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Xiaoli WEI

Problèmes de contrôle de type McKean-Vlasov et applications

Control of McKean-Vlasov systems and applications

Dirigée par Pr. **Huyên PHAM**

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Jean-François CHASSAGNEUX	Université Paris Diderot	examinateur
François DELARUE	Université Nice-Sophia Antipolis	rapporteur & examinateur
Xin GUO	University of California, Berkeley	rapporteure
Idris KHARROUBI	Sorbonne Université, UPMC	examinateur
Huyên PHAM	Université Paris Diderot	directeur
Agnès SULEM	Centre de Recherche Inria de Paris	examinatrice
Denis TALAY	Inria Sophia-Antipolis	président du jury
Xiaolu TAN	Université Paris Dauphine	examinateur

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Résumé

Cette thèse étudie le contrôle optimal de la dynamique de type McKean-Vlasov et ses applications en mathématiques financières. La thèse contient deux parties.

Dans la première partie, nous devéloppons la méthode de la programmation dynamique pour résoudre les problèmes de contrôle stochastique de type McKean-Vlasov. En utilisant les contrôles admissibles appropriés, nous pouvons reformuler la fonction valeur en fonction de la loi (resp. la loi conditionnelle) du processus comme seule variable d'état et obtenir la propriété du flot de la loi (resp. la loi conditionnelle) du processus, qui permettent d'obtenir en toute généralité le principe de la programmation dynamique. Ensuite nous obtenons l'équation de Bellman correspondante, en s'appuyant sur la notion de différentiabilité par rapport aux mesures de probabilité introduite par P.L. Lions [Lio12] et la formule d'Itô pour le flot de probabilité. Enfin nous montrons la propriété de viscosité et l'unicité de la fonction valeur de l'équation de Bellman. Dans le premier chaptire, nous résumons quelques résultats utiles du calcul différentiel et de l'analyse stochastique sur l'espace de Wasserstein. Dans le deuxième chapitre, nous considérons le contrôle optimal stochastique de système à champ moyen non linéaire en temps discret. Le troisème chapitre étudie le problème de contrôle optimal stochastique d'EDS de type McKean-Vlasov sans bruit commun en temps continu où les coefficients peuvent dépendre de la loi joint de l'état et du contrôle, et enfin dans le dernier chapitre de cette partie nous nous intéressons au contrôle optimal de la dynamique stochastique de type McKean-Vlasov en présence de bruit commun en temps continu.

Dans la deuxième partie, nous proposons un modèle d'allocation de portefeuille robuste permettant l'incertitude sur la rentabilité espérée et la matrice de corrélation des actifs multiples, dans un cadre de moyenne-variance en temps continu. Ce problème est formulé comme un jeu différentiel à champ moyen. Nous montrons ensuite un principe de séparation pour le problème associé. Nos résultats explicites permettent de justifier quantativement la sous-diversification, comme le montrent les études empiriques.

Mots-clefs

Équation de type McKean-Vlasov, EDS de type McKean-Vlasov, programmation dynamique, espace de Wasserstein, équation de Bellman, solution de viscosité , problème de Markowitz en temps continu, incertitude sur les modèles, drift et corrélation ambiguës, principe de séparation, sous-diversification

Abstract

This thesis deals with the study of optimal control of McKean-Vlasov dynamics and its applications in mathematical finance. This thesis contains two parts.

In the first part, we develop the dynamic programming (DP) method for solving McKean-Vlasov control problem. Using suitable admissible controls, we propose to reformulate the value function of the problem with the law (resp. conditional law) of the controlled state process as sole state variable and get the flow property of the law (resp. conditional law) of the process, which allow us to derive in its general form the Bellman programming principle. Then by relying on the notion of differentiability with respect to probability measures introduced by P.L. Lions [Lio12], and Itô's formula along measure-valued processes, we obtain the corresponding Bellman equation. At last we show the viscosity property and uniqueness of the value function to the Bellman equation. In the first chapter, we summarize some useful results of differential calculus and stochastic analysis on the Wasserstein space. In the second chapter, we consider the optimal control of nonlinear stochastic dynamical systems in discrete time of McKean-Vlasov type. The third chapter focuses on the stochastic optimal control problem of McKean-Vlasov SDEs without common noise in continuous time where the coefficients may depend upon the joint law of the state and control. In the last chapter, we are interested in the optimal control of stochastic McKean-Vlasov dynamics in the presence of common noise in continuous time.

