermanifold, to the critical sections of the corresponding Lagrangian, defined on a bosonic manifold with graded differential forms. This was done via a first version of what will be then called the Comparison Theorem.

In 1992 J. Monterde, in [113], showed that the Berezinian critical sections of an action defined with a Berezinian Lagrangian density must satisfy a super version of the Euler-Lagrange equations. He arrived to his results using a notation which I find a bit heavy.

In [114], Monterde and Muñoz Masqué made a step forward in building a geometric approach to superfield theories. While developing a version of supermechanics they present a theory defined on a base supermanifold of dimension $1|1$ and they show that the fields satisfy some Hamilton equations which are a generalization of the Hamilton-Volterra equations (2.7) to the case when the base manifolds has dimension $1|1$. They also presented for their theory the super version of the Poincaré-Cartan (or Cartan) form. Again it seems that these results have not been exploited by physicists nor they have yielded any significant new studies in the mathematical literature. This again may be due to the heavy notations used.

Note that a super Cartan form and its exterior derivative (a super symplectic form) were also introduced in papers on Lagrangian supermechanics and Hamiltonian supermechanics. For a geometric presentation of Lagrangian supermechanics, see L. A. Ibort and J. Marín-Solano [83]; for Hamiltonian supermechanics see J. F. Cariñena and H. Figueroa [26]. In those papers, however, the theories studied are defined on a base manifold which is the real one-dimensional time-line and so they cannot be considered as field theories.

In 2002, in [115], Monterde and Muñoz Masqué further developed their theory of Hamiltonian supermechanics: they study the supermanifold of solutions and they built on it a super symplectic structure. In 2003, in [117], Monterde and J. A. Vallejo presented the Lagrangian version of the same theory.

Once one disposes of a super version of the Poincaré-Cartan form, it is then natural to look for a fully geometric version of the corresponding Hamilton-Volterra theory for superfields theories.

In fact in 2006 Monterde, Muñoz Masqué and Vallejo published a paper, [116], in which they proposed a Hamilton-Cartan formalism for first-order Berezinian variational problems valid for fields defined on supermanifolds of any dimension. As it had already been done in [114], they achieved their purpose by studying, with the help of the Comparison Theorem, an associated higher-order graded variational problem, defined on a bosonic base manifold. They obtained a super Poincaré-Cartan form valid for theories on bases of any dimension. However they chose a notation which I judge not very much adapted to general proofs, neither to actual calculations. In some situations they were forced to present their results using examples of low dimensions (typically odd dimension equal to 2). They also obtained a very beautiful and important result, which is the generalization of first Noether theorem to super field theories (theorem 8.2 in [116]); but they obtained it using a rather technical assumption needed in the hypothesis.

To my knowledge neither Monterde, Muñoz Masqué or Vallejo, nor any other mathematician, tried to use the results on super Poincaré-Cartan forms to describe the general superfield theories with the multisymplectic approach.

Independently from the results obtained by the Spanish school, previously described, there has been, to my knowledge, only one attempt to extend the multisymplectic formalism to superfields. S. P. Hrabak in [81, 82] initiated a study of the formulation of the classical BRST symmetry within the framework of a multisymplectic theory. To do so, he needed to extend the multisymplectic formalism so that it works also for field theories whose base is a classical bosonic manifold, but whose space of fields is a supermanifold (with bosonic and fermionic sectors). He accomplished his task in [82]. He didn't however show how to eventually extend the formalism also to the case when the base itself is a supermanifold.

Here in the third part of my thesis I will present a full multisymplectic version of the superfield theories valid for any dimension (even and odd) of the base space and of the space of fields. My results are a generalization of those obtained by Hrabak in [82] and they are based on a full exploitation of the potential of the theory of superforms of Voronov and Zorich. They can also be considered a natural generalization of the results obtained in the geometrical presentations of finite dimensional supermechanics, which includes the use of a super symplectic form, like [83] and [26].

If one wants to build a multisymplectic superfield theory, he has to use objects (for example the multisymplectic form) which can be integrated on a supermanifold and which in the same time can be used for a Cartan calculus, including contraction by supervectors, external product by one forms and external derivation. This is the difficult point. In fact, before the articles of Voronov of 90's, [153, 154], no such object did exist. Before the appearance of superforms of Voronov and Zorich, the best candidates to play the role which in classical field theory is played by differential forms were Kostant forms or pseudodifferential and integral forms. Unfortunately Kostant forms can be integrated only on even base manifolds. On the other hand, pseudodifferential and integral forms are good for integration but do not admit a simple and natural version of Cartan calculus. In their works, Hernández Ruipérez, Muñoz Masqué, Monterde and Vallejo, in order to find a way to bypass this fundamental difficulty, treated the theories defined on a superbase, relating them to corresponding theories (of higher order) which can be understood as defined on an even base.

In my work, I use a different approach. I believe that Voronov Zorich superforms are the natural objects to use to build a multisymplectic theory because they can be integrated and, as we have seen in the second part of this thesis, they admit a full Cartan calculus. So I use them and, more specifically, I try and use, as far as possible, only fractional superforms. In this way all the proofs and calculations become, simpler, more transparent and directly comparable to the corresponding ones of classical field theory.

In chapter 7 I will show how to found superfield theories on an action principle, defining the action starting from the fractional Berezinian superforms presented in 5.3. I will obtain the same super version of Euler-Lagrange equations already obtained in [113], but using a lighter notation and a formalism which I judge more natural and which allows simpler and shorter proofs. In particular I show that there is no need to use an higher order Lagrangian in components for a theory which can be described by a first order Berezinian Lagrangian.

Chapter 8 is the most important part of my thesis: it contains the main results of this work. It consists in the presentation of the multisymplectic approach to superfield theories made with the help of the fractional superforms defined in 5.4. In section 8.1 I define the super-multimomenta space and the super version of the Legendre transform. In section 8.2 I present the super version of the Hamilton-Volterra equations. In section 8.3 I introduce the super Poincaré-Cartan form and the super multisymplectic form and I prove the theorem which relate them to the Hamiltonian surfaces solutions of the theory. In section 8.4 I build a super symplectic structure on the super covariant phase space (the space of solutions of the theory).

In chapter 9 I will show how the Comparison Theorem can be seen from the perspective of the formalism introduced in the two previous chapters. The chosen concrete approach hopefully will clarify the relations existing between the so called components theories and the so called superfield theories. In section 9.2, I will look at the comparison from the Hamiltonian point of view and I will make a first comparison of symplectic structures on the spaces of solutions of theories expressed in the superfield and in the components formalisms. These results are original.

In chapter 10 I will explain how the supermultisymplectic formalism can be used to define super Poisson brakets for super fields. In particular in section 10.1 I will study in more detail the simplest case of supermechanics; I will show how on the space of solutions of a supermechanic theory is naturally defined a super symplectic structure and I will relate my results to the already published results obtained by Khudaverdian [91] [] and by Monterde and Muñoz Masqué [115].

To my knowledge, nobody has tried yet to build covariantly a super symplectic struture on the space $\mathscr{G}$ of solutions of a superfield theory; apart Monterde, Muñoz Masqué and Vallejo, in [115] and [117] quoted above, who did it for the special case of base manifold of dimension $1|1$, which give rise to a supermechanics theory. In section 10.2 I show how the constructions expounded in chapter 4 for classical field theories, can be directly extended to the super case for super field theories defined on base supermanifold X of any even and odd dimension. The space $\mathscr{G}$ becomes then a truly super covariant phase space. From the super Poisson structure built on $\mathscr{G}$, I derive the super commutation rules to which Fermionic and Bosonic fields have to obey and I demonstrate that these rules are exactly those expected from a physical point of view. This will justify in a natural way the use of anticommutator for Fermionic fields.

In chapter 11 I will study the symmetries and supersymmetries of super field theories with the techniques offered by the formalism of fractional mixed forms and from the point of view of the super multisymplectic approach expounded in the previous chapters.

Some "super" versions of the first Noether theorem valid for supermechanics already exist: see for example Ibort and Marín-Solano [83] and Cariñena and Figueroa [25].

L. Fatibene and M. Francaviglia, in [48] and L. Fatibene, M. Ferraris, M. Francaviglia and R. G. McLenaghan, in [47], have tried to give a geometric interpretation of supersymmetries using the classical tools of classical Poincaré-Cartan form and generalized vector fields defined over a bosonic manifold for field theories whose field spaces are product of exterior powers of some vector spaces, such that spinors can be represented.

As already said, in 2006 Monterde, Muñoz Masqué and Vallejo, [116], obtained a version of the first Noether theorem valid for generic super field theories, but with the help of a rather technical assumption needed in the hypothesis.

Here in Chapter 11, section 11.2, I will show that my approach allows to have a super version of Noether theorem which is quite natural, simple to prove with my formalism, and quite general since it does not require any specific technical assumption of the kind used in [116]. In section 11.3, I will present a super extension of the multimomentum map introduced by Gotay, Isenberg, Marsden, Montgomery, Śniatycki and Yasskin in [63]. Both the super Noether theorem and the super multimomentum map will be presented with a formulation which will reveal to be very close to the corresponding one for classical theories.

Finally, in chapter 12, I will present some examples to show how all the theory can be implemented for some specific Lagrangians. I will treat with my formalism the superoscillator, the superparticle in a curved space and the 3-dimensional super σ-model.

The principal aim of this third part of my thesis is to show that fractional superforms are the natural objects to use, when one wants to give a variational and a geometric formulation of a super field theory, and specifically that they are the key ingredient in building the super version of the multisymplectic formalism.

Chapter 7

Lagrangian super field theories.

In this chapter I will present an extension of the classical Lagrangian field theory to superfields defined on supermanifolds and with value on supermanifolds.

An algebraic setting for the Lagrangian formalism over graded algebras has been proposed by A. Verbovetsky in [146], who was looking for a super- and non-commutative generalizations of the A. M. Vinogradov theory of C-spectral sequence, which is indeed a way to give an algebro-geometric foundation of Lagrangian field theory. I will not follow his ideas.

A geometrical setting has already been proposed by Hernández Ruipérez, Muñoz Masqué, and Monterde in their works during 80's and 90's, [76, 77, 78, 118, 119, 80, 113, 114], and by Monterde and Vallejo in [117].

Indeed I will use a different approach for a geometrical foundation of Lagrangian superfield theories. All the theory here presented will be based on the use of fractional (most of the time Berezinian) superforms, defined in sections 5.3 and 5.4. I will show that this choice allows for simpler proofs and calculations and make it transparent the analogy between the superfield and the classical field theory.

In fact I will follow here the path already undertaken in chapter 1, extending, whenever possible, the notions and the results there presented. Sometime, when the extension is trivial, I will consider it understood.

Let E, X and F be finite dimensional G^∞-supermanifolds, E and X being connected; and let (E, π, X, F) be a super fiber bundle with total space E, base X, type-fiber F and bundle G^∞-projection π.

A field Φ over X is a G^∞-section of the fiber bundle π and we write: $\Phi \in \Gamma\big(E\big)$. From now on and throughout all this section and the following one of this paper, all the maps between supermanifolds will be considered G^∞ if not otherwise stated.

With a construction totally analogous to the classical one, it is possible to define $J^1\pi$, the first order jet space of sections of E, see [133] chapter 10. It is a super fiber bundle whose super dimension depends on the superdimensions of X and F. For any section Φ of E, $j^1\Phi$ denotes its first order jet, and it is a lift to $J^1\pi$. For more details on super-jet-bundles, and another point of view, one can look at [76, 80, 113, 114] and [61] chapter 3, where jets of sections of bundles, whose base X is even, are considered. One can also read Bruce [19] for a categorical point of view on curves in supermanifolds and their jets.

There is no difficulty in defining the maps $j^1\pi$ and $j^1_0\pi$. Also the definition of the map j^1 between sections of E and sections of $J^1\pi$ doesn't pose any new problem, but the same one existing in the classical case, because both spaces of sections may be infinite dimensional.

On $J^1\pi$, on E and on X do exist adapted atlases of charts, so that, if on an open chart U of X we use the local coordinates (x^a, x^α), $a = 1, \cdots, n$, $\alpha = 1, \cdots, m$ (or the coordinates x^A with $A = 1, \cdots, n+m$), where $n|m$ is the superdimension of X, then, on an open chart V of E over U, it is possible to use, with a little abuse of notations, the coordinates $\left(x^a, x^\alpha; q^i, q^\iota\right)$ with $i = 1, \cdots, r$, $\iota = 1, \cdots, s$ (or the coordinates (x^A, q^I), with $I = 1, \cdots, r+s$), $r|s$ being the dimension of the fiber F; and on an open chart W of $J^1\pi$ over U it is possible to use the coordinates $\left(x^a, x^\alpha; q^i, q^\iota; \dot{q}^I_A\right)$. The degree of $\dot{q}^I_A$ being $\left|\dot{q}^I_A\right| = |A| + |I|$.

On $T(J^1\pi)$, I will use local adapted charts with coordinates $\left(x^A; q^I; \dot{q}^I_A; \overline{v^A}; \widetilde{v^A}; \overline{v^I}; \widetilde{v^I}; \overline{v^I_A}; \widetilde{v^I_A}\right)$, where the positions and the names given to the indices of coordinates v should be enough to identify them in a natural way; sometime, to be clearer, I will use the corresponding coordinates $\left(x^A; q^I; \dot{q}^I_A; \overline{v^{x^A}}; \widetilde{v^{x^A}}; \overline{v^{q^I}}; \widetilde{v^{q^I}}; \overline{v^{\dot{q}^I_A}}; \widetilde{v^{\dot{q}^I_A}}\right)$. If $r \in J^1\pi$, an even generic vector $\overline{v}$ over r ($\overline{v} \in T_{r,0}(J^1\pi)$) will be written in local coordinates as: $\overline{v} = \overline{v^A}\frac{\partial}{\partial x^A} + \overline{v^I}\frac{\partial}{\partial q^I} + \overline{v^I_A}\frac{\partial}{\partial \dot{q}^I_A}$. Note that $\left|\overline{v^A}\right| = |A|$ and so $|\overline{v^a}| = 0$; $|\overline{v^\alpha}| = 1$. The coordinates of an even vectors can be even or odd. A generic odd vector $\widetilde{v}$ over r ($\widetilde{v} \in T_{r,1}(J^1\pi)$) will be written in local coordinates as: $\widetilde{v} = \widetilde{v^A}\frac{\partial}{\partial x^A} + \widetilde{v^I}\frac{\partial}{\partial q^I} + \widetilde{v^I_A}\frac{\partial}{\partial \dot{q}^I_A}$. Note that $\left|\widetilde{v^A}\right| = |A| + 1$ and so $\left|\widetilde{v^a}\right| = 1$; $\left|\widetilde{v^\alpha}\right| = 0$. Also the coordinates of an odd vectors can be even or odd. A generic vector $v \in T_r(J^1\pi)$ will be $v = \overline{v} + \widetilde{v}$.

There are important 1-forms on $J^1\pi$, the contact forms:

Definition 92. *The contact $1|0$-extended-forms c^I on $J^1\pi$ are the local forms defined on local charts, with self-explaining notation, by:*

$$c^I = dq^I - dx^A \dot{q}^I_A \tag{7.1}$$

With the contact forms it is possible to identify the section of $J^1\pi$ which are lift of sections of E exactly as in the classical framework. We have indeed:

Proposition 93. *A section $s \in \Gamma(J^1\pi)$ is the lift of a section $\Phi \in \Gamma(E)$, and, if so, we write $s = j^1\Phi$, if and only if $\forall c^I$, $s^*c^I = 0$. If in coordinates s reads: $x = (x^A) \longrightarrow s(x) = (x^A, q^I, \dot{q}^I_A)$, then $\Phi : x = (x^A) \longrightarrow \Phi(x) = (x^A, q^I)$*

We have now all the ingredients to define a Lagrangian super field theory.

Definition 94. *Let's consider a superbundle (E, π, X) with base X of dimension $n|m$. The Lagrangian morphism $\mathbb{LAG}$ of a field theory defined on E is a superfiberbundle morphism between $J^1\pi$ and $\Lambda^{n|m}X$ where $\Lambda^{n|m}X$ is the bundle of $n|m$-forms on the base X.*

By pull-back performed with the G^∞ map $j^1\pi$ we can define the super-fiberbundle $j^1\pi^*\Lambda^{n|m}X$, which is a subbundle of $\Lambda^{n|m}\left(J^1\pi\right)$.

With the help of the morphism $\mathbb{LAG}$, we can define the Lagrangian form:

Definition 95. *The Lagrangian form $\mathcal{L} \in \Omega^{n|m}(J^1\pi)$ associated to the Lagrangian morphism $\mathbb{LAG}$ is the $n|m$-form over $J^1\pi$ defined by:*

$$\forall r \in J^1\pi : \quad \mathcal{L}|_r := j^1\pi^*\mathbb{LAG}(r)|_r$$

The form $\mathcal{L}$ is a section of the super-fiberbundle $j^1\pi^*\Lambda^{n|m}X$.

With the help of proposition 43, one can easily prove that:

Proposition 96. *Every Lagrangian $n|m$-form $\mathcal{L} \in \Gamma\left(j^1\pi^*\Lambda^{n|m}X\right)$ can be written on a local chart as:*

$$\mathcal{L} = L\left(x^A; q^I; \dot{q}^I_A\right)\beta$$

where $\beta = \frac{dx^1\wedge\cdots\wedge dx^n}{dx^{n+1}\odot\cdots\odot dx^{n+m}}$ is the Berezinian local $n|m$-form defined in 42.

In the following part of this thesis, when there is no risk of confusion, for brevity I will alternatively call Lagrangian the Lagrangian form $\mathcal{L}$ or the Lagrangian density L, which is a local function on $J^1\pi$, but not a global function on it.

If we pose the following:

Definition 97. *If $J^1\pi$ is the first order jet superbundle of a superbundle (E,π,X) over the $n|m$-dimensional basis X, we say that a $n|m$-form $\alpha \in \Omega^{n|m}(J^1\pi)$ is horizontal, if its value is null, or non defined, whenever applied to a multivector $v \in \Gamma(\underbrace{T_0J^1\pi\otimes\cdots T_0J^1\pi}_{n}\otimes\underbrace{T_1J^1\pi\otimes\cdots T_1J^1\pi}_{m})$ which is vertical in one of its components.*

Then it is immediate to prove that:

Proposition 98. *The Lagrangian $\mathcal{L}$ of our field theory is an horizontal $n|m$-dimensional differential form on $J^1\pi$.*

As we have seen in the section 5.5, Voronov and Zorich have shown that $n|m$-forms can be integrated over $n|m$-supermanifold with boundary, whence a suitable definition of boundary is given.

The action A is the integral of $\mathcal{L}$ on the $n|m$-dimensional surface $s(X)$, where s is a section of $J^1\pi$ and so $s(X) \subset J^1\pi$ is a $n|m$-dimensional submanifold of $J^1\pi$. As in section 1 we can define the action over a section Φ of the bundle π, using again the same name for two different functions defined on related spaces:

$$A(\Phi) := A(j^1\Phi) = \int_{j^1\Phi(X)} \mathcal{L} = \int_X j^1\Phi^*\mathcal{L} \tag{7.2}$$

We assume that we have given a differential super-structure to $\Gamma\big(E\big)$ and to analogous spaces that we will meet in the following. A is then a smooth super-functional on $\Gamma\big(E\big)$.

Remark 99. *For the integral appearing in (7.2) to be well defined, some conditions on X (X compact) or on Φ (boundary conditions) or on both are required. However in the following, to define a superfield theory, we will not need that the integral of $j^1\Phi^*\mathcal{L}$ is well defined on all the base X so we will not need to use (7.2).*

What we will need is to perform integrations over supermanifolds with boundary which are compact. As we are going to see, such kind of integrals are sufficient to define the action $A_{U,\partial U}(\Phi)$:

$$A_{U,\partial U}(\Phi) := A_{U,\partial U}(j^1\Phi) = \int_{U,\partial U} j^1\Phi^*\mathcal{L} \tag{7.3}$$

where $(U,\partial U)$ is a supermanifold with boundary such that $U\cup\partial U \subset X$ is compact.

This definition in turns allows to define superfield theories also on non-compact supermanifolds. In fact starting from (7.3) we will be able to define the solutions of a field theory as critical sections of the action.

We can proceed exactly as in chapter 1. I will underline only some differences coming from the super-structure.

First of all, to give full sense to (7.2), to (7.3) and to similar ones, we must first choose and fix an immersed body of X: see section 5.5. I will assume that this choice is made once and for all. I will not keep track of this choice in the notation of the integrals below.

Then we can proceed to define $A_{U,\partial U}(\Phi)$.

As in chapter 1, for every submanifold $U \subset X$ with boundary ∂U and for every section Φ, we can define the action over U. But in the super case, we must take care of the boundary, because integrals on supermanifolds depends on their boundaries and two supermanifolds with boundary can have the same internal part U but different boundaries. This will not prevent us from defining the action and its critical sections.

Remembering definitions 61, 62 and 64 we can define $A_{U,\partial U}(\Phi)$ with (7.3):

Definition 100. *Let $U \subset X$ be a submanifold of X with boundary ∂U and such that $U \cup \partial U \subset X$ is compact; let Φ be a section of the fields bundle (E, π, X); let $\mathcal{L}$ be the Lagrangian form on $J^1\pi$; then the action $A_{U,\partial U}(\Phi)$ on the section Φ over the supermanifold with boundary $(U, \partial U)$ is defined by:*

$$A_{U,\partial U}(\Phi) := A_{U,\partial U}(j^1\Phi) = \int_{U,\partial U} j^1\Phi^*\mathcal{L}$$

For the sake of simplicity I will sometime write $A_U(\Phi)$, understanding the dependence on ∂U.

Keeping in mind the dependence both on U and on ∂U and remembering that an immersed body has been fixed, we can then define the spaces of local sections $\mathscr{U}_\Phi$, which share the same values on the boundary ∂U and then we can define $j^1\mathscr{U}_\Phi$ in a way completely analogous to the one used in Chapter 1; and we can set:

Definition 101. *A solution Φ of the field theory with Lagrangian $\mathcal{L}$ over $J^1\pi$ is a section $\Phi \in \Gamma(\pi)$ such that, $\forall U$ submanifold of X and for every ∂U boundary of U such that $U \cup \partial U$ is compact, $dA_{U,\partial U}\big|_{\mathscr{U}_\Phi}(\Phi) = 0$ or, equivalently, $dA_{U,\partial U}\big|_{j^1\mathscr{U}_\Phi}(j^1\Phi) = 0$.*

We can define an even path trough Φ in $\Gamma(E)$, or in $\mathscr{U}_\Phi$, as a G^∞ superfunction p from an open set $I \subset \mathbb{R}^{1|0}$ containing the point 0 to the set $\Gamma(E)$, or $\mathscr{U}_\Phi$, and so that $p(0) = \Phi$. A G^∞ odd path in $\Gamma(E)$, or $\mathscr{U}_\Phi$, trough Φ will be a superfunction p from $\mathbb{R}^{0|1}$ to the set $\Gamma(E)$, or $\mathscr{U}_\Phi$, and so that $p(0) = \Phi$. Let's parametrize I with the coordinate l and $\mathbb{R}^{0|1}$ with λ; we have then:

$$\Phi \text{ is a solution of the theory} \iff \forall U, \ \forall \text{ even path } p \text{ in } \mathscr{U}_\Phi \text{ trough } \Phi, \ \left.\frac{dA_U}{dl}\right|_{l=0} = 0 \tag{7.4}$$

It can be shown that (7.4) is equivalent to

$$\Phi \text{ is a solution of the theory} \iff \forall U, \ \forall \text{ odd path } p \text{ in } \mathscr{U}_\Phi \text{ trough } \Phi, \ \left.\frac{dA_U}{d\lambda}\right|_{\lambda=0} = 0 \tag{7.5}$$

For proving one direction of the equivalence it is important to work with the Grassmann algebra with infinite generators.

It is also possible to find a result analogous to 1.7 of section 1. For every section Φ of E and for every $U \subset X$, we can define the vertical tangent bundle over $j^1\Phi(X)$, call it $V_{j^1\Phi}(J^1\pi)$, and then we can define $\mathscr{V}_{j^1\Phi}$ again in a way analogous to the one undertaken in section 1. Then we can repeat the arguments given in (1.6), with the shrewdness of considering only even paths which lead to even $w \in V_{j^1\Phi}(J^1\pi)$. The argument works because ∂U_l and V_l are indeed a $n|m$-

and a $n+1|m$-manifolds; it is possible to show that $j^1\phi_l(U)+\partial U_l - j^1\phi(U)$ is a boundary of V_l when suitable orientations are chosen and the analogous of Stokes theorem is valid for $n|m$-forms: see [155] and [152] for this last statement. We finally discover that:

$$\Phi \text{ is a solution of the theory } \iff \forall U,\ \forall \partial U,\ \forall w \in \mathscr{V}_{j^1\Phi} \text{ even, } \int_{j^1\Phi(U)} w \lrcorner d\mathcal{L} = 0 \tag{7.6}$$

where $d\mathcal{L}$ is the exterior differential of $r|s$-forms defined with (5.6) and $\lrcorner$ is the interior product defined with (5.51).

To prove the analogous of (7.6) when X is compact without boundary is even simpler: one can proceed as in (1.6), keeping in mind that $\partial X = \emptyset$, paying attention to the order of multiplications and using Voronov and Zorich version of Stokes theorem for $r|s$-forms.

Let $U \subset X$ be a local chart of an adapted atlas with coordinates as in the beginning of this section, then:

Theorem 102. *A superfield $\Phi \in \Gamma\big(E\big)$ is a solution of the Lagrangian theory with Lagrangian defined on local charts U by $\mathcal{L} = L\beta$, if and only if for every local chart U and $\forall x \in U$:*

$$(-1)^{|A||I|}\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\big(j^1\Phi(x)\big) - \frac{\partial L}{\partial q^I}\big(j^1\Phi(x)\big) = 0 \tag{7.7}$$

Proof. Let's consider a local chart U as a submanifold $U \subset X$ and let's consider one of its boundaries ∂U and an even path p in $\mathscr{U}_\Phi$ through Φ: $p\ :\ l \longrightarrow \Phi_l$ with $\Phi_0 = \Phi$. Locally on U, we can write:

$$A_{U,\partial U}(\Phi_l) = \int_{j^1\Phi_l(U)} \mathcal{L} = \int_U j^1\Phi_l^*\mathcal{L} = \int_U j^1\Phi_l^*(L\beta)$$

We understand that all integrals depend on the chosen boundaries, without writing it explicitly. On U we have:

$$\begin{aligned}
\frac{dA_{U,\partial U}}{dl} &= \frac{d}{dl}\int_U j^1\Phi_l^*(L\beta) = \int_U \frac{d}{dl} j^1\Phi_l^*(L\beta) = \int_U \left[\frac{d}{dl} j^1\Phi_l^*(L)\right] j^1\Phi_l^*(\beta) \\
&= \int_U \left[\frac{d}{dl} L(j^1\Phi_l)\right]\beta = \int_U \left[\frac{\partial q^I_l}{dl}(j^1\Phi_l)\frac{\partial L}{\partial q^I}(j^1\Phi_l) + \frac{\partial \dot{q}^I_{l,A}}{dl}(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right]\beta \\
&= \int_U \left[\frac{\partial q^I_l}{dl}(j^1\Phi_l)\frac{\partial L}{\partial q^I}(j^1\Phi_l) + \frac{\partial}{dl}\frac{\partial}{\partial x^A} q^I_l(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right]\beta \\
&= \int_U \left[\frac{\partial q^I_l}{dl}(j^1\Phi_l)\frac{\partial L}{\partial q^I}(j^1\Phi_l) + \frac{\partial}{\partial x^A}\left(\frac{\partial}{dl} q^I_l(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right)\right]\beta \\
&- \int_U \left[(-1)^{|A||I|}\frac{\partial}{dl} q^I_l(j^1\Phi_l)\frac{\partial}{\partial x^A}\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right]\beta
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_U \left[\frac{\partial}{\partial x^A}\left(\frac{\partial}{dl} q^I_l(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right)\right]\beta \\
&= \int_U \left[(-1)^{|A||L|}\frac{\partial}{\partial x^A}\left((-1)^{|A||L|}\frac{\partial}{dl} q^I_l(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}^I_A}(j^1\Phi_l)\right)\right]\beta = 0
\end{aligned}$$

where the last equality holds because of the "divergence" theorem for super-integrals (see [152] pag. 27); and because $\left|\frac{\partial}{dl}q_l^I(j^1\Phi_l)\frac{\partial L}{\partial \dot{q}_{l,A}^I}(j^1\Phi_l)\right| = |A+L|$ and $\frac{\partial}{dl}q_l^I = 0$ on $j^1\Phi_l\,(\partial U)$. So we have:

$$
\begin{aligned}
\left.\frac{dA}{dl}\right|_{l=0} &= \left.\frac{dA_U}{dl}\right|_{l=0} \\
&= \int_U \left[\left.\frac{\partial}{dl}q_l^I(j^1\Phi)\right|_{l=0}\frac{\partial L}{\partial q^I}(j^1\Phi) - (-1)^{|A||I|}\left.\frac{\partial}{dl}q_l^I(j^1\Phi)\right|_{l=0}\frac{\partial}{\partial x^A}\frac{\partial L}{\partial \dot{q}_A^I}(j^1\Phi)\right]\beta
\end{aligned}
$$

and, for the arbitrariness of U, ∂U and p, this imply that if Φ is a solution of the theory, then for every U and for every $x \in U$, must be:

$$
\frac{\partial L}{\partial q^I}(j^1\Phi(x)) - (-1)^{|A||I|}\frac{\partial}{\partial x^A}\frac{\partial L}{\partial \dot{q}_A^I}(j^1\Phi(x)) = 0
$$

After all this, the inverse is obvious and so the theorem is proved. ■

The condition (7.7) is the super Euler-Lagrange system of equations for Lagrangian super-field theories. It had already been found by Monterde in [113] (note that in the original paper there is a misprint in the corresponding formula in Remark 3 at the end of section 6). If one checks in [113], or for a more recent presentation in [116] (theorem 7.6), how (7.7) has been proved by Monterde, then one immediately realizes the advantage of using the formalism of Voronov Zorich superforms here adopted.

Note that formula (7.7) appears already in the works of Voronov and Zorich, but in a different context and it has there a different meaning.

We have originally built our field theory starting from the action principle 101. We could use instead (7.7) as a definition for a solution of a Lagrangian theory given by a Lagrangian $n|m$-density L on a supermanifold X. Theorem 102 proves that the two definitions are equivalent when they are both well posed.

The formulation starting from the action principle is the one used mostly by physicists when dealing with supersymmetric field theories treated from the super-fields point of view. Usually, once a super-action on super-fields is given, Berezin integration over odd variable is undertaken, to obtain a classical action on classical bosonic and fermionic fields defined on the body of the domain of the superaction. This leads to classical Euler-Lagrange equation for those fields. Since the starting points are the superaction, as the one I used here, and the same kind of extremal action principle, the approach used by physicists must be equivalent to the one here presented. In fact one could directly show that the super-Euler-Lagrange system of equations (7.7) is equivalent to a classical Euler-Lagrange system for the classical fields which are the coefficients of the superfield Φ in its expansion in the odd variable of the basis supermanifold X, calculated on the body of X itself. These fields are precisely the fermionic and the bosonic fields which appear in the classical Euler-Lagrangian system obtained by the physicists. I will not give this proof, which is unessential having already established with theorem 102 the equivalence between (7.7) and 101. I will treat with some more detail the subject in chapter 9 and I will show an example in section 12.

Chapter 8

Multisymplectic super field theories.

In this chapter I will try and extend the multisymplectic formalism to super field theories starting from the Lagrangian set up proposed in chapter 7 and following the same path undertaken in chapter 2 for classical field theories.

In section 8.1 I will begin to present the covariant Hamiltonian formalism for a generic regular super field theory. I will define the super multimomenta space and the super Legendre tranform.

In section 8.2 I will define the Hamiltonian and I'll show the super version of the Hamilton-Volterra system (2.7).

In section 8.3 I will define a multisymplectic fractional super-form analogous to the form defined in 16. I will prove a theorem analogous to proposition 17. I will define on the super multimomenta space the super Cartan (or Poincaré-Cartan) form and its pullback on the first jet space of the superfield bundle.

In section 8.4 I will then build, on the space of solutions of the theory (the super covariant phase space), the super analogous of the symplectic form (2.11).

8.1 The super-multimomenta space and the super Legendre transform

As in chapter 7, the super bundle of fields is (E, π, X) with base the $n|m$-dimensional supermanifold X and with fiber-type the supermanifold F of dimensions $r|s$. On X we can build the super bundle $\Lambda^{n-1|m}X$ as shown in section 5.2. On E we can build $V_\pi E$, using a construction analogous to the one exposed at the end of chapter 1 and adapted to supermanifolds. The fibers of both these bundles are superspaces which are the direct sum of the even and of the odd part of two supermodules; so in fact we have: $\Lambda^{n-1|m}X = \Lambda_0^{n-1|m}X \oplus \Lambda_1^{n-1|m}X$ and $V_\pi E = V_{\pi,0}E \oplus V_{\pi,1}E$.

In the following we will be interested in the subbundle $B^{n-1|m}X \subset \Lambda^{n-1|m}X$ whose fibers are spanned by the $n-1|m-$forms β_A which I am going to define. Remember that, if $v = (\overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}) \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{n} \times \underbrace{T_1X \times \cdots T_1X}_{m})$ and if $\beta = \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$ is the local canonical $n|m$-form defined with (5.30), then $\beta(v) = \beta(\overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}) \in G^\infty U$.

The superform β can be easyly extended by $\mathbb{R}_S$-linearity to $\Gamma(TX \times \underbrace{T_0X \times \cdots T_0X}_{n-1} \times \underbrace{T_1X \times \cdots T_1X}_{m})$

with the procedure described in proposition 33: by direct calculation it can be shown that this is achieved when in the formula (5.30) we admit that the entries in the first row can be of the opposite parity with-respect to the usual one. Note that, if you have a Berezinian $r|s$-superform ω defined by a superdeterminant on a $n|m$-manifold X, then it is not obvious that its $\mathbb{R}_S$ extension $\hat{\omega}$ is defined by the same superdeterminant defining ω, with first row extended to any parity entry. In fact this is in general false, but it is true for the canonical local $n|m$-form β. I will call this extended superform with the same name β, letting drop the $\widehat{\ }$, unless there is risk of confusion. Then I define locally:

$$\beta_a(\overline{v_1}, \cdots, \overline{v_{n-1}}, \widetilde{v_1}, \cdots, \widetilde{v_m}) := \beta(\frac{\partial}{\partial x^a}, \overline{v_1}, \cdots, \overline{v_{n-1}}, \widetilde{v_1}, \cdots, \widetilde{v_m}) \tag{8.1}$$

and

$$\beta_\alpha(\overline{v_1}, \cdots, \overline{v_{n-1}}, \widetilde{v_1}, \cdots, \widetilde{v_m}) := \beta(\frac{\partial}{\partial x^\alpha}, \overline{v_1}, \cdots, \overline{v_{n-1}}, \widetilde{v_1}, \cdots, \widetilde{v_m}) \tag{8.2}$$

To prove that β_a and β_α are indeed $n-1|m$-forms, we make use of lemma 48:

Proposition 103. *Each β_A is a local $n-1|m$-form and when a coordinates changing is performed we have the following transformation rules:*

$$\beta_{A'} = \mathrm{Ber}(\frac{\partial x'}{\partial x})\frac{\partial x^A}{\partial x^{A'}}\beta_A \tag{8.3}$$

Proof. β_A is a local $n-1|m$-form on the local chart U because obtained by contracting the canonical local $n|m$-form β with the vector field ∂_A and so Lemma 48 applies to it.

If U and U' are two overlapping local charts of X, β and β' are the corresponding local canonical $n|m$-forms and we call $\frac{\partial x'}{\partial x}$ the matrix of the coordinates change, then, by direct calculation, one can see that:

$$\forall v \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{n} \times \underbrace{T_1X \times \cdots T_1X}_{m}), \quad \beta'(v) = \mathrm{Ber}(\frac{\partial x'}{\partial x})\beta(v)$$

So we have:

$$\begin{aligned} &\forall w \in \Gamma(\underbrace{T_0X \times \cdots T_0X}_{n-1} \times \underbrace{T_1X \times \cdots T_1X}_{m}), \\ \beta'_{A'}(w) =& \beta'(\partial_{A'}, w) = \beta'(\frac{\partial x^A}{\partial x^{A'}}\partial_A, w) = \frac{\partial x^A}{\partial x^{A'}}\beta'(\partial_A, w) = \frac{\partial x^A}{\partial x^{A'}}\mathrm{Ber}(\frac{\partial x'}{\partial x})\beta(\partial_A, w) \\ =& \mathrm{Ber}(\frac{\partial x'}{\partial x})\frac{\partial x^A}{\partial x^{A'}}\beta_A(w) \end{aligned}$$

∎

I set by convention $|\beta_a| = 0$ and $|\beta_\alpha| = 1$. This is coherent with the convention used in the second part of this thesis. We can see that, in the same way as $\Lambda_x^{n-1|m}X$, also $B_x^{n-1|m}X$, for every x is a left and right supermodule of dimension $n|m$ which can be interpreted as a superspace of dimension $n+m|m+n$.

As in section 2.1, we can build $Hom_\pi(VE, B^{n-1|m}X)$, which is a fiber bundle over X. The fiber over a point $x \in X$ is the collection of all $\mathbb{R}_S$-linear maps between the supermodules V_eE and $B_x^{n-1|m}X$ for all e, such that $\pi(e) = x$. As explained in section 5.1, these $\mathbb{R}_S$-linear maps between supermodules can be considered as super linear maps between the corresponding superspaces. The space of these maps is itself a supermanifold and precisely it is superdiffeomorph to $l(r|s,n|m) \oplus l(r|s,m|n)$, where $l(r|s,n|m)$ is the set of super linear maps between a superspace of dimension $r|s$ and a superspace of dimension $m|n$.

We can see $Hom_\pi(VE, B^{n-1|m}X)$ as a super fiber bundle over E; then its fiber-type is the superspace $Z = Z_0 \oplus Z_1 \cong l(r|s,n|m) \oplus l(r|s,m|n)$.

We can call $Hom_{\pi,j}(VE, B^{n-1|m}X),\ j = 0,1$ the bundle over E obtained by restricting the fiber-type Z to Z_j. All these super fiber-bundles are defined in a natural way with G^∞ transition functions.

Our super-multimomenta-space (or minimal super-multiphase-space, to recall a terminology used in section 2.1) will be a super-bundle

$$P \subset Hom_\pi(VE, B^{n-1|m}X)$$

to be more precis we will have $P \subset Hom_{\pi,0}(VE, B^{n-1|m}X)$, when $|\mathcal{L}| = 0$, and $P \subset Hom_{\pi,1}(VE, B^{n-1|m}X)$, when $|\mathcal{L}| = 1$. Let's see all this in coordinates, to make it more clear.

On $l(r|s,n|m)$ we can use $(\overline{p_I^A})$ as local coordinates. On $l(r|s,m|n)$ we can use as local coordinates $(\widetilde{p_I^A})$. On $Hom_\pi(VE, B^{n-1|m}X)$ I use an adapted atlas with local charts U with coordinates $(x^A, q^I, \overline{p_I^A}, \widetilde{p_I^A})$. On $Hom_{\pi,0}(VE, B^{n-1|m}X)$ the local coordinates are $(x^A, q^I, \overline{p_I^A})$; on $Hom_{\pi,1}(VE, B^{n-1|m}X)$ the local coordinates are $(x^A, q^I, \widetilde{p_I^A})$. On the corresponding $U \subset X$ are defined the canonical $n|m$-form β and the $n-1|m$-forms β_A.

The point $p \in Hom_\pi(VE, B^{n-1|m}X)$ with local coordinates $(x^A, q^I, \overline{p_I^A}, \widetilde{p_I^A})$ represents the $\mathbb{R}_S$-linear map between the supermodule V_eE, with $e \in E$ with coordinates (x^A, q^I), and $B_x^{n-1|m}X$, with $x \in X$ with coordinates (x^A), which maps the generator $\frac{\partial}{\partial q^I} \in V_eE$ to $(\overline{p_I^A} + \widetilde{p_I^A})\beta_A\big|_x = p_I^A \beta_A\big|_x$, with the understandable notation $p_I^A = \overline{p_I^A} + \widetilde{p_I^A}$. Note that $\overline{p_I^A}$, $\widetilde{p_I^A}$ and p_I^A act on the left of β_A.