In the second part, we propose a robust portfolio selection model, which takes into account ambiguity about both expected rate of return and correlation matrix of multiply assets, in a continuous-time mean-variance setting. This problem is formulated as a mean-field type differential game. Then we derive a separation principle for the associated problem. Our explicit results provide an explanation to under-diversification, as documented in empirical studies.

Keywords

McKean-Vlasov equation, McKean-Vlasov SDEs, dynamic programming, Wasserstein space, Bellman equation, viscosity solution, continuous-time Markowtiz problem, model uncertainty, ambiguous drift and correlation, separation principle, under-diversification

Contents

Notations

I. Sets

- ⋄ $\mathbb{N}$ is the set of nonnegative integers.
- ⋄ $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ the set of positive real numbers.
- ⋄ $x.y$ denotes the scalar product of two Euclidian vectors x and y.
- ⋄ We denote by $\|\cdot\|_2$ the L_2-norm (Euclidian norm) in $\mathbb{R}^d$:

$$\|a\|_2 \quad = \quad \sqrt{a.a}.$$

- ⋄ For any matrix or vector M, we denote its transpose by $M^\intercal$ and its trace by $\mathrm{tr}(M)$. We denote identity matrix in $\mathbb{R}^{d\times d}$ by $I_{d\times d}$ and zero matrix in $\mathbb{R}^{d\times q}$ by $\mathbf{0}_{d\times q}$.
- ⋄ $\mathbb{S}^d$ is the set of symmetric matrices in $\mathbb{R}^{d\times d}$, $\mathbb{S}^d_+$ the set of nonnegative symmetric matrices in $\mathbb{R}^{d\times d}$ and $\mathbb{S}^d_{>+}$ the set of positive definite symmetric matrices in $\mathbb{R}^{d\times d}$. We define the order on $\mathbb{S}^d_{>+}$ as

$$A < B \Longleftrightarrow B - A \in \mathbb{S}^d_{>+}.$$

II. Functions

- ⋄ $1_A(x)$ is the indicator function on the set A.
- ⋄ Given a continuously differentiable function f on $\mathbb{R}^d$, we denote by ∇f the gradient vector in $\mathbb{R}^d$ with components $\frac{\partial f}{\partial x_i}$, $1 \le i \le d$.
- ⋄ $C([t,s],E)$ is the set of continuous functions from the interval $[t,s]$, $t < s$, into some Euclidian space E. The interval $[t,s]$ is often $\mathbb{R}_+$ or $\mathbb{R}$.

III. Integration and probability

Given a topological space E and a probability space $(\Omega,\mathcal{F},\mathbb{P})$

- ⋄ $\mathcal{B}(E)$: Borelian σ-filed generated by the open subsets of E.
- ⋄ δ_x denotes the Dirac measure at x. $\delta_x(B) = 1_B(x)$ for any $B \in \mathcal{B}(E)$.
- ⋄ $\mathbb{P}_X$ is the probability measure of random variables X on $(\Omega,\mathcal{F},\mathbb{P})$ under $\mathbb{P}$ and $\mathcal{L}(X)$ the law of X. $\mathbb{P}_{(X,Y)}$ is the joint distribution of random variables X and Y on $(\Omega,\mathcal{F},\mathbb{P})$.
- ⋄ $L^2(\Omega,\mathcal{F},\mathbb{P};E)$ is the space of square integrable random variable X, valued in $\mathbb{E}$, $\mathcal{F}$-measurable and

such that $\mathbb{E}|X|^2 < \infty$. We sometimes omit some arguments and write $L^2(\mathcal{F}; E)$ when there is no ambiguity.

- $\mathcal{P}(E)$, denotes the set of all probability measures on measurable space $(E, \mathcal{B}(E))$.
- $\mathcal{P}_p(E)$, $p \geq 1$, denotes the set of all probability measures on measurable space $(E, \mathcal{B}(E))$ with a finite p-order moment, i.e. $\mu \in \mathcal{P}(E)$, and the p-norm of μ: $\|\mu\|_p := (\int_E |x|^p)^{1/p} < \infty$. We often call $\mathcal{P}_p(E)$ Wasserstein space.
- For any $\mu \in \mathcal{P}_2(E)$, F Euclidian space(often $\mathbb{R}^q$ in our context), we denote by $L^2_\mu(F)$ the set of measurable functions $\varphi : E \to F$ which are square integrable with respect to μ, by $L^2_{\mu\otimes\mu}(F)$ the set of measurable functions $\psi : \mathbb{R}^d \times \mathbb{R}^d \to F$, which are square integrable with respect to the product measure $\mu \otimes \mu$, and we set

$$\mu(\varphi) \ = < \varphi, \mu >:= \int_{\mathbb{R}^d} \varphi(x)\, \mu(dx), \qquad \mu \otimes \mu(\psi) \ := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, x')\mu(dx)\mu(dx').$$

  We also define $L^\infty_\mu(F)$ (resp. $L^\infty_{\mu\otimes\mu}(F)$) as the subset of elements $\varphi \in L^2_\mu(F)$ (resp. $L^2_{\mu\otimes\mu}(F)$) which are bounded μ (resp. $\mu \otimes \mu$) a.e., and $\|\varphi\|_\infty$ is their essential supremum.
- f:$E \to F$ is a measurable mapping, we denote pushforward measure by $f \star \mu$: $f \star \mu(B) = \mu(f^{-1}(B))$, $\forall\ B \in \mathcal{B}(F)$.
- $\mathcal{P}$ is a family of probability measures. We say that a property holds $\mathcal{P}$ quasi-surely ($\mathcal{P}$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$.

Introduction

The main objective of this Ph.D. thesis is to investigate the optimal control of McKean-Vlasov systems and give several applications.

In the first part, we analyze in detail the DP method for the optimal control of McKean-Vlasov systems. Due to the dependence of the coefficients on the law of the state process, DP requires adaptation. Therefore, the key idea is to reformulate the problem as a distributed control problem with the probability distribution (resp. conditional probability distribution) of the controlled state process as sole state variable. Then we state a dynamic programming principle (DPP) for the value function in the space of probability measures, which follows from a flow property of the law (resp. conditional law) of the controlled state process. Moreover by relying on Wasserstein results mentioned in Chapter 1, we derive the Bellman equation, and prove the viscosity property together with a uniqueness result for the value function. We also deduce a verification theorem when the value function is smooth, and solve explicitly the LQ McKean-Vlasov control problem with applications to the mean-variance portfolio selection and an interbank systemic risk model. In Chapter 1, we present some results of Wasserstein differential calculus such as the notion of differentiability with respect to measures due to P.L. Lions and Itô's formula along a flow of probability measures. In Chapter 2, we are interested in the optimal control of McKean-Vlasov systems in discrete time. Chapter 3 is about the optimal control of McKean-Vlasov SDEs in continuous time. In Chapter 4, we study the optimal control of stochastic SDEs of McKean-Vlasov type in the presence of a common noise.

In the second part, we build a framework for the robust continuous-time mean-variance portfolio selection where the model uncertainty carries on both expected rate of return and correlation matrix of multiply risky assets. It is formulated as a McKean-Vlasov control problem under model uncertainty and solved by a weak version of martingale optimality principle. We deduce a separation principle for this associated robust problem, which allows to reduce the computation of robust optimal portfolio strategy to the parametric infimum computation of the risk premium function. We quantify explicitly the diversification effects in terms of Sharpe ratio and correlation parameter. In particular, our findings consist in no trading in risky assets with large expected return ambiguity and trading only one risky asset with high level of ambiguity about correlation.

0.1 Part I: The optimal control of McKean-Vlasov dynamics

McKean-Vlasov SDEs, whose coefficients may depend on the marginal distributions of the solutions, have a long history with the work initialed by [Kac56] and [McK67]. However, the optimal control of McKean-Vlasov dynamics, also called mean field type control problem, has notoriously been ignored in standard stochastic control literature for a while. Recently, it has been known a surge of renewed interest since the emergence of the MFG proposed by Lasry and Lions [LL07] and simultaneously by Huang et al. [HMC06].