$Hom_\pi(VE, B^{n-1|m}X)$ is a natural bundle over E. When coordinates are changed on E, the p coordinates change as follow:

$$p'^{A'}_{I'} = \mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}} p_I^A \frac{\partial x^{A'}}{\partial x^A} \tag{8.4}$$

or equivalently:

$$\begin{aligned} \overline{p'^{A'}_{I'}} &= \mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}} \overline{p_I^A} \frac{\partial x^{A'}}{\partial x^A} \\ \widetilde{p'^{A'}_{I'}} &= \mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}} \widetilde{p_I^A} \frac{\partial x^{A'}}{\partial x^A} \end{aligned} \tag{8.5}$$

so that $dq^I\big|_{VE} \otimes p_I^A \beta_A$ is globally well defined.

We can now define P. On a local chart U, I define the super Legendre transform $\mathbb{FL}$, a G^∞ map between $J^1\pi\big|_U$ and $Hom_\pi(VE\big|_U, B^{n-1|m}U)$. Suppose that on $J^1\pi\big|_U$ is defined a Lagrangian density $L = \overline{L} + \widetilde{L}$ so that $\overline{L}$ is even and $\widetilde{L}$ is odd, then:

$$\mathbb{FL}\left(x^A, q^I, \dot{q}_A^I\right) = \left(x^A, q^I, (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right), \frac{\partial \widetilde{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right)\right) \tag{8.6}$$

Keeping in mind the transformation rules of $\dot{q}_A^I$

$$\dot{q}_{A'}^{I'} = \frac{\partial x^A}{\partial x^{A'}} \dot{q}_A^I \frac{\partial q^{I'}}{\partial q^I} + \frac{\partial x^A}{\partial x^{A'}}\frac{\partial q^{I'}}{\partial x^A} \tag{8.7}$$

we find that:

Proposition 104. *Formula* (8.6) *is a well-posed definition of a G^∞ map* $\mathbb{FL} : J^1\pi \longrightarrow Hom_\pi(VE, B^{n-1|m}X)$.

Proof. It is enough to show that $(-1)^{|A|}\frac{\partial\overline{L}}{\partial\dot{q}^I_A}$ and $\frac{\partial\widetilde{L}}{\partial\dot{q}^I_A}$ transform like in (8.5). In fact, with a bit of calculations and using (8.7), one finds that:

$$\begin{aligned}
(-1)^{|A'|}\frac{\partial\overline{L'}}{\partial\dot{q}^{I'}_{A'}} &= (-1)^{|A'|}(-1)^{(|I'|+|A'|)(|A|+|A'|)}\,\mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial x^{A'}}{\partial x^A}\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\overline{L}}{\partial\dot{q}^I_A} \\
&= (-1)^{|A'|}(-1)^{(|A|+|A'|)(|A|+|A'|)}\,\mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\overline{L}}{\partial\dot{q}^I_A}\frac{\partial x^{A'}}{\partial x^A} \\
&= (-1)^{|A|}\,\mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\overline{L}}{\partial\dot{q}^I_A}\frac{\partial x^{A'}}{\partial x^A}
\end{aligned} \tag{8.8}$$

and

$$\begin{aligned}
\frac{\partial\widetilde{L'}}{\partial\dot{q}^{I'}_{A'}} &= (-1)^{(|I'|+|A'|)(|A|+|A'|)}\,\mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial x^{A'}}{\partial x^A}\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\widetilde{L}}{\partial\dot{q}^I_A} \\
&= (-1)^{(|A|+|A'|)(|A|+|A'|+1)}\,\mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\widetilde{L}}{\partial\dot{q}^I_A}\frac{\partial x^{A'}}{\partial x^A} \\
&= \mathrm{Ber}(\frac{\partial x}{\partial x'})\frac{\partial q^I}{\partial q^{I'}}\frac{\partial\widetilde{L}}{\partial\dot{q}^I_A}\frac{\partial x^{A'}}{\partial x^A}
\end{aligned} \tag{8.9}$$

and (8.8) and (8.9) are equivalent to (8.5).

Since L is G^∞, by (8.6) one can see that also FL is G^∞. ■

So I can give the following:

Definition 105. *Whenever $P := \mathbb{FL}(J^1\pi) \subset Hom_\pi(VE, B^{n-1|m}X)$ inherits from $Hom_\pi(VE, B^{n-1|m}X)$ the differential structure of a supermanifold, we say that the super-multimomenta-space, or minimal super-multiphase-space, of a super-field theory is the super-fiber-bundle $P = (P, \pi_P, X)$ with total space P, with base X and with projection π_P.*

One can compare (8.6) with the definition of super-Legendre transformation given in [26] for the case of supermechanics and one can notice the analogy.

Remark 106. *The conditions under which $\mathbb{FL}(J^1\pi)$ is indeed a sub-supermanifolds of $Hom_\pi(VE, B^{n-1|m}X)$ are not easy to be established and a future work in this direction can be foreseen. The difficult point here is that it is not immediate to establish when the image of a G^∞ map from a supermanifold to an other target supermanifold is indeed a sub-supermanifold of the target. From the theorem of the inverse function of G^∞ maps, see [133], it is however possible to deduce that if $\mathbb{FL}$ is invertible on its image and its tangent map is invertible too, then its inverse $\mathbb{FL}^{-1}$ is also G^∞ and therefore the image of the Legendre transform $\mathbb{FL}(J^1\pi)$ is a supermanifold. This fact is useful to establish that, when the Lagrangian is regular, then P is always well defined.*

The definition of the super Legendre transformation (8.6) explains the conventions chosen for the coordinates p^A_I: from (8.6) one finds that we must have $|p^A_I| = |A| + |I| + |L|$, so that in fact $p^A_I = \overline{p^A_I}$ when L is purely even and $p^A_I = \widetilde{p^A_I}$ when L is purely odd. Keeping in mind all the degree conventions used until now and using (8.6) one can directly check that:

Proposition 107. *If we consider P as a bundle over E, then its fiber-type is a submanifold of*

– $l(r|s, n|m)$ *when* $|\mathcal{L}| = |L| = 0$
– $l(r|s, m|n)$ *when* $|\mathcal{L}| = |L| = 1$

and we have that:

– $P \subset Hom_{\pi,0}(VE, B^{n-1|m}X)$*, when* $|\mathcal{L}| = 0$
– $P \subset Hom_{\pi,1}(VE, B^{n-1|m}X)$*, when* $|\mathcal{L}| = 1$

which is the proposition already anticipated above.

Sometime in the following, on a local chart of P, I will need the local functions (x^A, q^I, p_I^A), remembering that $\left|p_I^A\right| = |A| + |I| + |L|$ or the local functions $(x^A, q^I, \overline{p_I^A}, \widetilde{p_I^A})$, with $\left|\overline{p_I^A}\right| = |A| + |I|$ and $\left|\widetilde{p_I^A}\right| = |A| + |I| + 1$.

As in the classical case, from $\mathbb{FL}$ we can build a map between $\Gamma\left(J^1\pi\right)$ and $\Gamma(P)$.

8.2 The Hamiltonian and super Hamilton-Volterra equations

On P it is possible to define the Hamiltonian function H as we did in section 2.2 for classical theories:

$$H(x^A, q^I, p_I^A) := \dot{q}_A^I \widetilde{p_I^A} + (-1)^{|A|} \dot{q}_A^I \overline{p_I^A} - L\left(x^A, q^I, \dot{q}_A^I\right) \tag{8.10}$$

where we assume that $\dot{q}_A^I$ is a solution of

$$(-1)^{|A|(|L|+1)} \frac{\partial L}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right) = p_I^A \tag{8.11}$$

Or, equivalently, that it is at the same time a solution of the following two equations:

$$\begin{aligned} (-1)^{|A|} \frac{\partial \overline{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right) &= \overline{p_I^A} \\ \frac{\partial \widetilde{L}}{\partial \dot{q}_A^I}\left(x^A, q^I, \dot{q}_A^I\right) &= \widetilde{p_I^A} \end{aligned} \tag{8.12}$$

As in the classical case, with the help of the super version of the implicit function theorem, it can be shown that this is a good definition because the value of H does not depend on the choice of the particular solution $\dot{q}_A^I$, provided that some topological conditions of connectedness are satisfied.

Since L is G^∞, also H is G^∞ whenever $\dot{q}_A^I$ is a G^∞ function of $\left(x^A, q^I, p_I^A\right)$ (note that we are assuming that P has the structure of a G^∞ supermanifold, see definition 105 and remark 106).

H transforms in this way:

$$\begin{aligned}
&H'\left(x^{A'},q^{I'},\dot{q}^{I'}_{A'}\right)=\\
&=\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\dot{q}^I_A\widetilde{p^A_I}+(-1)^{|A|}\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\dot{q}^I_A\overline{p^A_I}+\\
&-\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)L\left(x^A,q^I,\dot{q}^I_A\right)+\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\frac{\partial q^{I'}}{\partial x^A}(x^A,q^I)\frac{\partial q^I}{\partial q^{I'}}(x^A,q^I)\widetilde{p^A_I}+\\
&+(-1)^{|A|}\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\frac{\partial q^{I'}}{\partial x^A}(x^A,q^I)\frac{\partial q^I}{\partial q^{I'}}(x^A,q^I)\overline{p^A_I}=\\
&=\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)H\left(x^A,q^I,\dot{q}^I_A\right)+\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\frac{\partial q^{I'}}{\partial x^A}(x^A,q^I)\frac{\partial q^I}{\partial q^{I'}}(x^A,q^I)\widetilde{p^A_I}+\\
&+(-1)^{|A|}\operatorname{sdet}\left(\frac{\partial x}{\partial x'}\right)(x^A)\frac{\partial q^{I'}}{\partial x^A}(x^A,q^I)\frac{\partial q^I}{\partial q^{I'}}(x^A,q^I)\overline{p^A_I}
\end{aligned}\tag{8.13}$$

H can be split in two components which are both G^∞:

$$H=\overline{H}+\widetilde{H}\tag{8.14}$$

where:

$$\begin{aligned}
\overline{H}&=(-1)^{|A|}\dot{q}^I_A\overline{p^A_I}-\overline{L}\left(x^A,q^I,\dot{q}^I_A\right)\\
\widetilde{H}&=\dot{q}^I_A\widetilde{p^A_I}-\widetilde{L}\left(x^A,q^I,\dot{q}^I_A\right)
\end{aligned}\tag{8.15}$$

By the definition of P every section $z\in\Gamma(P)$ is the image trough $\mathbb{FL}$ of a section $s\in\Gamma(J^1\pi)$. If both $\mathbb{FL}$ and its tangent map are invertible, we say that the Lagrangian is regular, see remark 106. This implies that the equation (8.11) has no more than one solution. Comparing this notion of regularity with the one given in [83] and [26] for supermechanics Lagrangians (*id est* for theories defined on a base manifold X of dimension $1|0$), we see that they are equivalent. My definition of regularity is also equivalent to the one given by Monterde and Muñoz Masqué in [114] and [115] for theories defined on $X=\mathbb{R}^{1|1}$.

The notion of regular Lagrangian is not, however, the most useful for what it follows. I define now some other related notions which reveal to be of some use:

Definition 108. *A Lagrangian L is said to be purely even if $L=\overline{L}$.*

A Lagrangian L is said to be purely odd if $L=\widetilde{L}$.

A Lagrangian L is said to be even-regular if the map $\overline{\mathbb{FL}}:\ \dot{q}^I_A\ \longrightarrow\ \overline{p^A_I}=(-1)^{|A|}\frac{\partial\overline{L}}{\partial\dot{q}^I_A}\left(x^A,q^I,\dot{q}^I_A\right)$ is invertible on its image and its inverse is G^∞.

A Lagrangian L is said to be odd-regular if the map $\overline{\mathbb{FL}}:\ \dot{q}^I_A\ \longrightarrow\ \frac{\partial\widetilde{L}}{\partial\dot{q}^I_A}\left(x^A,q^I,\dot{q}^I_A\right)$ is invertible on its image and its inverse is G^∞.

A Lagrangian L is said to be purely even regular if it is purely even and regular.

A Lagrangian L is said to be purely odd regular if it is purely odd and regular.

Note that when L is even-regular or odd-regular, then P is always well defined (see remark 106). When L is even-regular, $\overline{p^A_I}$ can be used as coordinates on the fiber of P. When L is odd-regular, $\widetilde{p^A_I}$ can be used as coordinates on the fiber of P.

Remark 109. *If a Lagrangian L is purely even regular, then the superdimension of $J^1\pi$ must be equal to the superdimension of $Hom_{\pi,0}(VE,B^{n-1|m}X)$.*

If a Lagrangian L is purely odd regular, then the superdimension of $J^1\pi$ must be equal to the superdimension of $Hom_{\pi,1}(VE, B^{n-1|m}X)$. This is quite a strong condition, because it immediately implies that either the dimension of the base X is $n|n$, or the dimension of the fiber F of the field bundle is $r|r$. This condition, nevertheless, doesn't prevent to have interesting purely odd regular Lagrangians: see for example the Lagrangian of the superparticle introduced in section 12.2.

We fix the following:

Definition 110. *A section $z \in \Gamma(P)$ is called a lifted-section if $\mathbb{FL}^{-1}z = j^1\Phi$ for a section $\Phi \in \Gamma(E)$.*

We can then characterize locally the lifted-sections of P with the help of H. We have indeed:

Theorem 111. *When the Lagrangian L is even-regular, a section $z \in \Gamma\left(\mathbb{FL}(J^1\pi)\right)$ is a lifted-section if and only if for every local chart U and for every $x \in U$:*

$$(-1)^{|I|}\frac{\partial q^I}{\partial x^A}\left(z(x)\right) = \frac{\partial \overline{H}}{\partial \overline{p^A_I}}\left(z(x)\right) \tag{8.16}$$

When the Lagrangian L is odd-regular, a section $z \in \Gamma\left(\mathbb{FL}(J^1\pi)\right)$ is a lifted-section if and only if for every local chart U and for every $x \in U$:

$$\frac{\partial q^I}{\partial x^A}\left(z(x)\right) = \frac{\partial \widetilde{H}}{\partial \widetilde{p^A_I}}\left(z(x)\right) \tag{8.17}$$

Proof. To make calculations shorter, in this proof I indicate with the symbol p^A_I a coordinate which can be alternatively $\overline{p^A_I}$ or $\widetilde{p^A_I}$.

For the definition of H, and because $\mathbb{FL}^{-1}z$ is well defined being L regular, we have that:

$$\begin{aligned}\frac{\partial H}{\partial p^A_I}\left(z(x)\right) &= (-1)^{|B|(|L|+1)}\frac{\partial \dot{q}^J_B}{\partial p^A_I}p^B_J\left(z(x)\right)\\ &+ (-1)^{|B|(|L|+1)}(-1)^{(|A|+|I|+|L|)(|B|+|J|)}\dot{q}^J_B\delta^I_J\delta^A_B\left(z(x)\right) - \frac{\partial \dot{q}^J_B}{\partial p^A_I}\frac{\partial L}{\partial \dot{q}^J_B}\left(\mathbb{FL}^{-1}z(x)\right)\end{aligned}$$

For (8.11), we have that $\frac{\partial L}{\partial \dot{q}^J_B}\left(\mathbb{FL}^{-1}z(x)\right) = (-1)^{|B|(|L|+1)}p^B_J(z(x))$ and so:

$$\begin{aligned}\frac{\partial H}{\partial p^A_I}\left(z(x)\right) &= (-1)^{|A|(|L|+1)}(-1)^{(|A|+|I|+|L|)(|A|+|I|)}\dot{q}^I_A\left(z(x)\right)\\ &= (-1)^{|I|(|L|+1)}\dot{q}^I_A\left(z(x)\right)\end{aligned}$$

But $\dot{q}^I_A\left(z(x)\right) = \dot{q}^I_A\left(\mathbb{FL}^{-1}z(x)\right)$.

If $\mathbb{FL}^{-1}z = j^1\Phi$ with $\Phi \in \Gamma(E)$, then $\dot{q}^I_A\left(\mathbb{FL}^{-1}z(x)\right) = \frac{\partial q^I}{\partial x^A}$ and so one implication is proved.

Inversely, if $\frac{\partial H}{\partial p^A_I}\left(z(x)\right) = (-1)^{(|I|)(|L|+1)}\frac{\partial q^I}{\partial x^A}\left(z(x)\right)$, then we must have $\dot{q}^I_A\left(\mathbb{FL}^{-1}z(x)\right) = \frac{\partial q^I}{\partial x^A}\left(z(x)\right)$; and the other implication is proved too. ∎

Note that, as in the classical case, H is not really a function defined on P, but a more complicated object, as shown by (8.13). As in the classical case (see section 2.2), one could first define a super-multiphase-space Z, find out that Z is a vector bundle over the minimal super-multiphase-space P, and then define H as a section of Z (see [82] and [24], [63], [108] for the construction of Z in the classical case).

On a local chart, with canonical horizontal $n|m$-form β, one can define the local Hamiltonian $n|m$-form $\mathcal{H} := H\beta$ but in general this will not lead to a well defined Hamiltonian $n|m$-form on P.

I can now prove the extension of Volterra theorem 12 to super-field-theories.

Theorem 112. *Let $\overline{L}$ be a purely even regular Lagrangian function on $J^1\pi$ and let $\overline{H}$ be its corresponding Hamiltonian function on P. If a field $\Phi \in \Gamma(\pi)$ is a solution of the Euler-Lagrange system of equations* (7.7)*, $\forall U$ local chart and $\forall x \in U \subset X$, id est if it is a solution of the Lagrangian field theory, then $\forall U$ local chart and $\forall x \in U \subset X$, $z = \mathbb{FL} \circ j^1\Phi$ is a solution of the system* (8.18)*.*

Conversely if $z \in \Gamma(P)$, $\forall U$ local chart and $\forall x \in U \subset X$, satisfies:

$$\begin{cases} (-1)^{|I|} \dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \overline{H}}{\partial \overline{p^A_I}}(z(x)) \\ (-1)^{|A|}(-1)^{|A||I|} \dfrac{\partial \overline{p^A_I}}{\partial x^A}(z(x)) = -\dfrac{\partial \overline{H}}{\partial q^I}(z(x)) \end{cases} \tag{8.18}$$

then there is a section $\Phi \in \mathscr{E}$, solution of the field theory, such that $z = \mathbb{FL}j^1\Phi$.

Let $\widetilde{L}$ be a purely odd regular Lagrangian function on $J^1\pi$ and let $\widetilde{H}$ be its corresponding Hamiltonian function on P, then if a field $\Phi \in \Gamma(\pi)$ is a solution of the Euler-Lagrange system of equations (7.7)*, $\forall U$ local chart and $\forall x \in U \subset X$, id est if it is a solution of the Lagrangian field theory, then $\forall U$ local chart and $\forall x \in U \subset X$, $z = \mathbb{FL} \circ j^1\Phi$ is a solution of the system* (8.19).

Connversely if $z \in \Gamma(P)$, $\forall U$ local chart and $\forall x \in U \subset X$, satisfies:

$$\begin{cases} \dfrac{\partial q^I}{\partial x^A}(z(x)) = \dfrac{\partial \widetilde{H}}{\partial \widetilde{p^A_I}}(z(x)) \\ (-1)^{|A||I|} \dfrac{\partial \widetilde{p^A_I}}{\partial x^A}(z(x)) = -\dfrac{\partial \widetilde{H}}{\partial q^I}(z(x)) \end{cases} \tag{8.19}$$

then there is a section $\Phi \in \mathscr{E}$, solution of the field theory, such that $z = \mathbb{FL}j^1\Phi$.

Proof. I will prove the theorem for $L = \overline{L}$ purely even regular. Similar arguments would give the proof for $L = \widetilde{L}$ purely odd regular.

For the first implication: by theorem 111 we already know that $\mathbb{FL}j^1\Phi(x)$ satisfies the first equation in (8.18). From the definition of H, let's now calculate:

$$\begin{aligned} \frac{\partial \overline{H}}{\partial q^I}\left(\mathbb{FL}j^1\Phi(x)\right) =& (-1)^{|A|} \frac{\partial \dot{q}^J_A}{\partial q^I}\left(j^1\Phi(x)\right) \overline{p^A_J}\left(\mathbb{FL}j^1\Phi(x)\right) - \frac{\partial \overline{L}}{\partial q^I}\left(j^1\Phi(x)\right) \\ &- \frac{\partial \dot{q}^J_A}{\partial q^I}\left(j^1\Phi(x)\right) \frac{\partial \overline{L}}{\partial \dot{q}^J_A}\left(j^1\Phi(x)\right) \end{aligned}$$

and since $(-1)^{|A|}\overline{p^A_J}\left(\mathbb{FL}j^1\Phi(x)\right) = \frac{\partial \overline{L}}{\partial \dot{q}^J_A}\left(j^1\Phi(x)\right)$, we have that:

$$\frac{\partial \overline{H}}{\partial q^I}\left(\mathbb{FL}j^1\Phi(x)\right) = -\frac{\partial \overline{L}}{\partial q^I}\left(j^1\Phi(x)\right)$$

and when $\Phi \in \mathscr{E}$, (7.7) holds and we have:

$$-\frac{\partial H}{\partial q^I}\left(\mathbb{F}\mathbb{L}j^1\Phi(x)\right) = (-1)^{|A||I|}\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\left(j^1\Phi(x)\right) = (-1)^{|A|}(-1)^{|A||I|}\frac{\partial p^A_I}{\partial x^A}\left(\mathbb{F}\mathbb{L}j^1\Phi(x)\right)$$

and the first implication is proved.
For the second implication: if z satisfies the first equation in (8.18), then, by theorem 111, it exists a section $\Phi \in \Gamma(\pi)$ so that $z = \mathbb{F}\mathbb{L}j^1\Phi$. Then:

$$-\frac{\partial \overline{H}}{\partial q^I}\left(z(x)\right) = -\frac{\partial \overline{H}}{\partial q^I}\left(\mathbb{F}\mathbb{L}j^1\Phi\right) = \frac{\partial \overline{L}}{\partial q^I}\left(j^1\Phi(x)\right)$$

and

$$\begin{aligned}(-1)^{|A|}(-1)^{|A||I|}\frac{\partial \overline{p^A_I}}{\partial x^A}\left(z(x)\right) &= (-1)^{|A|}(-1)^{|A||I|}\frac{\partial \overline{p^A_I}}{\partial x^A}\left(\mathbb{F}\mathbb{L}j^1\Phi\right) = \\ &= (-1)^{|A||I|}\frac{d}{dx^A}\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\left(j^1\Phi(x)\right)\end{aligned}$$

and if $z = \mathbb{F}\mathbb{L}j^1\Phi$ satisfies the second equation in (8.18), then $j^1\Phi$ must satisfy (7.7) and so Φ is a solution of the theory and $\Phi \in \mathscr{E}$. ■

A first version of an Hamilton-like system of equations valid in the case when $X = \mathbb{R}^{1|1}$, can be found in [115], where Monterde and Muñoz Masqué treat supermechanics as a theory on $\mathbb{R}^{1|1}$. The system of equations found there is however much more complicated than system (8.18) and its analogy with the classical system 2.7 or with the classical Hamilton equations for mechanics is not immediately apparent.

Note that, when L has not a pure parity, but it is a mix of even and odd components (let's call it a non-homogeneous Lagrangian), then the situation is rather more complicated. Indeed when L is at the same time even-regular and odd-regular, the Euler-Lagrange system is clearly in general overdetermined. When it is satisfied then the corresponding system of equations (8.18) and (8.19) are satisfied simultaneously. The converse holds when (8.18) and (8.19) are simultaneously satisfied.

In general when L is even-regular but not purely even, then in order to be a solution of the theory a section of P must satisfy (8.19) plus some other conditions that can be interpreted as constraints. A similar situation occurs when L is odd-regular but not purely odd.

An interesting situation could occur when L is regular without being neither even-regular, nor odd-regular. I will not treat these more complicated situations in this thesis.

I am not sure that non-homogeneous Lagrangians could be of interest for Physics, and I don't know if non-trivial, mathematically interesting, examples of non-homogeneous Lagrangians (regular or not) can be given, such that the corresponding P is a well defined supermanifold, see remarks 106 and 109. In the following I will not treat in details non-homogeneous theories.

8.3 The super-multisymplectic form

In this section I will give a geometric formulation of a superfield theory based on the use of a multisymplectic fractional superform.

The image of a section $z \in \Gamma(P)$ is a $n|m$-dimensional surface in the total space P. As we did in section 2.3 for the classical case, we can address ourselves the question: if we have a $n|m$-dimensional submanifold $G \subset P$, when does a $\Phi \in \mathscr{E}$ exist so that $G = \mathbb{F}\mathbb{L} \circ j^1\phi(X)$? What geometric conditions has G to satisfy?
Exactly as in the classical case:

1. G has to be the image of a section $z \in \Gamma\big(P\big)$, so: $\exists z \in \Gamma\big(P\big)$ so that $G = z(X)$
2. z has to be the image trough $\mathbb{FL}$ of a section $s \in \Gamma\big(J^1\pi\big)$, so: $\exists s \in \Gamma\big(J^1\pi\big)$ so that $z = \mathbb{FL}(s)$;
3. s must belong to $j^1\Gamma\big(E\big)$, so: $\exists \Phi \in \Gamma\big(E\big)$ so that $s = j^1\Phi$;
4. Φ must be a solution of the theory, so: $\Phi \in \mathscr{E}$; or, which is equivalent, Φ has to satisfy one of the conditions in (7.4), (7.5), (7.6), or (7.7).

We call Hamiltonian a $n|m$-submanifold of P which satisfies the above four conditions; we call $\mathscr{G}$ the space of all Hamiltonian submanifolds. Then $\mathscr{G}$ and $\mathscr{E}$ are in one-to-one correspondence and they are indeed diffeomorph if a suitable G^∞-differential structure is put on them.

As in the classical case, if we call i the immersion of G in P, we have:

Proposition 113. *Let U be a local chart on P and let β be the local canonical $n|m$-form on U, let $G \subset P$ be a simply connected $n|m$-dimensional submanifold of P and i its immersion map in P, then G is the image of a section $z \in \Gamma\big(P\big)$ if and only if $\forall U$ local chart, $i^*\beta \neq 0$.*

Condition 2 is automatically satisfied because $P = \mathbb{FL}(J^1\pi)$.

If L is even-regular or odd-regular then, with understandable notations, theorem 111 shows that:

$$\frac{\partial H}{\partial p_I^A}\left(\mathbb{FL}s(x)\right) = \big(-1\big)^{(|L|+1)(|I|)}\dot{q}_A^I\left(s(x)\right)$$

Therefore we have that:

$$g^I := \mathbb{FL}_* c^I = dq^I - \big(-1\big)^{(|L|+1)(|I|)} dx^A \frac{\partial H}{\partial p_I^A} \tag{8.20}$$

where c^I are the contact local extended-forms on $J^1\pi$ defined in chapter 7 with (7.1). Identity (8.20) is a shortcut for one of the two identities:

$$g^I = dq^I - dx^A \frac{\partial \widetilde{H}}{\partial \widetilde{p_I^A}} \tag{8.21}$$

and

$$g^I = dq^I - \big(-1\big)^{(|I|)} dx^A \frac{\partial \overline{H}}{\partial \overline{p_I^A}} \tag{8.22}$$

The g^I can be considered as "naive" local 1-forms on P or local $1|0$-extended-forms on P: I call them contact forms.

If we call section-submanifolds those $n|m$-submanifolds $G \subset P$ which satisfy condition 1 , and we call lifted-submanifolds those $n|m$-submanifolds $G \subset P$ which satisfy conditions 1, 2 and 3, then Condition 3 translates to the following:

Proposition 114. *Let L be an even-regular or an odd-regular Lagrangian function on $J^1\pi$ and H be its corresponding Hamiltonian function on the super-multimomenta-space P, let g^I be the contact local forms on $\mathbb{FL}(J^1\pi)$, then a section-submanifold $G \subset P$ with $G = z(X)$ for $z \in \Gamma\big(\mathbb{FL}(J^1\pi)\big)$ is a lifted-submanifolds of P if and only if $\forall\ g^I :\ z^*g^I = 0$.*

Note that this condition is equivalent to the first equation of 8.18 or 8.19.

I now show what is the super-correspondent of the multisymplectic form.

Definition 115. *Let L be an even-regular or an odd-regular Lagrangian function on $J^1\pi$ and H be its corresponding Hamiltonian function on the super-multimomenta-space P, let U be a local chart of P. I call the super-multisymplectic form the fractional $n+1|m$-local-form ω, which is a G^∞-map from $\underbrace{T_0U \times \cdots \times T_0U}_{n+1} \times \underbrace{T_1U \times \cdots \times T_1U}_{m}$ to $\mathbb{R}_S$, defined locally by:*

$$\forall\,(\overline{v_0},\ \overline{v_1},\cdots,\overline{v_n},\widetilde{v}_1,\cdots,\widetilde{v_m}) \in \underbrace{T_0U \times \cdots \times T_0U}_{n+1} \times \underbrace{T_1U \times \cdots \times T_1U}_{m}$$

$$\omega\,(\overline{v_0},\ \overline{v_1},\cdots,\overline{v_n},\widetilde{v}_1,\cdots,\widetilde{v_m}) =$$

$$= -\operatorname{sdet}_{n+2|m}\begin{pmatrix}
\overline{v_0^{q^I}} & \overline{v_0^{p_I^a}} & \overline{v_0^{x^1}} & \cdots & \overline{v_0^{x^a}} & \cdots & \overline{v_0^{x^n}} & \overline{v_0^{x^{n+1}}} & \cdots & \overline{v_0^{x^{n+m}}} \\
\overline{v_1^{q^I}} & \overline{v_1^{p_I^a}} & \overline{v_1^{x^1}} & \cdots & \overline{v_1^{x^a}} & \cdots & \overline{v_1^{x^n}} & \overline{v_1^{x^{n+1}}} & \cdots & \overline{v_1^{x^{n+m}}} \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\overline{v_2^{q^I}} & \overline{v_2^{p_I^a}} & \overline{v_2^{x^1}} & \cdots & \overline{v_2^{x^a}} & \cdots & \overline{v_2^{x^n}} & \overline{v_2^{x^{n+1}}} & \cdots & \overline{v_2^{x^{n+m}}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{v_n^{q^I}} & \overline{v_n^{p_I^a}} & \overline{v_n^{x^1}} & \cdots & \vdots & \cdots & \overline{v_n^{x^n}} & \overline{v_n^{x^{n+1}}} & \cdots & \overline{v_n^{x^{n+m}}} \\
\widetilde{v_1^{q^I}} & \widetilde{v_1^{p_I^a}} & \widetilde{v_1^{x^1}} & \cdots & \vdots & \cdots & \widetilde{v_1^{x^n}} & \widetilde{v_1^{x^{n+1}}} & \cdots & \widetilde{v_1^{x^{n+m}}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{v_m^{q^I}} & \widetilde{v_m^{p_I^a}} & \widetilde{v_m^{x^1}} & \cdots & \vdots & \cdots & \widetilde{v_m^{x^n}} & \widetilde{v_m^{x^{n+1}}} & \cdots & \widetilde{v_m^{x^{n+m}}}
\end{pmatrix} +$$

$$-(-1)^{|L|+1}\frac{\partial}{\partial\eta}\operatorname{sdet}_{n+2|m}\begin{pmatrix}
\overline{v_0^{q^I}} & \overline{v_0^{p_I^\alpha}} & \overline{v_0^{x^1}} & \cdots & \overline{v_0^{x^n}} & \overline{v_0^{x^{n+1}}} & \cdots & \overline{v_0^{x^{n+\alpha}}} & \cdots & \overline{v_0^{x^{n+m}}} \\
\overline{v_1^{q^I}} & \overline{v_1^{p_I^\alpha}} & \overline{v_1^{x^1}} & \cdots & \overline{v_1^{x^n}} & \overline{v_1^{x^{n+1}}} & \cdots & \overline{v_1^{x^{n+\alpha}}} & \cdots & \overline{v_1^{x^{n+m}}} \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \eta & \cdots & 0 \\
\overline{v_2^{q^I}} & \overline{v_2^{p_I^\alpha}} & \overline{v_2^{x^1}} & \cdots & \overline{v_2^{x^n}} & \overline{v_2^{x^{n+1}}} & \cdots & \overline{v_2^{x^{n+\alpha}}} & \cdots & \overline{v_2^{x^{n+m}}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\overline{v_n^{q^I}} & \overline{v_n^{p_I^\alpha}} & \overline{v_n^{x^1}} & \cdots & \overline{v_n^{x^n}} & \overline{v_n^{x^{n+1}}} & \cdots & \vdots & \cdots & \overline{v_n^{x^{n+m}}} \\
\widetilde{v_1^{q^I}} & \widetilde{v_1^{p_I^\alpha}} & \widetilde{v_1^{x^1}} & \cdots & \widetilde{v_1^{x^n}} & \widetilde{v_1^{x^{n+1}}} & \cdots & \vdots & \cdots & \widetilde{v_1^{x^{n+m}}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\widetilde{v_m^{q^I}} & \widetilde{v_m^{p_I^\alpha}} & \widetilde{v_m^{x^1}} & \cdots & \widetilde{v_m^{x^n}} & \widetilde{v_m^{x^{n+1}}} & \cdots & \vdots & \cdots & \widetilde{v_m^{x^{n+m}}}
\end{pmatrix} +$$

$$-\operatorname{sdet}_{n+1|m}\begin{pmatrix}
\overline{v_0^{q^I}}\,\frac{\partial H}{\partial q^I} & \overline{v_0^{x^1}} & \cdots & \overline{v_0^{x^n}} & \overline{v_0^{x^{n+1}}} & \cdots & \overline{v_0^{x^{n+m}}} \\
\overline{v_1^{q^I}}\,\frac{\partial H}{\partial q^I} & \overline{v_1^{x^1}} & \cdots & \overline{v_1^{x^n}} & \overline{v_1^{x^{n+1}}} & \cdots & \overline{v_1^{x^{n+m}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{v_n^{q^I}}\,\frac{\partial H}{\partial q^I} & \overline{v_n^{x^1}} & \cdots & \overline{v_n^{x^n}} & \overline{v_n^{x^{n+1}}} & \cdots & \overline{v_n^{x^{n+m}}} \\
\widetilde{v_1^{q^I}}\,\frac{\partial H}{\partial q^I} & \widetilde{v_1^{x^1}} & \cdots & \widetilde{v_1^{x^n}} & \widetilde{v_1^{x^{n+1}}} & \cdots & \widetilde{v_1^{x^{n+m}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{v_m^{q^I}}\,\frac{\partial H}{\partial q^I} & \widetilde{v_m^{x^1}} & \cdots & \widetilde{v_m^{x^n}} & \widetilde{v_m^{x^{n+1}}} & \cdots & \widetilde{v_m^{x^{n+m}}}
\end{pmatrix} +$$

$$
-\operatorname{sdet}_{n+1|m}\begin{pmatrix}
\overline{v_0^{p_I^A}}\frac{\partial H}{\partial p_I^A} & v_0^{x^1} & \cdots & v_0^{x^n} & v_0^{x^{n+1}} & \cdots & v_0^{x^{n+m}} \\
\overline{v_1^{p_I^A}}\frac{\partial H}{\partial p_I^A} & \overline{v_1^{x^1}} & \cdots & \overline{v_1^{x^n}} & \overline{v_1^{x^{n+1}}} & \cdots & \overline{v_1^{x^{n+m}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{v_n^{p_I^A}}\frac{\partial H}{\partial p_I^A} & \overline{v_n^{x^1}} & \cdots & \overline{v_n^{x^n}} & \overline{v_n^{x^{n+1}}} & \cdots & \overline{v_n^{x^{n+m}}} \\
\widetilde{v_1^{p_I^A}}\frac{\partial H}{\partial p_I^A} & \widetilde{v_1^{x^1}} & \cdots & \widetilde{v_1^{x^n}} & \widetilde{v_1^{x^{n+1}}} & \cdots & \widetilde{v_1^{x^{n+m}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{v_m^{p_I^A}}\frac{\partial H}{\partial p_I^A} & \widetilde{v_m^{x^1}} & \cdots & \widetilde{v_m^{x^n}} & \widetilde{v_m^{x^{n+1}}} & \cdots & \widetilde{v_m^{x^{n+m}}}
\end{pmatrix} \tag{8.23}
$$

where the symbols $v^{p_I^A}$ are shortcuts for $v^{\overline{p_I^A}}$ or $v^{\widetilde{p_I^A}}$, the symbols $\frac{\partial H}{\partial p_I^A}$ are shortcuts for $\frac{\partial \overline{H}}{\partial \overline{p_I^A}}$ or $\frac{\partial \widetilde{H}}{\partial \widetilde{p_I^A}}$, according to the parity of L; and where sum over repeated indexes (also when in different columns) is understood as usual.

A much more concise expression for ω will be given in Formula (8.24).

It is still to be proven that ω is in fact a $n+1|m$-form.

Proposition 116. *The super-multisymplectic function ω is a local $n+1|m$-form.*

Proof. Note that in each of the four matrices defining ω all entries respect the parity required by a classical supermatrix with the exceptions of at most the entries in the first and the second columns. Then this proposition is a direct consequence of theorem 38 and Lemma 48. ∎

Proposition 117. *The super-multisymplectic function ω is well defined globally.*

Proof. The proof relies on the fact that, with the use of (5.40), (5.52) and (5.53) and using the proprieties of the wedge product (5.55) and (5.60) and of the exterior derivative (5.48) and (5.62), one can prove that:

$$
\omega = -dq^I \wedge dp_I^A \wedge \beta_A - dH \wedge \beta = d(dq^I \wedge p_I^A \beta_A - H\beta) \tag{8.24}
$$

But the form $dq^I \wedge p_I^A \beta_A - H\beta$ is globally well defined. To prove it, we first have to note that from formula (5.56) it follows immediately that:

$$
dx^A \wedge \beta_A = (-1)^{|A|}\beta \tag{8.25}
$$

where there is no sum on repeated indexes. More generally we have that:

$$
dx^B \wedge \beta_A = (-1)^{|A|}\beta\delta_A^B \tag{8.26}
$$

So, changing coordinates and remembering (8.3), (8.5) and (8.13), we find:

$$
\begin{aligned}
&dq^{I'} \wedge p_{I'}^{A'} \beta'_{A'} - H'\beta' \\
&= \left(dq^I \frac{dq^{I'}}{dq^I} + dx^B \frac{dq^{I'}}{dx^B}\right) \wedge \operatorname{Ber}\left(\frac{\partial x}{\partial x'}\right) \frac{\partial q^J}{\partial q^{I'}} p_J^A \frac{\partial x^{A'}}{\partial x^A} \operatorname{Ber}\left(\frac{\partial x'}{\partial x}\right) \frac{\partial x^C}{\partial x^{A'}} \beta_C \\
&- \left[\operatorname{Ber}\left(\frac{\partial x}{\partial x'}\right) H + (-1)^{|A|(|L|+1)} \operatorname{Ber}\left(\frac{\partial x}{\partial x'}\right) \frac{\partial q^{I'}}{\partial x^A} \frac{\partial q^I}{\partial q^{I'}} p_I^A\right] \operatorname{Ber}\left(\frac{\partial x'}{\partial x}\right) \beta \\
&= dq^I \wedge p_I^A \beta_A - H\beta + (-1)^{|A||L|}(-1)^{|A|} \frac{dq^{I'}}{dx^A} \frac{\partial q^J}{\partial q^{I'}} p_J^A \beta - (-1)^{|A|(|L|+1)} \frac{dq^{I'}}{dx^A} \frac{\partial q^J}{\partial q^{I'}} p_J^A \beta \\
&= dq^I \wedge p_I^A \beta_A - H\beta
\end{aligned} \tag{8.27}
$$

so also its exterior derivative is globally well defined and theorem is proved. ■

Note that the calculation presented above with Formula (8.27) strongly relies on the properties of Voronov-Zorich superforms and their Cartan Calculus, introduced in the Second Part of this thesis, and on the definition of the super multimomenta space P and of the Hamiltonian H, given above in this Chapter. It is precisely having in mind Proposition 117 that most of the preliminary work of Chapters 5 and of Sections 8.1 and 8.2 has been made.