The study of optimal control of McKean-Vlasov dynamics is motivated by the description of the asymptotic behavior of a large but finite numbers of players (financial agents, firms) with mean field interactions, which can be described loosely as follows: imagine there are N state processes interacting through their empirical measures via the following SDE system:

$$\begin{aligned} dX_t^i &= b(X_t^i, \bar{\rho}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\rho}_t^N, \alpha_t^i)dB_t^i + \sigma_0(X_t^i, \bar{\rho}_t^N, \alpha_t^i)dW_t^0, \\ \bar{\rho}_t^N &= \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}. \end{aligned}$$

Here W^0, B^1, …, B^N are independent Brownian motions, and α^i is the control of player i. As the player i feels only B^i directly , we call B^1, …, B^N the idiosyncratic noises, and we call W^0 common noise, since each player feels W^0 equally. The cost functional associated to the player i of the strategy profile $(\alpha^1, \dots, \alpha^N)$ is

$$J^i(\alpha^1, \dots, \alpha^N) \quad = \quad \mathbb{E}\Big[\int_0^T f(X_t^i, \bar{\rho}_t^N, \alpha_t^i) + g(X_T^i, \bar{\rho}_T^N)\Big].$$

As the number of players increases, this system of N-player games with the interaction of mean field type is in high dimension of order N and not tractable in general. The required computational effort for equilibrium typically involves a system of N PDEs and is usually prohibitively large. From a practical point of view, one tends to study the asymptotic behavior of N-player games. The hope is to reduce the analysis of the whole system to the analysis of a single representative player from the theory of propagation of chaos. A natural question is in which sense this N-player stochastic differential games as $N \to \infty$ converge.

Two different notions of equilibrium may be considered. One notion is Nash equilibrium, which usually takes two forms: open-loop Nash equilibrium and closed-loop Nash equilibrium. When the N-player equilibrium is open-loop, compactness arguments yield that every limit point of N-player Nash equilibrium is characterized as a weak solution of MFG, formulated as a standard control problem and a fixed point problem over the space of probability measures, see [CD18, Vol II, Chapter 6], [Lac16]; when the N-player equilibrium is closed-loop, the convergence to the MFG equilibrium is known in [CD18, Vol II, Chapter 6] when the associated Master equation has a unique smooth solution. There are two approaches to solve MFG. In the original work [LL07], Lasry and Lions followed an analytic PDE approach. The recent book by Carmona and Delarue [CD18, Vol II, Chapter 6] offers a comprehensive treatment of the probabilistic approach to MFG. It brings numerous applications in economics and finance such as [GLL11], [CFS15] among others.

The other notion is Pareto optimality where a social player chooses the control policy to minimize the averaged objective

$$J(\alpha^1, \dots, \alpha^N) := \frac{1}{N}\sum_{i=1}^N J^i(\alpha^1, \dots, \alpha^N).$$

In [Lac17], the authors restricted themselves to the case without common noise and introduced the relaxed formulation of McKean-Vlasov control problem, meaning that the state equation for N-player games and the McKean-Vlasov control problem are formulated as controlled martingale problems and with relaxed (i.e. measure-valued) controls. They showed that the empirical measure sequence $(\bar{\rho}_t^N)_{t\in[0,T]}$ of near-optimally controlled N-player games admits in distribution limit points, and that every limit point is supported on the set of measure flows $(\mathbb{P}_{X_t})_{t\in[0,T]}$, where X is an optimally controlled state in the McKean-Vlasov control problem. In particular, whenever the McKean-Vlasov control problem admits a unique optimal control, this is a proper convergence theorem, i.e. propagation of chaos. The strong formulation (compared to the relaxed formulation in [Lac17]) of the McKean-Vlasov control problem is given by

$$dX_t \quad = \quad b(X_t, \mathbb{P}_{X_t}^{W^0}, \alpha_t)dt + \sigma(X_t, \mathbb{P}_{X_t}^{W^0}, \alpha_t)dB_t + \sigma_0(X_t, \mathbb{P}_{X_t}^{W^0}, \alpha_t)dW_t^0,$$

where $\mathbb{P}_{X_t}^{W^0}$ is the distribution of X_t given W^0, and the objective of the representative player is the minimization of the cost functional

$$J(\alpha) \quad := \quad \mathbb{E}\Big[\int_0^T f\big(X_t^\alpha, \mathbb{P}_{X_t^\alpha}^{W^0}, \alpha_t\big)dt + g\big(X_T, \mathbb{P}_{X_T}^{W^0}\big)\Big].$$

The similarities and differences between MFG and optimal control of McKean-Vlasov dynamics have been discussed in [CDLL15] and [CD18, Vol I, Chapter 6]. Being a counterpart of MFG, the optimal control of McKean-Vlasov dynamics also has numerous potential applications in several areas besides the traditional domain of statistical physics, like economics and finance, social interactions, engineering and so on. Actually, its applications in economics and finance include a portfolio liquidation problem with trade crowding and a substitutable production goods model in [BP17], an interbank systematic risk with partial observation model in [BHL+18] and a problem of interaction between customers and firms on renewable energy [ABP17].

In the literature there are essentially two approaches to solve the optimal control of McKean-Vlasov dynamics. The first, more probabilistic, approach is Pontryagain maximum principle. The early attempts at tackling this problem with maximum principle focused on a class of models where the dynamics depends solely upon moments of the distribution, see [AD10], [BDL11] and [BFY13]. By taking good advantage of Lion's differential calculus over the Wasserstein space, [CD15] and [CD18] developed an appropriate version of Pontryagin maximum principle in its full generality. They introduced adjoint processes by differentiating the Hamiltonian function with respect to both the state and its marginal distribution. Then they characterized the optimal trajectories as the solutions of a forward-backward SDE of McKean-Vlasov type.

The second approach is DPP. An important feature of DPP is that it does not require any convexity on the coefficients. However, the difficulty of using DPP lies in the fact that (i) the dynamics described by the McKean-Vlasov SDEs is non-Markovian and (ii) the dependence of the cost functional on the law of the state induces time inconsistency. For example, in the mean-variance problem, the optimal control at time t is precommittment, meaning that the solution depends not only on the current state but also on the initial state, see [LZ00], [BC10]. To restore time consistency, [LP14] and [BFY15] used the closed-loop feedback controls and reformulated the value function into the deterministic distributed control problem together with a Fokker-Planck equation. Then they derived the dynamic programming equation by calculus of variation. However, their work relies heavily on the assumption that the probability measure admits at all times square integrable probability density function.

Inspired by the works [LP14] and [BFY15], we drop the density assumption and extend the DP approach of these two papers over the McKean-Vlasov control problem, in discrete time and continuous time, with and without common noise developed below.

0.1.1 Some differential calculus on Wasserstein space

In Chapter 1, we review the Wasserstein space and its topological structure, which will be often used implicitly in the remaining chapters. Then we recall the notion of differentiability on the Wasserstein space introduced by Lions in his seminar lectures at Collège de France [Lio12]: structure of the derivative of first order and second order. Finally, we recall the Itô's formula along a flow of deterministic or random probability measure and their lifted form, which will be useful for obtaining HJB equation and viscosity characterization of the value function.

0.1.2 Dynamic programming for discrete time McKean-Vlasov control problem

In Chapter 2, we are interested in the optimal control of McKean-Vlasov equations in discrete time. It can be regarded as an approximation of the continuous time problem discussed in the next chapter.

We are given two normed spaces $(E, |.|)$ and $(A, |.|)$ representing respectively the state space and the control space. The dynamics of state valued in E (typically $\mathbb{R}^d$) is described by the equation of McKean-Vlasov type

$$X_{k+1}^\alpha \;=\; F_{k+1}(X_k^\alpha, \mathbb{P}_{X_k^\alpha}, \alpha_k, \mathbb{P}_{\alpha_k}, \varepsilon_{k+1}), \quad k \in \mathbb{N}, \quad X_0^\alpha \;=\; \xi,$$

for some deterministic measurable functions F_k defined from $E \times \mathcal{P}(E) \times A \times \mathcal{P}(A) \times \mathbb{R}^d$ to E, where $(\xi_k)_k$ is a sequence of i.i.d random variables, independent of the square integrable initial random value ξ, and we denote by $\mathbb{F} = (\mathcal{F}_k)_k$ the filtration generated by $\{\xi, \xi_1, \ldots, \xi_k\}$. The control $(\alpha_k)_k$ is a square integrable $\mathbb{F}$-adapted process valued in A (typically a subset of $\mathbb{R}^m$). For any $(x, \mu, a, \lambda) \in E \times \mathcal{P}(E) \times A \times \mathcal{P}(A)$, we denote by $P_{k+1}(x, \mu, a, \lambda, dx')$ the probability distribution of random variables $F_{k+1}(x, \mu, a, \lambda, \xi_{k+1})$. The cost functional over a finite horizon $n \in \mathbb{N} \setminus \{0\}$ for a control process α, is

$$J(\alpha) \;=\; \mathbb{E}\Big[\sum_{k=0}^{n-1} f_k(X_k^\alpha, \mathbb{P}_{X_k^\alpha}, \alpha_k, \mathbb{P}_{\alpha_k}) \;+\; g(X_n^\alpha, \mathbb{P}_{X_n^\alpha})\Big],$$