Definition 118. *The global $n|m$-form defined locally by:*

$$\theta := dq^I \wedge p_I^A \beta_A - H\beta \tag{8.28}$$

is called the Cartan form on the super-multimomenta-space P.

Note that $\theta = \mathbb{FL}_* \mathcal{L} + g^I \wedge p_I^A \beta_A$.

Note that both θ and ω are fractional superforms; indeed:

$$\theta := dq^I \wedge p_I^A \beta_A - H\beta = dq^I \wedge p_I^A \left(\partial_A \lrcorner \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}} \right) - H \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$$

$$\omega = -dq^I \wedge dp_I^A \wedge \beta_A - dH \wedge \beta = -dq^I \wedge dp_I^A \wedge \left(\partial_A \lrcorner \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}} \right) - dH \wedge \frac{dx^1 \wedge \cdots \wedge dx^n}{dx^{n+1} \odot \cdots \odot dx^{n+m}}$$

It is important to note that the equality (8.24) establishes a direct link between the super-multisymplectic form ω and its classical counterpart defined with 16.

The superforms defined here with (8.28) and (8.24) can be compared with the analogous Cartan forms defined in [83] and [26] for the supermechanics, which can be considered as a special case for $X = \mathbb{R}^{1|0}$, and with the super Poincaré-Cartan forms defined in [114], [117] and [115] for their supermechanics on $\mathbb{R}^{1|1}$, which can be considered as the integral of my θ taken on an odd submanifold of $\mathbb{R}^{1|1}$ transverse to the even time line.

My super Cartan form can be also compared with the super Cartan form defined by Monterde, Muñoz Masqué and Vallejo in [116] and valid for every dimension of X. One sees that my θ defined with (8.28) has a closest and more transparent connection with the classical Cartan form. I think that the notation introduced in chapter 5 for fractional superforms helps in keeping formula lighter; moreover the very use of Voronov Zorich superforms, in my approach, allows to make evident the parallelism between classical and super theory.

I can now prove the main result of this thesis:

Theorem 119. *Let L be a purely even-regular or a purely odd-regular Lagrangian function on $J^1\pi$ and H be its corresponding Hamiltonian function on the super-multimomenta-space P, then a section-submanifold $G \subset P$, of the form $G = z(X)$ for $z \in \Gamma(\mathbb{FL}(J^1\pi))$, is a Hamiltonian submanifold of P if and only if $\forall U$ local chart of P, with corresponding local super-multisymplectic $n+1|m$-form ω, and $\forall u \in \Gamma(T_0U)$, $z^*(u \lrcorner \omega) = 0$; where T_0U is the even tangent space to U, and*
$u \lrcorner \omega\, (\overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}) := \omega\, (u,\ \overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m})$.

Proof. What we have to prove is that:

$$\forall u \in \Gamma(T_0U), \quad \forall v = (\overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}) \in \Gamma\left(\underbrace{T_0X \times \cdots \times T_0X}_{n} \times \underbrace{T_1X \times \cdots \times T_1X}_{m} \right)$$

with $(\overline{v_1}, \cdots, \overline{v_n})$ and $(\widetilde{v_1}, \cdots, \widetilde{v_m})$, $\mathbb{R}_S$-linearly independent,

$$\omega\, (u, z_*\overline{v_1}, \cdots, z_*\overline{v_n}, z_*\widetilde{v_1}, \cdots, z_*\widetilde{v_m}) = 0$$

Indeed, if $(\overline{v_1}, \cdots, \overline{v_n})$ are linearly dependent, then their images trough z_* are linear dependent too and it is easy to see that the value of their contraction with ω is 0. If $(\widetilde{v_1}, \cdots, \widetilde{v_m})$ are linearly dependent, it is easy to see that ω is not defined on their image through z_*. We can then extend ω to value 0 there.

I chose v so that, on the local chart U, $\overline{v_a} = \partial_a$ and $\widetilde{v_\alpha} = \partial_\alpha$. Then:

$$\forall x \in U,\ z_* \partial_A\big|_x = \frac{\partial}{\partial x^A}\big|_{z(x)} + \frac{\partial q^I}{\partial x^A}(z(x)) \frac{\partial}{\partial q^I}\big|_{z(x)} + \frac{\partial p_I^B}{\partial x^A}(z(x)) \frac{\partial}{\partial p_I^B}\big|_{z(x)}$$

I want now to calculate $\omega(u, z_*\partial_1, \cdots, z_*\partial_{n+m})(z(x))$ for a generic $u \in \Gamma(T_0 U)$ and a generic $x \in X$. Note that any other v', with the required characteristics of linear independence, is linked to the chosen v by $v' = gv$ where $g \in GL(n|m)$ and consequently $z_* v' = g z_* v$; so, because of Lemma 48, $\omega(u, z_* v') = \operatorname{Ber} g\, \omega(u, z_* v)$. If I can show that $\omega(u, z_*\partial_1, \cdots, z_*\partial_{n+m})(z(x)) = 0$ for a generic u, then the same holds for any v'.

I'll carry on the calculations separately for the terms appearing in (8.23).

From the first term we have:

$$-\operatorname{sdet}_{n+2|m}\begin{pmatrix}
u^{q^I} & u^{p_I^a} & u^{x^1} & \cdots & u^{x^a} & \cdots & u^{x^n} & u^{x^{n+1}} & \cdots & u^{x^{n+m}} \\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^a}{\partial x^1} & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\frac{\partial q^I}{\partial x^2} & \frac{\partial p_I^a}{\partial x^2} & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^a}{\partial x^n} & 0 & \cdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\
\frac{\partial q^I}{\partial x^{n+1}} & \frac{\partial p_I^a}{\partial x^{n+1}} & 0 & \cdots & \vdots & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial q^I}{\partial x^{n+m}} & \frac{\partial p_I^a}{\partial x^{n+m}} & 0 & \cdots & \vdots & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

$$= -\det_{n+2}\left[\begin{pmatrix}
0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
u^{q^I} & u^{p_I^a} & u^{x^1} & \cdots & u^{x^a} & \cdots & u^{x^n} \\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^a}{\partial x^1} & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^a}{\partial x^n} & 0 & \cdots & \vdots & \cdots & 1
\end{pmatrix} - \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
u^{x^{n+\alpha}} \frac{\partial q^I}{\partial x^{n+\alpha}} & u^{x^{n+\alpha}} \frac{\partial p_I^a}{\partial x^{n+\alpha}} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}\right]$$

$$= -\det_{n+2}\begin{pmatrix}
0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
u^{q^I} - u^{x^{n+\alpha}} \frac{\partial q^I}{\partial x^{n+\alpha}} & u^{p_I^a} - u^{x^{n+\alpha}} \frac{\partial p_I^a}{\partial x^{n+\alpha}} & u^{x^1} & \cdots & u^{x^a} & \cdots & u^{x^n} \\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^a}{\partial x^1} & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^a}{\partial x^n} & 0 & \cdots & \vdots & \cdots & 1
\end{pmatrix}$$

$$\begin{aligned}
&= -\left(u^{q^I} - u^{x^{n+\alpha}} \frac{\partial q^I}{\partial x^{n+\alpha}}\right) \frac{\partial p_I^a}{\partial x^a} + \sum_{b \neq a} \frac{\partial q^I}{\partial x^b} \frac{\partial p_I^a}{\partial x^a} u^{x^b} \\
&\quad + \frac{\partial q^I}{\partial x^a}\left(u^{p_I^a} - u^{x^{n+\alpha}} \frac{\partial p_I^a}{\partial x^{n+\alpha}}\right) - \sum_{b \neq a} \frac{\partial q^I}{\partial x^a} \frac{\partial p_I^a}{\partial x^b} u^{x^b}
\end{aligned} \tag{8.29}$$

From the second term we have:

$$
-(-1)^{|L|+1}\frac{\partial}{\partial\eta}\operatorname{sdet}_{n+2|m}\begin{pmatrix}
u^{q^I} & u^{p_I^\alpha} & u^{x^1} & \cdots & u^{x^n} & u^{x^{n+1}} & \cdots & u^{x^\alpha} & \cdots & u^{x^{n+m}}\\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^\alpha}{\partial x^1} & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \eta & \cdots & 0\\
\frac{\partial q^I}{\partial x^2} & \frac{\partial p_I^\alpha}{\partial x^2} & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^\alpha}{\partial x^n} & 0 & \cdots & 1 & 0 & \cdots & \vdots & \cdots & 0\\
\frac{\partial q^I}{\partial x^{n+1}} & \frac{\partial p_I^\alpha}{\partial x^{n+1}} & 0 & \cdots & 0 & 1 & \cdots & \vdots & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^{n+m}} & \frac{\partial p_I^\alpha}{\partial x^{n+m}} & 0 & \cdots & 0 & 0 & \cdots & \vdots & \cdots & 1
\end{pmatrix}
$$

$$
\begin{aligned}
&= -(-1)^{|L|+1}\frac{\partial}{\partial\eta}\det_{n+2}\left[\begin{pmatrix}
0 & 0 & 0 & \cdots & 0\\
u^{q^I} & u^{p_I^\alpha} & u^{x^1} & \cdots & u^{x^n}\\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^\alpha}{\partial x^1} & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^\alpha}{\partial x^n} & 0 & \cdots & 1
\end{pmatrix} - \begin{pmatrix}
\eta\frac{\partial q^I}{\partial x^{n+\alpha}} & \eta\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} & 0 & \cdots & 0\\
u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}} & u^{x^{n+\beta}}\frac{\partial p_I^\alpha}{\partial x^{n+\beta}} & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}\right]\\
&= -(-1)^{|L|+1}\frac{\partial}{\partial\eta}\det_{n+2}\begin{pmatrix}
-\eta\frac{\partial q^I}{\partial x^{n+\alpha}} & -\eta\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} & 0 & \cdots & 0\\
u^{q^I} - u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}} & u^{p_I^\alpha} - u^{x^{n+\beta}}\frac{\partial p_I^\alpha}{\partial x^{n+\beta}} & u^{x^1} & \cdots & u^{x^n}\\
\frac{\partial q^I}{\partial x^1} & \frac{\partial p_I^\alpha}{\partial x^1} & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^n} & \frac{\partial p_I^\alpha}{\partial x^n} & 0 & \cdots & 1
\end{pmatrix}\\
&= -(-1)^{|L|+1}\frac{\partial}{\partial\eta}\left(u^{q^I} - u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\right)\eta\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} + (-1)^{|L|+1}\frac{\partial}{\partial\eta}\frac{\partial q^I}{\partial x^b}\eta\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}}u^{x^b}\\
&\quad + (-1)^{|L|+1}\frac{\partial}{\partial\eta}\eta\frac{\partial q^I}{\partial x^{n+\alpha}}\left(u^{p_I^\alpha} - u^{x^{n+\beta}}\frac{\partial p_I^\alpha}{\partial x^{n+\beta}}\right) - (-1)^{|L|+1}\frac{\partial}{\partial\eta}\eta\frac{\partial q^I}{\partial x^{n+\alpha}}\frac{\partial p_I^\alpha}{\partial x^b}u^{x^b}\\
&= -(-1)^{|L|+|I|+1}\left(u^{q^I} - u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\right)\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} + (-1)^{|L|+|I|+1}\frac{\partial q^I}{\partial x^b}\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}}u^{x^b}\\
&\quad + (-1)^{|L|+1}\frac{\partial q^I}{\partial x^{n+\alpha}}\left(u^{p_I^\alpha} - u^{x^{n+\beta}}\frac{\partial p_I^\alpha}{\partial x^{n+\beta}}\right) - (-1)^{|L|+1}\frac{\partial q^I}{\partial x^{n+\alpha}}\frac{\partial p_I^\alpha}{\partial x^b}u^{x^b}
\end{aligned}
\tag{8.30}
$$

From the third and the fourth term we have:

$$
-\operatorname{sdet}_{n+1,m}\begin{pmatrix}
u^{q^I}\frac{\partial H}{\partial q^I} + u^{p_I^A}\frac{\partial H}{\partial p_I^A} & u^{x^1} & \cdots & u^{x^n} & u^{x^{n+1}} & \cdots & u^{x^{n+m}}\\
\frac{\partial q^I}{\partial x^1}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^1}\frac{\partial H}{\partial p_I^A} & 1 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^n}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^n}\frac{\partial H}{\partial p_I^A} & 0 & \cdots & 1 & 0 & \cdots & 0\\
\frac{\partial q^I}{\partial x^{n+1}}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^{n+1}}\frac{\partial H}{\partial p_I^A} & 0 & \cdots & 0 & 1 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\frac{\partial q^I}{\partial x^{n+m}}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^{n+m}}\frac{\partial H}{\partial p_I^A} & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

$$= -\det_{n+1}\left[\begin{pmatrix} u^{q^I}\frac{\partial H}{\partial q^I} + u^{p_I^A}\frac{\partial H}{\partial p_I^A} & u^{x^1} & \cdots & u^{x^n} \\ \frac{\partial q^I}{\partial x^1}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^1}\frac{\partial H}{\partial p_I^A} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q^I}{\partial x^n}\frac{\partial H}{\partial q^I} + \frac{p_I^A}{\partial x^n}\frac{\partial H}{\partial p_I^A} & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\frac{\partial H}{\partial q^I} + u^{x^{n+\beta}}\frac{p_I^A}{\partial x^{n+\beta}}\frac{\partial H}{\partial p_I^A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right]$$

$$\begin{aligned} &= -\,u^{q^I}\frac{\partial H}{\partial q^I} + u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\frac{\partial H}{\partial q^I} + \frac{\partial q^I}{\partial x^b}\frac{\partial H}{\partial q^I}u^{x^b} \\ &\quad -\left(u^{p_I^A}\frac{\partial H}{\partial p_I^A}\right) + u^{x^{n+\beta}}\left(\frac{\partial p_I^A}{\partial x^{n+\beta}}\frac{\partial H}{\partial p_I^A}\right) + \left(\frac{\partial p_I^A}{\partial x^b}\frac{\partial H}{\partial p_I^A}\right)u^{x^b} \end{aligned} \tag{8.31}$$

Adding (8.29), (8.30) and (8.31), we find that $\omega\left(u, z_*\partial_1, \cdots, z_*\partial_{n+m}\right)(z(x))$ is equal to:

$$\begin{aligned} & u^{q^I}\left(-\frac{\partial p_I^a}{\partial x^a} - (-1)^{|L|+|I|+1}\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} - \frac{\partial H}{\partial q^I}\right) \\ & + \frac{\partial q^I}{\partial x^a}\left(u^{p_I^a}\right) + (-1)^{|L|+1}\frac{\partial q^I}{\partial x^{n+\alpha}}\left(u^{p_I^\alpha}\right) - \left(u^{p_I^A}\frac{\partial H}{\partial p_I^A}\right) \\ & + \sum_{b\neq a}\left(\frac{\partial q^I}{\partial x^b}\frac{\partial p_I^a}{\partial x^a} - \frac{\partial q^I}{\partial x^a}\frac{\partial p_I^a}{\partial x^b}\right)u^{x^b} \\ & + (-1)^{|L|+|I|+1}\frac{\partial q^I}{\partial x^b}\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}}u^{x^b} - (-1)^{|L|+1}\frac{\partial q^I}{\partial x^{n+\alpha}}\frac{\partial p_I^\alpha}{\partial x^b}u^{x^b} \\ & + \frac{\partial q^I}{\partial x^b}\frac{\partial H}{\partial q^I}u^{x^b} + \frac{\partial p_I^A}{\partial x^b}\frac{\partial H}{\partial p_I^A}u^{x^b} \\ & + u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\frac{\partial p_I^a}{\partial x^a} - u^{x^{n+\beta}}(-1)^{|I|}\frac{\partial q^I}{\partial x^a}\frac{\partial p_I^a}{\partial x^{n+\beta}} \\ & + (-1)^{|L|+|I|+1}\left(u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\frac{\partial p_I^\alpha}{\partial x^{n+\alpha}} + u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\alpha}}\frac{\partial p_I^\alpha}{\partial x^{n+\beta}}\right) \\ & + u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\frac{\partial H}{\partial q^I} + u^{x^{n+\beta}}\left(\frac{\partial p_I^A}{\partial x^{n+\beta}}\frac{\partial H}{\partial p_I^A}\right) \end{aligned}$$

$$\begin{aligned} = {} & u^{q^I}\left[-(-1)^{|A|(|L|+1)}(-1)^{|A||I|}\frac{\partial p_I^A}{\partial x^A} - \frac{\partial H}{\partial q^I}\right] \\ & + (-1)^{(|L|+|I|)|I|}u^{p_I^a}\frac{\partial q^I}{\partial x^a} + (-1)^{(|L|+|I|)|I|}u^{p_I^\alpha}\frac{\partial q^I}{\partial x^{n+\alpha}} - \left(u^{p_I^A}\frac{\partial H}{\partial p_I^A}\right) \\ & + \frac{\partial q^I}{\partial x^b}\left[(-1)^{|A|(|L|+|I|+1)}\frac{\partial p_I^A}{\partial x^A} + \frac{\partial H}{\partial q^I}\right]u^{x^b} \\ & - \left[(-1)^{|A|(|L|+1)}\frac{\partial q^I}{\partial x^A} - (-1)^{(|A|+|I|)(|A|+|I|+|L|)}\frac{\partial H}{\partial p_I^A}\right]\frac{\partial \overline{p_I^A}}{\partial x^b}u^{x^b} \\ & + \ u^{x^{n+\beta}}\frac{\partial q^I}{\partial x^{n+\beta}}\left((-1)^{|A|(|L|+|I|+1)}\frac{\partial p_I^A}{\partial x^A} + \frac{\partial H}{\partial q^I}\right) \\ & - u^{x^{n+\beta}}(-1)^{|I|(|I|+|L|)}\left(\frac{\partial p_I^A}{\partial x^{n+\beta}}\frac{\partial q^I}{\partial x^A}\right) + u^{x^{n+\beta}}\left(\frac{\partial p_I^A}{\partial x^{n+\beta}}\frac{\partial H}{\partial p_I^A}\right) \end{aligned} \tag{8.32}$$

Then, being u^{q^I}, $u^{p_I^A}$, u^{x^b} and $u^{x^{n+\beta}}$ arbitrary, 8.32, compared with (8.18) and (8.19), proves the theorem. ■

Note that part of the proof of theorem 119 could be rewritten exploiting the properties of the internal and external products for superforms. I chose to present instead explicit calculations, to not hide what is behind the definitions of fractional superforms.

The arguments used in proving theorem 119 can be easily adapted to be used when the vector field u is odd and we make use of the extension in the first argument of ω. We obtain this way the following theorem:

Theorem 120. *Let L be a purely even-regular or a purely odd-regular Lagrangian function on $J^1\pi$ and H be its corresponding Hamiltonian function on the super-multimomenta-space P, then a section-submanifold $G \subset P$, with $G = z(X)$ for $z \in \Gamma\big(\mathbb{FL}(J^1\pi)\big)$, is a Hamiltonian submanifold of P if and only if $\forall U$ local chart of P, with corresponding local super-multisymplectic $n+1|m$-extended-form $\hat{\omega}$, and $\forall u \in \Gamma\left(T_1 U\right)$, $z^*(u \lrcorner \hat{\omega}) = 0$; where $T_1 U$ is the odd tangent space to U, and*
$u \lrcorner \hat{\omega}\left(\overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}\right) := \hat{\omega}\left(u,\ \overline{v_1}, \cdots, \overline{v_n}, \widetilde{v_1}, \cdots, \widetilde{v_m}\right)$.

Again, as in the classical case, we can note that in theorems 119 and 120 it is requested in the hypothesis that G is a section-submanifold: the fact that G ends out to be also a lifted-submanifold is a consequence of the condition imposed on it. This can be seen if one notes that ω can also be written locally as

$$\omega = -g^I \wedge dp_I^A \wedge \beta_A - dq^I \frac{\partial H}{\partial q^I} \wedge \beta$$

If a section-submanifold G satisfy the condition required in proposition 119 and 120, then this last one can be applied to $u = \frac{\partial}{\partial p_J^B}$ and this yields that $\forall B, \forall J,\ z^*(g^J \wedge \beta_B) = 0$ which in turns yields that $\forall J,\ z^*(g^J) = 0$.

The study of the purely even and purely odd regular cases can be used as the starting point for the study of more complicated non-regular and/or non-homogeneous Lagrangians.

8.4 The symplectic structure of the super covariant phase space

Having at hand the super-multisymplectic form, we can follow the path undertaken in the classical case in section 2.4.

We call again $\mathscr{G}$ the space of Hamiltonian submanifold of P and we build on it a symplectic structure. I will then sometime call $\mathscr{G}$ the super covariant phase space.

The super covariant phase space is in general infinite-dimensional. I will not treat here the problem of how to give it a superdifferential structure such that it can be treated as an infinite-dimensional supermanifold. For a possible solution of this problem in a categorical framework of supermathematics, one can see A. Alldridge, [1] or C. Sachse, [138] and F Hanish, [67], who worked on older ideas of V. Molotkov, [112]. One can also read the introductions and consult the bibliographies of the works quoted above to find about other possible approaches to this subject.

Suppose that on $\mathscr{G}$ a suitable super-differential structure is given, such that we can speak about its tangent module $T_G\mathscr{G}$ at one of its points $G \in \mathscr{G}$. Let $\delta_u G \in T_G\mathscr{G}$ be an even vector over G: it is a vector tangent to an even path in $\mathscr{G}$ and it is represented by some $u \in \Gamma\left(i^*\left(VP\right)\right)$, id est a section over G of the pull-back image of the vertical (with respect to the projection π_P of the total space P onto the base X) tangent bundle VP by the embedding map $i\ :\ G \longrightarrow P$. The section u can be both even or odd. As in the classical case, u can be seen as a vector field (even or odd) on G, "following" which, each point $g \in G$ is moved to a point $g' \in G'$, being $G' \in \mathscr{G}$ an other Hamiltonian $n|m$-curve. u deforms an Hamiltonian $n|m$-curve G into another Hamiltonian $n|m$-curve G'. Considerations similar to the ones made at the end of sections 2 could be made. What is most interesting is the definition of a symplectic form on $T\mathscr{G}$.

As in the classical case, we can consider a slice Σ of co-dimension 1 in P, with the property that for any Hamiltonian $n|m$-curve $G \in \mathscr{G}$ the intersection of Σ with G is transverse. In the super case we have two possible choices:

1. the codimension of Σ is even;
2. the codimension of Σ is odd.

We pick up the first choice.

I don't treat here the problems involved with the orientation of Σ.

Once a Σ with a suitable orientation is chosen, we can then define Ω_Σ to be a functional acting on couples of vectors of $T\mathscr{G}$ and ma$^{p}pingthemto\mathbb{R}_S$ in the following way:

Definition 121. *Let be $\overline{\delta_1 G}$, $\overline{\delta_2 G} \in T_{G,0}\mathscr{G}$ two even vectors at $G \in \mathscr{G}$, and let $\overline{u_1}, \overline{u_2} \in \Gamma\left(i^*\left(V_0 P\right)\right)$ be the corresponding even vector fields over G, then we pose:*

$$\left.\Omega_\Sigma\right|_G\left(\overline{\delta_1 G}, \overline{\delta_2 G}\right) := \int_{\Sigma\cap G}\left(\overline{u_1}\wedge\overline{u_2}\right)\lrcorner\,\omega \tag{8.33}$$

Definition 122. *Let be $\delta_1 G$, $\delta_2 G \in T_G\mathscr{G}$ two vectors on $G \in \mathscr{G}$ of generic parities, and let be $u_1, u_2 \in \Gamma\left(i^*\left(VP\right)\right)$ the corresponding vector fields over G, then we pose:*

$$\left.\Omega_\Sigma\right|_G\left(\delta_1 G, \delta_2 G\right) := \int_{\Sigma\cap G}\left(u_1\wedge u_2\right)\lrcorner\,\hat{\omega} = \int_{\Sigma\cap G}\hat{\omega}\left(u_1, u_2, \cdot\right) \tag{8.34}$$

where $\hat{\omega}$ is the extension of ω in the first two arguments. Ω_Σ is our symplectic $2|0$-extended-form on the infinite dimensional supermanifold $\mathscr{G}$.

Note that, by the definition of $\hat{\omega}$, 33, we have that:

$$\left.\Omega_\Sigma\right|_G\left(\delta_1 G, \delta_2 G\right) = (-1)^{|\delta_1 G||\delta_2 G|}\left.\Omega_\Sigma\right|_G\left(\delta_2 G, \delta_1 G\right)$$

and Ω_Σ can be seen as a 2-form à la Kostant on $\mathscr{G}$.

Note that, since $|\omega| = |L|$, we have that $|\Omega| = |L| + m$, where m is the odd dimension of X and where the shift is due to the Berezinian integration in (8.34).

In the special case of $X = \mathbb{R}^{1|0}$, the integral of ω over $\Sigma \cap G$ reduces to an evaluation at a fixed time t_0 and the super symplectic form Ω_Σ reduces substantially to ω evaluated at t_0. In this case the symplectic form Ω coincides with the one found for supermechanics in [83]. The symplectic structure presented in [26] is more difficult to compare to mine, due to the more involute notation.

A symplectic structure on the space of solutions of their theory has been constructed also by Monterde, Muñoz Masqué and Vallejo in [115]. It corresponds to the symplectic structure on the space of solutions constructed here for theories built on a base $X = \mathbb{R}^{1|1}$. Note that, in [115], the authors cite the fact their symplectic superform has parity opposite to the one of the Lagrangian: this agrees with what we have seen above when $m = 1$.

It is important to note that, for Ω_Σ to be well defined, Σ has to be compact; otherwise it could happen that Ω is defined only on a subset of vectors of $T\mathscr{G}$. Finding a compact Σ which satisfies the conditions exposed above is always possible when X itself is compact or when X is the product of a compact supermanifolds times $\mathbb{R}^{1|0}$.

The symplectic form Ω on $\mathscr{G}$ can be pull-back on $\mathscr{E}$. We can pull-back ω by $\mathbb{FL}$ on $J^1\pi$.

Definition 123. *$o := \mathbb{FL}^*\widehat{\omega}$ is called the super-multisymplectic form on $(J^1\pi)$*

Using (8.24) one can see that:

Proposition 124. *If $L = \overline{L}$ is purely even regular, then:*

$$\begin{aligned} o &= \mathbb{FL}^* d(dq^I \wedge \overline{p_I^A}\beta_A - \overline{H}\beta) \\ &= d\left[dq^I \wedge (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\beta_A - (-1)^{|A|}\dot{q}^I_A(-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\beta + \overline{L}\beta\right] \\ &= d\left[c^I \wedge (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\beta_A + \overline{L}\beta\right] \end{aligned} \tag{8.35}$$

If $L = \widetilde{L}$ is purely odd regular, then:

$$\begin{aligned} o &= \mathbb{FL}^* d(dq^I \wedge \widetilde{p_I^A}\beta_A - \widetilde{H}\beta) \\ &= d\left[dq^I \wedge \frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\beta_A - \dot{q}^I_A\frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\beta + \widetilde{L}\beta\right] \\ &= d\left[c^I \wedge \frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\beta_A + \widetilde{L}\beta\right] \end{aligned} \tag{8.36}$$

In general:

$$\begin{aligned} o &= \mathbb{FL}^* d(dq^I \wedge p_I^A\beta_A - H\beta) \\ &= d\left[dq^I \wedge (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\beta_A - \dot{q}^I_A\frac{\partial \overline{L}}{\partial \dot{q}^I_A}\beta + dq^I \wedge \frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\beta_A - \dot{q}^I_A\frac{\partial \widetilde{L}}{\partial \dot{q}^I_A}\beta + L\beta\right] \\ &= d\left[(-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A}\beta_A + L\beta\right] \end{aligned} \tag{8.37}$$

On $J^1\pi$ we consider a compact slice Σ of co-dimension $1|0$ with the property that, for any $s = \Gamma(J^1\pi)$, the intersection of Σ with $s(X)$ is transverse. We will then have $j^1\pi(\Sigma) = \Sigma_X$ where Σ_X is a $n-1|m$-submanifold of X. If we call i the embedding map $i\ :\ s(X)\ \hookrightarrow J^1\pi$, we can then define O_Σ to be a functional acting on couples of vectors on $s \in \Gamma(J^1\pi)$ and sending them to $\mathbb{R}_S$ in the following way:

Definition 125. *Let be $\delta_1 s,\ \delta_2 s \in T_s\Gamma(J^1\pi)$ two vectors on $s \in \Gamma(J^1\pi)$, and let be $u_1, u_2 \in \Gamma\left(i^*\left(V_{j^1\pi}J^1\pi\right)\right)$ the corresponding vertical vector fields over $s(X)$, then we pose:*

$$O_\Sigma\big|_s(\delta_1 s, \delta_2 s)\ :=\ \int_{\Sigma\cap s(X)} (u_1 \wedge u_2) \lrcorner\, \hat{o} \tag{8.38}$$

and O_Σ is our symplectic $2|0$-form on $\Gamma(J^1\pi)$.

Again there is almost no difference from the classical case. O_Σ can be restricted to $j^1\mathscr{E} \subset j^1\Gamma(E) \subset \Gamma(J^1\pi)$. Let be $i\ :\ j^1\Phi(X)\ \hookrightarrow J^1\pi$ the embedding map of $j^1\Phi(X)$. Let be $\delta_1 j^1\Phi,\ \delta_2 j^1\Phi \in T_{j^1\phi}j^1\mathscr{E}$ two vectors on $j^1\Phi \in j^1\mathscr{E}$, and let be $u,\ v \in \Gamma\left(i^*\left(V_{j^1\pi}J^1\pi\right)\right)$ the corresponding vertical vector fields over $j^1\Phi(X)$, then we pose:

$$O_\Sigma\big|_{j^1\Phi}\left(\delta_1 j^1\Phi, \delta_2 j^1\Phi\right)\ :=\ \int_{\Sigma\cap j^1\Phi(X)} (u \wedge v) \lrcorner\, \hat{o} \tag{8.39}$$

To have an idea of how O can be computed on local charts, let's consider a local chart U so that Σ is defined locally by $x^1 = 0$, then for $\delta_1 j^1\Phi$ and $\delta_2 j^1\Phi$ even we have:

$$O_{\Sigma,U}\big|_{j^1\Phi}\left(\delta_1 j^1\Phi, \delta_2 j^1\Phi\right) := \int_{\Sigma\cap j^1\Phi(U)} v\lrcorner u\lrcorner o =$$
$$= \int_{\Sigma\cap j^1\Phi(U)} \left[-u^{q^I}\left(v^{q^J}\frac{\partial^2 L}{\partial q^J\partial\dot{q}^I_1} + v^{\dot{q}^J_B}\frac{\partial^2 L}{\partial\dot{q}^J_B\partial\dot{q}^I_1}\right) + v^{q^I}\left(u^{q^J}\frac{\partial^2 L}{\partial q^J\partial\dot{q}^I_1} + u^{\dot{q}^J_B}\frac{\partial^2 L}{\partial\dot{q}^J_B\partial\dot{q}^I_1}\right)\right]\beta_1 \tag{8.40}$$

As in the classical case, it has to be noted that calling O_Σ a symplectic form may be an abuse. It is indeed a $2|0$-form on its, possibly infinite dimensional, supermanifold of definition, but it may not be non-degenerate, unless some hypotheses of regularity on the Lagrangian L are assumed.

Chapter 9

Comparison theorems

Supersymmetric field theories are defined in the Physics literature with the superfield on supermanifolds formalism or with the so called components formalism, where the bosonic and fermionic fields are defined over conventional even manifolds. In this chapter I want to prove the existence of an isomorphism between the space of fields of a supersymmetric theory defined with the superfield formalism and the space of fields of a corresponding theory presented with the components formalism. I want to prove also the existence of an isomorphism between the spaces of solutions of the corresponding theories, and I want to compare the symplectic structures naturally arising from the two theories.

It is sometime said that, to give sense to the isomorphisms above, it is necessary to use the functor of points formalism for treating the supermanifolds: for such an assertion see for example Freed [54]; for a presentation of the functor of points formalism see [35]. What follows shows how to make sense of them within a concrete formalism which allows a more intuitive interpretation in terms of geometrical objects.

The material presented in this chapter can also be interpreted as the translation using my formalism of the Comparison Theorem formulated with other formalisms. The Comparison Theorem establishes indeed an equivalence between a first order Lagrangian super theory defined on a base manifold X of dimension $n|m$ and a corresponding $m+1$-th order Lagrangian supertheory defined in a suitable way. The two theories have, at least classically, the same physical meaning, since they have spaces of solutions which are diffeomorph. The theorem was formulated in a first version in [80] and reformulated in [113] and [114].

In my version, the theories compared are a first order Lagrangian theory defined on a base manifold X of dimension $n|m$ and a corresponding first order Lagrangian theory defined on its n-dimensional body $\underline{X}$. The fact that both theories are first order is a consequence of a suitable choice of the fields target manifolds used. This fact, by the way, also clarifies why both theories admit (whence the Lagrangian are chosen) a canonical Poincaré-Cartan form (which is not obvious for $m+1$-th order Lagrangian theories).

I will treat with some more details the case when the starting superfield theory has for field space an even supermanifold. The Comparison Theorem is valid in the more general case, which I will not treat in details, but which can easily be dealt with.

In section 9.1 I will treat the Lagrangian approach obtaining results analogous to the one found in [80], [113] and [114].

In section 9.2 I will treat the Hamiltonian approach and I will make a first comparison of symplectic structures on the spaces of solutions of theories expressed in the so called superfield and in the so called components formalisms. The results there presented are original.

I will sacrifice some rigor and precision for a shorter presentation.

9.1 The comparison theorem

Suppose that the fields-super-bundle E has base X with dimension $n|m$, has fiber-type F with dimension $r|0$ and has projection π. Then F has only even coordinates; on a local chart the index I, of local coordinates q^I, is only even. From now on I will indicate that index with i. I call $\underline{X} = \epsilon(X)$ the body of X and $b(\underline{X})$ its immersed body, see section 5.5. I will suppose from now on that we work with a subatlas of X such that in all local charts U with coordinates (x^a, x^α) and $\forall x \in U$, $x^a [b(\underline{x})] = \underline{x}^a \in \mathbb{R}$ and $x^\alpha [b(\underline{x})] = 0$.

As in section 7, I define the Lagrangian field theory starting from a Lagrangian horizontal $n|m$-form defined on $J^1\pi$.

It is possible to define two other important fiber bundles over E.

Definition 126. *I call $J^{0|m}\pi$ the sub-bundle of $J^m\pi$ where only jets with odd derivatives are considered.*
I call $J^{1|m}\pi$ the sub-bundle of $J^{m+1}\pi$ where only jets with odd derivatives and at most one even derivative are considered.

Note that all these bundles are G^∞-bundles over E and also over X. They can be pull-back on $\underline{X}$ by the map b which is C^∞ obtaining C^∞-bundles over the real manifold $\underline{X}$ with fiber-type which may be supermanifolds. This is somehow an hybrid situation; let's schematize it:

$$
\begin{tikzcd}
 & J^{m+1}\pi \arrow[d, "j^{m+1}_{1|m}\pi"] & & & & \\
 & J^{1|m}\pi \arrow[dl, "j^{1|m}_{0|m}\pi"'] \arrow[dr, "j^{1|m}_{1}\pi"] & & & b^*\left(J^{m+1}\pi\right) \arrow[d] & \\
J^{0|m}\pi \arrow[dr, "j^{0|m}_{0}\pi"'] & & J^1\pi \arrow[dl, "j^1_0\pi"] \arrow[rr, dotted] & & b^*\left(J^{1|m}\pi\right) \arrow[dl] \arrow[dr] & \\
 & E \arrow[rr, dotted] \arrow[dd, "\pi"] & & b^*\left(J^{0|m}\pi\right) \arrow[dr] & & b^*\left(J^1\pi\right) \arrow[dl] \\
 & & & & b^*(E) \arrow[d] & \\
 & X \arrow[rrr, "\epsilon"] & & & \underline{X} \arrow[lll, bend left, "b"] &
\end{tikzcd}
\tag{9.1}
$$

On $J^{0|m}\pi$ we can use $\left(x^A, \dot{q}^i_\Lambda\right)$ as coordinates on a local chart. We have that $A = 1, \cdots, n+m$, $x^A \in \mathbb{R}_S$. The Greek capital letter Λ stands for a multiindex which can be 0 or can be a sequence of ordered integer numbers α_j running from $n+1$ to $n+m$, without repetition of the same number; the order goes from the bigger α_j to the smaller α_j. So for example it can be $\Lambda = \alpha_5\alpha_2\alpha_1$ where $\alpha_j = n + j \leq n + m$. We have also that $\dot{q}^i_0 = q^i$. We define the length of the multiindex Λ as $l(\Lambda) = 0$ if $\Lambda = 0$ and $l(\Lambda)$ is equal to the number of α occurring in the sequence when $\Lambda \neq 0$; so for example $l(\alpha_5\alpha_2\alpha_1) = 3$. By the definition, the maximum length of a multiindex Λ can be

m. We have that $\dot{q}^i_\Lambda \in \mathbb{R}_S$ and $\left|\dot{q}^i_\Lambda\right| = |\Lambda| = l(\Lambda)$, where the last equality is taken modulo 2, as usual when dealing with degrees.

On $J^{1|m}\pi$ we can use as local coordinates $(x^A, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda})$, where $a = 1, \cdots, n$, and $\dot{q}^i_{a,0} = \dot{q}^i_a$. We have that $\dot{q}^i_{a,\Lambda} \in \mathbb{R}_S$ and $|\dot{q}^i_{a,\Lambda}| = |\Lambda| = l(\Lambda)$.

On $b^*\left(J^{0|m}\pi\right)$ we can use as local coordinates $\left(\underline{x}^a, \dot{q}^i_\Lambda\right)$, where $a = 1, \cdots, n$, $\underline{x}^a \in \mathbb{R}$ and $\dot{q}^i_\Lambda \in \mathbb{R}_S$.

On $b^*\left(J^{1|m}\pi\right)$ we can use as local coordinates $(\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda})$.

The transition functions of all these bundles are defined as one would expect.

The shift between the two sides of the diagram (9.1) is made on purpose. We have in fact the following:

Proposition 127. *There exists a one-to-one correspondence between the space $\Gamma(E)$ of G^∞ sections of E and the space $\Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ of C^∞ sections of $b^*\left(J^{0|m}\pi\right)$.*

Proof. Let U be a local chart on E with coordinates $\left(x^A, q^i\right)$. Let $\underline{U}$ be the naturally associated chart on $b^*\left(J^{0|m}\pi\right)$, with local coordinates $\left(\underline{x}^a, \dot{q}^i_\Lambda\right)$.
To every $\Phi \in \Gamma(E)$ locally defined by:

$$\Phi \;:\; x^A \;\longrightarrow\; \left(x^A, q^i\left(x^A\right)\right)$$

we associate the section $\underline{\Phi} \in \Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ locally defined by:

$$\underline{\Phi} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right)\right) = \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}\right)$$

where $\partial_\Lambda := \partial_{\alpha_j} \cdots \partial_{\alpha_k}$ when $\Lambda = \alpha_j \cdots \alpha_k$.
To every $\underline{\Phi} \in \Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ locally defined by:

$$\underline{\Phi} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right)\right)$$

we associate the section $\Phi \in \Gamma(E)$ locally defined by:

$$\Phi \;:\; x^A \;\longrightarrow\; \left(x^A, q^i\left(x^A\right)\right) = \left(x^A, x^\Lambda \widehat{\dot{q}^i_\Lambda}\left(x^a\right)\right)$$

where $x^\Lambda = \left(x^{\alpha_k} - x^{\alpha_k}\left(b(x^A)\right)\right)\cdots\left(x^{\alpha_j} - x^{\alpha_j}\left(b(x^A)\right)\right) = x^{\alpha_k}\cdots x^{\alpha_j}$ when $\dot{q}^i_\Lambda = \dot{q}^i_{\alpha_j\cdots\alpha_k}$ (note the different order of α_j when the index Λ is an apex and when it is a subscript) and $\widehat{\dot{q}^i_\Lambda}\left(x^a\right)$ is the Grassmann analytic continuation of $\dot{q}^i_\Lambda\left(\underline{x}^a\right)$ (see Rogers chapter 4 for its definition).