where $f_k : E \times \mathcal{P}_2(E) \times A \times \mathcal{P}_2(A) \to \mathbb{R}$ and $g : E \times \mathcal{P}_2(E) \to \mathbb{R}$ are measurable real-valued functions with square growth in each of their inputs. The objective is to minimize over all admissible controls the cost functional

$$V_0 \;=\; \inf_\alpha J(\alpha).$$

What does the admissible control set look like? In our context, we restrict our attention to controls α in feedback form

$$\alpha_k \;=\; \tilde{\alpha}_k(X_k^\alpha), \quad \text{for} \quad \tilde{\alpha}_k \in A^E, \;\; k = 0, \ldots, n-1, \tag{0.1.1}$$

where A^E is the set of deterministic measurable functions: $E \to A$, satisfying a linear growth condition. Then the admissible control set $\mathcal{A}$ consists of all $\mathbb{F}$-adapted square integrable controls α given by (0.1.1). Note that by Fubini's theorem, we rewrite the cost functional as, for $\alpha \in \mathcal{A}$

$$J(\alpha) \;=\; \sum_{k=0}^{n-1} \hat{f}_k(\mathbb{P}_{X_k^\alpha}, \tilde{\alpha}_k) + \hat{g}(\mathbb{P}_{X_n^\alpha}),$$

where the functions $\hat{f}_k$, $k = 0, \ldots, n-1$, defined on $\mathcal{P}_2(E) \times A^E$ and $\hat{g}$ defined on $\mathcal{P}_2(E)$ are given by:

$$\hat{f}_k(\mu, \tilde{\alpha}) := \int_E f_k(x, \mu, \tilde{\alpha}(x), \tilde{\alpha} \star \mu)\mu(dx), \qquad \hat{g}(\mu) := \int_E g(x, \mu)\mu(dx).$$

We transform the initial stochastic control problem into a deterministic control problem involving only the marginal distribution of the state process $\mathbb{P}_{X_k^\alpha}$, and then define the dynamic value processes V_k^α at time $k = 0, \ldots, n$,

$$V_k^\alpha := \inf_{\beta \in \mathcal{A}_k(\alpha)} \sum_{j=k}^{n-1} \hat{f}_j(\mathbb{P}_{X_j^\beta}, \tilde{\beta}_j) + \hat{g}(\mathbb{P}_{X_n^\beta}), \quad \text{for } \alpha \in \mathcal{A},$$

where $\mathcal{A}_k(\alpha) = \{\beta \in \mathcal{A} : \beta_j = \alpha_j, j = 0, \ldots, k-1\}$, with the convention that $\mathcal{A}_0(\alpha) = \mathcal{A}$. Now the initial value function $V_0 = \inf_{\alpha \in \mathcal{A}} J(\alpha) = V_0^\alpha$. When $V_k^\alpha > -\infty$, it is elementary to obtain the DPP for V_k^α, which takes the following form

$$\begin{cases} V_n^\alpha = \hat{g}(\mathbb{P}_{X_n^\alpha}) \\ V_k^\alpha = \inf_{\beta \in \mathcal{A}_k(\alpha)} \hat{f}_k(\mathbb{P}_{X_k^\beta}, \tilde{\beta}_k) + V_{k+1}^\beta, \quad k = 0, \ldots, n-1. \end{cases} \tag{0.1.2}$$

Actually, the value processes V_k^α, $k = 0, \ldots, n-1$, are deterministic measurable functions of $\mathbb{P}_{X_k^\alpha}$. The DPP (0.1.2) is reduced to the recursive computation of a sequence of deterministic functions $(v_k)_k$ (called value functions) on $\mathcal{P}(E)$, which is described in the following Theorem.

Theorem 0.1.1. *When $V_k^\alpha > -\infty$, we have for any $\alpha \in \mathcal{A}$, $V_k^\alpha = v_k(\mathbb{P}_{X_k^\alpha})$, $k = 0, \ldots, n$, where $(v_k)_k$ is the sequence of value functions defined recursively on $\mathcal{P}_2(E)$ by:*

$$\begin{cases} v_n(\mu) = \hat{g}(\mu) \\ v_k(\mu) = \inf_{\tilde{\alpha} \in A^E} \Big[\hat{f}_k(\mu, \tilde{\alpha}) + v_{k+1}\big(\Phi_{k+1}(\mu, \tilde{\alpha})\big)\Big] \end{cases} \tag{0.1.3}$$

for $k = 0, \ldots, n-1$, $\mu \in \mathcal{P}_2(E)$, where Φ_{k+1} is the measurable function $\mathcal{P}_2(E) \times A^E \to \mathcal{P}_2(E)$ defined by

$$\Phi_{k+1}(\mu, \tilde{\alpha})(dx') = \int_E \mu(dx) P_{k+1}(x, \mu, \tilde{\alpha}(x), \tilde{\alpha} \star \mu, dx').$$

The proof of Theorem (0.1.1) is an immediate consequence of backward induction and the flow property of controlled marginal distribution $\mathbb{P}_{X_k^\alpha}$ given by

$$\mathbb{P}_{X_{k+1}^\alpha} = \Phi_{k+1}\big(\mathbb{P}_{X_k^\alpha}, \tilde{\alpha}_k\big), \quad k \in \mathbb{N}, \ \mathbb{P}_{X_0^\alpha} = \mathbb{P}_\xi.$$

In view of the DPP (0.1.3), the verification theorem not only asserts that a solution to (0.1.3) equals the value functions $(v_k)_k$ but also provides a sufficient condition for a feedback optimal control realizing the infimum of (0.1.3).

We then test verification results on several cases: (i) the case without mean-field interaction; in this case, the infimum in (0.1.3) can be reduced to the infimum on the finite-dimensional control space. (ii) the case where the interaction is of order 1 (integral of function with respect to probability measure); in this case, the value functions are defined on n-tuples and thus the size of this problem is reduced. (iii) finally, the case of multivariate LQ McKean-Vlasov type (the coefficients only depend on the expectation and variance of the state) has an explicit solution under suitable coercivity assumptions. DPP provides another approach to the problem of LQ McKean-Vlasov control solved in [ELN13] via four different approaches.

0.1.3 Bellman equation and viscosity solutions for continuous time McKean-Vlasov control problem

In Chapter 3, we focus on the optimal control of McKean-Vlasov dynamics in continuous time, which can be viewed as the continuous version of the discrete time problem in the previous chapter. Our framework takes into account a more general class allowing for the mean-field interactions through controls, in addition to the mean-field interactions through states.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, $B = (B_t)_{t\geq 0}$ a n-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the filtration generated by B, and $\mathcal{F}_0$ a sub-σ-algebra of $\mathcal{F}$ independent of B. We assume that $\mathcal{F}_0$ is rich enough in the sense that $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi, \xi \in L^2(\mathcal{F}_0; \mathbb{R}^d)\}$. Then, the controlled McKean-Vlasov SDE is given by

$$dX_t^\alpha \;=\; b(t, X_t^\alpha, \alpha_t, \mathbb{P}_{(X_t^\alpha,\alpha_t)})dt + \sigma(t, X_t^\alpha, \alpha_t, \mathbb{P}_{(X_t^\alpha,\alpha_t)})dB_t, \;\; X_0^\alpha \;=\; X_0 \in L^2(\mathcal{F}_0, \mathbb{R}^d), \quad (0.1.4)$$

where the control process α is progressively measurable, square integrable with values in a subset A of $\mathbb{R}^m$, and $\mathbb{P}_{(X_t^\alpha,\alpha_t)}$ the joint distribution of the state X_t^α and the control α_t. On the coefficients b: $[0, T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R}^{d\times d}$, we impose standard lipschitz and linear growth conditions, which guarantee the existence and uniqueness of a square integrable solution to (0.1.4). The control problem consists in minimizing over all admissible control processes the following functional,

$$J(\alpha) \;:=\; \mathbb{E}\Big[\int_0^T f(t, X_t, \alpha_t, \mathbb{P}_{(X_t,\alpha_t)})dt + g(X_T, \mathbb{P}_{X_T})\Big], \quad (0.1.5)$$

where the deterministic measurable functions $f : [0, T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R}$ and $g : R^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfy the quadratic growth conditions.