The proof that these associations are well made, and that they are one the inverse of the other one, is straightforward. ■

Analogously, we have:

Proposition 128. *There exists a surjective correspondence between the space $\Gamma\left(J^1\pi\right)$ of G^∞ sections of $J^1\pi$ and the space $\Gamma\left(b^*\left(J^{1|m}\pi\right)\right)$ of C^∞ sections of $b^*\left(J^{1|m}\pi\right)$.*

Proof. Let U be a local chart on $J^1\pi$ with coordinates $\left(x^A, q^i, \dot{q}^i_A\right)$. Let $\underline{U}$ be the naturally associated chart on $b^*\left(J^{1|m}\pi\right)$, with local coordinates $\left(\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda}\right)$.
To every $s \in \Gamma\left(J^1\pi\right)$ locally defined by:

$$s \;:\; x^A \;\longrightarrow\; \left(x^A, q^i\left(x^A\right), \dot{q}^i_A\left(x^A\right)\right)$$

we associate the section $\underline{s} \in \Gamma\left(b^*\left(J^{1|m}\pi\right)\right)$ locally defined by:

$$\underline{s} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), \dot{q}^i_{a,\Lambda}\left(\underline{x}^a\right)\right) = \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, \partial_\Lambda \dot{q}^i_a\big|_{b(\underline{x}^a)}\right)$$

To every $\underline{s} \in \Gamma\left(b^*\left(J^{1|m}\pi\right)\right)$ locally defined by:

$$\underline{s} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), \dot{q}^i_{a,\Lambda}\left(\underline{x}^a\right)\right)$$

we associate the section $s \in \Gamma\left(J^1\pi\right)$ locally defined by:

$$s \;:\; x^A \;\longrightarrow\; \left(x^A, q^i\left(x^A\right), \dot{q}^i_a\left(x^A\right), \dot{q}^i_\alpha\left(x^A\right)\right) = \left(x^A, x^\Lambda \widehat{\dot{q}^i_\Lambda}\left(x^a\right), x^\Lambda \widehat{\dot{q}^i_{a,\Lambda}}\left(x^a\right), \partial_\alpha x^\Lambda \widehat{\dot{q}^i_\Lambda}\left(x^a\right)\right)$$

where $\widehat{\dot{q}^i_{a,\Lambda}}\left(x^a\right)$ is the Grassmann analytic continuation of $\dot{q}^i_{a,\Lambda}\left(\underline{x}^a\right)$.

The application of the first correspondence, followed by the application of the second one, do not lead to the identity in $\Gamma\left(J^1\pi\right)$. The sections which are fixed points for that map are those which satisfy the condition: $\dot{q}^i_\alpha = \partial_\alpha q^i$ id est those who are lifted in the odd sector. ∎

We can define a lift map $\underline{j^{1|m}}$ between $\Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ and $\Gamma\left(b^*\left(J^{1|m}\pi\right)\right)$.
To every $\underline{\Phi} \in \Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$, locally defined by:

$$\underline{\Phi} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right)\right)$$

we associate the section $\underline{j^{1|m}}\underline{\Phi} \in \Gamma\left(b^*\left(J^{1|m}\pi\right)\right)$, locally defined by:

$$\underline{j^{1|m}}\underline{\Phi} \;:\; \underline{x}^a \;\longrightarrow\; \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), \dot{q}^i_{a,\Lambda}\left(\underline{x}^a\right)\right) = \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), \partial_a \dot{q}^i_\Lambda\left(\underline{x}^a\right)\right)$$

Note that, if we call $\underline{j^{0|m}}$ the projection from $b^*\left(J^{0|m}\pi\right)$ to $\underline{X}$, then we have that $b^*\left(J^{1|m}\pi\right) = J^1\underline{j^{0|m}}$. We could then simply set $\underline{j^1} := \underline{j^{1|m}}$.

We have the following:

Proposition 129. *For every $\Phi \in \Gamma(E)$, $\underline{j^1\Phi} = \underline{j^{1|m}}\underline{\Phi} = \underline{j^1}\underline{\Phi}$.*

Proof. It follows directly from the equality $\partial_a \partial_\Lambda = \partial_\Lambda \partial_a$. ∎

We therefore have the following commutative diagram:

$$\begin{tikzcd}
\Gamma\left(J^1\pi\right) \arrow[r, "ub"] & \Gamma\left(b^*\left(J^{1|m}\pi\right)\right) \\
\Gamma(E) \arrow[u, "j^1"] \arrow[r, leftrightarrow, "ub"] & \Gamma\left(b^*\left(J^{0|m}\pi\right)\right) \arrow[u, "\underline{j^{1|m}}"']
\end{tikzcd} \tag{9.2}$$

where ub are the maps between spaces of sections denoted before by the underbar.

We can do more. To every point $\underline{e} \in b^*\left(J^{0|m}\pi\right)$, we can associate a $0|m$-dimensional submanifold $S_{\underline{e}}$ of E in this way:

Definition 130. *If $\underline{e} \in b^*\left(J^{0|m}\pi\right)$ has local coordinates $\left(\underline{x}^a, \dot{q}^i_\Lambda\right)$, then $S_{\underline{e}}$ is the $0|m$-submanifold of E parametrized by the coordinates x^α defined by:*

$$S_{\underline{e}} \;:\; \left(x^\alpha\right) \;\longrightarrow\; \left(x^a\left(b\left(\underline{x}^a\right)\right), x^\alpha, q^i\left(x^\alpha\right)\right) = \left(\underline{x}^a, x^\alpha, x^\Lambda \dot{q}^i_\Lambda\right) \tag{9.3}$$

where $x^\Lambda = x^{\alpha_k} \cdots x^{\alpha_j}$ when $\dot{q}^i_\Lambda = \dot{q}^i_{\alpha_j \cdots \alpha_k}$.

We can do something similar between $b^*\left(J^{1|m}\pi\right)$ and $J^1\pi$.

Definition 131. *If $\underline{r} \in b^*\left(J^{1|m}\pi\right)$ has local coordinates $(\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda})$, we associate to it the $0|m$-dimensional submanifold $S_{\underline{r}} \subset J^1\pi$ parametrized by the coordinates x^α and defined by:*

$$\begin{aligned} S_{\underline{r}} \ : \ (x^\alpha) \ \longrightarrow \ & \left(x^a\left(b\left(\underline{x}^a\right)\right), x^\alpha, q^i\left(x^\alpha\right), \dot{q}^i_a\left(x^\alpha\right), \dot{q}^i_\alpha\left(x^\alpha\right)\right) = \\ & = \left(\underline{x}^a, x^\alpha, x^\Lambda \dot{q}^i_\Lambda, x^\Lambda \dot{q}^i_{a,\Lambda}, \partial_\alpha\left(x^\Lambda\right)\dot{q}^i_\Lambda\right) \end{aligned} \tag{9.4}$$

Suppose that on $J^1\pi$ is defined a Lagrangian $n|m$-density L, so that $\mathcal{L}$, defined locally by $\mathcal{L} = \beta L$ (with β the local canonical $n|m$-form), is the Lagrangian $n|m$-horizontal-form of our field theory. We can then define a Lagrangian n-form $\underline{\mathcal{L}}$ on $b^*\left(J^{1|m}\pi\right)$ by:

$$\underline{\mathcal{L}}\left(\underline{r}\right) := \int_{S_{\underline{r}}} \beta L \tag{9.5}$$

I can now define a functional action $\underline{A}$ acting on sections $\underline{\Phi} \in \Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ in the following way:

$$\underline{A}\left(\underline{\Phi}\right) := \int_{\underline{X}} \underline{j^{1|m}}\underline{\Phi}^*\underline{\mathcal{L}} \tag{9.6}$$

And I can prove the following theorem:

Theorem 132. *A field $\Phi \in \Gamma(E)$ is a solution of the field theory with Lagrangian $\mathcal{L}$ if and only if the corresponding field $\underline{\Phi} \in \Gamma\left(b^*\left(J^{0|m}\pi\right)\right)$ is a solution of the field theory with Lagrangian $\underline{\mathcal{L}}$.*

Proof. Let's consider a local chart $\underline{U} \subset \underline{X}$ which is a submanifold with boundary $\partial\underline{U}$; there always exists a chart $U \subset X$ which is a submanifold of X with coordinates (x^a, x^α) and with a boundary ∂U so that $\epsilon(U) = \underline{U}$, $\epsilon(\partial U) = \partial\underline{U}$ and so that the boundary ∂U is defined by a function v which is real on $b(\underline{U})$ and so that $\forall\alpha,\ \partial_\alpha v = 0$.

For such a U and ∂U, and for every $\Phi \in \Gamma(E)$, we have that:

$$A_{U,\partial U}\left(\Phi\right) = \int_U j^1\Phi^*\mathcal{L} = \int_{\underline{U}} \underline{j^{1|m}}\underline{\Phi}^*\underline{\mathcal{L}} = \underline{A}_{\underline{U}}\left(\underline{\Phi}\right)$$

In fact:

$$\begin{aligned} &\int_U j^1\Phi^*\mathcal{L} = \int_{\mathbb{R}^{n|m}} Dx^a Dx^\alpha L\left(x^A, q^i(\Phi(x)), \partial_A q^i(\Phi(x))\right)\hat{\theta}\left(v(x)\right) \\ &= \int_{\mathbb{R}^n} D\underline{x}^a \int_{\mathbb{R}^{0|m}} Dx^\alpha L\left(x^a\left(b(\underline{x})\right), x^\alpha, x^\Lambda\partial_\Lambda q^i(\Phi(b(\underline{x}))), x^\Lambda\partial_a\partial_\Lambda q^i(\Phi(b(\underline{x}))), \partial_\alpha(x^\Lambda)\partial_\Lambda q^i(\Phi(b(\underline{x})))\right)\theta\left(v(\underline{x})\right) \\ &= \int_{\mathbb{R}^n} D\underline{x}^a \int_{S_{\underline{j^{1|m}}\underline{\Phi}(\underline{x})}} Dx^\alpha L\,\theta(v(\underline{x})) \\ &= \int_{\mathbb{R}^n}\int_{S_{\underline{j^{1|m}}\underline{\Phi}(\underline{x})}} \beta L\,\theta(v(\underline{x})) \\ &= \int_{\underline{U}} \underline{j^{1|m}}\underline{\Phi}^*\underline{\mathcal{L}} \end{aligned}$$

If we have an even path in $\mathscr{U}_\Phi$, it induces a real path in $\underline{\mathscr{U}}_{\underline{\Phi}}$; and inversely. We can parametrize by $l \in \mathbb{R}^{1|0}$ the first path and by $\underline{l} = \epsilon(l) \in \mathbb{R}$ the second path (being ϵ the body map of $\mathbb{R}^{1|0}$). We have then:

$$\frac{\partial}{\partial l}A_{U,\partial U}\left(\Phi_l\right) = \frac{\partial}{\partial\underline{l}}\underline{A}_{\underline{U}}\left(\underline{\Phi}_l\right)$$

It follows immediately that, if Φ is a solution of the theory with Lagrangian $\mathcal{L}$, then $\underline{\Phi}$ is a solution of the theory with Lagrangian $\underline{\mathcal{L}}$.

For establishing the converse, it is enough to prove that a field Φ is a solution of the theory if $\frac{\partial}{\partial l} A_{U,\partial U}(\Phi_l)\Big|_{l=0} = 0$ for every U and for every ∂U defined by a v as above (which are quite a special kind of boundaries); the proof is straightforward, because this last condition implies easily that Φ satisfy (7.7). ■

Note that the definitions 126, 130 and 131 can be extended verbatim to the case when the fiber-type F of the super fiber bundle E is of any super dimension $r|s$. The local coordinates would then be $\left(x^A, \dot{q}^I_\Lambda\right)$ on $J^{0|m}\pi$ and $(x^A, \dot{q}^I_\Lambda, \dot{q}^I_{a,\Lambda})$ on $J^{1|m}\pi$; with $\dot{q}^I_\Lambda, \dot{q}^I_{a,\Lambda} \in \mathbb{R}_S$, $\left|\dot{q}^I_\Lambda\right| = |\Lambda| + |I| = l(\Lambda) + |I|$ and $|\dot{q}^I_{a,\Lambda}| = |\Lambda| + |I| = l(\Lambda) + |I|$.

Propositions 127, 128 and 129 still hold in this more general case: the proofs must just be adapted to the parities of the elements involved.

All previous diagrams are still valid.

Definitions 9.5 and 9.6 can be taken as they are. Then theorem 132 can be adapted in an easy way to become my version of the Comparison Theorem.

I presented here the simplest case where the odd dimension of F is $s = 0$, because this case is adapted to treat the comparison between the so called superfield and the so called components formalisms.

In the next section I will continue to study this special case. I will prove some other comparisons theorems some of which involve the symplectic structures on the spaces of solutions of the theories treated. The results there obtained could be extended to the more general case of fiber-type of general odd dimension.

9.2 Comparison of symplectic structures in superfield and components formalisms

We have thus established a one-to-one correspondence between the space $\mathscr{E}$ of solutions of the field theory on E with Lagrangian $\mathcal{L}$ and the space $\underline{\mathscr{E}}$ of solutions of the field theory on $b^*\left(J^{0|m}\pi\right)$ with Lagrangian $\underline{\mathcal{L}}$. I stress one more time the fact that fields in $\underline{\mathscr{E}}$ are classical fields, because defined on a classical manifolds, $\underline{X}$, but they take value on a supermanifold, the fiber-type of $b^*\left(J^{0|m}\pi\right)$. The situation is somehow hybrid. If on a local chart, with local canonical n-form $\underline{\beta} = d\underline{x}^1 \wedge \cdots \wedge d\underline{x}^n$, we have that $\underline{\mathcal{L}} = \underline{\beta}\underline{L}$, it is nevertheless possible to prove that the action principle for the action $\underline{A}$ leads in local coordinates to the Euler-Lagrange system:

$$\frac{d}{d\underline{x}^a}\frac{\partial \underline{L}}{\partial \dot{q}^i_{a,\Lambda}}\left(\underline{j}^1\underline{\Phi}(\underline{x})\right) - \frac{\partial \underline{L}}{\partial \dot{q}^i_\Lambda}\left(\underline{j}^1\underline{\Phi}(\underline{x})\right) = 0$$

Note that by (9.5) we have that $|\underline{L}| = |L| + m$.

We can define a Legendre transform from $b^*\left(J^{1|m}\pi\right)$ to a space $\underline{P}$. The construction is analogous to the ones of section 2 and 5: we substitute E with $b^*\left(J^{0|m}\pi\right)$, π with $\underline{j^{0|m}}$ and X with $\underline{X}$. So we have that

$$\underline{P} := Hom_{\underline{j^{0|m}}}\left(Vb^*(J^{0|m}\pi), \mathbb{R}_S \otimes \Lambda^{n-1}T^*\underline{X}\right)$$

where the tensor product is necessary because $\underline{L}$ may take value in $\mathbb{R}_S$. As usual we can consider $\mathbb{R}_S = \mathbb{R}_{S,0} \oplus \mathbb{R}_{S,1}$ and consider the fiber type of $\underline{P}$ as a supermanifold. In local coordinates:

$$\underline{\mathbb{FL}} \; : \; (\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda}) \longrightarrow (\underline{x}^a, \dot{q}^i_\Lambda, p^{a,\Lambda}_i) = \left(\underline{x}^a, \dot{q}^i_\Lambda, \frac{\partial \underline{L}}{\partial \dot{q}^i_{a,\Lambda}}\left(\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda}\right)\right) \tag{9.7}$$

where $p_i^{a,\Lambda} = \overline{p_i^{a,\Lambda}} + \widetilde{p_i^{a,\Lambda}} = \frac{\partial \underline{\overline{L}}}{\partial \dot{q}^i_{a,\Lambda}} + \frac{\partial \underline{\widetilde{L}}}{\partial \dot{q}^i_{a,\Lambda}}$. Or:

$$\underline{\mathbb{FL}} \ : \ \left(\underline{x}^a, \dot{q}^i_\Lambda, \dot{q}^i_{a,\Lambda}\right) \ \longrightarrow \ \left(\underline{x}^a, \dot{q}^i_\Lambda, \overline{p_i^{a,\Lambda}}, \widetilde{p_i^{a,\Lambda}}\right) = \left(\underline{x}^a, \dot{q}^i_\Lambda, \frac{\partial \underline{\overline{L}}}{\partial \dot{q}^i_{a,\Lambda}}, \frac{\partial \underline{\widetilde{L}}}{\partial \dot{q}^i_{a,\Lambda}}\right) \tag{9.8}$$

Note that $\left|\overline{p_i^{a,\Lambda}}\right| = |\underline{L}| + l(\Lambda) = |L| + m + l(\Lambda)$ whereas $\left|\widetilde{p_i^{a,\Lambda}}\right| = |\underline{L}| + l(\Lambda) + 1 = |L| + m + l(\Lambda) + 1$.

A corresponding map (which again I call with the same name) $\underline{\mathbb{FL}}$, between sections of the spaces involved, can be defined naturally.

We can therefore add a line to the diagram (9.2) and we obtain:

$$\begin{tikzcd}
\Gamma(P) & \Gamma(\underline{P}) \\
\Gamma\left(J^1\pi\right) \arrow[u, "\mathbb{FL}"] \arrow[r, "ub_{J^1\pi}"] & \Gamma\left(b^*\left(J^{1|m}\pi\right)\right) \arrow[u, "\underline{\mathbb{FL}}"'] \\
\Gamma(E) \arrow[u, "j^1"] \arrow[r, leftrightarrow, "ub_E"] & \Gamma\left(b^*\left(J^{0|m}\pi\right)\right) \arrow[u, "\underline{j^{1|m}}"']
\end{tikzcd} \tag{9.9}$$

When $\mathcal{L}$ is regular, $\mathbb{FL}$ is one-to-one, and we can easily complete the diagram with an arrow between $\Gamma(P)$ and $\Gamma(\underline{P})$, and obtain a commutative diagram. This is also possible in general. In fact:

Proposition 133. *There exists an onto correspondence, noted with the underline* $\underline{\ \ }$, *between the space* $\Gamma(P)$ *of* G^∞ *sections of* P *and the space* $\Gamma(\underline{P})$ *of* C^∞ *sections of* $\underline{P}$, *which makes diagram* (9.9) *commutative. Moreover, when* $\mathcal{L}$ *is regular,* $\underline{\ \ }_P = \underline{\mathbb{FL}} \circ \underline{\ \ }_{J^1\pi} \circ \mathbb{FL}^{-1}$

Proof. Let U be a local chart on P with coordinates $\left(x^A, q^i, p_i^A\right)$. Let $\underline{U}$ be the naturally associated chart on $\underline{P}$, with local coordinates $\left(\underline{x}^a, \dot{q}^i_\Lambda, p_i^{a,\Lambda}\right)$. To every $z \in \Gamma(P)$ locally defined by:

$$z \ : \ x^A \ \longrightarrow \ \left(x^A, q^i\left(x^A\right), p_i^A\left(x^A\right)\right)$$

we associate the section $\underline{z} \in \Gamma(\underline{P})$ locally defined by:

$$\begin{aligned}
\underline{z} \ : \ \underline{x}^a \ \longrightarrow \ & \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), p_i^{a,\Lambda}\left(\underline{x}^a\right)\right) \\
= & \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha x^\Lambda p_i^a\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right)\right)
\end{aligned}$$

Let $s \in \Gamma\left(J^1\pi\right)$ be locally defined by:

$$s \ : \ x^A \ \longrightarrow \ \left(x^A, q^i\left(x^A\right), \dot{q}^i_A\left(x^A\right)\right) \ .$$

Then:

$$\mathbb{FL} \cdot s : \ x^A \ \longrightarrow \ \left(x^A, q^i\left(x^A\right), \frac{\partial L}{\partial \dot{q}^i_A}\left(x^A, q^i\left(x^A\right), \dot{q}^i_A\left(x^A\right)\right)\right)$$

and

$$\underline{\mathbb{FL}\cdot s}:\ \underline{x}^a \ \longrightarrow\ \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, p_i^{a,\Lambda}(\underline{x}^a)\right), \tag{9.10}$$

where:

$$p_i^{a,\Lambda}(\underline{x}^a) = (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha x^\Lambda \frac{\partial L}{\partial \dot{q}_a^i}\left(x^a\left(b(\underline{x}^a)\right), x^\alpha, q^i\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right), \dot{q}_A^i\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right)\right) \tag{9.11}$$

On the other hand:

$$\underline{s}\ :\ \underline{x}^a \ \longrightarrow\ \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, \partial_\Lambda \dot{q}_a^i\big|_{b(\underline{x}^a)}\right)$$

and

$$\mathbb{FL}\cdot\underline{s}\ :\ \underline{x}^a \ \longrightarrow\ \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, p_i^{a,\Lambda}(\underline{x}^a)\right) \tag{9.12}$$

where:

$$\begin{aligned}
&p_i^{a,\Lambda}(\underline{x}^a) = \frac{\partial \underline{L}}{\partial \dot{q}_{a,\Lambda}^i}\left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, \partial_\Lambda \dot{q}_a^i\big|_{b(\underline{x}^a)}\right)\\
&= \frac{\partial}{\partial \dot{q}_{a,\Lambda}^i}\left[\int Dx^\alpha L\left(x^a\left(b\left(\underline{x}^a\right)\right), x^\alpha, x^\Lambda \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, x^\Lambda \partial_\Lambda \dot{q}_a^i\big|_{b(\underline{x}^a)}, \partial_\alpha\left(x^\Lambda\right)\partial_\Lambda q^i\big|_{b(\underline{x}^a)}\right)\right]\\
&= (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha x^\Lambda \frac{\partial L}{\partial \dot{q}_a^i}\left(x^a\left(b(\underline{x}^a)\right), x^\alpha, x^\Lambda \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, x^\Lambda \partial_\Lambda \dot{q}_a^i\big|_{b(\underline{x}^a)}, \partial_\alpha\left(x^\Lambda\right)\partial_\Lambda q^i\big|_{b(\underline{x}^a)}\right)\\
&= (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha\\
&x^\Lambda \frac{\partial L}{\partial \dot{q}_a^i}\left(x^a\left(b(\underline{x}^a)\right), x^\alpha, q^i\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right), \dot{q}_a^i\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right), \dot{q}_\alpha^i\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right)\right)
\end{aligned} \tag{9.13}$$

Comparing (9.11) and (9.13), we achieve the first part of the assertion. The second then follows immediately because, when $\mathcal{L}$ is regular, $\mathbb{FL}$ is invertible. ∎

For every multiindex Λ I define $c\Lambda$ the complement multiindex containing all the α_j which do not appear in Λ and only them, with the usual decreasing order. I also call $g(\Lambda)$ the sign defined trough the equality

$$g(\Lambda)\Lambda\cdot c\Lambda = \alpha_m \cdots \alpha_1$$

For example, if $m = 7$ and $\Lambda = \alpha_5\alpha_3\alpha_1$, then $c\Lambda = \alpha_7\alpha_6\alpha_4\alpha_2$ and $g(\Lambda) = (-1)^9 = -1$.
If we write $p_i^A\left(x^A\right) = x^\Gamma \partial_\Gamma p_i^A\left(b(x^A)\right)$, then we can note that the expression for $p_i^{a,\Lambda}\left(\underline{x}^a\right)$ appearing in the proof of proposition 133 can be simplified:

$$\begin{aligned}
p_i^{a,\Lambda}\left(\underline{x}^a\right) &= (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha x^\Lambda p_i^a\left(x^a\left(b(\underline{x}^a)\right), x^\alpha\right)\\
&= (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^\alpha x^\Lambda x^\Gamma \partial_\Gamma p_i^a\left(b(\underline{x}^a)\right)\\
&= (-1)^{l(\Lambda)(l(\Lambda)+m)} g(\Lambda)\partial_{c\Lambda} p_i^a\left(b(\underline{x}^a)\right)\\
&= h(\Lambda)\partial_{c\Lambda} p_i^a\left(b(\underline{x}^a)\right)
\end{aligned} \tag{9.14}$$

if we set $h(\Lambda) := (-1)^{l(\Lambda)(l(\Lambda)+m)} g(\Lambda)$.

We can therefore reformulate the action of $\underline{\ \ }_P$ in coordinates by saying that:

Definition 134. *The map* $\underline{\quad}_P$ *sends every* $z \in \Gamma(P)$ *locally defined by:*

$$z \ : \ x^A \ \longrightarrow \ \left(x^A, q^i\left(x^A\right), p_i^A\left(x^A\right)\right)$$

to the section $\underline{z} \in \Gamma(\underline{P})$ *locally defined by:*

$$\begin{aligned}\underline{z} \ : \ \underline{x}^a \ \longrightarrow \ & \left(\underline{x}^a, \dot{q}^i_\Lambda\left(\underline{x}^a\right), p_i^{a,\Lambda}\left(\underline{x}^a\right)\right) \\ & = \left(\underline{x}^a, \partial_\Lambda q^i\big|_{b(\underline{x}^a)}, h(\Lambda)\partial_{c\Lambda} p_i^a\big|_{b(\underline{x}^a)}\right)\end{aligned}$$

with $h(\Lambda) = \pm 1$ *according to definition given above.*

When L is regular we have the following commutative diagram:

$$\begin{tikzcd}
\Gamma(P) \arrow[rr, "\underline{\quad}_P"] \arrow[d, leftrightarrow, "\mathbb{FL}"'] & & \Gamma(\underline{P}) \arrow[d, leftrightarrow, "\underline{\mathbb{FL}}"] \\
\Gamma\left(J^1\pi\right) \arrow[rr, "\underline{\quad}_{J^1\pi}"] & & \Gamma\left(b^*\left(J^{1|m}\pi\right)\right) \\
\Gamma(E) \arrow[u, "j^1"] \arrow[rr, leftrightarrow, "\underline{\quad}_E"] & & \Gamma\left(b^*\left(J^{0|m}\pi\right)\right) \arrow[u, "\underline{j^1}"']
\end{tikzcd} \tag{9.15}$$

On $\underline{P}$ we define the Hamiltonian $\underline{H}$ with:

$$\underline{H} := \dot{q}^i_{a,\Lambda} p_i^{a,\Lambda} - \underline{L} := \dot{q}^i_{a,\Lambda}\overline{p_i^{a,\Lambda}} - \overline{L} + \dot{q}^i_{a,\Lambda}\widetilde{p_i^{a,\Lambda}} - \widetilde{\underline{L}} \tag{9.16}$$

We have that $|\underline{H}| = |\underline{L}| = |L| + m$.

With proposition 133 and theorem 132 we have established that $ub_P := \underline{\quad}_P$ sends Hamiltonian section of P to Hamiltonian section of $\underline{P}$. It is then possible to pull back the symplectic form $\underline{\Omega}$ from the space of Hamiltonian sections $\underline{\mathscr{G}}$ to the space $\mathscr{G}$ and compare it with the symplectic form Ω on it.

Let's consider two even paths in $\mathscr{G}$: p_1 and p_2 parametrized by the variables L_1 and L_2:

$$\begin{aligned} p_1 \ : \ L_1 \ \longrightarrow \ & \left[z_{L_1} : \ x^A \ \longrightarrow \ \left(x^A, q^i(x^A, L_1), p_i^A(x^A, L_1)\right)\right] \\ p_2 \ : \ L_2 \ \longrightarrow \ & \left[z_{L_2} : \ x^A \ \longrightarrow \ \left(x^A, q^i(x^A, L_2), p_i^A(x^A, L_2)\right)\right]\end{aligned}$$

The map ub_P sends p_1 and p_2 to two paths, $ub_P(p_1)$ and $ub_P(p_2)$, in $\underline{\mathscr{G}}$:

$$\begin{aligned} & ub_P(p_1) \ : \ L_1 \ \longmapsto \\ & \left[\underline{z_{L_1}} : \ \underline{x^a} \ \longmapsto \ \left(\underline{x^a}, \partial_\Lambda q^i(b(x^a), L_1), (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^{n+1} \cdots Dx^{n+m} x^\Lambda p_i^a(\underline{x^a}, x^\alpha, L_1)\right)\right] \\ & ub_P(p_2) \ : \ L_2 \ \longmapsto \\ & \left[\underline{z_{L_2}} : \ \underline{x^a} \ \longmapsto \ \left(\underline{x^a}, \partial_\Lambda q^i(b(x^a), L_2), (-1)^{l(\Lambda)(l(\Lambda)+m)} \int Dx^{n+1} \cdots Dx^{n+m} x^\Lambda p_i^a(\underline{x^a}, x^\alpha, L_2)\right)\right]\end{aligned}$$

If we choose $\underline{\Sigma}$ to be the surface of even codimension 1 locally defined by the equation $\underline{x}^1 = 0$, we can calculate:

$$\begin{aligned} & ub_P^* \underline{\Omega}_{\underline{\Sigma}} \left(\left.\frac{\partial p_1}{\partial L_1}\right|_{L_1=0}, \left.\frac{\partial p_2}{\partial L_2}\right|_{L_2=0} \right) \\ & = \underline{\Omega}_{\underline{\Sigma}} \left(\left.\frac{\partial ub_P(p_1)}{\partial L_1}\right|_{L_1=0}, \left.\frac{\partial ub_P(p_2)}{\partial L_2}\right|_{L_2=0} \right) \end{aligned} \tag{9.17}$$

and since:

$$\begin{aligned} \left.\frac{\partial \underline{z}_L}{\partial L}\right|_{L=0} &= \left.\frac{\partial}{\partial L} \partial_\Lambda q^i(b(\underline{x}^a), L)\right|_{L=0} \frac{\partial}{\partial q^i_\Lambda} \\ &+ (-1)^{l(\Lambda)(l(\Lambda)+m)} \left.\frac{\partial}{\partial L} \int Dx^{n+1} \cdots Dx^{n+m} x^\Lambda p^a_i(\underline{x}^a, x^\alpha, L)\right|_{L=0} \frac{\partial}{\partial p^{a,\Lambda}_i} \end{aligned}$$

we have that:

$$\begin{aligned} & ub_P^* \underline{\Omega}_{\underline{\Sigma}} \left(\left.\frac{\partial p_1}{\partial L_1}\right|_{L_1=0}, \left.\frac{\partial p_2}{\partial L_2}\right|_{L_2=0} \right) \\ &= \int_{\underline{\Sigma}} \left\{ \left.\frac{\partial}{\partial L_1} \partial_\Lambda q^i(b(\underline{x}^a), L_1)\right|_{L_1=0} (-1)^{l(\Lambda)(l(\Lambda)+m)} \left.\frac{\partial}{\partial L_2} \int Dx^{n+1} \cdots Dx^{n+m} x^\Lambda p^a_i(\underline{x}^a, x^\alpha, L_2)\right|_{L_2=0} \underline{\beta}_a \right. \\ & \left. - \left.\frac{\partial}{\partial L_2} \partial_\Lambda q^i(b(\underline{x}^a), L_2)\right|_{L_2=0} (-1)^{l(\Lambda)(l(\Lambda)+m)} \left.\frac{\partial}{\partial L_1} \int Dx^{n+1} \cdots Dx^{n+m} x^\Lambda p^a_i(\underline{x}^a, x^\alpha, L_1)\right|_{L_1=0} \underline{\beta}_a \right\} \\ &= \int_\Sigma \beta_a \left\{ \left.\frac{\partial}{\partial L_1} q^i(x^A, L_1)\right|_{L_1=0} \left.\frac{\partial}{\partial L_2} p^a_i\left(x^A, L_2\right)\right|^{L_2=0} - \left.\frac{\partial}{\partial L_2} q^i(x^A, L_2)\right|_{L_2=0} \left.\frac{\partial}{\partial L_1} p^a_i\left(x^A, L_1\right)\right|^{L_1=0} \right\} \end{aligned} \tag{9.18}$$

where the last equality holds because L_1 and L_2 are even variables.

So we have that, when L_1 and L_2 are even:

$$ub_P^* \underline{\Omega}_{\underline{\Sigma}} \left(\left.\frac{\partial p_1}{\partial L_1}\right|_{L_1=0}, \left.\frac{\partial p_2}{\partial L_2}\right|_{L_2=0} \right) = \Omega_\Sigma \left(\frac{\partial p_1}{\partial L_1}|_{L_1=0}, \frac{\partial p_2}{\partial L_2}|_{L_2=0} \right) \tag{9.19}$$

where Σ is defined locally by $x^1 = 0$.

If L_1 or L_2 is odd, or if both are odd, the calculation is not valid anymore, but we should pass to the extended forms $\widehat{\Omega_\Sigma}$ and $ub_P^*\widehat{\underline{\Omega}_{\underline{\Sigma}}}$.

Note that it is possible that L is degenerate whereas $\underline{L}$ is regular, which allows to build $\underline{\Omega}$ and $ub_P^*\underline{\Omega}$, but doesn't allow to build up Ω.

Chapter 10

Super Poisson Brackets

Fields brackets are the fundamental object in classical canonical formulation of field theories and they constitute the starting point for canonical quantization.

Even if this has not been often noticed, the definition of fields brackets for classical Fermionic fields constituted also the motivation for one of the first uses of anticommuting variables in Physics. The work of Peierls in 1952, [122], can be considered the origin of supermathematics in Physics. In [122], facing the problem of defining the brackets of classical Fermionic fields, R. E. Peierls puts in front of them some anticommuting parameters and then carries on a classical analysis as if he was dealing with classical, commuting, quantities.

This procedure, which seems a bit arbitrary, has since then been, implicitly or explicitly, standard for physicists working with the commutation laws of fields.

In this chapter I want to show how the super-Poisson commutation laws of superfields arise naturally in the formalism of fractional superform.

In section 8.4 I showed how to build a symplectic structure on the super covariant phase space space $\mathscr{G}$, *id est* the space of Hamiltonian $n|m$-submanifolds of the super-multimomenta-space P, and on the space $\mathscr{E}$, isomorphic to $\mathscr{G}$, of solutions of the Lagrangian superfield theory. When the Lagrangian density L is non degenerate, the symplectic structure is non degenerate as well and it allows the construction of a corresponding Poisson structure on the same spaces.

In section 10.1 I show how to define the super Poisson brackets in the simplest case of supermechanics. That simple case will be nonetheless fundamental to understand the mechanism which leads to the correct commutation laws of superfields.

I will also comment some interesting properties of the super Poisson brackets, deriving from the nature of the symplectic structure Ω_Σ on $\mathscr{G}$.

In section 10.2 I will very briefly treat the more general case of higher dimensional super field theories.

10.1 Supermechanics and Poisson brackets for super-functions.

In this section I will study the special case of supermechanics. This arises when the fiberbundle of fields E has base X of dimension $1|0$ and fiber-type F of dimension dimension $r|s$ or, as well, when X has dimension $1|m$ and the fiber-type F has dimension $r|0$, which is a special case of the situation studied in detail in the previous section.

Let's first consider the case when X has dimension $1|0$.

On E we can use as local coordinates (t, q^I); on $J^1\pi$ we can use local coordinates $(t, q^I, \dot{q}^I)$; if the Lagrangian L is regular we can build the corresponding super-covariant-phase-space P and on P we can use as coordinates (t, q^I, p_I) , where $p_I = \overline{p_I}$ if $L = \overline{L}$ is even and $p_I = \widetilde{p_I}$ if $L = \widetilde{L}$ is odd.

A section $s \in \Gamma(J^1\pi)$ can be written as:

$$s:\ X \longrightarrow J^1\pi\ ;\ s:\ t \longrightarrow (t, q_s^I(t), \dot{q}_s^I(t))$$

A section $z \in \Gamma(P)$ can be written as:

$$z:\ X \longrightarrow P\ ;\ z:\ t \longrightarrow (t, q_z^I(t), p_{z,I}(t)) \tag{10.1}$$

The spaces $\mathscr{E}$, of solutions of the field theory, and $\mathscr{G}$, of Hamiltonian surfaces, are in this case finite dimensional and they are diffeomorphic when L is regular. If we choose on P the surface Σ of codimension 1 defined locally by the equation $t = 0$, then the space $\mathscr{G}$ is well parametrized by the coordinates $(q_0^I, p_{I,0})$ where a section $z \in \mathscr{G}$ has coordinates $q_0^I(z) = q_z^I(0)$ and $p_{I,0}(z) = p_{z,I}(0)$. A vector $\delta z \in T_z\mathscr{G}$ can be written as $\delta z = u^{q_0^I} \left.\frac{\partial}{\partial q_0^I}\right|_z + u^{p_{I,0}} \left.\frac{\partial}{\partial p_{I,0}}\right|_z$. It is easy to see that, when $\overline{\delta_1 z}, \overline{\delta_2 z}, \in T_{z,0}\mathscr{G}$ are even vectors, then equation (8.33) becomes:

$$\begin{aligned} \Omega_\Sigma\big|_z\left(\overline{\delta_1 z}, \overline{\delta_2 z}\right) &:= \int_{\Sigma\cap G} \overline{u_1} \wedge \overline{u_2} \lrcorner\, \omega = -dq_0^I \wedge dp_{I,0}\big|_z(\overline{u_1}, \overline{u_2}) \\ &= -\overline{u_1^{q_0^I}}\,\overline{u_2^{p_{I,0}}} + \overline{u_2^{q_0^I}}\,\overline{u_1^{p_{I,0}}} \end{aligned} \tag{10.2}$$

For the sake of simplicity I will from now on let the subsript 0 drop, so I will rename the coordinates on $\mathscr{G}$ as (q^I, p_I), I will write $\delta z = u^{q^I} \left.\frac{\partial}{\partial q^I}\right|_z + u^{p_I} \left.\frac{\partial}{\partial p_I}\right|_z$ and:

$$\begin{aligned} \Omega_\Sigma\big|_z\left(\overline{\delta_1 z}, \overline{\delta_2 z}\right) &= -dq^I \wedge dp_I\big|_z(\overline{u_1}, \overline{u_2}) \\ &= -\overline{u_1^{q^I}}\,\overline{u_2^{p_I}} + \overline{u_2^{q^I}}\,\overline{u_1^{p_I}} \end{aligned} \tag{10.3}$$

When $\delta_1 z, \delta_2 z, \in T_{z,}\mathscr{G}$ are vectors of generic parity, then, remembering (5.9), equation (8.34) becomes:

$$\begin{aligned} \widehat{\Omega_\Sigma}\big|_z(\delta_1 z, \delta_2 z) &= -d\widehat{q^I \wedge d}p_I\big|_z(u_1, u_2) \\ &= -\overline{u_1^{q^I}}\,\overline{u_2^{p_I}} + \overline{u_2^{q^I}}\,\overline{u_1^{p_I}} - (-1)^{|I|}\,\overline{u_1^{q^I}}\,\widetilde{u_2^{p_I}} + \widetilde{u_2^{q^I}}\,\overline{u_1^{p_I}} + \\ &- \widetilde{u_1^{q^I}}\,\overline{u_2^{p_I}} + (-1)^{|I|}\,\overline{u_2^{q^I}}\,\widetilde{u_1^{p_I}} - (-1)^{|I|}\,\widetilde{u_1^{q^I}}\,\widetilde{u_2^{p_I}} + (-1)^{|I|+1}\,\widetilde{u_2^{q^I}}\,\widetilde{u_1^{p_I}} \end{aligned} \tag{10.4}$$

Similar considerations can be done for the space $\mathscr{E}$ and the form O, using as local coordinates on $\mathscr{E}$ the Cauchy data $(q_0^I, \dot{q}_0^I)$ defined as expected with the help of a surface Σ again defined locally by $t = 0$.

When $\widehat{\Omega_\Sigma}$ is non degenerate, it defines on P a supersymplectic structure. Note that Ω_Σ is even when $L = \overline{L}$ is even and it is odd when $L = \widetilde{L}$ is odd, because $\left|dq^I\right| = |I|$ and $|dp_I| = |I| + |L|$. This is coherent with what we have seen in section 8.4 for the case when the odd dimension of the base manifold X is $m = 0$.

If $\widehat{\Omega_\Sigma}$, defined by (10.4), is non degenerate, it can be used to define a Poisson bracket on $\mathcal{F}(\mathscr{G})$ which is the space of function on $\mathscr{G}$. Remember that $\widehat{\Omega_\Sigma} \in \Omega^{2|0}\mathscr{G}$.

Let's consider $f \in \mathcal{F}(\mathscr{G})$, then $df \in \Omega^{1|0}\mathscr{G}$. We associate to f a vector field $u_f \in X(\mathscr{G}) = \Gamma(T\mathscr{G})$ defining it with:

$$\widehat{\Omega_\Sigma}(\cdot, \delta z) = df(\cdot) \tag{10.5}$$

where δz is associated to u_f.

Then from (10.4), we obtain that it must be:

$$\begin{aligned} -\overline{u_f^{p_I}} - (-1)^{|I|}\, \widetilde{u_f^{p_I}} &= \frac{\partial f}{\partial q^I} \\ (-1)^{|I|+|I||L|}\, \overline{u_f^{q^I}} + (-1)^{|L|+|I||L|}\, \widetilde{u_f^{q^I}} &= \frac{\partial f}{\partial p_I} \end{aligned} \tag{10.6}$$

Note that it is essential to put u_f as second argument of $\widehat{\Omega_\Sigma}$, otherwise we could not obtain a $1|0$-form.

If $f, g \in \mathcal{F}(\mathscr{G})$, then we set:

$$\{f, g\} := \widehat{\Omega_\Sigma}\,(u_f, u_g) \tag{10.7}$$

Since from (10.6) we have that:

$$|u_f| = |f| + |L|$$

it follows that:

$$\{f, g\} = \widehat{\Omega_\Sigma}\,(u_f, u_g) = -\,(-1)^{|u_f||u_g|}\,\widehat{\Omega_\Sigma}\,(u_g, u_f) = -\,(-1)^{(|f|+|L|)(|g|+|L|)}\,\{g, f\} \tag{10.8}$$

With some calculations from (10.7), (10.4) and (10.6) one obtains that:

$$\{f, g\} = (-1)^{(|I|+|L|)(|f|+1)}\,\frac{\partial f}{\partial p_I}\frac{\partial g}{\partial q^I} - (-1)^{(|I|+|L|)(|g|+1)}\,(-1)^{(|f|+|L|)(|g|+|L|)}\,\frac{\partial g}{\partial p_I}\frac{\partial f}{\partial q^I} \tag{10.9}$$

Formula (10.9) should be consider a shortcut for the formula obtained summing the two below (10.11) and (10.12), which are more interesting since we are more interested to theories which stems from Lagrangians of pure degree.

Formula (10.9) is obviously coherent with (10.8); in fact from (10.9) we obtain by direct calculation that:

$$\{f, g\} = -\,(-1)^{(|f|+|L|)(|g|+|L|)}\,\{g, f\} \tag{10.10}$$

When $|L| = 0$, we have that $\mathscr{G} \cong \mathscr{E} \cong T_0^*F$, where F is the target space, id est the fiber-type of our fields bundle or the space of fields. Remember that, if F has dimension $r|s$, then T_0^*F has dimension $2r|2s$. When $|L| = 0$, also $|H| = 0$ and Ω_Σ is even and it defines an even symplectic structure on $\mathscr{G} \cong \mathscr{E} \cong T_0^*F$. The Hamiltonian supermechanics in this case can be completely described in terms of this even symplectic structure on T_0^*F. The even Poisson brackets (10.9) are:

$$\{f, g\} = (-1)^{|I|(|f|+1)}\,\frac{\partial f}{\partial \overline{p_I}}\frac{\partial g}{\partial q^I} - (-1)^{|I|(|g|+1)}\,(-1)^{|f||g|}\,\frac{\partial g}{\partial \overline{p_I}}\frac{\partial f}{\partial q^I} \tag{10.11}$$

When $|L| = 1$, we have that $\mathscr{G} \cong \mathscr{E} \cong T_1^*F$. Remember that, if F has dimension $r|s$, then T_1^*F has dimension $r+s|r+s$. When $|L| = 1$, also $|H| = 1$ and Ω_Σ is odd and it defines an odd symplectic structure on $\mathscr{G} \cong \mathscr{E} \cong T_1^*F$. The Hamiltonian supermechanics in this case can be completely described in terms of an odd symplectic structure on T_1^*F. In the literature an odd form of the type of Ω_Σ is also called sometime periplectic, see Rogers [133].

The odd Poisson brackets (10.9) are:

$$\{f, g\} = (-1)^{(|I|+1)(|f|+1)}\,\frac{\partial f}{\partial \widetilde{p_I}}\frac{\partial g}{\partial q^I} - (-1)^{(|I|+1)(|g|+1)}\,(-1)^{(|f|+1)(|g|+1)}\,\frac{\partial g}{\partial \widetilde{p_I}}\frac{\partial f}{\partial q^I} \tag{10.12}$$

Note that in this case equation (10.10), become:

$$\{f, g\} = -\,(-1)^{(|f|+1)(|g|+1)}\,\{g, f\}$$

The odd Poisson brackets are an instance of *Buttin brackets*, from C. Buttin who first studied them in a general setting during the 60's, see [20]. Sometime they take the name of *antibrackets*.

Odd symplectic structures are closely related to Batalin-Vilkovisky formalism as one can learn by reading Schwarz [142] and the papers of O. M. Khudaverdian, in particular [90], [92], with A. P. Neressian, and [91]. An extensive bibliography on the subjects can be found in [91]. On odd Poisson brackets, and their connection with BV formalism, is also interesting to read Witten [162] and Y. Kosmann-Schwarzbach and J. Monterde [100] and to consult the bibliography therein.

It would be interesting to reformulate the results obtained by Khudaverdian and by other authors with the formalism of mixed fractional forms introduced in chapter 6, starting from the work already done in [93]. I will not follow this path in this thesis.

The analysis just made for the case when X has dimension $1|0$, can be extended to the case when X has dimension $1|m$ and the fiber-type F has dimension $1|0$. In this case, following the considerations presented in section 6, I can build an equivalent theory on the bundle $b^*\left(J^{0|m}\pi\right)$ whose base $\underline{X}$ is a one dimensional real manifold (so it can be treated as if it had dimension $1|0$). On the space of solutions $\underline{\mathscr{G}}$, which is diffeomorph to the spaces $\mathscr{G}$ and $\mathscr{E}$ when L is regular, we can build the symplectic form $\underline{\Omega}$ which I used in (9.17), (9.18) and (9.19) starting from a surface $\underline{\Sigma}$ to be defined by the local equation $\underline{t}=0$, where $\underline{t}$ is the only local even coordinate.

Formulas (10.2), (10.3) and (10.4) can be rewritten substituting the index I with the index Λ. $\underline{\Omega}$ assume the form:

$$\underline{\underline{\Omega_{\underline{\Sigma}}}}\Big|_G\left(\overline{\delta_1 G}, \overline{\delta_2 G}\right) \;=\; \int_{\underline{\Sigma}\cap G} \overline{v_1}\wedge\overline{v_2}\lrcorner\underline{\omega} = -dq_\Lambda\wedge dp^\Lambda(\overline{v_1},\overline{v_2})|_{t=0} \tag{10.13}$$

Equation (10.4), remembering that $\left|q^\Lambda\right| = l(\Lambda)$ becomes:

$$\begin{aligned}
\widehat{\underline{\underline{\Omega_{\underline{\Sigma}}}}}\Big|_z(\delta_1 z,\delta_2 z) &= -d\widehat{q_\Lambda\wedge dp}^\Lambda\Big|_z(u_1,u_2)\\
&= -\overline{u_1^{q_\Lambda}}\,\overline{u_2^{p^\Lambda}} + \overline{u_2^{q_\Lambda}}\,\overline{u_1^{p^\Lambda}} - (-1)^{l(\Lambda)}\,\overline{u_1^{q_\Lambda}}\,\widetilde{u_2^{p^\Lambda}} + \widetilde{u_2^{q_\Lambda}}\,\overline{u_1^{p^\Lambda}}\\
&\quad - \widetilde{u_1^{q_\Lambda}}\,\overline{u_2^{p^\Lambda}} + (-1)^{l(\Lambda)}\,\overline{u_2^{q_\Lambda}}\,\widetilde{u_1^{p^\Lambda}} - (-1)^{l(\Lambda)}\,\widetilde{u_1^{q_\Lambda}}\,\widetilde{u_2^{p^\Lambda}} + (-1)^{l(\Lambda)+1}\,\widetilde{u_2^{q_\Lambda}}\,\widetilde{u_1^{p^\Lambda}}
\end{aligned} \tag{10.14}$$

If $\underline{\underline{\Omega_{\underline{\Sigma}}}}$ is non degenerate, it can be used to define a Poisson bracket and equations (10.8) and (10.9) become:

$$\{f,g\} = \widehat{\Omega_\Sigma}(u_f,u_g) = (-1)^{|u_f||u_g|}\,\widehat{\Omega_\Sigma}(u_g,u_f) = (-1)^{(|f|+|\underline{L}|)(|g|+|\underline{L}|)}\{g,f\} \tag{10.15}$$

and

$$\{f,g\} = (-1)^{(|\Lambda|+|\underline{L}|)(|f|+1)}\frac{\partial f}{\partial p^\Lambda}\frac{\partial g}{\partial q_\Lambda} - (-1)^{(|\Lambda|+|\underline{L}|)(|g|+1)}(-1)^{(|f|+|\underline{L}|)(|g|+|\underline{L}|)}\frac{\partial g}{\partial p^\Lambda}\frac{\partial f}{\partial q_\Lambda} \tag{10.16}$$

with $|\underline{L}| = |L| + m$.

You can see the example given in section 7 for understanding what happens.

When X has dimension $1|m$ and F has dimension $r|s$ the spaces $\mathscr{G}\cong\mathscr{E}$ are still finite dimensional. Formula (10.13) can be generalized to this case; it becomes:

$$\underline{\underline{\Omega_{\underline{\Sigma}}}}\Big|_G\left(\overline{\delta_1 G},\overline{\delta_2 G}\right) \;=\; -dq^I_\Lambda\wedge dp^\Lambda_I(\overline{v_1},\overline{v_2})|_{t=0} \tag{10.17}$$

with $\left|q^I_\Lambda\right| = |I| + l(\Lambda)$ and $\left|p^\Lambda_I\right| = |I| + l(\Lambda) + |L| + m$.

Equivalently formula (10.14), (10.15) and (10.16) could be generalized as well.

In the special case when X has dimension $1|1$ and F has dimension $r|s$ all these formula can be then compared to the corresponding ones appearing in [114], [115] and [117], to find that they agree with them.

10.2 Super Poisson brackets for superfields

Phenomena similar to the ones described in section 10.1 occur when X has even dimension greater than 1 and Poisson brackets are defined on functions of superfields like we did in chapter 4.

The path followed in chapter 4 can be repeated without to many changes, paying attention to the parity of the quantities involved and using extended superforms as we have done in the previous section. If A and B are functionals defined on $\mathscr{G}$, formula (4.3) can be generalized and it becomes:

$$\begin{aligned}
\{A,B\}(G) &= \int_{\Sigma_X} -\left({}^{A}V_G^{q^I}(\vec{x})\,{}^{B}V_G^{\pi_I}(\vec{x}) - {}^{B}V_G^{q^I}(\vec{x})\,{}^{A}V_G^{\pi_I}(\vec{x})\right) d\vec{x} \\
&= \int_{\Sigma_X} \Bigg[(-1)^{(|I|+|L|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta B}{\delta q^I}\right|_G (\vec{x}) \\
&\qquad - (-1)^{(|I|+|L|)(|B|+1)} (-1)^{(|A|+|L|)(|B|+|L|)} \left.\frac{\delta B}{\delta \pi_I}\right|_G (\vec{x}) \left.\frac{\delta A}{\delta q^I}\right|_G (\vec{x})\Bigg] d\vec{x} \\
&= \int_{\underline{\Sigma_X}} \Bigg[(-1)^{(|I|+|\Lambda|+|\underline{L}|)(|A|+1)} \left.\frac{\delta A}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta B}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}}) \\
&\qquad - (-1)^{(|I|+|\Lambda|+|\underline{L}|)(|B|+1)} (-1)^{(|A|+|\underline{L}|)(|B|+|\underline{L}|)} \left.\frac{\delta B}{\delta \pi_I^\Lambda}\right|_G (\underline{\vec{x}}) \left.\frac{\delta A}{\delta q_\Lambda^I}\right|_G (\underline{\vec{x}})\Bigg] d\underline{\vec{x}}
\end{aligned} \tag{10.18}$$

where Σ_X is a Cauchy slice in X of codimension $1|0$, $(\vec{x})$ are the restriction of the Cauchy coordinates on the $n-1|m$ dimensional supermanifold Σ_X, $d\vec{x}$ is the canonical volume $n-1|m$-form defined by the Cauchy coordinates on Σ_X; $\underline{\Sigma_X}$ is the body of Σ_X, $(\underline{\vec{x}})$ are the Cauchy coordinates on $\underline{\Sigma_X}$ and $d\underline{\vec{x}}$ is the canonical volume $n-1$-form defined by the Cauchy coordinates on $\underline{\Sigma_X}$. Note that $\left|\pi_I^\Lambda\right| = |I| + l(\Lambda) + |L| + m$.

Note that the last equality in (10.18) is a consequence of the equivalence of the symplectic structures built on $\mathscr{G}$ with Ω_Σ and with $\Omega_{\underline{\Sigma}}$. Of this equivalence I gave a sketch of a proof in section 9.2.

The components analysis of formula (10.18), analogous to the one undertaken in the previous section in the case when $n=1$, shows that the components fields obey the expected commutation and anticomutation relations of Bosonic and Fermionic fields of a Physics field theories whenever $|\underline{L}| = |L| + m = 0$.

When $|\underline{L}| = |L| + m = 1$, equation (10.18) defines an odd Poisson structure on the space of functions over $\mathscr{G}$ (the observables).

What should be retained here is that, if we are studying a field theory with $|\underline{L}| = |L| + m = 0$, and we define the brackets of functional of fields in the super multisymplectic framework, then we obtain (10.18), and the brackets satisfy the expected supercoummutation rules.

It is interesting to understand better where the correct parities of the commutators arise from. Peierls in his paper of 1952, [122] pag. 149, had to introduce an ad hoc assumption in order to justify the presence of anticommutators for Fermions. He justifies it with these words:

"...To obtain the usual commutation rules for Fermi statistics, one has to resort to a special device, which can be made plausible in the following way. Fermion wave functions [$\psi_\alpha(x)$] are well known not to be single-valued, since a rotation of the co-ordinate system by 2π will change their sign. ... To avoid this, one can postulate the existence of some operator θ, which is itself of such a character as to change sign if the co-ordinate system is rotated by 2π, and which anticommutes with all components of the spinor field at all points in space-time. Then $\theta\psi_\alpha(x)$ is a single-valued quantity which may reasonably occur in the action principle. One obtains then the correct commutation laws...". Peierls puts in front of Fermionic fields some anticommuting parameters and then carries on a classical analysis as if he was dealing with classical quantities.

Many other authors, after him, won't even justify the fact that Fermions need anticommutators instead of commutators, they will just introduce them and use them.

With my geometric approach, the presence of anticommutators for Fermions is a consequence of two assumptions:

- that the classical fields associated to Fermions are odd coordinates of a supermanifold;
- that the multisymplectic Voronov Zorich form ω is extended to a Kostant-like form $\widehat{\omega}$ and so is the symplectic form Ω on the super covariant phase space, which is extended to $\widehat{\Omega}$.

The latter assumption links the parity of commutators to the symmetry properties of some forms (the Kostant forms) which are indeed quite natural objects in supergeometry.

As long as the extension of the Voronov Zorich forms to extended forms (which share symmetry properties with Kostant forms) is natural, then the only relevant assumption made is the first one.

So, with this approach, at the end of the story, the correct parity of the classical commutators arise from the fact that we assume that classical Fermion fields are odd objects. This in my opinion is not so clear if one uses other approaches to field theory.

Of course it could be interesting to investigate the possibility to extend Voronov Zorich forms in another way, so that they satisfy other type of symmetries, and then eventually to study what would be the consequences for the new multisymplectic superform, for the new symplectic superform on the covariant phase space and for the new supercommutators.

I close this chapter with a remark. As I wrote in chapter 4, Kijowski [94, 95], Kijowski and Szczyrba [97], I. Kanatchikov [85, 86, 87], Hélein [74], Forger and H. Römer [52], Forger, C. Paufler and Römer [50, 51], M. O. Salles [139], Baez, Hoffnung and C. L. Rogers [3], Baez and Rogers [4], Richter [131], worked on specific subclasses of functionals on $\mathscr{G}$: namely those functionals which arise from the integration on some submanifold of X of the pullback of, possibly generalized, forms defined on P. (Here generalized roughly means that their coefficients in local coordinates may depends on the derivative of the coordinates as well as on the coordinates themselves; generalized forms can be considered then as the restrictions on P of forms defined on some jet space of P.) For those functionals the Poisson brackets defined with (10.18) turn out to be linked to some other brackets defined directly on the space of forms with the help of the multisymplectic structure.

It would be interesting to study similar settings for the super field theories (and super string theories) using my formalism and studying functionals arising from generalized fractional superforms. In fact most of the results found in the papers quoted above can be easily transposed to the super case by using the definitions and the calculation techniques expounded in this third part of my thesis. I will use similar calculation techniques in the fourth part of this thesis while proving a super version of Noether first theorem and defining a super multimomentum map.

Chapter 11

Super Noether theorem and super multimomentum map

Classical symmetries are one of the most important features of field theories and, at least since the birth of supersymmetric field theories during the 70's, supersymmetries as well are of the greatest importance in Physics. It is then not surprising that also the mathematical literature on the subject is vast. However, a satisfactory framework to describe supersymmetries for superfield theories in a geometrical way has not yet been completely set up.

The studies of supersymmetries from a geometrical point of view have reached for supermechanical theories more advanced results. Various different super version of first Noether theorem for supermechanics have been proposed: see for example Ibort and Marín-Solano [83] and Cariñena and Figueroa [25]. A super version of reduction theorems of Poisson manifolds have been studied, see for example F. Cantrijn and L. A. Ibort [21].

L. Fatibene and M. Francaviglia, [48] and L. Fatibene, M. Ferraris, M. Francaviglia and R. G. McLenaghan, [47], studied supersymmetric field theories in the components formulation and showed that the supersymmetry can be treated as a generalized symmetry in classical theories, with just some attention to the parity of the fields. So they can give a version of Noether theorem valid for the supersymmetries of supersymmetric field theories in the components formulation. Their ultimate aim is to show that the standard and the supersymmetric frameworks admit a unifying mathematical language. Unfortunately their approach doesn't help in treating symmetries of theories defined with the super-field formulation and therefore it doesn't allow to fully exploit the power of that formulation.

In 2006 Monterde, Muñoz Masqué and Vallejo, [116], obtained a geometric formulation of a super version of first Noether theorem valid for super field theories defined on base manifolds of any dimension, even or odd. They achieved this beautiful and important result by studying, with the help of the Comparison Theorem, an associated higher-order graded variational problem, defined on a bosonic base manifold. They although needed a rather technical assumption added to the hypothesis to prove Noether theorem. Moreover their notation doesn't make really transparent the analogy between their theory and the classical geometrical theory of symmetry for Lagrangian field theories.

In this chapter, I will show that the formalism introduced in this thesis allows a very simple formulation of Noether first theorem for super fields theories, which doesn't require any additional hypothesis. This formulation will be strictly analogue to the one used in geometrical interpretations of classical field theories and therefore will achieve the aim of describing the standard and the supersymmetric frameworks with a unified mathematical language. Moreover my formulation will allow to fully exploit the super-field approach to supersymmetric field theories. I will also show how to obtain a super version of the multimomentum map introduced in [63].

In section 11.1 I will explain what are even and odd symmetries of a super field theory and I will show how they are linked to even and odd (generalized) vector fields on the space of configurations.

In section 11.2 I will exhibit and prove my super version of first Noether theorem.

In section 11.3 I will build the super multimomentum map for a supergroup of symmetry.

11.1 Symmetries and supersymmetries of a super field theory

Let E, X and F be finite dimensional G^∞-supermanifolds of dimensions $n+r|m+s$, $n|m$ and $r|s$, E and X being connected; and let (E,π,X,F) be a super fiber bundle with total space E, base X, type-fiber F and bundle G^∞-projection π. In this chapter I will call sometime E the configuration space and F the space of fields or the target space. On X we suppose that is fixed an immersed body (see Section 5.5 for the definition of immersed body).

A field Φ over X is a G^∞-section of the fiber bundle π: $\Phi \in \Gamma\big(E\big)$. As in the previous chapters, we will deal with the space $\Gamma\big(E\big)$ of all possible sections of E supposing that it has been given a super differential structure. The same assumption will be understood for other infinite-dimensional space of sections here studied.

Let's consider the superfield theory defined by the Lagrangian $\mathcal{L}$ (which is an horizontal $n|m$-dimensional differential form on J^1E). The action A over a field Φ on an $n|m$-dimensional submanifold $U \subset X$ with boundary ∂U is defined by the integral of $\mathcal{L}$:

$$A_{U,\partial U}(\Phi) := \int_{(U,\partial U)} j^1\Phi^*\mathcal{L} \tag{11.1}$$

where the integral is supposed to be performed with respect the fixed immersed body.

For every $(U,\partial U)$, $A_{U,\partial U}$ is a smooth super-functional on $\Gamma\big(E\big)$.

I give now a version of the action principle slightly different from the one given in Chapter 7:

Definition 135. *A solution Φ of the field theory with Lagrangian $\mathcal{L}$ over J^1E is a section $\Phi \in \Gamma\big(E\big)$ such that, $\forall U$ submanifold of X with ∂U boundary of U, $dA_U\big|_{\mathscr{U}_\Phi}(\Phi) = 0$; where $\mathscr{U}_\Phi := \{\phi \in \Gamma(E) \mid \forall k \in \mathbb{N},\ j^k\phi|_{\partial U} = j^k\Phi|_{\partial U}\}$.*

Definition 135 differs from (101) because of the different definition of $\mathscr{U}_\Phi$. Definition 135 can be generalized without changes to theories with Lagrangian of arbitrary order; moreover definition 135 and definition (101) give the same space of solutions for Lagrangian of the first orders, although definition 135 allows an easiest treating of symmetries.

Let's call $\mathscr{E}$ the space of all solutions of the field theory. We have that $\mathscr{E} \subset \Gamma(E)$. We suppose that also $\mathscr{E}$ has been given a super differential structure.

Definition 136. *An even projectable* flow map $\mathfrak{R}$ *on E is a G^∞-map from the supermanifolds $\mathbb{R}^{1|0} \times E$ to E which satisfies the usual conditions for a* flow map *and such that:*

$$\forall c_1,\ c_2 \in E,\ \forall k \in \mathbb{R}^{1|0} : \pi\left(c_1\right) = \pi\left(c_2\right) \Longrightarrow \pi\left(\mathfrak{R}(k,c_1)\right) = \pi\left(\mathfrak{R}(k,c_2)\right)$$

An odd projectable flow map $\mathfrak{R}$ *on E is a G^∞-map from the supermanifolds $\mathbb{R}^{0|1} \times E$ to E which satisfies the usual conditions for a* flow map *and such that:*

$$\forall c_1,\ c_2 \in E,\ \forall \kappa \in \mathbb{R}^{0|1} : \pi\left(c_1\right) = \pi\left(c_2\right) \Longrightarrow \pi\left(\mathfrak{R}(\kappa,c_1)\right) = \pi\left(\mathfrak{R}(\kappa,c_2)\right)$$

A projectable flow map on the fiber bundle E can be projected onto a flow map on its base X.

On a local chart U of E, the flow map $\mathfrak{R}$ can be written in this way:

$$\mathfrak{R}:\ (K, x^A, q^I) \longmapsto (x^A_K, q^I_K)$$

where the parameter $K \in \mathbb{R}_S$ is an even or an odd element of the Grassmann algebra with infinite generators $\mathbb{R}_S$. Since $\mathfrak{R}$ is projectable, we have that:

$$\begin{aligned} x^A_K &= x^A_K\left(K, x^A\right) \\ q^I_K &= \left(K, x^A, q^I\right) \end{aligned}$$

The first of the two equations defines the projected flow map on X.

To a projectable flow map $\mathfrak{R}$ corresponds a projectable vector field on E which I will call with the name χ. Locally χ can be written:

$$\begin{aligned} \chi\left(x^A, q^I\right) =& \chi^A\left(x^A\right)\frac{\partial}{\partial x^A} + \chi^I\left(x^A, q^I\right)\frac{\partial}{\partial q^I} \\ =& \frac{\partial x^A_K}{\partial K}\left(0, x^A\right)\frac{\partial}{\partial x^A} + \frac{\partial q^I_K}{\partial K}\left(0, x^A, q^I\right)\frac{\partial}{\partial q^I} \end{aligned} \tag{11.2}$$

When the flow map $\mathfrak{R}$ is even, its corresponding vector field χ is even too; when $\mathfrak{R}$ is odd, also χ is odd.

The flow map $\mathfrak{R}$ induces also a flow map $\mathfrak{R}_\Gamma$ on the space $\Gamma(E)$. In fact, if $\Phi \in \Gamma(E)$, $x \in X$ and

$$\Phi:\ x\ \longmapsto\ \Phi(x)$$

then for every K, we can define $\Phi_K \in \Gamma(E)$ so that $\forall x \in X$:

$$\Phi_K:\ x\ \longrightarrow\ \mathfrak{R}\left(K, \Phi\left(x_{-K}\right)\right) \tag{11.3}$$

The flow map $\mathfrak{R}$ and its corresponding projectable vector field χ can be lifted to a flow map and a projectable vector field $j^1\mathfrak{R}$ and $j^1\chi$ on J^1E in a way analogous to the one used for classical vector fields on classical jet bundles (see for example [63]). What one wants is that:

$$\forall K \in \mathbb{R}_S,\ \forall \Phi \in \Gamma(E):\quad j^1\mathfrak{R}\left(K, j^1\Phi(x)\right) = j^1\left[\Phi_K\right](x_K) \tag{11.4}$$

On a local chart U of J^1E, the flow map $j^1\mathfrak{R}$ can be written in this way:

$$j^1\mathfrak{R}:\ (K, x^A, q^I, \dot{q}^I_A) \longrightarrow (x^A_K, q^I_K, \dot{q}^I_{K,A}) \tag{11.5}$$

Equation (11.4) implies that $\forall \Phi \in \Gamma(E)$:

$$j^1\mathfrak{R}\left(K, x^A, \Phi^I(x), \frac{\partial \Phi^I}{\partial x^A}(x)\right) = \left(x^A_K, \Phi^I_K(x_K), \frac{\partial \Phi^I_K}{\partial x^A_K}(x_K)\right) \tag{11.6}$$

Since:

$$\frac{\partial \Phi^I_K}{\partial x^A_K}(x_K) = \frac{\partial x^B}{\partial x^A_K}(x_K)\,\frac{d\Phi^I_K}{dx^B}\left[x\left(x_K\right)\right] \tag{11.7}$$

where

$$\frac{d\Phi^I_K}{dx^B}(x) := \frac{\partial \Phi^I_K}{\partial x^B}(x) + \frac{\partial \Phi^J}{\partial x^B}(x)\frac{\partial \Phi^I_K}{\partial \Phi^J}(x)$$

and since (11.6) must be satisfied for any $\Phi \in \Gamma(E)$, we have that $\dot{q}^I_{K,A}$ in equation (11.5) must satisfy the following rule:

$$\dot{q}^I_{K,A} = \frac{\partial x^B}{\partial x^A_K}\left[x_K(x)\right]\left[\frac{\partial q^I_K}{\partial x^B}(x,q) + \dot{q}^J_B \frac{\partial q^I_K}{\partial q^J}(x,q)\right] \tag{11.8}$$

The projectable vector field $j^1\chi$ can be written on a chart:

$$\begin{aligned} j^1\chi\left(x^A,q^I\right) &= \chi^A\left(x^A\right)\frac{\partial}{\partial x^A} + \chi^I\left(x^A,q^I\right)\frac{\partial}{\partial q^I} + \chi^I_B\left(x^A,q^I,\dot{q}^I_B\right)\frac{\partial}{\partial \dot{q}^I_B} \\ &= \frac{\partial x^A_K}{\partial K}\left(0,x^A\right)\frac{\partial}{\partial x^A} + \frac{\partial q^I_K}{\partial K}\left(0,x^A,q^I\right)\frac{\partial}{\partial q^I} + \frac{\partial \dot{q}^I_{K,B}}{\partial K}\left(0,x^A,q^I,\dot{q}^I_B\right)\frac{\partial}{\partial \dot{q}^I_B} \end{aligned} \tag{11.9}$$

From (11.8) and (11.2) we have that:

$$\begin{aligned} \frac{\partial \dot{q}^I_{K,A}}{\partial K} =& \frac{\partial}{\partial K}\left(\frac{\partial x^B}{\partial x^A_K}\right)\left[x_K(x)\right]\left[\frac{\partial q^I_K}{\partial x^B}(x,q) + \dot{q}^J_B\frac{\partial q^I_K}{\partial q^J}(x,q)\right] \\ &+ (-1)^{|K|(|A|+|B|)}\frac{\partial x^B}{\partial x^A_K}\left[x_K(x)\right]\frac{\partial}{\partial K}\left[\frac{\partial q^I_K}{\partial x^B}(x,q) + \dot{q}^J_B\frac{\partial q^I_K}{\partial q^J}(x,q)\right] \\ =& \frac{\partial}{\partial K}\left(\frac{\partial x^B}{\partial x^A_K}\right)\left[x_K(x)\right]\left[\frac{\partial q^I_K}{\partial x^B}(x,q) + \dot{q}^J_B\frac{\partial q^I_K}{\partial q^J}(x,q)\right] \\ &+ (-1)^{|K|(|A|+|B|)}(-1)^{|K||B|}\frac{\partial x^B}{\partial x^A_K}\left[x_K(x)\right]\left[\frac{\partial^2 q^I_K}{\partial x^B\partial K}(x,q) + \dot{q}^J_B\frac{\partial^2 q^I_K}{\partial q^J\partial K}(x,q)\right] \end{aligned} \tag{11.10}$$

Moreover, remembering that:

$$\frac{\partial}{\partial K}\left[\frac{\partial x^B_K}{\partial x^C}\left(x(x_K)\right)\frac{\partial x^A}{\partial x^B_K}(x_K)\right] = \frac{\partial}{\partial K}\left(\delta^A_C\right) = 0$$

we have that:

$$\frac{\partial^2 x^B_K}{\partial K\partial x^C}\left(x(x_K)\right)\frac{\partial x^A}{\partial x^B_K}(x_K) = -(-1)^{|K|(|B|+|C|)}\frac{\partial x^B_K}{\partial x^C}\left(x(x_K)\right)\frac{\partial^2 x^A}{\partial K\partial x^B_K}(x_K)$$

and

$$\frac{\partial^2 x^A}{\partial K\partial x^D_K}(x_K) = -(-1)^{|K|(|D|+|C|)}\frac{\partial x^C}{\partial x^D_K}(x_K)\frac{\partial^2 x^B_K}{\partial K\partial x^C}\left(x(x_K)\right)\frac{\partial x^A}{\partial x^B_K}(x_K)$$

So (11.10) becomes:

$$\begin{aligned} \frac{\partial \dot{q}^I_{K,A}}{\partial K} =& (-1)^{|K||A|}\frac{\partial x^B}{\partial x^A_K}\left[x_K(x)\right]\left[\frac{\partial^2 q^I_K}{\partial x^B\partial K}(x,q) + \dot{q}^J_B\frac{\partial^2 q^I_K}{\partial q^J\partial K}(x,q)\right] \\ &- (-1)^{|K|(|A|+|C|)}\frac{\partial x^C}{\partial x^A_K}(x_K(x))\frac{\partial^2 x^D_K}{\partial K\partial x^C}(x)\frac{\partial x^B}{\partial x^D_K}(x_K(x))\left[\frac{\partial q^I_K}{\partial x^B}(x,q) + \dot{q}^J_B\frac{\partial q^I_K}{\partial q^J}(x,q)\right] \\ =& (-1)^{|K||A|}\frac{\partial x^B}{\partial x^A_K}\left[x_K(x)\right]\left[\frac{\partial^2 q^I_K}{\partial x^B\partial K}(x,q) + \dot{q}^J_B\frac{\partial^2 q^I_K}{\partial q^J\partial K}(x,q)\right] \\ &- (-1)^{|K|(|A|+|C|)}\frac{\partial x^C}{\partial x^A_K}(x_K(x))\frac{\partial^2 x^D_K}{\partial K\partial x^C}(x)\,\dot{q}^I_{K,D} \end{aligned} \tag{11.11}$$

and

$$\begin{aligned}\chi^I_A\left(x^A, q^I, \dot{q}^I_B\right) = \left.\frac{\partial \dot{q}^I_{K,A}}{\partial K}\right|_{K=0} &= (-1)^{|K||A|}\, \frac{\partial x^B}{\partial x^A_K}\left.\left[x_K\left(x\right)\right]\right|_{K=0}\left[\frac{\partial \chi^I}{\partial x^B}\left(x,q\right) + \dot{q}^J_B\frac{\partial \chi^I}{\partial q^J}\left(x,q\right)\right] \\ &- (-1)^{|K||A|}\, \frac{\partial x^C}{\partial x^A_K}\left.\left(x_K(x)\right)\right|_{K=0}\frac{\partial \chi^D}{\partial x^C}\left(x\right)\dot{q}^I_{K,D}|_{K=0} \\ &= (-1)^{|K||A|}\left[\frac{\partial \chi^I}{\partial x^A}\left(x,q\right) + \dot{q}^J_A\frac{\partial \chi^I}{\partial q^J}\left(x,q\right)\right] - (-1)^{|K||A|}\,\frac{\partial \chi^B}{\partial x^A}\left(x\right)\dot{q}^I_B\end{aligned} \tag{11.12}$$

Suppose there is a flow map $\mathfrak{R}$ on E such that:

$$\forall U \subset X,\ \forall \Phi \in \Gamma\left(E\right) : \int_U j^1\Phi^* \operatorname{Lie}_{j^1\chi}\mathcal{L} = 0 \tag{11.13}$$

then with standard techniques it is easy to see that the corresponding flow map $\mathfrak{R}_\Gamma$ on $\Gamma\left(E\right)$ sends $\Phi \in \mathscr{E}$ to $\Phi_K \in \mathscr{E}$, *id est* it sends solutions of the theory to solutions of the theory. It is possible to restrict $\mathfrak{R}_\Gamma$ to $\mathscr{E}$, building so a flow map $\mathfrak{R}_\mathscr{E}$ on $\mathscr{E}$; in this situation $\mathfrak{R}$ defines a symmetry of the theory.

One can ask himself if, given a flow map $\mathfrak{M}$ on $\mathscr{E}$, it is always possible to find a flow map $\mathfrak{R}$ on E such that $\mathfrak{M} = \mathfrak{R}_\mathscr{E}$ and such that $j^1\Phi^* \operatorname{Lie}_{j^1\chi}\mathcal{L} = 0$, where χ is the vector field on E associated to $\mathfrak{R}$. The answer is no for two reasons.

First of all it is possible to build symmetries by adding a divergence-like term to the right hand side of (11.13).

So we have for example that:

Proposition 137. *Let $\mathfrak{R}$ be a flow map on E and χ its associated vector field. If:*

$$\begin{aligned}&\exists k \in \mathbb{N},\ \exists \alpha \in \Omega^{n-1|m}\left(J^kC\right) : \ \forall U \subset X,\ \forall \Phi \in \Gamma\left(C\right) : \\ &\int_U j^1\Phi^* \operatorname{Lie}_{j^1\chi}\mathcal{L} = \int_U j^k\Phi^* d\alpha\end{aligned} \tag{11.14}$$

then the corresponding flow map $\mathfrak{R}_\Gamma$ on $\Gamma\left(E\right)$ sends $\Phi \in \mathscr{E}$ to $\Phi_K \in \mathscr{E}$, id est it sends solutions of the theory to solutions of the theory and $\mathfrak{R}$ defines a symmetry of the theory.

When $\alpha \neq 0$, it is often said in the physicists literature that χ generate a non manifest symmetry: see for example [54]. When (11.14) holds with $\alpha = 0$, the symmetry is called manifest.

But there are also flow maps $\mathfrak{M}$ on $\mathscr{E}$ which do not correspond to any $\mathfrak{R}$ on E, with or without divergence terms.

Under certain conditions it is nevertheless possible to show that the flow map $\mathfrak{M}$ can be generated by a so called generalized vertical vector field on E. For understanding what a generalized vector field is see P. J. Olver [121], L. Fatibene and M. Francaviglia [48] and L. Fatibene, M. Ferraris, M. Francaviglia and R. G. McLenaghan [47], who treated the classical case and the super case when the base manifold is classic.

Note that in the special case of supermechanics every flow map $\mathfrak{M}$ can be generated by a generalized vector field on E, as is stated by the super version of the inverse of Noether theorem proved in [83] and [25].

In the classical case, generalized vector fields can be though of as special vector fields on the jet bundle $J^k E$ for a certain $k \in \mathbb{N}$ (although this is not entirely correct, see [47]). In [83] and [25] the authors extended the definition of generalized vector fields to supermechanical theories. In [47] the definition has been extended to theories defined on even base manifolds with odd field space. The definition given in [48] can be directly extended to the most general super case (possibly with odd base manifold) using our formalism: I will not do so here. It will be sufficient to know that, locally, a generalized vertical vector field on E can be written as:

$$K = K^I \left(x^A, q^I, \dot{q}^I_A, \ddot{q}^I_{AB}, \dots\right) \frac{\partial}{\partial q^I} \tag{11.15}$$

where $\left(x^A, q^I, \dot{q}^I_A, \ddot{q}^I_{AB}, \dots\right)$ are local adapted coordinates in the infinite-jets space $J^\infty E$. The component $K^I \left(x^A, q^I, \dot{q}^I_A, \ddot{q}^I_{AB}, \dots\right)$ obviously cannot be seen as the derivative of a flow map on E. Nonetheless Lie-dragging and the Lie derivative can be defined along generalized vector fields. For each section $\Phi \in \Gamma(E)$, the composition $K_\Phi = K \circ \Phi$ is a vertical vector field defined on the image of Φ and it can be used to drag the section itself. Moreover generalized vector fields can be lifted to $J^1 E$. So formulas similar to (11.13) and (11.14) can be written to define a symmetry which in turns defines the flow map on $\mathscr{E}$ see [47] and [48] for the case when X has even dimension. I call these symmetries "generalized manifest" or "generalized non manifest" symmetries (depending on the absence or presence of divergence like terms like $d\alpha$ in their defining formula).

Adopting the point of view of P. Deligne and D. Freed, [36, 54], a flow map $\mathfrak{M}$ generated by a generalized vertical vector field on E corresponds to a local vector field on $\Gamma(E)$. With that point of view there is no need to distinguish between generalized and conventional vector fields for treating symmetries. In fact Freed in [54], unlike what I am doing here, uses the term "generalized" as a synonymous of "non manifest". Nevertheless I think that it is worth to distinguish generalized vector fields from conventional ones and generalized symmetries from non-generalized ones.

Symmetries which are interesting for physical theories are generated by vector fields or generalized vector fields on the space of configurations E.

In [47] and [48] the authors show that, if one defines a supersymmetric field theory with the components formulation, so that the base manifold X is an even manifold, then the supersymmetries are generated by generalized vertical vector fields on E of the form of (11.15).

In the next section, with the help of fractional forms formalism, I will give a super version of Noether theorem valid for base manifolds of any degree, even or odd. In the last chapter of this thesis, for some examples, I will show that, if one defines a supersymmetric field theory with the super-field formulation (so that the base manifold X has an odd section different than 0), then it is possible that the supersymmetry is generated by non-generalized vector fields on the space of configurations E. This will give a simpler geometrical interpretation of supersymmetry.

11.2 Noether theorem for super fields: supercurrents and conserved quantities.

In this section I will prove a version of Noether theorem valid for super field theories. I will then show that it implies the presence of supercurrents and supercharges.

The superform locally defined on $J^1 E$ by the formula

$$\theta^* = L\beta + (-1)^{|A|(|A|+|L|)} c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \tag{11.16}$$

is indeed well defined globally and it is in fact the pull back by the Legendre transform of the Cartan superform defined on the covariant phase space P:

$$\theta^* = \mathbb{FL}^* \theta. \tag{11.17}$$

This form θ^* is the Poincaré-Cartan form of our superfield theory in the Lagrangian representation. I will show here how it is involved in symmetries.

Theorem 138. *Consider a field theory defined by a configurations bundle E with fiber type the supermanifold F over a base supermanifold X and by a Lagrangian form $\mathcal{L}$ which locally is $\mathcal{L} = L\beta$. Let $\mathscr{E}$ be the space of solutions of the field theory. Let χ be a projectable vector field on E such that:*

$$\forall \Phi \in \mathscr{E} \ \exists \alpha_\Phi \in \Omega^{n-1|m} X : \ \forall U \subset X \ : \int_U j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = \int_U d\alpha_\Phi \tag{11.18}$$

where U is a compact submanifold with boundary of X with the same dimension of X; then, for every $\Phi \in \mathscr{E}$, the form $\left[j^1\Phi^* \left(j^1\chi \lrcorner \theta^* \right) - \alpha_\Phi \right]$ is closed; id est:

$$\begin{aligned} &\forall U \subset X, \ \forall \Phi \in \mathscr{E} \ : \\ &\int_U d \left\{ j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)} c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \right\} = 0 \end{aligned} \tag{11.19}$$

Proof. We first note that:

$$\begin{aligned} &(-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[j^1\chi \lrcorner \left(c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] \\ &= (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[c^I \left(j^1\chi \right) \frac{\partial L}{\partial \dot{q}^I_A} \beta_A + (-1)^{|\chi||I|} c^I \wedge \left(j^1\chi \lrcorner \beta_A \right) \right] \\ &= (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[c^I \left(j^1\chi \right) \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right] \end{aligned} \tag{11.20}$$

So from (11.18) and (11.20) we have that:

$$\begin{aligned} &\int_U j^1\Phi^* \operatorname{Lie}_{j^1K} \mathcal{L} - \int_U d\alpha_\Phi + (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[j^1\chi \lrcorner \left(c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] \\ &- (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[c^I \left(j^1\chi \right) \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right] = 0 \end{aligned} \tag{11.21}$$

and using Cartan formula for superforms (5.63) proved by Voronov and Zorich we see that:

$$\begin{aligned} &\int_U d \cdot j^1\Phi^* \left[j^1\chi \lrcorner \mathcal{L} \right] + \int_U j^1\Phi^* \left[j^1\chi \lrcorner d\mathcal{L} \right] - \int_U d\alpha \\ &+ (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[j^1\chi \lrcorner \left(c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] \\ &- (-1)^{|A|(|A|+|L|)} \int_U d \cdot j^1\Phi^* \left[c^I \left(j^1\chi \right) \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right] = 0 \end{aligned} \tag{11.22}$$

So, remembering formula (11.12), remembering that $\left| K^A \right| = |K| + |A|$ and using the Euler

Lagrange equations for superfields (7.7), we have:

$$
\begin{aligned}
&\int_U d\left\{ j^1\Phi^*\left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A}\beta_A \right)\right] - \alpha_\Phi \right\} \\
&= \int_U j^1\Phi^*\left[j^1\chi \lrcorner d\mathcal{L}\right] - (-1)^{|A|(|A|+|L|)} \int_U d\cdot j^1\Phi^*\left[c^I\left(j^1\chi\right)\frac{\partial L}{\partial \dot{q}^I_A}\beta_A\right] \\
&= \int_U \chi^I \frac{\partial L}{\partial q^I}\beta + \chi^I_B \frac{\partial L}{\partial \dot{q}^I_B}\beta \\
&\quad + \int_U -(-1)^{|\chi||L|}\, dx^C \frac{\partial q^I}{\partial X^C}\frac{\partial L}{\partial q^I}\chi^A\beta_A - (-1)^{|\chi||L|}\, dx^C \frac{\partial^2 q^I}{\partial x^C \partial x^B}\frac{\partial L}{\partial \dot{q}^I_B}\chi^A\beta_A \\
&\quad + \int_U -(-1)^{|A||\chi|}\frac{d}{dx^A}\left(j^1\Phi^* c^I\left(j^1\chi\right)\frac{\partial L}{\partial \dot{q}^I_A}\right)\beta \\
&= \int_U \left\{ \chi^I \frac{\partial L}{\partial q^I} + \chi^I_B \frac{\partial L}{\partial \dot{q}^I_B} - (-1)^{|\chi||L|}(-1)^{|A|(|A|+|L|+|\chi|)}\frac{\partial q^I}{\partial x^A}\frac{\partial L}{\partial q^I}\chi^A \right. \\
&\quad - (-1)^{|\chi||L|}(-1)^{|A|(|A|+|L|+|\chi|)}\frac{\partial^2 q^I}{\partial x^A \partial x^B}\frac{\partial L}{\partial \dot{q}^I_B}\chi^A \\
&\quad \left. - (-1)^{|A||\chi|}\frac{d}{dx^A}\left(j^1\Phi^* c^I\left(j^1\chi\right)\right)\frac{\partial L}{\partial \dot{q}^I_A} - (-1)^{|A||I|} j^1\Phi^* c^I\left(j^1\chi\right)\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\right\}\beta \\
&= \int_U \left\{ \chi^I \frac{\partial L}{\partial q^I} + (-1)^{|\chi||B|}\frac{d\chi^I}{dx^B}\frac{\partial L}{\partial \dot{q}^I_B} - (-1)^{|\chi||B|}\frac{d\chi^A}{dx^B}\dot{q}^I_A\frac{\partial L}{\partial \dot{q}^I_B} \right. \\
&\quad - (-1)^{|\chi||L|}(-1)^{|A|(|A|+|L|+|\chi|)}\frac{\partial q^I}{\partial x^A}\frac{\partial L}{\partial q^I}\chi^A \\
&\quad - (-1)^{|\chi||L|}(-1)^{|A|(|A|+|L|+|\chi|)}\frac{\partial^2 q^I}{\partial x^A \partial x^B}\frac{\partial L}{\partial \dot{q}^I_B}\chi^A \\
&\quad \left. - (-1)^{|A||\chi|}\left[\frac{d\chi^I}{dx^A} - \frac{d}{dx^A}\left(\chi^B\frac{\partial q^I}{\partial x^B}\right)\right]\frac{\partial L}{\partial \dot{q}^I_A} - (-1)^{|A||I|}\left[\chi^I - \chi^B\frac{\partial q^I}{\partial x^B}\right]\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\right\}\beta
\end{aligned}
$$

$$
\begin{aligned}
= \int_U &\left\{ \chi^I\left[\frac{\partial L}{\partial q^I} - (-1)^{|A||I|}\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\right] + (-1)^{|\chi||B|}\frac{d\chi^I}{dx^B}\frac{\partial L}{\partial \dot{q}^I_B} - (-1)^{|\chi||B|}\frac{d\chi^A}{dx^B}\dot{q}^I_A\frac{\partial L}{\partial \dot{q}^I_B} + \right. \\
&- \chi^A\frac{\partial q^I}{\partial x^A}\left[\frac{\partial L}{\partial q^I} - (-1)^{|A||I|}\frac{d}{dx^A}\frac{\partial L}{\partial \dot{q}^I_A}\right] - (-1)^{|A||B|}\chi^A\frac{\partial^2 q^I}{\partial x^B\partial x^A}\frac{\partial L}{\partial \dot{q}^I_B} + \\
&\left. -(-1)^{|A||\chi|}\left[\frac{d\chi^I}{dx^A} - \frac{d}{dx^A}\left(\chi^B\frac{\partial q^I}{\partial x^B}\right)\right]\frac{\partial L}{\partial \dot{q}^I_A}\right\}\beta = 0
\end{aligned}
\tag{11.23}
$$

And the theorem is proved. ■

Note that theorem 138 is also true if we replace the vector field χ by a generalized vector field. To prove it one has just to pay attention to the lift of the generalized vector field and to definition of the Lie derivative along it. These concepts are in the super framework analogous to the corresponding ones in the classical framework; see for the classical case [47], [48] and [121].

It is also true the converse of theorem 138:

Theorem 139. *Consider a field theory defined by a configurations bundle E with fiber type the supermanifold F over a base supermanifold X and by a Lagrangian form $\mathcal{L}$ which locally is $\mathcal{L} = L\beta$. Let $\mathscr{E}$ be the space of solutions of the field theory. Let χ be a projectable vector field on*

E such that:

$$\begin{gathered}\forall U \subset X,\ \forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \\ \int_U d\left\{ j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \right\} = 0\end{gathered} \tag{11.24}$$

then we have that:

$$\forall \Phi \in \mathscr{E} \ : \ \forall U \subset X \ : \int_U j^1\Phi^* \,\mathrm{Lie}_{j^1\chi}\, \mathcal{L} = \int_U d\alpha_\Phi \tag{11.25}$$

Proof. The proof is analogous to the one of theorem 138 ■

The quantity $\gamma := j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi$ can be called a supercurrent.

Remember that:

$$j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] = j^1\Phi^* \left[j^1\chi \lrcorner \mathbb{FL}^* (\theta) \right] \tag{11.26}$$

where $\mathbb{FL}$ is the super Legendre transformation between J^1E and the covariant multiphase space P and θ is the Cartan form on P all defined in Chapter 8.

Suppose that U is a $n|m$-dimensional submanifold of X with boundary ∂U, then, by Stokes theorem for superforms 68, we find that, under the hypotheses of theorem 138:

$$\begin{aligned}&\int_U d\left\{ j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \right\} \\ &= \int_{\partial U} j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi = 0\end{aligned} \tag{11.27}$$

If ∂U is the disjoint union of two submanifolds Σ_1 and Σ_2 of dimension $n-1|m$, then, after a suitable choice of orientations, (11.27) implies that:

$$\begin{aligned}&\int_{\Sigma_1} j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi \\ &= \int_{\Sigma_2} j^1\Phi^* \left[j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A} \beta_A \right) \right] - \alpha_\Phi := Q\end{aligned} \tag{11.28}$$

Hence on every solution Φ of the theory, Q is a conserved quantity and γ is its density.

Remember that at the beginning of section 11.1 we have fixed an immersed body on the base manifold X. It is then automatically fixed an immersed body for every submanifold of X. All the integrals performed until now and in the following are understood to be performed with respect of the fixed immersed body.

As a consequence of Batchelor theorem, [7] (see Remark 60), if X has dimension $n|m$, it can be seen as a super vector bundle whose fiber type Π has dimension $0|m$ and whose base $X_{n|0}$ has dimension $n|0$. Then it make sense to define the form $\gamma_{n|0}$, over the base $X_{n|0}$, integrating γ over all the odd variables, id est over the fibers Π_x:

$$\gamma_{n|0} := \int_{\Pi_x} \gamma \tag{11.29}$$

If an immersion b of the body $\underline{X}$ into X is fixed, see Definition 59, then we can define:

$$\underline{\gamma} := b^* \gamma_{n|0} \tag{11.30}$$

which is a C^∞ differential form over the body $\underline{X}$ of X.

Suppose again that U is a $n|m$-dimensional submanifold of X with boundary ∂U, then, performing the Berezin integrals with respect of the fixed immersed body, we have:

$$\int_U d\gamma = 0 \Longrightarrow \int_{\partial U} \gamma = 0 \Longrightarrow \int_{\partial \underline{U}} b^* \int_{\Pi_x} \gamma = 0 \Longrightarrow \int_{\partial \underline{U}} \underline{\gamma} = \int_{\partial \underline{U}} \underline{\gamma} = 0$$

Therefore $\underline{\gamma}$ is a current on $\underline{X}$, the body of X.

The discussion of this section can be extended to the case when the supergroup of symmetries of the superfield theory is not a one parameter supergroup but an higher dimensional supergroup acting on the field bundle E with a projectable action. In this more general case, the role of the *flow map* $\mathfrak{R}$ on E is taken by an action map. Each action map on E can be lifted to j^1E with the help of formula completely analogous to (11.3) and (11.4). The role of the projectable vector field χ on E is then played by the infinitesimal generators of the symmetries. For each of them it can be defined a lift to j^1E using the lifted action map and then for each infinitesimal generator of the supergroup of symmetries theorem, 138 and formula 11.28 are valid. Examples of such a situation will be given in sections 12.2 and 12.3.

11.3 Momentum maps for multisymplectic super field theories.

Suppose that a Lie group G acts smoothly on E respecting the bundle structure and suppose that $\mathfrak{g}$ is its Lie algebra. If $k \in \mathfrak{g}$ generates the flow map $\mathfrak{K}$ on E with associated vector field χ, then the map

$$\begin{array}{rcl} J \; : \; \mathfrak{g} & \longrightarrow & \Omega^{n-1|m}\left(J^1E\right) \\ k & \longmapsto & j^1\chi \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \dfrac{\partial L}{\partial \dot{q}^I_A} \beta_A\right) \end{array} \tag{11.31}$$

is the superfield extension of the map called by Gotay, Isenberg, Marsden, Montgomery, Śniatycki and Yasskin in [63] the *covariant momentum map* in the Lagrangian representation.

From what we have seen above we have that:

Theorem 140. *If G is a group of manifest symmetries for the field theory defined by the Lagrangian L and $\mathfrak{g}$ is its Lie algebra, then for every $k \in \mathfrak{g}$ and for every solution of the theory $\Phi \in \mathscr{E}$, we have that:*

$$d\left\{j^1\Phi^*\left[J(k)\right]\right\} = 0$$

Let's now study how symmetries can be seen on the multimomenta space P. On P we can use local coordinates $\left(x^A, q^I, p^A_I\right)$, see section 8.1.

I will suppose from now that the Lagrangian $\mathcal{L}$ is even-regular or odd-regular, see section 8.2. In both case the Legendre super transform $\mathbb{FL}$ between J^1E and P is invertible. Remember that the transform $\mathbb{FL}$ sends $\dot{q}^I_A$ to $p^A_I = (-1)^{|A|(|A|+|L|)} \frac{\partial L}{\partial \dot{q}^I_A}$.

We have that:

Theorem 141. *Let χ be a projectable vector field on E, then:*

$$\begin{aligned} &\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \\ &j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} = d\alpha_\Phi \end{aligned} \tag{11.32}$$

if and only if:

$$\begin{aligned} &\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \\ &\left(\mathbb{FL}_* j^1\Phi\right)^* \operatorname{Lie}_{\mathbb{FL}_* j^1\chi} \theta = d\alpha_\Phi \end{aligned} \tag{11.33}$$

where θ is the Cartan form on P defined locally by $\theta := dq^I \wedge p_I^A \beta_A - H\beta$.

Proof. We remember that locally:

$$\mathbb{FL}^*\theta = \mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}_A^I}\beta_A$$

and

$$\theta = \mathbb{FL}_*\mathcal{L} + g^I \wedge p_I^A \beta_A$$

where the g^I are the local contact forms on P defined with (8.20).

We remember also that:

$$d\theta = \omega$$

where ω is the multisymplectic superform defined in 115.

From the properties of the pull back and from theorems 138 and 139, we have that the following propositions are equivalent:

1. $$\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \ j^1\Phi^* \operatorname{Lie}_{j^1\chi} \mathcal{L} - d\alpha_\Phi = 0$$

2. $$\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \ j^1\Phi^* \left[d\left(j^1\chi \lrcorner \mathbb{FL}^*\theta\right)\right] - d\alpha_\Phi = 0$$

3. $$\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \ \left(\mathbb{FL}_* j^1\Phi\right)^* \left[d\left(\mathbb{FL}_* j^1\chi \lrcorner \theta\right)\right] - d\alpha_\Phi = 0$$

But from Theorem 119, we have that:

$$\forall \Phi \in \mathscr{E},\ \forall u \in \Gamma(TP) \ : \ \left(\mathbb{FL}_* j^1\Phi\right)^* [u \lrcorner d\theta] = 0$$

And so propositions 1, 2 and 3 are equivalent to

$$\begin{aligned} &\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \\ &\left(\mathbb{FL}_* j^1\Phi\right)^* \left[d\left(\mathbb{FL}_* j^1\chi \lrcorner \theta\right) + \mathbb{FL}_* j^1\chi \lrcorner d\theta\right] - d\alpha_\Phi = 0 \end{aligned}$$

Which in turn is equivalent to:

$$\forall \Phi \in \mathscr{E},\ \exists \alpha_\Phi \in \Omega^{n-1|m} X \ : \ \left(\mathbb{FL}_* j^1\Phi\right)^* \operatorname{Lie}_{\mathbb{FL}_* j^1\chi} \theta - d\alpha_\Phi = 0$$

and the theorem is proved. ■

For the sake of brevity, we can pose: $\chi_P := \mathbb{FL}_* j^1\chi$ and $\sigma_\Phi := \mathbb{FL}_* j^1\Phi$.

Note that we could state and prove a similar theorem for generalized vector fields on P.

Corollary 142. *Let G be a Lie supergroup acting on P with a lifted covariant action, id est so that for each element $k \in \mathfrak{g}$, there exists a corresponding action on E generated by the projectable vector field χ, such that* $\mathrm{Lie}_{\chi_P}\,\omega = 0$*, and suppose that there exists $J \in Hom\left(\mathfrak{g}, \Omega^{n-1|m}(P)\right)$, such that for each $k \in \mathfrak{g}$:*

$$\chi_P \lrcorner \omega = d\,[J(k)]$$

then, $\forall \Phi \in \mathscr{E}$:

$$j^1\Phi^*\,\mathrm{Lie}_{j^1\chi}\,\mathcal{L} = d\,[J(k) + \chi_P \lrcorner \theta] \tag{11.34}$$

In this case we say that the map:

$$\begin{aligned} J \ : \ \mathfrak{g} &\longrightarrow \Omega^{n-1|m}(P) \\ k &\longrightarrow J(k) \end{aligned} \tag{11.35}$$

is the super covariant momentum map *of the action.*

Note that the corollary can be very easily extended to the case when the action of the Lie group G on P is not a lifted action. The super covariant momentum map defined with (11.35) is the extension to superfield theories of the covariant momentum map defined for classical field theories in [63], [108] and [29].

Suppose that a Lie group G acts on P with a special covariant action, *id est* so that for each element $k \in \mathfrak{g}$, there exists a corresponding action on E generated by the projectable vector field χ, such that $\mathrm{Lie}_{\mathbb{FL}_* j^1\chi}\,\theta = 0$, then we are in a special case of application of theorem 141 and corollary 142. Following [63], I can then define a special super covariant momentum map J, with:

$$\begin{aligned} J \ : \ \mathfrak{g} &\longrightarrow \Omega^{n-1|m}(P) \\ k &\longmapsto -\chi_P \lrcorner \theta \end{aligned} \tag{11.36}$$

Note that my definitions of momentum maps slightly differ from the ones given in [63] and [108] because I wanted to keep the notation a bit lighter. Moreover there exists a difference in the conventional mutual sign between θ and ω (for me $\omega := d\theta$, whereas in those papers is $\omega := -d\theta$). Following [21], I could have called my maps *super comomentum maps*. In fact one can see that they are the direct generalizations of the super comomentum maps defined in [21], which can be used in treating symmetries of supermechanical theories, where the base manifold of the theory has dimension $1|0$.

Chapter 12

Examples

In this chapter I will present some examples to illustrate the theory expounded in the chapters above.

The theory developed in the previous chapters is fully exploitable when one deals with regular Lagrangian super field theories. However, almost always, physical interesting super-Lagrangians are not regular. The natural continuation of the study undertaken in this thesis, is the extension to non regular theories, which can be done using techniques similar to the one used to treat classical field theories with constraints or in general non regular Lagrangians. This work has not been done yet.

Nonetheless, some of the considerations made above, can lead to interesting results also in non-regular cases. For this reason some of the examples here exhibited consist of theories whose Lagrangian is not regular.

In section 12.1 I will present a mechanical theory which is a super version of the harmonic oscillator which doesn't exhibit any manifest supersymmetry.

In section 12.2 I will treat the superparticle in a flat space. This simple example will allow me to show how supersymmetries of a supermechanical system are treated with my formalism, following what presented in Chapter 11.

In section 12.3 I will study a 3-dimensional σ-model with 2 supersymmetries and I'll derive with my formalism the current corresponding to one of its odd symmetries.

12.1 A super oscillator

In this section I will treat a super version of the harmonic oscillator which doesn't exhibit any manifest supersymmetry. I want to use this toy model to show that one can think of super field theories (in this case a super mechanic theory) outside of the context of supersymmetry. Moreover this example will allow to illustrate some of the techniques introduced in Chapter 9.

Let's take the following fields bundle:

$$\begin{array}{c} E := \mathbb{R}^{1|2} \times \mathbb{R}^{1|0} \\ {\scriptstyle \pi}\Big\downarrow \\ X := \mathbb{R}^{1|2} \end{array}$$

On X we use the coordinates $(x^A) \equiv (t;\tau^1,\tau^2)$, then we have that every field can be locally expressed as:

$$\Phi(x) = q(t) + \tau^1\psi_1(t) + \tau^2\psi_2(t) + \tau^1\tau^2 F(t)$$

where q and F are even and ψ_1 and ψ_2 are odd.

We define the field theory by the Lagrangian:

$$\mathcal{L} = L\left(x^A; q; \dot{q}_A\right)\beta := \left(\frac{1}{2}\sum_A \dot{q}_A\dot{q}_A - \frac{1}{2}q^2\right)\beta = \left(\frac{1}{2}\dot{q}_t\dot{q}_t - \frac{1}{2}q^2\right)\beta$$

The fields equation for a field Φ is:

$$\frac{\partial L}{\partial q}(j^1\Phi(x)) - \frac{\partial}{\partial x^A}\frac{\partial L}{\partial \dot{q}_A}(j^1\Phi(x)) = 0$$

So:

$$-\Phi(x) - \sum_A \frac{\partial}{\partial x^A}\frac{\partial}{\partial x^A}\Phi(x) = 0 \tag{12.1}$$

Equation (12.1) becomes:

$$q(t)+\tau^1\psi_1(t)+\tau^2\psi_2(t)+\tau^1\tau^2F(t) = -\frac{\partial^2}{\partial t^2}q(t)-\tau^1\frac{\partial^2}{\partial t^2}\psi_1(t)-\tau^2\frac{\partial^2}{\partial t^2}\psi_2(t)-\tau^1\tau^2\frac{\partial^2}{\partial t^2}F(t) \tag{12.2}$$

or:

$$\begin{cases} q(t) = -\dfrac{\partial^2}{\partial t^2}q(t) \\ \psi_1(t) = -\dfrac{\partial^2}{\partial t^2}\psi_1(t) \\ \psi_2(t) = -\dfrac{\partial^2}{\partial t^2}\psi_2(t) \\ F(t) = -\dfrac{\partial^2}{\partial t^2}F(t) \end{cases} \tag{12.3}$$

Using the notation of section 9.1, we have that $\dot{q}_0 = q$, $\dot{q}_1 = \psi_1$, $\dot{q}_2 = \psi_2$ and $\dot{q}_{21} = F$; and we can set $\dot{q}_{t,0} = \dot{q}$, $\dot{q}_{t,1} = \dot{\psi}_1$, $\dot{q}_{t,2} = \dot{\psi}_2$ and $\dot{q}_{t,21} = \dot{F}$. We have then:

$$\underline{L\beta} = \left(\dot{q}\dot{F} - \dot{\psi}_1\dot{\psi}_2 - qF + \psi_1\psi_2\right)d\underline{t} \tag{12.4}$$

where $\underline{t} \in \mathbb{R}$. The corresponding Euler-Lagrange equations are precisely (12.3).

The Lagrangian L is purely even but not regular. We can anyway define a Legendre transform and an Hamiltonian "function", which, remembering (8.6) and (8.10), are:

$$\begin{aligned}\mathbb{FL}\left(x^A, q, \dot{q}_A\right) =& \left(x^A, q, \overline{p^A}\left(x^A, q, \dot{q}_A\right)\right) = \left(x^A, q, \overline{p^t}\left(x^A, q, \dot{q}_A\right), \overline{p^{\tau_1}}\left(x^A, q, \dot{q}_{\tau_1}\right), \overline{p^{\tau_2}}\left(x^A, q, \dot{q}_{\tau_2}\right)\right) \\ =& \left(x^A, q, (-1)^{|A|}\frac{\partial \overline{L}}{\partial \dot{q}_A}\left(x^A, q, \dot{q}_A\right)\right) = \left(x^A, q, \dot{q}_t, 0, 0\right)\end{aligned} \tag{12.5}$$

and:

$$H(x^A, q^I, p_I^A) := (-1)^{|A|}\dot{q}_A\overline{p^A} - L\left(x^A, q^I, \dot{q}_A^I\right) = \frac{1}{2}\overline{p^t p^t} + \frac{1}{2}q^2 \tag{12.6}$$

On the space P defined by $\overline{p^{\tau_1}} = 0, \overline{p^{\tau_2}} = 0, \widetilde{p^t} = 0, \widetilde{p^{\tau_1}} = 0, \widetilde{p^{\tau_2}} = 0$, we can also define the multisymplectic form. On P we can use as local coordinates $\left(t, \tau_1, \tau_2, q, \overline{p^t}\right)$, and we have locally:

$$\omega = -\,dq \wedge d\overline{p^t} \wedge \left(\partial_t \lrcorner \frac{dt}{d\tau^1 \odot d\tau^2}\right) - \left(\overline{p^t}d\overline{p^t} + qdq\right) \wedge \frac{dt}{d\tau^1 \odot d\tau^2} \tag{12.7}$$

Since L is not regular, the hypothesis of theorems 112 and 119 are not satisfied and we don't have an Hamiltonian description as in (8.18).

On the other hand $\underline{L}$ is even and regular. We have then:

$$\begin{aligned}\underline{\mathbb{FL}}\left(\underline{t}, q, \psi_1, \psi_2, F; \dot{q}, \dot{\psi}_1, \dot{\psi}_2, \dot{F}\right) &= \left(\underline{t}, q, \psi_1, \psi_2, F; \overline{p^q}, \overline{p^{\psi_1}}, \overline{p^{\psi_2}}, \overline{p^F}\right) \\ &= \left(\underline{t}, q, \psi_1, \psi_2, F; \frac{\partial \underline{L}}{\partial \dot{q}}, \frac{\partial \underline{L}}{\partial \dot{\psi}_1}, \frac{\partial \underline{L}}{\partial \dot{\psi}_2}, \frac{\partial \underline{L}}{\partial \dot{F}}\right) \\ &= \left(\underline{t}, q, \psi_1, \psi_2, F; \dot{F}, -\dot{\psi}_2, \dot{\psi}_1, \dot{q}\right)\end{aligned} \tag{12.8}$$

and:

$$\underline{H} = \overline{p^F}\,\overline{p^q} + \overline{p^{\psi_2} p^{\psi_1}} + qF - \psi_1\psi_2 = \dot{q}\dot{F} - \dot{\psi}_1\dot{\psi}_2 + qF - \psi_1\psi_2 \tag{12.9}$$

Hamilton system becomes:

$$\begin{cases} (-1)^{l(\Lambda)} \dfrac{\partial q_\Lambda}{\partial t}\left(\mathbb{FL}j^1\Phi(x)\right) = \dfrac{\partial \overline{H}}{\partial \overline{p^\Lambda}}\left(\mathbb{FL}j^1\Phi(x)\right) \\ \dfrac{\partial \overline{p^\Lambda}}{\partial t}\left(\mathbb{FL}j^1\Phi(x)\right) = -\dfrac{\partial \overline{H}}{\partial q_\Lambda}\left(\mathbb{FL}j^1\Phi(x)\right) \end{cases} \tag{12.10}$$

or:

$$\begin{cases} \dfrac{\partial q}{\partial t} = \overline{p^F} \\ -\dfrac{\partial \psi_1}{\partial t} = -\overline{p^{\psi_2}} \\ -\dfrac{\partial \psi_2}{\partial t} = \overline{p^{\psi_1}} \\ \dfrac{\partial F}{\partial t} = \overline{p^q} \end{cases} \quad \begin{cases} \dfrac{\partial \overline{p^q}}{\partial t} = -F \\ \dfrac{\partial \overline{p^{\psi_1}}}{\partial t} = \psi_2 \\ \dfrac{\partial \overline{p^{\psi_2}}}{\partial t} = -\psi_1 \\ \dfrac{\partial \overline{p^F}}{\partial t} = -q \end{cases} \tag{12.11}$$

or:

$$\begin{cases} \dfrac{\partial q}{\partial t} = \dot{q} \\ -\dfrac{\partial \psi_1}{\partial t} = -\dot{\psi}_1 \\ -\dfrac{\partial \psi_2}{\partial t} = -\dot{\psi}_2 \\ \dfrac{\partial F}{\partial t} = \dot{F} \end{cases} \quad \begin{cases} \dfrac{\partial^2 F}{\partial t^2} = -F \\ -\dfrac{\partial^2 \psi_2}{\partial t^2} = \psi_2 \\ \dfrac{\partial^2 \psi_1}{\partial t^2} = -\psi_1 \\ \dfrac{\partial^2 q}{\partial t^2} = -q \end{cases} \tag{12.12}$$

which is equivalent to (12.3).

We can also build $\underline{\Omega}$. The space of solutions $\mathscr{G} \cong \mathscr{E}$ is a supermanifold of finite dimension $4|4$ which can be well parametrized by $(q_0, F_0, \dot{q}_0, \dot{F}_0; \psi_{1\,0}, \psi_{2\,0}, \dot{\psi}_{1\,0}, \dot{\psi}_{2\,0})$, where the subscript 0 means that the fields are calculated at $t = 0$. If $G \in \mathscr{G}$ is an Hamiltonian surface, then a vector $\delta_1 G \in T_G\mathscr{G}$ can be written as $\delta_1 G = v_1^{q_0}\frac{\partial}{\partial q_0} + v_1^{F_0}\frac{\partial}{\partial F_0} + v_1^{\dot{q}_0}\frac{\partial}{\partial \dot{q}_0} + \cdots$ and it corresponds to the vector field defined on G whose value at $t = 0$ is: $v_1(0) = v_1^{q_0}\frac{\partial}{\partial q} + v_1^{F_0}\frac{\partial}{\partial F} + v_1^{\dot{q}_0}\frac{\partial}{\partial \dot{q}} + \cdots$. I choose the surface Σ of codimension 1 to be defined by the equation $t = 0$. Then we have, when $\delta_1 G$

and $\delta_2 G$ are even:

$$
\begin{aligned}
\underline{\Omega}_{\Sigma}\big|_G\left(\overline{\delta_1 G}, \overline{\delta_2 G}\right) &= \int_{\Sigma\cap G} \overline{v_1}\wedge\overline{v_2} \lrcorner\, \omega = -dq_\Lambda \wedge dp^\Lambda(\overline{v_1},\overline{v_2})|_{t=0} = \\
&= -v_1^{q_0} v_2^{\dot{F}_0} + v_2^{q_0} v_1^{\dot{F}_0} - v_1^{F_0} v_2^{\dot{q}_0} + v_2^{F_0} v_1^{\dot{q}_0} + \\
&+ v_1^{\psi_{1\,0}} v_2^{\dot{\psi}_{2_0}} - v_2^{\psi_{1\,0}} v_1^{\dot{\psi}_{2\,0}} - v_1^{\psi_{2\,0}} v_2^{\dot{\psi}_{1_0}} + v_2^{\psi_{2\,0}} v_1^{\dot{\psi}_{1\,0}}
\end{aligned}
\tag{12.13}
$$

When one, or both, of the vectors involved, is odd, the value of $\underline{\widehat{\Omega}}$ can be obtained by replacing ω with $\widehat{\omega}$ in the integral appearing in formula (12.13); where $\widehat{\omega}$ is the extension of ω in its two first arguments, performed according to definition 33, so that the new form $\widehat{\omega}$ can be contracted by odd vectors too.

The form $\underline{\widehat{\Omega}}$ is non degenerate and it can be used to define a Poisson bracket on $\mathcal{F}(\mathscr{G})$, the space of functions on $\mathscr{G}$.

I want to stress here one interesting property of that Poisson bracket which follows from the definition of $\underline{\widehat{\Omega}}$. Let's consider $f \in \mathcal{F}(\mathscr{G})$, then we associate to it the vector X_f defined by:

$$\underline{\widehat{\Omega}}\,(\cdot, X_f) = df(\cdot) \tag{12.14}$$

If $f, g \in \mathcal{F}(\mathscr{G})$, then we set:

$$\{f, g\} := \underline{\widehat{\Omega}}\,(X_f, X_g) \tag{12.15}$$

Let's consider for our example the functions ψ_1 and $-\dot{\psi}_2$ (I let the subscript 0 drop for the sake of simplicity) which are canonically conjugate and which are both odd. Remembering 33, so that:

$$
\begin{aligned}
\widehat{\underline{\Omega}_{\Sigma}}\big|_G(\delta_1 G, \delta_2 G) &= -d\widehat{q_\Lambda \wedge d}p^\Lambda(v_1, v_2)|_{t=0} = \\
= \overline{v_1^{\psi_1}}\,\overline{v_2^{\dot{\psi}_2}} - \overline{v_2^{\psi_1}}\,\overline{v_1^{\dot{\psi}_2}} + \widetilde{v_1^{\psi_1}}\,\overline{v_2^{\dot{\psi}_2}} &+ \overline{v_2^{\psi_1}}\,\widetilde{v_1^{\dot{\psi}_2}} - \overline{v_1^{\psi_1}}\,\widetilde{v_2^{\dot{\psi}_2}} - \widetilde{v_2^{\psi_1}}\,\overline{v_1^{\dot{\psi}_2}} - \widetilde{v_1^{\psi_1}}\,\widetilde{v_2^{\dot{\psi}_2}} - \widetilde{v_2^{\psi_1}}\,\widetilde{v_1^{\dot{\psi}_2}} + \cdots
\end{aligned}
$$

We have that:

$$
\begin{aligned}
X_{\psi_1} &= -\frac{\partial}{\partial \dot{\psi}_2} \\
X_{-\dot{\psi}_2} &= \frac{\partial}{\partial \psi_1}
\end{aligned}
$$

and:

$$\{\psi_1, -\dot{\psi}_2\} = \underline{\widehat{\Omega}}\left(X_{\psi_1}, X_{-\dot{\psi}_2}\right) = \underline{\widehat{\Omega}}\left(X_{-\dot{\psi}_2}, X_{\psi_1}\right) = \{-\dot{\psi}_2, \psi_1\} = 1$$

And, as expected, the Poisson bracket is symmetric when acting on two odd functions.

12.2 The superparticle

In this section I present an example taken from supermechanics: the superparticle moving in a flat space (for which I will give detailed calculations) or in a curved space (for which I will skip some details). The situation can be modeled by a superfield theory defined on the supertime.

I will use this simple model to show how the supersymmetry generator, with my formalism, can be represented by an odd (non-generalized) vector field on the bundle of configurations. At the end of the section I will describe how the supersymmetry leads to a corresponding conserved quantity: the supercharge.

Let me first show how the Lagrangian of the superparticle moving in a generic curved space can be built with my formalism. Let be $X = \mathbb{R}^{1|1}$, $F = N$, where N is a generic even supermanifold of dimension $r|0$, an let $E = \mathbb{R}^{1|1} \times N$ be the configurations bundle. Let g be an even G^∞ Euclidean metric on N, which is the Grassmann continuation of a C^∞ Euclidean metric $\underline{g}$ on the body of N. On a local chart $U \subset E$, we can use $\left(t, \tau; q^i\right)$ as local coordinates. On an adapted chart of J^1E we can use the coordinates $\left(t, \tau; q^i; \dot{q}^i_t, \dot{q}^i_\tau\right)$. A section Φ of E can be locally written as:

$$\Phi^i(t,\tau) := q^i(\Phi(T)) = x^i(t) + \tau\psi^i(t) \tag{12.16}$$

where, since all Φ^i must be even, the x^i are even and ψ^i odd. From (12.16) we see that the same theory could be described using the fields x^i and ψ^i defined on the time line $\mathbb{R}^{1|0}$.

The lift of Φ to J^1E can be written as:

$$\begin{aligned} j^1\Phi :\ T = (t,\tau) \ \longmapsto\ & \left(t, \tau; q^i\left(j^1\Phi(T)\right); \dot{q}^i_t\left(j^1\Phi(T)\right), \dot{q}^i_\tau\left(j^1\Phi(T)\right)\right) \\ &= \left(t, \tau; \Phi^i(t,\tau); \frac{\partial}{\partial t}\Phi^i(t,\tau), \frac{\partial}{\partial \tau}\Phi^i(t,\tau)\right) \\ &= \left(t, \tau; x^i(t) + \tau\psi^i(t); \dot{x}^i(t) + \tau\dot{\psi}^i(t), \psi^i(t)\right) \end{aligned} \tag{12.17}$$

The Lagrangian of the theory is locally $\mathcal{L} = L\beta$, where:

$$L\left(t, \tau; q^i; \dot{q}^i_t, \dot{q}^i_\tau\right) = \frac{1}{2}\left(\tau\dot{q}^i_t - \dot{q}^i_\tau\right) g_{ij}\left(q^i\right)\dot{q}^j_t \tag{12.18}$$

Note that $|\mathcal{L}| = |L| = 1$.

Let's now study the simplest case occurring when N is the flat even r-dimensional space. All the considerations made below about this simplest case can be extended to the most general case with a bit more complicated calculations.

If $N = \mathbb{R}^{r|0}$, then an atlas with a single chart can be used and the coordinates of the chart described above can be taken as global ones.

When the metric g is the flat Euclidean metric, we have that $g_{ij}\left(q^i\right) = \delta_{ij}$ and L reduces to:

$$L\left(t, \tau; q^i; \dot{q}^i_t, \dot{q}^i_\tau\right) = \frac{1}{2}\tau\dot{q}^i_t\dot{q}^i_t - \frac{1}{2}\dot{q}^i_\tau\dot{q}^i_t \tag{12.19}$$

So the Euler Lagrange equations have the simple form:

$$\tau\frac{d^2\Phi^i}{dt^2} - \frac{1}{2}\frac{d^2\Phi^i}{dt\,d\tau} - \frac{1}{2}\frac{d^2\Phi^i}{d\tau\,dt} = 0 \tag{12.20}$$

or:

$$\tau\ddot{x}^i(t) - \frac{1}{2}\dot{\psi}^i(t) - \frac{1}{2}\dot{\psi}^i(t) = 0 \tag{12.21}$$

or:

$$\begin{cases} \ddot{x}^i(t) = 0 \\ \dot{\psi}^i(t) = 0 \end{cases} \tag{12.22}$$

The Lagrangian L is purely odd, $L = \widetilde{L}$, and it is regular. Remembering (8.6) and (8.10), we can define a Legendre transform and an Hamiltonian "function":

$$\begin{aligned} \mathbb{FL}\left(t, \tau; q^i; \dot{q}^i_t, \dot{q}^i_\tau\right) &= \left(t, \tau; q^i; \widetilde{p^t_i}, \widetilde{p^\tau_i}\right) \\ &= \left(t, \tau; q^i; \frac{\partial\widetilde{L}}{\partial\dot{q}^i_t}, \frac{\partial\widetilde{L}}{\partial\dot{q}^i_\tau}\right) = \left(t, \tau; q^i; \tau\dot{q}^i_t - \frac{1}{2}\dot{q}^i_\tau, -\frac{1}{2}\dot{q}^i_t\right) \end{aligned} \tag{12.23}$$

and:

$$\begin{aligned} H\left(t,\tau;q^i;\widetilde{p_i^t},\widetilde{p_i^\tau}\right) =&\widetilde{H}\left(t,\tau;q^i;\widetilde{p_i^t},\widetilde{p_i^\tau}\right)\\ =&\dot{q}_t^i\widetilde{p_i^t}+\dot{q}_\tau^i\widetilde{p_i^\tau}-L=-2\widetilde{p_i^\tau}\widetilde{p_i^t}+\left(-4\widetilde{p_i^\tau}-2\widetilde{p_i^t}\right)\widetilde{p_i^\tau}-2\tau\widetilde{p_i^\tau}^2-\left(-4\widetilde{p_i^\tau}-2\widetilde{p_i^t}\right)\widetilde{p_i^\tau}\\ =&-2\widetilde{p_i^\tau}\widetilde{p_i^t}-2\tau\widetilde{p_i^\tau}^2 \end{aligned} \tag{12.24}$$

The Hamilton-Volterra system is then:

$$\begin{cases} \dfrac{\partial\Phi^i}{\partial t}=\dfrac{\partial\widetilde{H}}{\partial\widetilde{p_i^t}}\\ \dfrac{\partial\Phi^i}{\partial\tau}=\dfrac{\partial\widetilde{H}}{\partial\widetilde{p_i^\tau}}\\ \dfrac{\partial\widetilde{p_i^t}}{\partial t}+\dfrac{\partial\widetilde{p_i^\tau}}{\partial\tau}=-\dfrac{\partial\widetilde{H}}{\partial q^i} \end{cases} \tag{12.25}$$

or:

$$\begin{cases} \dfrac{\partial\Phi^i}{\partial t}=-2\widetilde{p_i^\tau}\\ \dfrac{\partial\Phi^i}{\partial\tau}=-2\widetilde{p_i^t}-4\tau\widetilde{p_i^\tau}\\ \dfrac{\partial\widetilde{p_i^t}}{\partial t}+\dfrac{\partial\widetilde{p_i^\tau}}{\partial\tau}=0 \end{cases} \tag{12.26}$$

or:

$$\begin{cases} \dfrac{\partial\Phi^i}{\partial t}=-2\widetilde{p_i^\tau}\\ \dfrac{\partial\Phi^i}{\partial\tau}=-2\widetilde{p_i^t}-4\tau\widetilde{p_i^\tau}\\ -\dfrac{1}{2}\dfrac{\partial^2\Phi^i}{\partial t\partial\tau}+\tau\dfrac{\partial^2\Phi^i}{\partial t^2}-\dfrac{1}{2}\dfrac{\partial^2\Phi^i}{\partial\tau\partial t}=0 \end{cases} \tag{12.27}$$

which is equivalent to (12.20).

We can also define the multisymplectic form:

$$\omega=-dq^i\wedge d\overline{p_i^t}\wedge\frac{1}{d\tau}-dq^i\wedge d\overline{p_i^\tau}\wedge\left(\partial_\tau\lrcorner\frac{dt}{d\tau}\right)-\left(-2\widetilde{p_i^\tau}d\widetilde{p_i^t}-2\widetilde{p_i^t}d\widetilde{p_i^\tau}-4\tau\widetilde{p_i^\tau}d\widetilde{p_i^\tau}\right)\wedge\frac{dt}{d\tau} \tag{12.28}$$

In the generic case of metric depending on the fields q^i, the Euler Lagrange equations, the Hamilton-Volterra system and the multisymplectic forms are a bit more complicated and I will not exhibit them here.

Nevertheless it is not difficult to prove with my formalism that the generic Lagrangian (12.18) is invariant under the action of the group of supertranslations of the supertime whose generators are the vector fields:

$$\begin{aligned} \chi_t=&\frac{\partial}{\partial t}\\ \chi_\tau=&\tau\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau} \end{aligned} \tag{12.29}$$

whose lifts to J^1E are:

$$\begin{aligned} j^1\chi_t =& \frac{\partial}{\partial t} \\ j^1\chi_\tau =& \tau\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + \dot{q}^i_t \frac{\partial}{\partial \dot{q}^i_\tau} \end{aligned} \tag{12.30}$$

Note that χ_t and its lift are even whereas χ_τ and its lift are odd.

Let's study better the symmetry generated by χ_τ. The map on E generated by χ_τ can be parametrized by an odd parameter κ and it is defined by:

$$\begin{cases} t_\kappa = t + \kappa\tau \\ \tau_\kappa = \tau + \kappa \\ q^i_\kappa = q^i \end{cases} \tag{12.31}$$

A lifted map on J^1E is defined by:

$$\begin{cases} t_\kappa = t + \kappa\tau \\ \tau_\kappa = \tau + \kappa \\ q^i_\kappa = q^i \\ \dot{q}^i_{t,\kappa} = \dot{q}^i_t \\ \dot{q}^i_{\tau,\kappa} = \dot{q}^i_\tau + \kappa\dot{q}^i_t \end{cases} \tag{12.32}$$

Note that:

$$q^i_\kappa(t,\tau) = x^i_\kappa(t) + \tau\psi_\kappa(t)$$

and

$$q^i_\kappa(t_\kappa,\tau_\kappa) = x^i_\kappa(t_\kappa) + \tau_\kappa\psi(t_\kappa) \tag{12.33}$$

Therefore, since all the functions here involved are G^∞ and admit Taylor expansions, taking into account the parities of variables, (12.31) implies that:

$$\begin{cases} x^i_\kappa(t) = x^i(t) - \kappa\psi^i(t) \\ \psi^i_\kappa(t) = \psi^i(t) + \kappa\dot{x}^i(t) \end{cases} \tag{12.34}$$

Note that (12.34) is obviously coherent with (12.32).

If one describes the superparticle using the component fields x^i and ψ^i defined on the real time line, then one falls necessarily on the transformation (12.34), which involves derivatives of the fields and whose generator can be only a generalized vector field. Moreover the corresponding component Lagrangian defined in terms of x^i and ψ^i would show a non manifest symmetry.

Describing the superparticle with the superfields q^i defined on the supertime $\mathbb{R}^{1|1}$ allows us to treat the supersymmetry (12.31) as generated by a conventional vector field and, as we are going to see, the supersymmetry in this formalism is manifest.

In fact we still have to see that (12.31) and the corresponding generator (12.29) define a symmetry of the theory.

Let's calculate $\mathrm{Lie}_{j^1\chi_\tau}\mathcal{L}$. We have that:

$$\begin{aligned} \mathrm{Lie}_{j^1\chi_\tau}\mathcal{L} =& d\left(j^1\chi_\tau \rfloor \mathcal{L}\right) + j^1\chi_\tau \rfloor d\mathcal{L} = d\left(-L\cdot j^1\chi_\tau \rfloor \beta\right) + j^1\chi_\tau \rfloor d\mathcal{L} \\ =& -dL\wedge\left(j^1\chi_\tau \rfloor \beta\right) + j^1\chi_\tau \rfloor d\mathcal{L} = j^1\chi_\tau \rfloor dL\cdot\beta - j^1\chi_\tau \rfloor \left(dL\wedge\beta\right) + j^1\chi_\tau \rfloor d\mathcal{L} \\ =& j^1\chi_\tau \rfloor dL\cdot\beta = \left(\frac{\partial L}{\partial\tau} + \dot{q}^i_t\frac{\partial L}{\partial\dot{q}^i_\tau}\right)\beta = 0 \end{aligned} \tag{12.35}$$

Where the result is obtained remembering (12.18) and (12.30), remembering that $|L| = |dL| = 1$ and $\left|j^1\chi_\tau\right| = 1$, using the properties of fractional superforms, and specifically (5.56), and noting that:

$$d\left(j^1\chi_\tau \lrcorner \beta\right) = 0$$

So χ_τ defines indeed a manifest symmetry of the system.

Using the super Poincaré-Cartan form, we can therefore calculate the corresponding super-current γ on a solution Φ. We have that:

$$\begin{aligned}\gamma &= j^1\Phi^*\left[j^1\chi_\tau \lrcorner \left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I \wedge \frac{\partial L}{\partial \dot{q}^I_A}\beta_A\right)\right] \\ &= j^1\Phi^*\left[j^1\chi_\tau \lrcorner \left(\mathcal{L} + c^i \wedge \frac{\partial L}{\partial \dot{q}^i_t}\beta_t + c^i \wedge \frac{\partial L}{\partial \dot{q}^i_\tau}\beta_\tau\right)\right] \\ &= j^1\Phi^*\left[\left(\tau\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) \lrcorner \left(\mathcal{L} + c^i \wedge \frac{\partial L}{\partial \dot{q}^i_t}\beta_t + c^i \wedge \frac{\partial L}{\partial \dot{q}^i_\tau}\beta_\tau\right)\right]\end{aligned} \tag{12.36}$$

Since $\mathbb{R}^{1|1}$ is a split supermanifold, we can integrate over the odd variable τ: id est we chose the submanifold Σ of codimension 1 defined by $t = t_0$, we integrate γ over Σ, and we obtain the current $\underline{\gamma}$ defined on $\mathbb{R}^{1|0}$ and depending on the variable t_0. Letting drop the subscript 0 of t_0, we obtain:

$$\begin{aligned}\underline{\gamma} = \int_\Sigma \gamma &= \int j^1\Phi^*\left[-\frac{1}{2}\tau\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t + \frac{1}{2}\tau\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t + \frac{1}{2}\tau\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t\right]\beta_t \\ &= \int \frac{1}{d\tau}\left[\frac{1}{2}\tau\psi^i(t) g_{ij}(x,\psi,\tau)\left(\dot{x}^j(t) + \tau\dot{\psi}^j(t)\right)\right] \\ &= \frac{1}{2}\psi^i(t) g_{ij}(x,\psi,0)\dot{x}^j(t)\end{aligned} \tag{12.37}$$

Where the result is obtained by remembering (12.18), using the properties of fractional superforms, and specifically (5.56), and noting that:

$$\begin{aligned}&\int_\Sigma \beta_\tau = 0 \\ &\int_\Sigma \frac{\partial}{\partial t} \lrcorner \frac{\partial}{\partial \tau} \lrcorner \left(c^i \wedge \beta\right) = 0 \\ &\int_\Sigma \frac{\partial}{\partial \tau} \lrcorner \frac{\partial}{\partial \tau} \lrcorner \left(c^i \wedge \beta\right) = 0\end{aligned}$$

Since $\underline{\gamma}$ is a current on $\mathbb{R}^{1|0}$, we have that on a solution of the theory $\frac{d\underline{\gamma}}{dt} = 0$. In other word on a solution we have:

$$\frac{d}{dt}\left[\psi^i(t) g_{ij}(x,\psi,0)\dot{x}^j(t)\right] = 0$$

The corresponding conserved quantity is therefore:

$$Q = \psi^i g_{ij}(x,\psi,0)\dot{x}^j \tag{12.38}$$

Note that, since g is G^∞ and it depends on $(x^i + \tau\psi^i)$, in fact $g_{ij}(x,\psi,0)$ does not depend on ψ. When g doesn't depend on q, (12.38) is an immediate consequence of (12.22).

To see that χ_t in (12.29) is the generator of a symmetry is even simpler; from (12.30) we have in fact that:

$$\begin{aligned}\mathrm{Lie}_{j^1\chi_t}\,\mathcal{L} =& d\left(j^1\chi_t \rfloor \mathcal{L}\right) + j^1\chi_t \rfloor d\mathcal{L} = d\left(\frac{\partial}{\partial t}\rfloor \mathcal{L}\right) + \frac{\partial}{\partial t}\rfloor d\mathcal{L} \\ =& dL\wedge\beta_t + \frac{\partial}{\partial t}\rfloor\left(dL\wedge\beta\right) = -\frac{\partial}{\partial t}\rfloor\left(dL\wedge\beta\right) + \frac{\partial}{\partial t}\rfloor\left(dL\wedge\beta\right) = 0\end{aligned} \tag{12.39}$$

Where again use has been made of (5.56).

The corresponding supercurrent γ_t on a solution Φ is:

$$\begin{aligned}\gamma_t &= j^1\Phi^*\left[j^1\chi_t\rfloor\left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}\, c^I\wedge\frac{\partial L}{\partial \dot{q}^I_A}\beta_A\right)\right] \\ &= j^1\Phi^*\left[\frac{\partial}{\partial t}\rfloor\left(\mathcal{L} + c^i\wedge\frac{\partial L}{\partial \dot{q}^i_t}\beta_t + c^i\wedge\frac{\partial L}{\partial \dot{q}^i_\tau}\beta_\tau\right)\right] \\ &= j^1\Phi^*\left[L\beta_t + \frac{\partial}{\partial t}\rfloor\left(c^i\wedge\frac{\partial L}{\partial \dot{q}^i_t}\beta_t + c^i\wedge\frac{\partial L}{\partial \dot{q}^i_\tau}\beta_\tau\right)\right]\end{aligned} \tag{12.40}$$

With considerations similar to the ones made above, one can see that:

$$\begin{aligned}\underline{\gamma_t} = \int_\Sigma \gamma_t =& \int j^1\Phi^*\left[\frac{1}{2}\tau\dot{q}^i_t g_{ij}(q)\dot{q}^j_t - \frac{1}{2}\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t + \frac{1}{2}\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t + \frac{1}{2}\dot{q}^i_\tau g_{ij}(q)\dot{q}^j_t\right]\beta_t \\ =& \int \frac{1}{d\tau}\left[\frac{1}{2}\tau\dot{q}^i_t g_{ij}(q)\dot{q}^j_t\right] \\ =& \int \frac{1}{d\tau}\left[\frac{1}{2}\tau\left(\dot{x}^i(t) + \tau\dot{\psi}^i(t)\right)g_{ij}(x,\psi,\tau)\left(\dot{x}^j(t) + \tau\dot{\psi}^j(t)\right)\right] \\ =& \frac{1}{2}\dot{x}^i(t)g_{ij}(x,\psi,0)\dot{x}^j(t)\end{aligned} \tag{12.41}$$

And the corresponding conserved quantity is the energy:

$$E = \dot{x}^i g_{ij}(x,\psi,0)\dot{x}^j$$

where E in fact doesn't depend on ψ; as it was for Q.

12.3 The 3-dimensional σ-model with two supersymmetries

In this section I will study a 3-dimensional σ-model with 2 supersymmetries to show how the formalism of superfields, fractional forms and super Poincaré-Cartan form, described above, allows to calculate in a simple and geometrically transparent way the supercurrents of the theory and the conserved supercharghes.

For this field model I will use as base manifold $X = \mathbb{R}^{3|2}$ and as target space $F = \mathbb{R}^{1|0}$. The configuration bundle is $E = \mathbb{R}^{3|2}\times\mathbb{R}^{1|0}$. The model could be extended to more general base manifolds whose even parts are Riemaniann manifolds and to a more general target space of dimension $1|0$ (see for example [54], [69] and [37]). To keep things easier, I will use this simpler model; moreover I will not carry on all the computations. The techniques of calculations would not change in the more general case. I judge that the simpler case and the few calculations presented here will be sufficient to illustrate the theory introduced in the previous chapters and specifically to show how the super Poincaré-Cartan form can be exploited.

On E we can use an atlas with a single global chart with coordinates $\left(t, x, y, \theta^1, \theta^2; q\right)$. On an adapted chart of J^1E we can use the coordinates $\left(t, x, y, \theta^1, \theta^2; q; \dot{q}_t, \dot{q}_x, \dot{q}_y, \dot{q}_{\theta^1}, \dot{q}_{\theta^2}\right)$. A section Φ of E can be written as:

$$\Phi(t, x, y, \theta^1, \theta^2) := q(\Phi(X)) = \varphi(t, x, y) + \theta^1\psi_1(t, x, y) + \theta^2\psi_2(t, x, y) + \theta_1\theta^2 F(t, x, y) \quad (12.42)$$

where, since Φ must be even, φ and F are even and ψ_1 and ψ_2 are odd. The same theory could be described using the fields φ, ψ_1, ψ_2 and F defined on the bosonic spacetime $\mathbb{R}^3$ which is the body of $\mathbb{R}^{3|2}$.

The Lagrangian of the theory is $\mathcal{L} = L\beta$, where:

$$\begin{aligned} &L\left(t, x, y, \theta^1, \theta^2; q; \dot{q}_t, \dot{q}_x, \dot{q}_y, \dot{q}_{\theta^1}, \dot{q}_{\theta^2}\right) \\ &= \frac{1}{2}\left(\dot{q}_{\theta^1} - \theta^1\dot{q}_t - \theta^1\dot{q}_x - \theta^2\dot{q}_y\right)\left(\dot{q}_{\theta^2} - \theta^2\dot{q}_t + \theta^2\dot{q}_x - \theta^1\dot{q}_y\right) + h(q) \end{aligned} \quad (12.43)$$

and where h is an even G^∞ function from $\mathbb{R}^{1|0}$ to $\mathbb{R}^{1|0}$.

Note that $|\mathcal{L}| = |L| = 0$.

From (12.43), we obtain that:

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_t} &= -\frac{1}{2}\theta^1\left(\dot{q}_{\theta^2} - \theta^2\dot{q}_t + \theta^2\dot{q}_x - \theta^1\dot{q}_y\right) - \frac{1}{2}\left(\dot{q}_{\theta^1} - \theta^1\dot{q}_t - \theta^1\dot{q}_x - \theta^2\dot{q}_y\right)\theta^2 \\ &= -\frac{1}{2}\theta^1\dot{q}_{\theta^2} - \frac{1}{2}\dot{q}_{\theta^1}\theta^2 + \theta^1\theta^2\dot{q}_t \\ \frac{\partial L}{\partial \dot{q}_x} &= -\frac{1}{2}\theta^1\left(\dot{q}_{\theta^2} - \theta^2\dot{q}_t + \theta^2\dot{q}_x - \theta^1\dot{q}_y\right) + \frac{1}{2}\left(\dot{q}_{\theta^1} - \theta^1\dot{q}_t - \theta^1\dot{q}_x - \theta^2\dot{q}_y\right)\theta^2 \\ &= -\frac{1}{2}\theta^1\dot{q}_{\theta^2} + \frac{1}{2}\dot{q}_{\theta^1}\theta^2 - \theta^1\theta^2\dot{q}_x \\ \frac{\partial L}{\partial \dot{q}_y} &= -\frac{1}{2}\theta^2\left(\dot{q}_{\theta^2} - \theta^2\dot{q}_t + \theta^2\dot{q}_x - \theta^1\dot{q}_y\right) - \frac{1}{2}\left(\dot{q}_{\theta^1} - \theta^1\dot{q}_t - \theta^1\dot{q}_x - \theta^2\dot{q}_y\right)\theta^1 \\ &= -\frac{1}{2}\theta^2\dot{q}_{\theta^2} - \frac{1}{2}\dot{q}_{\theta^1}\theta^1 - \theta^1\theta^2\dot{q}_y \\ \frac{\partial L}{\partial \dot{q}_{\theta^1}} &= \frac{1}{2}\left(\dot{q}_{\theta^2} - \theta^2\dot{q}_t + \theta^2\dot{q}_x - \theta^1\dot{q}_y\right) \\ \frac{\partial L}{\partial \dot{q}_{\theta^2}} &= -\frac{1}{2}\left(\dot{q}_{\theta^1} - \theta^1\dot{q}_t - \theta^1\dot{q}_x - \theta^2\dot{q}_y\right) \end{aligned} \quad (12.44)$$

Therefore the super Euler-Lagrangian equation is the following:

$$\begin{aligned} &\frac{1}{2}\theta^1\frac{d^2\Phi}{dtd\theta^2} + \frac{1}{2}\frac{d^2\Phi}{dtd\theta^1}\theta^2 - \theta^1\theta^2\frac{d^2\Phi}{dt^2} \\ &+ \frac{1}{2}\theta^1\frac{d^2\Phi}{dxd\theta^2} - \frac{1}{2}\frac{d^2\Phi}{dxd\theta^1}\theta^2 + \theta^1\theta^2\frac{d^2\Phi}{dx^2} \\ &+ \frac{1}{2}\theta^2\frac{d^2\Phi}{dyd\theta^2} + \frac{1}{2}\frac{d^2\Phi}{dyd\theta^1}\theta^1 + \theta^1\theta^2\frac{d^2\Phi}{dy^2} \\ &- \frac{1}{2}\frac{d^2\Phi}{d\theta^1 d\theta^2} - \frac{1}{2}\theta^2\frac{d^2\Phi}{d\theta^1 dt} + \frac{1}{2}\theta^2\frac{d^2\Phi}{d\theta^1 dx} + \frac{1}{2}\frac{d\Phi}{dy} - \frac{1}{2}\theta^1\frac{d^2\Phi}{d\theta^1 dy} \\ &+ \frac{1}{2}\frac{d^2\Phi}{d\theta^2 d\theta^1} + \frac{1}{2}\theta^1\frac{d^2\Phi}{d\theta^2 dt} + \frac{1}{2}\theta^1\frac{d^2\Phi}{d\theta^2 dx} - \frac{1}{2}\frac{d\Phi}{dy} + \frac{1}{2}\theta^2\frac{d^2\Phi}{d\theta^2 dy} + h'(\Phi) = 0 \end{aligned} \quad (12.45)$$

which is equivalent to:

$$\begin{aligned}0 =&\theta^1 \frac{d^2\Phi}{dtd\theta^2} + \frac{d^2\Phi}{dtd\theta^1}\theta^2 + \theta^1 \frac{d^2\Phi}{dxd\theta^2} - \frac{d^2\Phi}{dxd\theta^1}\theta^2 + \theta^2 \frac{d^2\Phi}{dyd\theta^2} + \frac{d^2\Phi}{dyd\theta^1}\theta^1 \\ &+ \frac{d^2\Phi}{d\theta^2 d\theta^1} - \theta^1\theta^2 \frac{d^2\Phi}{dt^2} + \theta^1\theta^2 \frac{d^2\Phi}{dx^2} + \theta^1\theta^2 \frac{d^2\Phi}{dy^2} + h'(\Phi)\end{aligned} \tag{12.46}$$

where h' is the derived function of h. One can compare equation (12.46) with the equivalent one in [37] (Formula 4.31), obtained using integral forms on superspace.

To derive the equivalent Euler-Lagrange equations of the components of the field Φ, one can follow two ways. Either one first finds the Lagrangian for the theory formulated in the components formalism (using for example the techniques presented in Chapter 9) and then derives the corresponding Euler-Lagrange equations, or one derive these equations directly from (12.45). In Chapter 9 I proved that the two way are equivalent. Here (as I have already done in the previous two sections) I will follow only the second path, which is shorter having already at hand equation (12.46). Rememebering (12.42), we find that (12.46) is equivalent to:

$$\begin{aligned}0 =&\theta^1 \frac{d\psi_2}{dt} + \frac{d\psi_1}{dt}\theta^2 + \theta^1 \frac{d\psi_2}{dx} - \frac{d\psi_1}{dx}\theta^2 + \theta^2 \frac{d\psi_2}{dy} + \frac{d\psi_1}{dy}\theta^1 \\ &+ F - \theta^1\theta^2 \frac{d^2\varphi}{dt^2} + \theta^1\theta^2 \frac{d^2\varphi}{dx^2} + \theta^1\theta^2 \frac{d^2\varphi}{dy^2} + h'\left[\varphi + \theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right]\end{aligned} \tag{12.47}$$

where φ, ψ_1, ψ_2 and F depend on the variables (t, x, y).

Since h is a G^∞ function, we have that:

$$\begin{aligned}h'\left[\varphi + \theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right] =&h'[\varphi] + \left[\theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right] h''[\varphi] \\ &+ \frac{1}{2}\left[\theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right]^2 h'''[\varphi] \\ =&h'[\varphi] + \left[\theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right] h''[\varphi] - \theta^1\theta^2\psi_1\psi_2 h'''[\varphi]\end{aligned} \tag{12.48}$$

where h'' and h''' are the second and the third derivative of the function h and where in the Taylor expansion (12.48), the higher order terms disappear because higher powers of the nilpotent term $\left[\theta^1\psi_1(t,x,y) + \theta^2\psi_2(t,x,y) + \theta_1\theta^2 F(t,x,y)\right]$ vanish.

Note that this kind of manipulation of the function h makes sense because we are working with the concrete Rogers-DeWitt approach to supermanifolds and because h is taken to be G^∞; see [133] for more details on Taylor expansions of G^∞-functions.

Equation (12.47) is then equivalent to:

$$\begin{aligned}0 =&\theta^1 \frac{d\psi_2}{dt} + \frac{d\psi_1}{dt}\theta^2 + \theta^1 \frac{d\psi_2}{dx} - \frac{d\psi_1}{dx}\theta^2 + \theta^2 \frac{d\psi_2}{dy} + \frac{d\psi_1}{dy}\theta^1 + F - \theta^1\theta^2 \frac{d^2\varphi}{dt^2} + \theta^1\theta^2 \frac{d^2\varphi}{dx^2} + \theta^1\theta^2 \frac{d^2\varphi}{dy^2} \\ &+ h'[\varphi] + \left[\theta^1\psi_1 + \theta^2\psi_2 + \theta_1\theta^2 F\right] h''[\varphi] - \theta^1\theta^2\psi_1\psi_2 h'''[\varphi]\end{aligned} \tag{12.49}$$

which in turn is equivalent to the Euler-Lagrange system of equations:

$$\begin{cases} \dfrac{d\psi_2}{dt}+\dfrac{d\psi_2}{dx}-\dfrac{d\psi_1}{dy}=-\psi_1 h''[\varphi] \\ \dfrac{d\psi_1}{dt}-\dfrac{d\psi_1}{dx}-\dfrac{d\psi_2}{dy}=\psi_2 h''[\varphi] \\ F=-h'[\varphi] \\ -\dfrac{d^2\varphi}{dt^2}+\dfrac{d^2\varphi}{dx^2}+\dfrac{d^2\varphi}{dy^2}=-Fh''[\varphi]+\psi_1\psi_2 h'''[\varphi] \end{cases} \tag{12.50}$$

This system can be compared with the equivalent ones found with other techniques in [37] (Formula 4.22, 4.34 and 4.35) and [69] (Formula 43).

Since the Lagrangian (12.43) is not regular, I can't give an Hamiltonian description of the theory using the tools developed in sections 8.1, 8.2 and 8.3. I can however study the symmetries of the theory from the Lagrangian point of view, using the techniques explained in Chapter 11 and I can build a symplectic structure on the covariant phase space using the pullback of the multisymplectic form on J^1E as explained at the end of section 8.4. Let's first see this construction.

Let's fix the surface $\Sigma \subset J^1E$ of codimension $1|0$ defined by the equation $t=0$. Let $\Phi \in \mathscr{E}$ be a solution of the theory and let be $\delta_1\Phi,\ \delta_2\Phi \in T_\Phi\mathscr{E}$ two vectors over Φ; let be $u_1, u_2 \in \Gamma\left(i^*\left(V_{j^1\pi}J^1E\right)\right)$ the corresponding vertical Jacobi vector fields over $j^1\Phi(X)$, then we pose:

$$O_\Sigma\big|_\Phi(\delta_1\Phi,\delta_2\Phi) \ := \ \int_{\Sigma\cap j^1\Phi(X)} (u_1\wedge u_2)\lrcorner\,\hat{o} \tag{12.51}$$

where $o=-dq\wedge d\left(\frac{\partial L}{\partial \dot{q}_A}\right)\wedge\beta_A - dH\wedge\beta$.

Because of the definition of Σ and of $\hat{o}$, we can simplify (12.52), which becomes:

$$O_\Sigma\big|_\Phi(\delta_1\Phi,\delta_2\Phi) \ := \ -\int_{\Sigma\cap j^1\Phi(X)} (u_1\wedge u_2)\lrcorner\, dq\wedge d\widehat{\left(\frac{\partial L}{\partial \dot{q}_t}\right)}\wedge\beta_t \tag{12.52}$$

where the hat $\widehat{\ }$ indicate the extension of the superform in its two first arguments (see section 5.2.2).

Let's now study the symmetries of the theory from the Lagrangian point of view. One can prove that the theory is invariant with respect of the transformations generated by the following vector fields on E:

$$\begin{aligned} &\chi_t=\frac{\partial}{\partial t} \qquad \chi_x=\frac{\partial}{\partial x} \qquad \chi_y=\frac{\partial}{\partial y} \\ &\chi_1=\frac{\partial}{\partial\theta^1}+\theta^1\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)+\theta^2\frac{\partial}{\partial y} \qquad \chi_2=\frac{\partial}{\partial\theta^2}+\theta^2\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)+\theta^1\frac{\partial}{\partial y} \end{aligned} \tag{12.53}$$

where $|\chi_t|=|\chi_x|=|\chi_y|=0$ and $|\chi_1|=|\chi_2|=1$.

Each one of the generators listed in (12.53) gives rise to a supercurrent (even or odd). I will treat only the symmetry generated by χ_1. Similar techniques can be applied to all other generators.

Let's show that χ_1 does indeed generate a symmetry. First of all from (11.9) and (11.12) we obtain that:

$$j^1\chi_1=\frac{\partial}{\partial\theta^1}+\theta^1\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)+\theta^2\frac{\partial}{\partial y}+(\dot{q}_t+\dot{q}_x)\frac{\partial}{\partial\dot{q}_{\theta^1}}+\dot{q}_y\frac{\partial}{\partial\dot{q}_{\theta^2}}$$

So we have that:

$$\begin{aligned}
\mathrm{Lie}_{j^1\chi_1}\mathcal{L} =& d\left(j^1\chi_1\lrcorner\mathcal{L}\right) + j^1\chi_1\lrcorner d\mathcal{L} = d\left(L\cdot j^1\chi_1\lrcorner\beta\right) + j^1\chi_1\lrcorner d\mathcal{L}\\
=& dL\wedge\left(j^1\chi_1\lrcorner\beta\right) + j^1\chi_1\lrcorner d\mathcal{L} = \left(j^1\chi_1\lrcorner dL\right)\cdot\beta - j^1\chi_1\lrcorner\left(dL\wedge\beta\right) + j^1\chi_1\lrcorner d\mathcal{L}\\
=& \left(j^1\chi_1\lrcorner dL\right)\cdot\beta = \left[\frac{\partial L}{\partial\theta^1} + \left(\dot q_t + \dot q_x\right)\frac{\partial L}{\partial\dot q_{\theta^1}} + \dot q_y\frac{\partial L}{\partial\dot q_{\theta^2}}\right]\beta\\
=& \left[-\frac{1}{2}\left(\dot q_t+\dot q_x\right)\left(\dot q_{\theta^2} - \theta^2\dot q_t + \theta^2\dot q_x - \theta^1\dot q_y\right) + \frac{1}{2}\left(\dot q_{\theta^1} - \theta^1\dot q_t - \theta^1\dot q_x - \theta^2\dot q_y\right)\dot q_y\right]\beta\\
&+\left[\left(\dot q_t+\dot q_x\right)\frac{1}{2}\left(\dot q_{\theta^2} - \theta^2\dot q_t + \theta^2\dot q_x - \theta^1\dot q_y\right)\right]\beta\\
&+\left[-\dot q_y\frac{1}{2}\left(\dot q_{\theta^1} - \theta^1\dot q_t - \theta^1\dot q_x - \theta^2\dot q_y\right)\right]\beta = 0
\end{aligned}\tag{12.54}$$

Where the result is obtained remembering (12.43), remembering that $|L| = |dL| = 0$, using the properties of fractional superforms, and specifically (5.56), and noting that:

$$d\left(j^1\chi_1\lrcorner\beta\right) = 0$$

So χ_1 generates indeed a manifest symmetry of the system. For every solution Φ of the theory, χ_1 defines a supercurrent γ_1:

$$\begin{aligned}
\gamma_1 =& j^1\Phi^*\left[j^1\chi_1\lrcorner\left(\mathcal{L} + (-1)^{|A|(|A|+|L|)}c\wedge\frac{\partial L}{\partial\dot q_A}\beta_A\right)\right]\\
=& j^1\Phi^*\left[j^1\chi_1\lrcorner\left(\mathcal{L} + c\wedge\frac{\partial L}{\partial\dot q_a}\beta_a - c\wedge\frac{\partial L}{\partial\dot q_\alpha}\beta_\alpha\right)\right]\\
=& j^1\Phi^*\left[j^1\chi_1\lrcorner\mathcal{L} + \left(j^1\chi_1\lrcorner c\right)\cdot\frac{\partial L}{\partial\dot q_a}\beta_a - \left(j^1\chi_1\lrcorner c\right)\cdot\frac{\partial L}{\partial\dot q_\alpha}\beta_\alpha\right]\\
&- j^1\Phi^*\left[c\wedge\left(j^1\chi_1\lrcorner\frac{\partial L}{\partial\dot q_a}\beta_a - j^1\chi_1\lrcorner\frac{\partial L}{\partial\dot q_\alpha}\beta_\alpha\right)\right]\\
=& j^1\Phi^*\left[\left(\frac{\partial}{\partial\theta^1} + \theta^1\frac{\partial}{\partial t} + \theta^1\frac{\partial}{\partial x} + \theta^2\frac{\partial}{\partial y}\right)\lrcorner\mathcal{L}\right]\\
&+ j^1\Phi^*\left\{\left[\left(\frac{\partial}{\partial\theta^1} + \theta^1\frac{\partial}{\partial t} + \theta^1\frac{\partial}{\partial x} + \theta^2\frac{\partial}{\partial y}\right)\lrcorner c\right]\cdot\frac{\partial L}{\partial\dot q_a}\beta_a\right\}\\
&- j^1\Phi^*\left\{\left[\left(\frac{\partial}{\partial\theta^1} + \theta^1\frac{\partial}{\partial t} + \theta^1\frac{\partial}{\partial x} + \theta^2\frac{\partial}{\partial y}\right)\lrcorner c\right]\cdot\frac{\partial L}{\partial\dot q_\alpha}\beta_\alpha\right\}
\end{aligned}\tag{12.55}$$

where c is the contact form on J^1E.

If $\Pi\subset X$ is the surface defined by $t = t_0$, $x = x_0$ and $y = y_0$, then the form:

$$\underline{\gamma_1} := \int_\Pi\gamma_1$$

depends on the variables t_0, x_0 and y_0 and it is a current on the body of X.

If we fix a time slice $\Sigma\subset X$ of codimension 1, then the quantity:

$$Q_1 = \int_\Sigma\gamma_1 = \int_{\underline{\Sigma}}\underline{\gamma_1}$$

is a conserved quantity.

Let's define Σ by the equation $t = t_0$. To compute Q_1 we don't need to explicitely compute all the components of the superform γ_1. We have in fact:

$$
\begin{aligned}
Q_1 &= \int_\Sigma j^1\Phi^*\left(\theta^1 \frac{\partial}{\partial t}\lrcorner L\beta\right) + \int_\Sigma j^1\Phi^*\left\{\left[\left(\frac{\partial}{\partial\theta^1}+\theta^1\frac{\partial}{\partial t}+\theta^1\frac{\partial}{\partial x}+\theta^2\frac{\partial}{\partial y}\right)\lrcorner c\right]\cdot\frac{\partial L}{\partial\dot q_t}\beta_t\right\} \\
&= \int_\Sigma j^1\Phi^*\left(\theta^1 L\beta_t\right) - \int_\Sigma j^1\Phi^*\left[\left(\dot q_{\theta^1}+\theta^1\dot q_t+\theta^1\dot q_x+\theta^2\dot q_y\right)\cdot\frac{\partial L}{\partial\dot q_t}\beta_t\right] \\
&= \int_\Sigma \left(\frac{1}{2}\theta^1\theta^2\psi_1\frac{\partial\varphi}{\partial t}-\frac{1}{2}\theta^1\theta^2\psi_1\frac{\partial\varphi}{\partial x}-\frac{1}{2}\theta^1\theta^2\psi_2\frac{\partial\varphi}{\partial y}+\frac{1}{2}\theta^1\theta^2\psi_2 F+\theta^1\theta^2\psi_2 h'(\varphi)\right)\frac{dx\wedge dy}{d\theta^1\odot\theta^2} \\
&\quad - \int_\Sigma \left[\left(\psi_1+\theta^2 F+\theta^1\frac{\partial\varphi}{\partial t}+\theta^1\theta^2\frac{\partial\psi_2}{\partial t}+\theta^1\frac{\partial\varphi}{\partial x}+\theta^1\theta^2\frac{\partial\psi_2}{\partial x}+\theta^2\frac{\partial\varphi}{\partial y}-\theta^1\theta^2\frac{\partial\psi_1}{\partial y}\right)\cdot\right. \\
&\qquad\qquad \left.\cdot\left(-\frac{1}{2}\theta^1\psi_2-\frac{1}{2}\psi_1\theta^2+\theta^1\theta^2\frac{\partial\varphi}{\partial t}\right)\right]\frac{dx\wedge dy}{d\theta^1\odot\theta^2} \\
&= \int_{\underline{\Sigma}} \left(\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial t}-\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial x}-\frac{1}{2}\psi_2\frac{\partial\varphi}{\partial y}+\frac{1}{2}\psi_2 F+\psi_2 h'(\varphi)\right)dx\wedge dy \\
&\quad - \int_\Sigma \left[\theta^1\theta^2\psi_1\frac{\partial\varphi}{\partial t}+\frac{1}{2}\theta^1\theta^2\psi_2 F+\frac{1}{2}\theta^1\theta^2\psi_1\frac{\partial\varphi}{\partial t}+\frac{1}{2}\theta^1\theta^2\psi_1\frac{\partial\varphi}{\partial x}+\frac{1}{2}\theta^1\theta^2\psi_2\frac{\partial\varphi}{\partial y}\right]\frac{dx\wedge dy}{d\theta^1\odot\theta^2} \\
&= \int_{\underline{\Sigma}} \left(\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial t}-\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial x}-\frac{1}{2}\psi_2\frac{\partial\varphi}{\partial y}+\frac{1}{2}\psi_2 F+\psi_2 h'(\varphi)\right)dx\wedge dy \\
&\quad - \int_{\underline{\Sigma}} \left[\psi_1\frac{\partial\varphi}{\partial t}+\frac{1}{2}\psi_2 F+\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial t}+\frac{1}{2}\psi_1\frac{\partial\varphi}{\partial x}+\frac{1}{2}\psi_2\frac{\partial\varphi}{\partial y}\right]dx\wedge dy \\
&= \int_{\underline{\Sigma}} \left(-\psi_1\frac{\partial\varphi}{\partial t}-\psi_1\frac{\partial\varphi}{\partial x}-\psi_2\frac{\partial\varphi}{\partial y}+\psi_2 h'(\varphi)\right)dx\wedge dy
\end{aligned}
\tag{12.56}
$$

Formula (12.56) is equivalent to the one obtained with different techniques in [54] (first of Formula 7.36).

Part IV

Conclusion and appendix

Conclusion and perspectives

The *fractional forms* introduced in the second part of this thesis allowed the *multisymplectic approach to superfield theories* developed in the third part. This offers an Hamiltonian point of view for superfield theories with several space-time variables. The formalism gives also new tools to study supermechanical theories.

As in the classical case, the super-multisymplectic approach provides a way to build a symplectic structure on the covariant phase space, which can be even or odd depending on the parity of the Lagrangian defining the superfield theory. From the symplectic structure a direct procedure leads to the fields brackets needed by physicists as the starting point of the canonical quantization.

I see some possible paths to follow in order to complete, to exploit and/or to extend the results found with the work of this thesis. I list here some of them.

- First of all it is necessary to extend the constructions and the theorems here presented to the case where the Lagrangian defining the superfield theory is not regular. This is indeed the most common case for superfield theories. Very likely the techniques used in classical field theory to treat singular Lagrangians, can be easily extended to the super case, so that the dynamic can be described with the super-multisymplectic formalism. This will give an Hamiltonian point of view on superfield theories where gauge symmetries are present and on theories which can be described with the use of constraints.
- It is then necessary to study the infinite dimensional super covariant phase space providing it with a rigorously defined super-differential structure in order to prove rigorously the properties of the super-symplectic form built with the help of the super-multisymplectic formalism.
- A more systematic and complete study of *fractional forms* can be foreseen, which may also clarify their relations with generic Voronov-Zorich superforms.
- A promising path to follow is in my opinion the one leading to a description of the BRST symmetry with the multisymplectic formalism, completing the work already initiated by S. P. Hrabak. BRST symmetry can indeed be seen as an odd symmetry, therefore the tools here developed for treating odd symmetries may provide new insights into the matter.
- The coforms defined in chapter 6 seem to me the natural objects to use to study the BV approach to field theories. Some of their properties, expounded here, may help in understanding better, and from a new point of view, some features of the BV theory.
- I think that the use of fractional forms may be useful in exploiting better the picture changing operators which appear in string theory. The formalism introduced in this thesis should be applied to some superfield theories of interest to physicists, like for example super Chern-Simons theories, where picture changing operators play an important role.
- Finally this formalism could be applied to other supersymmetric theories to exploit the power of its geometric point of view.

Appendix A

Computation of symplectic forms.

A.1 Computation of the components of Ω and $\bar{\Omega}$ for theories on bundles.

I exhibit here the calculations necessary to compute the values of the components of Ω and $\bar{\Omega}$ for the field theories defined in section 3.1.4.

We have:

$$\begin{aligned}
\Omega_{k1r,k'1r} &= \int_{\Sigma\cap G} \xi_{k1r} \wedge \xi_{k'1r} \lrcorner\, \omega \\
&= \int_0^b \left\{ -2\pi \left(\frac{\alpha}{a} + \frac{k}{a} \right) \sin \left[2\pi \left(\frac{\beta}{b} + \frac{j}{b} \right) x \right] \cos \left[2\pi \left(\frac{\beta}{b} + \frac{j'}{b} \right) x \right] \right\} dx \\
&\quad + \int_0^b \left\{ 2\pi \left(\frac{\alpha}{a} + \frac{k'}{a} \right) \sin \left[2\pi \left(\frac{\beta}{b} + \frac{j'}{b} \right) x \right] \cos \left[2\pi \left(\frac{\beta}{b} + \frac{j}{b} \right) x \right] \right\} dx \\
&= \int_0^b -2\pi \left(\frac{\alpha}{a} + \frac{k}{a} \right) \frac{1}{2} \left\{ \sin \left[2\pi x \left(\frac{2\beta}{b} + \frac{j+j'}{b} \right) \right] + \sin \left[2\pi x \left(\frac{j-j'}{b} \right) \right] \right\} dx \\
&\quad + \int_0^b 2\pi \left(\frac{\alpha}{a} + \frac{k'}{a} \right) \frac{1}{2} \left\{ \sin \left[2\pi x \left(\frac{2\beta}{b} + \frac{j+j'}{b} \right) \right] + \sin \left[2\pi x \left(\frac{j'-j}{b} \right) \right] \right\} dx = 0
\end{aligned} \tag{A.1}$$

because j, j' and 2β are integer. In the same way we can calculate:

$$\Omega_{k1r,k'3r} = \Omega_{k1r,k'4r} = \Omega_{k2r,k'2r} = \Omega_{k2r,k'3r} = \Omega_{k2r,k'4r} = \Omega_{k3r,k'3r} = \Omega_{k4r,k'4r} = 0 \tag{A.2}$$

On the other hand we have:

$$
\begin{aligned}
\Omega_{k1r,k'2r} =& \int_{\Sigma\cap G} \xi_{k1r} \wedge \xi_{k'2r} \lrcorner \omega \\
=& \int_0^b \left\{ -2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right) \sin\left[2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right] \sin\left[2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)x\right]\right\} dx \\
& - \int_0^b \left\{ 2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right) \cos\left[2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)x\right] \cos\left[2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\right\} dx \\
=& \int_0^b -2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\frac{1}{2}\left\{\cos\left[2\pi x\left(\frac{j-j'}{b}\right)\right] - \cos\left[2\pi x\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\right]\right\} dx \\
& - \int_0^b 2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right)\frac{1}{2}\left\{\cos\left[2\pi x\left(\frac{j-j'}{b}\right)\right] + \cos\left[2\pi x\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\right]\right\} dx \qquad (A.3)\\
=& -\pi\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\int_0^b \cos\left[2\pi x\left(\frac{j-j'}{b}\right)\right] dx \\
& + \pi\left(\frac{k-k'}{a}\right)\int_0^b \cos\left[2\pi x\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\right] dx \\
=& -\pi\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\delta_{jj'} b + \pi\left(\frac{k-k'}{a}\right)\int_0^b \cos\left[2\pi x\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\right] dx \\
=& -2\pi\frac{b}{a}(\alpha+k)\,\delta_{kk'} + 0
\end{aligned}
$$

because $j = j'$ only when $k = k'$, and because the last integral equal to 0 being in general $\left(\frac{\beta}{b}+\frac{j}{b}\right) \geq 0$ and $\left(\frac{\beta}{b}+\frac{j'}{b}\right) \geq 0$, and more specifically, when there is no degeneracy (as it is the case we are now studying): $\left(\frac{\beta}{b}+\frac{j}{b}\right) > 0$ and $\left(\frac{\beta}{b}+\frac{j'}{b}\right) > 0$. If in the sum (3.56) there is one specific $\tilde{k}$ with a degeneracy of the corresponding $\mathscr{E}_{\tilde{k}}$, then all our reasoning is still valid for every k and k' except $\tilde{k}$ which could be easily treated separately.

Finally we have:

$$\begin{aligned}
\Omega_{k3r,k'4r} =& \int_{\Sigma\cap G} \xi_{k3r} \wedge \xi_{k'4r} \lrcorner\, \omega \\
=& \int_0^b \left\{ -2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\sin\left[-2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\sin\left[-2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)x\right]\right\}dx \\
&- \int_0^b \left\{ 2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right)\cos\left[-2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)x\right]\cos\left[-2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)x\right]\right\}dx \\
=& \int_0^b -2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)\frac{1}{2}\left\{\cos\left[2\pi x\left(\frac{j'-j}{b}\right)\right]-\cos\left[2\pi x\left(-\frac{2\beta}{b}-\frac{j+j'}{b}\right)\right]\right\}dx \\
&- \int_0^b 2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right)\frac{1}{2}\left\{\cos\left[2\pi x\left(\frac{j'-j}{b}\right)\right]+\cos\left[2\pi x\left(-\frac{2\beta}{b}-\frac{j+j'}{b}\right)\right]\right\}dx \\
=& -\pi\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\int_0^b \cos\left[2\pi x\left(\frac{j'-j}{b}\right)\right]dx \\
&+ \pi\left(\frac{k-k'}{a}\right)\int_0^b \cos\left[2\pi x\left(-\frac{2\beta}{b}-\frac{j+j'}{b}\right)\right]dx \\
=& -\pi\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\delta_{jj'}b+\pi\left(\frac{k-k'}{a}\right)\int_0^b \cos\left[2\pi x\left(-\frac{2\beta}{b}-\frac{j+j'}{b}\right)\right]dx \\
=& -2\pi\frac{b}{a}\left(\alpha+k\right)\delta_{kk'}+0
\end{aligned} \tag{A.4}$$

We compute $\bar{\Omega}$ integrating on the slice $\bar{\Sigma}$, and we find with analogous techniques:

$$\bar{\Omega}_{k1r,k'3r} = \bar{\Omega}_{k1r,k'4r} = \bar{\Omega}_{k2r,k'2r} = \bar{\Omega}_{k2r,k'3r} = \bar{\Omega}_{k2r,k'4r} = \bar{\Omega}_{k3r,k'3r} = \bar{\Omega}_{k4r,k'4r} = 0 \tag{A.5}$$

and:

$$\begin{aligned}
\bar{\Omega}_{k1r,k'2r} =& \int_{\bar{\Sigma}\cap G} \xi_{k1r} \wedge \xi_{k'2r} \lrcorner\, \omega \\
=& \int_0^a \left\{ -2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\sin\left[2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)t\right]\sin\left[2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right)t\right]\right\}dt \\
&- \int_0^a \left\{ 2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)\cos\left[2\pi\left(\frac{\alpha}{a}+\frac{k'}{a}\right)t\right]\cos\left[2\pi\left(\frac{\alpha}{a}+\frac{k}{a}\right)t\right]\right\}dt \\
=& \int_0^a -2\pi\left(\frac{\beta}{b}+\frac{j}{b}\right)\frac{1}{2}\left\{\cos\left[2\pi t\left(\frac{k-k'}{a}\right)\right]-\cos\left[2\pi t\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\right]\right\}dt \\
&- \int_0^a 2\pi\left(\frac{\beta}{b}+\frac{j'}{b}\right)\frac{1}{2}\left\{\cos\left[2\pi t\left(\frac{k-k'}{b}\right)\right]+\cos\left[2\pi t\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\right]\right\}dt \\
=& -\pi\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\int_0^a \cos\left[2\pi t\left(\frac{k-k'}{a}\right)\right]dt \\
&+ \pi\left(\frac{j-j'}{b}\right)\int_0^a \cos\left[2\pi t\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\right]dt = \\
=& -\pi\left(\frac{2\beta}{b}+\frac{j+j'}{b}\right)\delta_{kk'}a+\pi\left(\frac{j-j'}{b}\right)\int_0^a \cos\left[2\pi t\left(\frac{2\alpha}{a}+\frac{k+k'}{a}\right)\right]dt \\
=& -2\pi\frac{a}{b}\left(\beta+j\right)\delta_{kk'}+0
\end{aligned} \tag{A.6}$$

because the last integral is equal to 0 being in general $k \geq -\alpha$ and $k' \geq -\alpha$, and more specifically, when there is no degeneracy (as it is the case we are now studying): $k > -\alpha$ and $k' > -\alpha$. If in the sum (3.56) there is one specific $\tilde{k}$ with a degeneracy of the corresponding $\mathscr{E}_{\tilde{k}}$, then all our reasoning is still valid for every k and k' except $\tilde{k}$ which could be easily treated separately.
We have then:

$$\begin{aligned}
\bar{\Omega}_{k3r,k'4r} =& \int_{\bar{\Sigma}\cap G} \xi_{k3r} \wedge \xi_{k'4r} \lrcorner \omega \\
=& \int_0^a \left\{ 2\pi \left(\frac{\beta}{b} + \frac{j}{b} \right) \sin \left[2\pi \left(\frac{\alpha}{a} + \frac{k}{a} \right) t \right] \sin \left[2\pi \left(\frac{\alpha}{a} + \frac{k'}{a} \right) t \right] \right\} dt \\
& \int_0^a \left\{ 2\pi \left(\frac{\beta}{b} + \frac{j'}{b} \right) \cos \left[2\pi \left(\frac{\alpha}{a} + \frac{k'}{a} \right) t \right] \cos \left[2\pi \left(\frac{\alpha}{a} + \frac{k}{a} \right) t \right] \right\} dt \\
=& \int_0^a 2\pi \left(\frac{\beta}{b} + \frac{j}{b} \right) \frac{1}{2} \left\{ \cos \left[2\pi t \left(\frac{k-k'}{a} \right) \right] - \cos \left[2\pi t \left(\frac{2\alpha}{a} + \frac{k+k'}{a} \right) \right] \right\} dt \\
& \int_0^a 2\pi \left(\frac{\beta}{b} + \frac{j'}{b} \right) \frac{1}{2} \left\{ \cos \left[2\pi t \left(\frac{k-k'}{a} \right) \right] + \cos \left[2\pi t \left(\frac{2\alpha}{a} + \frac{k+k'}{a} \right) \right] \right\} dt \qquad \text{(A.7)} \\
=& \pi \left(\frac{2\beta}{b} + \frac{j+j'}{b} \right) \int_0^a \cos \left[2\pi t \left(\frac{k-k'}{a} \right) \right] dt \\
& + \pi \left(\frac{j'-j}{b} \right) \int_0^a \cos \left[2\pi t \left(\frac{2\alpha}{a} + \frac{k+k'}{a} \right) \right] dt \\
=& \pi \left(\frac{2\beta}{b} + \frac{j+j'}{b} \right) \delta_{kk'} a + \pi \left(\frac{j'-j}{b} \right) \int_0^a \cos \left[2\pi t \left(\frac{2\alpha}{a} + \frac{k+k'}{a} \right) \right] dt \\
=& 2\pi \frac{a}{b} (\beta + j) \delta_{kk'} + 0
\end{aligned}$$

As already noted, we would obtain the same results if we calculated Ω and $\bar{\Omega}$ on the imaginary section of the theory.

Bibliography

[1] Alldridge A., *A convenient category of supermanifolds*, 2011, arXiv:1109.3161v1 [math.DG].

[2] Alldridge A., Hilgert J. and Palzer W., *Integration on non-compact supermanifolds*, 2011, arXiv:1106.1967v1 [math.DG].

[3] Baez J. C., Hoffnung A. E. and Rogers C. L., *Categorified Symplectic Geometry and the Classical String*, Commun. Math. Phys. (2010) 293: 701; arXiv:0808.0246v1 [math-ph].

[4] Baez J. C. and Rogers C. L., *Categorified Symplectic Geometry and the string Lie 2-Algebra*, Homol. Homotopy Appl. 12 (2010), 221-236; arXiv:0901.4721v1 [math-ph].

[5] Baranov M. A. and Schwarz A. S., *Cohomology of supermanifolds*, Funk. Anal. Pril., 18(3) (1984), 69-70.

[6] Bartocci C., Bruzzo U. and Hernández Ruipérez D., *The geometry of supermanifolds*, in Mathematics and its Applications, Vol. 71, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[7] Batchelor M., *The structure of supermanifolds*, Transactions of the American Mathematical Society 253 (1979), pp. 329-338.

[8] Batchelor M., *Two approaches to supermanifolds*, Transactions of the American Mathematical Society 258 (1980), pp. 257-270.

[9] Belopolsky A., *New geometrical approach to superstrings*, hep-th/9703183.

[10] Belopolsky A., *Picture changing operators in supergeometry and superstring theory.* hep-th/9706033.

[11] Berezin F. A., *The method of Second Quantization*, Academic Press, 1966.

[12] Berezin F. A., *Supermanifolds*, Usp. Mat. Nauk., 32 (1977), 154.

[13] Berezin F. A., *Mathematical foundations of supersymmetric field theories*, Yad. Fiz., 29 (1979), No. 6, 1670-1687.

[14] Berezin F. A. and Kac G. I., *Lie groups with commuting and anticommuting parameters*, Mat. Sbornik., 82 (1970), No. 3, 343-359.

[15] Berezin F. A. and Leites D. A., *Supermanifolds*, Doklady Akad. Nauk. SSSR, 224(1975), No. 3, 505-508.

[16] Bernstein J.N. and Leites D.A., *Integral forms and Stokes formula on supermanifolds*, Funk. Anal. Pril., 11(1), (1977), 55-56.

[17] Bernstein J.N. and Leites D.A., *How to integrate differential forms on supermanifolds*, Funk. Anal. Pril., 11(3), (1977), 70-71.

[18] Bruce A. J., Grabowska K. and Moreno G., *On a geometric framework for Lagrangian supermechanics*, 2016, arXiv: 1606.02604v1.

[19] Bruce A. J., *On curves and jets of curves on supermanifolds*, Archivum Mathematicum 050.2 (2014), 115-130, http://dml.cz/dmlcz/143785.

[20] Buttin C., *Théorie des opérateurs différentiels gradués sur les formes différentielles*, Bull. Soc. Math. Fr. 102 (1974), 49-73.

[21] Cantrijn F. and Ibort L. A., *Reduction of Poisson supermanifolds*, C. R. Acad. Sci. Paris, t. 311 (1990), Série II, p. 567-572.

[22] Cantrijn F., Ibort L. A. and de León M., *Hamiltonian Structures on Multisymplectic Manifolds*, Rend. Sem. Mat. Univ. Pol. Torino, 54 (1996), 225-236.

[23] Cantrijn F., Ibort L. A. and de León M., *On the geometry of multisymplectic manifolds*, J. Austral. Math. Soc. Ser. A66 (1999), 303.

[24] Cariñena J. F., Crampin M. and Ibort L. A. *On the multisymplectic formalism for first order field theories*, Diff. Geom. Appl.1 (1991), 345.

[25] Cariñena J. F. and Figueroa H. *A geometrical version of Noether's theorem in supermechanics*, Reports on Mathematical Physics, Vol. 34 (1994), No. 3, 277-302.

[26] Cariñena J. F. and Figueroa H. *Hamiltonian versus Lagrangian formulations of supermechanics*, J. Phys. A: Math. Gen. 30 (1997), 2705, arXiv:dg-ga/9703016v1.

[27] Cartan É., *Leçons sur les invariants intégraux*, Hermann, 1922.

[28] Crnković Č., Witten E., *Covariant description of canonical formalism in geometrical theories*, in Three hundred years of gravitation, 1986, 676-684.

[29] de León M., Martín de Diego D. and Santamaría-Merini A., *Symmetries in classical field theory*, Int. J. Geom. Methods Mod. Phys. 1 (2004), 651; arXiv:math-ph/0404013.

[30] de León M., McLean M., Norris L. K., Rey-Roca A. and Salgado M., *Geometric Structures in Field Theory*, 2002; arXiv:math-ph/0208036v1.

[31] de León M., Merino E., Oubiña J. A., Rodrigues P. and Salgado M., *Hamiltonian systems on kcosymplectic manifolds*, J. Math. Phys. 39(2) (1998) 876â“893; arXiv:.

[32] De Donder T., *Théorie invariantive du calcul des variations*, Gauthiers-Villars, Paris, 1930.

[33] Dedecker P., *Calcul des variations, formes différentielles et champs géodésiques*, in *Géométrie différentielle, Colloq. Intern. du CNRS LII, Strasbourg, 1953*, Publ. du CNRS, Paris, 1953, p. 17-34.

[34] Dedecker P., *On the generalization of symplectic geometry to multiple integrals in the calculus of variations*, in *Differential Geometrical Methods in Mathematical Physics*, eds. K. Bleuler and A. Reetz, Lect. Notes Maths. vol. 570, Springer-Verlag, Berlin, 1977, p. 395-456.

[35] Deligne P. and Morgan J. W., *Notes on supersymmetry (following Joseph Bernstein)*, in *Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, Part 1: Classical Fields and Supersymmetry*, P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds., American Mathematical Society, Providence, RI, 1999, pp 41-87.

[36] Deligne P. and Freed D., *Classical field theory* in *Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, Part 1: Classical Fields and Supersymmetry*, P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds., American Mathematical Society, Providence, RI, 1999, pp 137-225.

[37] Deligne P. and Freed D., *Supersolutions* in *Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, Part 1: Classical Fields and Supersymmetry*, P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds., American Mathematical Society, Providence, RI, 1999, pp 137-225.

[38] Demoures F., Gay-Balmaz F. and Ratiu T.S., *Multisymplectic variational integrators and space/time symplecticity*, to appear in Analysis and Applications; arXiv:1310.4772v1 [math.NA].

[39] Demoures F., Gay-Balmaz F., Kobilarov M. and Ratiu T.S., *Multisymplectic Lie group variational integrator for a geometrically exact beam in* $\mathbb{R}^3$, Commun. Nonlin. Sci. Numer. Simulat., 19 (10), 3492-3512; arXiv:1403.5410v1 [math.NA].

[40] DeWitt B., *Invariant Commutators for the Quantized Gravitational Field*, Phys. Rev. Lett. 4 (1960) 317-320.

[41] DeWitt B., *Dynamical Theory of Groups and Fields*, in B. DeWitt and C. DeWitt (eds.), *Relativity, Groups and Topology, 1963 Les Houches Lectures*, pp. 585-820, Gordon and Breach, New York 1964.

[42] DeWitt B., *The Spacetime Approach to Quantum Field Theory*, in B. DeWitt and R. Stora (eds.), *Relativity, Groups and Topology II, 1983 Les Houches Lectures*, pp. 382-738, Elsevier, Amsterdam 1984.

[43] DeWitt B., *Supermanifolds*, Cambridge University Press, 1984, 1992.

[44] Echeverría-Enríquez A., López C., Marín Solano J., Muñoz-Lecanda M. C. and Román-Roy N., *Lagrangian-Hamiltonian Unified Formalsm for Field Theory*, J. Math. Phys. 45(1) (2004) 360-380; arXiv:math-ph/0212002v2.

[45] Echeverría-Enríquez A., Muñoz-Lecanda M. and Román-Roy N., *On the Multimomentum bundles and the Legendre Maps in field theories*, Rep. Math. Phys. 45 (2000), 85-105; arXiv:math-ph/9904007.

[46] Echeverría-Enríquez A., Muñoz-Lecanda M. and Román-Roy N., *Geometry of multisymplectic Hamiltonian first-order field theories*, J. Math. Phys. 41 (2002), 7402; arXiv:math-ph/00040005.

[47] Fatibene L., Feraris M., Francaviglia M. and McLenaghan R. G., *Generalized symmetries in mechanics and field theories*, J. Math. Phys. 43, 3147 (2002).

[48] Fatibene L. and Francaviglia M., *Natural and gauge natural formalism for classical field theories. A Geometric Perspective including Spinors and Gauge Theories*, Kluwer Academic Publishers Boston/Dordrecht/London, 2003.

[49] Forger M. and Gomes L., *Multisymplectic and Polysymplectic Structures on Fiber Bundles*, 2007; arXiv:0708.1586.

[50] Forger M., Paufler C. and Römer H., *The Poisson Bracket for Poisson Forms in Multisymplectic Field Theory*, Rev. Math. Phys. 15 (2003), 705-744; arXiv:math-ph/0202043.

[51] Forger M., Paufler C. and Römer H., *Hamiltonian Multivector Fields and Poisson Forms in Multisymplectic Field Theory*, J. Math. Phys. 46 (2005) 112903; arXiv:math-ph/0407057.

[52] Forger M. and Römer H., *A Poisson Bracket on Multisymplectic Phase Space*, Rep. Math. Phys. 48 (2001), 211-218; arXiv:math-ph/0009037.

[53] Forger M. and Romero S. V., *Covariant Poisson Brackets in Geometric Field Theory*, Communications in Mathematical Physics 256 (2005), 375-410, arXiv:math-ph/0408008.

[54] Freed D. S., *Classical Field Theory and Supersymmetry*, IAS/Park City Mathematics Series Vol. 11, 2001.

[55] Gajduk A. V., Khudaverdian O. M. and Schwarz A S., *Integration on surfaces in superspace*, Teor. Mat. Fiz., 52(3) (1982), 375-383; Theor. Math. Phys., 52 (1983), 862.

[56] García P. L., *Geometría simplética en la teoria de campos*, Collect. Math. 19, 1-2, 73, 1968.

[57] García P. L., Pérez-Rendón A., *Symplectic approach to the theory of quantized fields, I*, Commun. Math. Phys. 13 (1969), 24-44.

[58] García P. L., Pérez-Rendón A., *Symplectic approach to the theory of quantized fields, II*, Arch. Rational Mech. Anal. 43 (1971), 101-124.

[59] Giachetta G., Mangiarotti L. and Sardanashvily G., *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific, Singapore, 1997.

[60] Giachetta G., Mangiarotti L. and Sardanashvily G., *Covariant Hamilton equations for field theory*, J. Phys. A32 (1999), 6629-6642.

[61] Giachetta G., Mangiarotti L. and Sardanashvily G., *Advanced classical field theory*, World Scientific Publishing Company, Singapore, 2009.

[62] Goldschmidt H., Sternberg S., *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier Grenoble 23, 1 (1973), 203-267.

[63] Gotay M. J., Isenberg J., Marsden J. E., Montgomery R., Śniatycki J. and Yasskin P. B., *Momentum maps and Classical Fields. Part I: Covariant Field Theory*, (1998), `arXiv:physics/9801019v2` [math-ph].

[64] Gotay M. J., Isenberg J., Marsden J. E., Montgomery R., Śniatycki J. and Yasskin P. B., *Momentum maps and Classical Fields. Part II: Canonical Analysis of Field Theories*, (2004), `arXiv:math-ph/0411032v1`.

[65] Grassi P. A. and Marescotti M., *Integration of Superforms and Super-Thom Class*, arXiv:0712.2600 [hep-th].

[66] Günther C., *The polysymplectic Hamiltonian formalism in field theory and calculus of variations, I: The local case*, J. Diff. Geom. 25 (1987), 23.

[67] Hanisch F., *A Supermanifold structure on Spaces of Morphisms between Supermanifolds*, 2014, arXiv:1406.7484v1 [math-DG].

[68] Hélein F., *A representation formula for maps on supermanifolds*, J. Math. Phys. 49 (2008),

[69] Hélein F., *An introduction to supermanifolds and supersymmetry*, in *Systèmes intégrables et théorie des champs quantiques*, edited by Baird, P., Hélein, F., Kouneiher, J., Pedit, F., Roubtsov, V., Hermann, 2009, pp. 103-157.

[70] Hélein F., *Multisymplectic formalism and the covariant phase space*, in *Variational Problems in Differential Geometry. Leeds 2009.*, edited by Bielawski, Houston and Speight, London Mathematical Society Lecture Note Series: 394, Cambridge University Press, 2012. `arXiv:1106.2086` [math-ph]

[71] Hélein F., *Multisymplectic formulation of Yang-Mills equations and Ehresmann connections*, Advances in Theoretical and Mathematical Physics, Volume 19, Number 4 (2015), 805-835, arXiv:1406.3641,

[72] Hélein F., Kouneiher J., *Finite dimensional Hamiltonian formalism for gauge theory and quantum field theory*, Journal of Mathematical Physics, Volume 43, Number 5, May 2002.

[73] Hélein F., Kouneiher J., *Covariant Hamiltonian formalism for the calculus of variations with several variables: Lepage-Dedecker versus De Donder-Weyl*, Adv. Theor. Math. Phys. 8 (2004), 565-601.

[74] Hélein F., Kouneiher J., *The notion of observable in the covariant Hamiltonian formalism for the calculus of variations with several variables*, Advances in Theoretical and Mathematical Physics 8 (2004), 735-777.

[75] Hélein F., Vey D., *Curved space-times by crystallization of liquid fiber bundles*, (2015), arXiv:1508.07765.

[76] Hernández Ruipérez D., Muñoz Masqué J., *Global variational calculus on graded manifolds, I: graded jet bundles, structure 1-form, and graded infinitesimal contact transformations*, J. Math. Pures et App., 63 (1984), 283-309.

[77] Hernández Ruipérez D., Muñoz Masqué J., *Infinitesimal functoriality of graded Poincaré-Cartan forms*, in Differential Geometric Methods in Theoretical Physics (Shumen, 1984), pp. 126-132, World Sci. Publishing, Singapore, 1986.

[78] Hernández Ruipérez D., Muñoz Masqué J., *Global variational calculus on graded manifolds, II*, J. Math. Pures et App., 64 (1985), 87-104.

[79] Hernández Ruipérez D., Muñoz Masqué J., *Construction intrinséque du faisceau de Berezin d'une variété graduée*, C.R. Acad. Sc. Paris, t. 301, Série I, 20, 1985, pp. 915-918.

[80] Hernández Ruipérez D., Muñoz Masqué J., *Variational Berezinian problems and their relationship with graded variational problems*, In Lecuture Notes in Mathematics, Vol. 1251, P. L. García and A. Pérez-Rendón, eds., Springer-Verlag, New York, (1987), pp. 137-149. Proceedings of the Conference on Differential Geometric Methods in Mathematical Physics, Salamanca, 1985.

[81] Hrabak S. P., *On a Multisymplectic Formulation of the Classical BRST Symmetry for First Order Field Theories Part I: Algebraic Structures*, 1999, arXiv:math-ph/9901012v1.

[82] Hrabak S. P., *On a Multisymplectic Formulation of the Classical BRST Symmetry for First Order Field Theories Part II: Geometric Structures*, 1999, arXiv:math-ph/9901013v1.

[83] Ibort L. A. and Marín-Solano J., *Geometrical foundations of Lagrangian supermechanics and supersymmetry*, Reports on Mathematical Physics, Vol. 32 (1993) No. 3, 385-409.

[84] Ibort A. and Spivak A., *Covariant Hamiltonian field theories on manifolds with boundary: Yang-Mills theories*, 2015; arXiv:1506.00338v2 [math-ph].

[85] Kanatchikov I., *On Field Theoretic Generalizations of a Poisson Algebra*, Rep. Math. Phys. 40 (1997), 225-234, hep-th/9710069.

[86] Kanatchikov I., *Canonical structure of classical field theory in the polymomentum phase space*, Rep. Math. Phys. vol. 41, No. 1 (1998), hep-th/9709229.

[87] Kanatchikov I., *Towards the Born-Weyl quantization of fields*, in *Proceedings of the International Quantum Structures Association 1996 (Berlin)*, Internat. J. Theoret. Phys. 37 (1998), no. 1, 333-342; arXiv:quant-ph/9712058v1.

[88] Kanatchikov I., *Precanonical quantization and the Schrödinger wave functional revisited*, Adv. Theor. Math. Phys. 18 (2014), no. 6, 1249-1265.

[89] Khavkine I., *Covariant phase space, constraints, gauge and the Peierls formula*, arXiv:1402.1282v2 [math-ph].

[90] Khudaverdian O. M., *Geometry of superspace with even and odd brackets*, preprint of the Geneva University, UGVA-DPT 1989/05-613, 1989; published in: J. Math. Phys. 32 (1991), 1934â“1937.

[91] Khudaverdian O. M., *Batalin-Vilkovisky formalism and odd symplectic geometry*, in P.N. Pyatov and S.N. Solodukhin, editors, Proceedings of the Workshop "Geometry and Integrable Models", Dubna, Russia, 4-8 October 1994, World Scientific Publ., (1995). hep-th 9508174.

[92] Khudaverdian H. M. and Neressian A. P., *On the geometry of Batalin-Vilkovisky formalism*, Mod. Phys. Lett. A 8 (1993), 2377-2385.

[93] Khudaverdian H. M. and Voronov T. T., *Differential forms, and odd Symplectic Geometry*, 2006, arXiv:math/0606560v4.

[94] Kijowski J., *A finite-dimensional canonical formalism in the classical field theory*, Comm. Math. Phys. **30** (1973), 99-128.

[95] Kijowski J., *Multiphase spaces and gauge in the calculus of variations*, Bull. de l'Acad. Polon. des Sci., Série sci. Math., Astr. et Phys. XXII (1974), 1219-1225.

[96] Kijowski J. and Moreno G., *Symplectic structures related with higher order variational problems*, Int. J. Geom. Methods Mod. Phys. Vol. 12 (2015), no. 9, 1550084, 45 pp.

[97] Kijowski J. and Szczyrba W., *Multisymplectic Manifolds and the Geometrical Construction of the Poisson Brackets in the Classical Field Theory*, in: J.-M. Souriau (ed.), *Géometrie Symplectique et Physique Mathématique, Colloques Internationaux du CNRS, Aix-en-Provence, 24-28 Juin 1974*, pp. 347-379, C.N.R.S., Paris 1975.

[98] Kijowski J. and Szczyrba W., *A canonical structure for classical field theories*, Comm. Math. Phys. **46** (1976), 183-206.

[99] Kijowski J. and Tulczyjew W. M., *A Symplectic Framework for Field Theories*, Lecture Notes in Physics Vol. 107, Springer-Verlag, Berlin, 1979.

[100] Kosmann-Schwarzbach Y. and Monterde J., *Divergence operators and odd Poisson brackets*, Ann. Inst. Fourier, 52, 419â“456, 2002; arXiv:math.QA/0002209.

[101] Kostant B., *Graded manifolds, graded Lie theory and prequantization*, in K. Bleuler and A. Reetz eds., Differential geometrical methods in mathematical physics : proceedings of the symposium held at the University of Bonn, July 1-4, 1975, Lecture Notes in Mathematics, Vol. 570 (1977), pp. 177-306, Berlin, Springer.

[102] Kourambaeva S. and Shkoller S., *A variational approach to second-order multisymplectic field theory*, Ann. Inst. Fourier, 52, 419â“456, 2002; arXiv:math/9909100v1 [math.DG].

[103] Leites D., *Introduction to the theory of supermanifolds*, Russian Math. Surveys 35 (1980), pp. 1-64.

[104] Lepage T., *Sur les champs géodésiques du calcul des variations*, Bull. Acad. Roy. Belg., Cl. Sci. 27 (1936), 716-729, 1036-1046.

[105] Manin Y., *Gauge Field theory and complex geometry*, Springer-Verlag, 1988.

[106] Marsden J. E., Patrick G. W. and Shkoller S., *Multisymplectic Geometry, Variational Integrators, and Nonlinear PDEs*, Comm. Math. Phys., 199, 351â“395; arXiv:math/9807080v1 [math.DG].

[107] Marsden J. E., Pekarsky S., Shkoller S. and West M., *Variational methods, multisymplectic geometry and continuum Mechanics*, J. Geometry and Physics, 38 (2001), 253â“284; arXiv:math/0005034v1 [math.DG].

[108] Marsden J. E. and Shkoller S., *Multisymplectic Geometry, Covariant Hamiltonians and Water Waves*, Math. Proc. Camb. Phil. Soc., 125 (1999), 553; arXiv:math/9807086v1 [math.DG].

[109] Martin J., *Generalized classical dynamics, and the 'classical analogue' of a Fermi oscillator*, Proceedings of the Royal Society A251 (1959), pp. 536-542.

[110] Martin J., *The Feynman principle for a Fermi system*, Proceedings of the Royal Society A251 (1959), pp. 543-549.

[111] McLean M. and Norris L. K., *Covariant field theory on frame bundles of fibered manifolds*, J. Math. Phys. 41(10) (2000) 6808â“6823.

[112] Molotkov V., *Infinite-dimensional* $\mathbb{Z}_2^k$*-supermanifolds*, ICTP preprints, IC/84/183, 1984. Scanned version available at http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?198506284 .

[113] Monterde J., *Higher order Graded and Berezinian Lagrangian densities and their Euler-Lagrange equations*, Ann. Inst. H. Poincaré Phys. Théor., 57(1) (1992), 3-26.

[114] Monterde J. and Muñoz Masqué J., *Variational problems on graded manifolds*, Contemporary Matematics, Vol. 132 (1992), 551-571.

[115] Monterde J. and Muñoz Masqué J., *Hamiltonian formalism in supermechanics*Int. J. Theoret. Phys., 41(3) (2002), 429-458.

[116] Monterde J., Muñoz Masqué J. and Vallejo J. A., *The Poincaré-Cartan form in superfield theory*, International Journal of Geometric Methods in Modern Physics, Vol. 3, No. 4 (2006), 775-822.

[117] Monterde J. and Vallejo J. A., *The symplectic structure of Euler-Lagrange superequations and Batalin-Vilkovisky formalism* J. Phys. A: Math. Gen., 36 (2003), 4993-5009.

[118] Muñoz Masqué J., *Formes de structure et transformations infinitésimales de contact d'ordre supérieur.*, C.R. Acad. Sc. Paris, t. 298, Série I, 8, 1984, pp. 185-188.

[119] Muñoz Masqué J., *Some applications of graded variational calculus*, IV Meeting in Math. Phys. Coimbra, 1984.

[120] Norris L. K., *Generalized symplectic geometry on the frame bundle of a manifold*, Proc. Symp. Pure Math. 54, Part 2 (Amer. Math. Soc., Providence RI, 1993), 435-465.

[121] Olver P. J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[122] Peierls R. E., *The commutation laws of relativistic field theory*, Proc. Roy. Soc. London, Ser. A, Vol. 214, No. 1117 (1952), 143-157.

[123] Poincaré H., *Les méthodes nouvelles de la mécanique céleste*, t. III, Paris, Gauthier-Villars, 1899.

[124] Prieto-Martínez P. D. and Román-Roy N., *A new multisymplectic unified formalism for second order classical field theories*, 2014; arXiv:1402.4087v3 [math-ph].

[125] Rejzner K., *Batalin-Vilkovisky formalism in locally covariant field theory*, PhD thesis, Hamburg (November 2011); arXiv:1111.5130v2 [math-ph].

[126] Rejzner K., *Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians*, Springer, 2016.

[127] Rey A., Román-Roy N. and Salgado M., *Günther' s formalism (k-symplectic formalism) in classical field theory: Skinner-Rusk approach and the evolution operator*, J. Math. Phys. 46 (2005), 052901.

[128] Rey A., Román-Roy N., Salgado M. and Vilariño S., *k-Cosymplectic Classical Field Theories: Tulckzyjew, Skinner-Rusk and Lie algebroid formulations*, Math. Phys. Anal. Geom. (2012) 15: 85; arXiv:math-ph/0602038v2.

[129] Richter M., *A Lie infinity-algebra of Hamiltonian forms in n-plectic geometry*, 2012; arXiv:1212.4596v2 [math.DG].

[130] Richter M., *Higher symplectic structure on torsionless Lie Rinehart pairs*, 2014; arXiv:1312.7752v2 [math.DG].

[131] Richter M., *Homotopy Poisson-n Algebras from n-plectic structures*, 2015; arXiv:1506.01129v1 [math.DG].

[132] Rogers A., *A global theory of supermanifolds*, J. Math. Phys., 21, No. 6 (1980),1352-1365.

[133] Rogers A., *Supermanifolds: Theory and Applications*, World Scientific, Singapore, 2007.

[134] Rogers C. L., *Higher Symplectic Geometry*, Ph.D. thesis, Univ. of California Riverside, 2001; arxiv:1106.4068 [math-ph].

[135] Román-Roy N., *Multisymplectic Lagrangian and Hamiltonian Formalisms of Classical Field Theories*, Symmetry, Integrability and Geometry: Methods and Applications, SIGMA 5 (2009), 100, 25 pages; arXiv:math-ph/0506022v3.

[136] Rosly A. A., Khudaverdian O. M. and Schwarz A S., *Supersymmetry and complex geometry*, volume 9 of Itogi Nauki i Tekhn. Sovrem. Problemy Matem. Fundam. Napravl. VINITI, Moscow, 1986, 247-284.

[137] Rothstein M. J., *Integration on noncompact supermanifolds*, Trans. Amer. Math. Soc. 299 (1987), no. 1, 387-396.

[138] Sachse C., *A categorical formulation of superalgebra and supergeometry*, 2008, arXiv:0802.4067 [math.AG].

[139] Salles M. O., *Campos Hamiltonianos e Colchete de Poisson na Teoria GeomÂ´etrica dos Campos*, PhD thesis, IME-USP, June 2004; arXiv: .

[140] Sardanashvily G., *Generalized Hamiltonian Formalism for Field Theory. Constraint Systems.*, World Scientific, Singapore, 1995.

[141] Schwarz A. S., *Supergravity, Complex Geometry and G-structures*, Commun. Math. Phys., 87, 37-63, (1982).

[142] Schwarz A. S., *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys., 155(2), 249-260, 1993; arXiv:hep-th/9205098v1 [math.AG].

[143] Schwinger J., *Particles and sources*, Gordon and Breach, New York, 1969.

[144] Segal I., *Quantization of nonlinear systems*, J. Math. Phys. vol. 1, N. 6 (1960), 468-488.

[145] Stolarsky K. B., *Algebraic numbers and Diophantine approximation*, Marcel Dekker Inc., New York, 1974.

[146] Verbovetsky A., *Lagrangian formalism over graded algebras*, J. Geom. Phys. 18 (1996), 195-214, hep-th/9407037 .

[147] Vitagliano L., *Secondary calculus and the covariant phase space*, J. Geom. Phys. 59 (2009), no. 4, 426â“447.

[148] Vitagliano L., *The Lagrangian-Hamiltonian Formalism for Higher Order Field Theories*, Preprint DIPSÂ2/2009, May 2009.

[149] Volkov D. and Akulov V., *Is the neutrino a Goldstone particle?*, Physics Letters B 46 (1973), pp. 109-110.

[150] Volterra V., *Sulle equazioni differenziali che provengono da questioni di calcolo delle variazioni*, Rend. Cont. Acad. Lincei, ser. IV, vol. VI, 1890, 42-54.

[151] Volterra V., *Sopra una estensione della teoria Jacobi-Hamilton del calcolo delle variazioni*, Rend. Cont. Acad. Lincei, ser. IV, vol. VI, 1890, 127-138.

[152] Voronov Th., *Geometric integration theory on supermanifolds*, Sov. Sci. Rev. C. Math. Phys., Vol. 9, pp.1-138. Harwood Academic Publ. (1992).

[153] Voronov Th., *Supermanifold forms and integration. A dual theory*, in V.M.Buchstaber and S.P.Novikov, editors, Solitons, Geometry and Topology: On the Crossroad, pages 153-172. AMS Translations, ser. 2, vol. 179, 1997. Advances in the Math. Sciences. dg-ga/9603009.

[154] Voronov Th., *Dual Forms on Supermanifolds and Cartan Calculus*, Comm. Math. Phys., June 2002, Vol. 228, Issue 1, pp 1-16. arXiv:math/9912192v1

[155] Voronov Th. and Zorich A., *Complexes of forms on supermanifolds*, Funktsion. Anal. Pril. 20 (2) (1986), 58-59.

[156] Voronov Th. and Zorich A., *Integral transformations of pseudodifferential forms*, Uspekhi Mat. Nauk 41 (6) (1986), 167-168.

[157] Voronov Th. and Zorich A., *Bordism theory and homotopy properties of supermanifolds*, Funktsion. Anal. Pril. 21 (3) (1988), 77-78.

[158] Voronov Th. and Zorich A., *Cohomology of supermanifolds and integral geometry*, Doklady Akademii Nauk SSSR 298 (3) (1988), 528-533.

[159] Wess J. and Zumino B., *Supergauge transformations in four dimensions*, Nuclear Physics B 70 (1974), p. 39.

[160] Weyl H., *Geodesic fields in the calculus of variation for multiple integrals*, Ann. Math. (3) 36 (1935), 607-629.

[161] Witten E., *Interacting field theory of open superstrings*, Nucl. Phys., B276 (1986), 291.

[162] Witten E., *A note on the antibracket formalism*, Mod. Phys. Lett. A5 (1990), 487-494.

[163] Witten E., *Notes On Supermanifolds And Integration*, 2012, arXiv:1209.21996v2 [hep.th].

[164] Zuckerman G., *Action functional and global geometry*, in Mathematical aspects of string theory, S.T. Yau, eds., Advanced Series in Mathematical Physics, vol 1, World Scientific, 1987, 259-284.