Even though the probability measure is deterministic, the dynamics X_t^α described by the controlled McKean-Vlasov SDE is genuinely non-Markovian. Like in the previous chapter, we restrict ourselves to admissible controls $\alpha \in \mathcal{A}$ in feedback form:

$$\alpha_t \;=\; \tilde{\alpha}(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}), \quad (0.1.6)$$

for some deterministic measurable function $\tilde{\alpha} : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to A$ which is linear growth and lipschitz in $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. We consider the dynamic version of (0.1.4) starting from $\xi \in L^2(\mathcal{F}_t, \mathbb{R}^d)$ at time $t \in [0, T]$, and written as

$$\begin{aligned} X_s^{t,\xi} \;=\; \xi \;+\; & \int_t^s b(r, X_r^{t,\xi}, \tilde{\alpha}(r, X_r^{t,\xi}, \mathbb{P}_{X_r^{t,\xi}}), Id\tilde{\alpha}(r, ., \mathbb{P}_{X_r^{t,\xi}}) \star \mathbb{P}_{X_r^{t,\xi}})dr \quad (0.1.7)\\ &+ \int_t^s \sigma(r, X_r^{t,\xi}, \tilde{\alpha}(r, X_r^{t,\xi}, Id\tilde{\alpha}(r, ., \mathbb{P}_{X_r^{t,\xi}}) \star \mathbb{P}_{X_r^{t,\xi}})dB_r, \quad t \leq s \leq T. \end{aligned}$$

for any $\tilde{\alpha} \in L(\mathbb{R}^d; A)$, where the map $\mathrm{Id}\tilde{\alpha} : \mathbb{R}^d \to \mathbb{R}^d \times A$ is defined by $Id\tilde{\alpha}(x) := (x, \tilde{\alpha}(x))$, and $L(\mathbb{R}^d; A)$ is the set of Lipschitz functions from $\mathbb{R}^d$ into A. The existence and uniqueness of a solution to (0.1.7) yield the flow property of $X_s^{t,\xi}$

$$X_s^{t,\xi} \;=\; X_s^{\theta, X_\theta^{t,\xi}}, \quad \forall\, 0 \leq t \leq \theta \leq s \leq T,\; \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d).$$

Notice that the law distribution of $X_s^{t,\xi}$ depends on ξ only through its law $\mathbb{P}_\xi$. We are then allowed to define

$$\mathbb{P}_s^{t,\mu} \;:=\; \mathbb{P}_{X_s^{t,\xi}}, \quad \text{for} \quad 0 \leq t \leq s \leq T, \;\mu \;=\; \mathbb{P}_\xi \;\in\; \mathcal{P}_2(\mathbb{R}^d),$$

and immediately get the flow property for the marginal distribution process:

$$\mathbb{P}^{t,\mu}_s \;=\; \mathbb{P}^{\theta,\mathbb{P}^{t,\mu}_\theta}_s, \quad \forall\, 0 \le t \le \theta \le s \le T,\; \mu \in \mathcal{P}_2(\mathbb{R}^d). \tag{0.1.8}$$

Therefore, we now rewrite the cost functional (0.1.5) in the deterministic form in terms of marginal distribution in $\mathcal{P}_2(\mathbb{R}^d)$ as sole state variable:

$$J(\alpha) \;=\; \int_0^T \hat f(t, \mathbb{P}_{X_t}, \tilde\alpha(t,.,\mathbb{P}_{X_t}))dt + \hat g(\mathbb{P}_{X_T}),$$

where $\hat f$: $[0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times L(\mathbb{R}^d;A) \to \mathbb{R}$ and $\hat g$: $\mathcal{P}_2(\mathbb{R}^d) \to R$ are defined by

$$\hat f(t,\mu,\tilde\alpha) \;:= < f(t,.,\tilde\alpha(.), Id\tilde\alpha \star \mu), \mu >, \qquad \hat g(\mu) \;:= < g(.,\mu), \mu > .$$

Now it is natural to define the dynamic cost functional and value function from the flow property of the marginal distribution process (0.1.8)

$$\begin{aligned} J(t,\mu,\alpha) &= \Big[\int_t^T \hat f(s, \mathbb{P}^{t,\mu}_s, \tilde\alpha(s,.,\mathbb{P}^{t,\mu}_s))ds + \hat g(\mathbb{P}^{t,\mu}_T)\Big], \\ v(t,\mu) &= \inf_{\alpha\in\mathcal{A}} J(t,\mu,\alpha) \quad t \in [0,T],\; \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{aligned} \tag{0.1.9}$$

Since (0.1.9) is a deterministic control problem that does not involve any measurable selection arguments, we obtain immediately the DPP.

Theorem 0.1.2. *When $v(t,\mu) > -\infty$, we have for all $0 \le t \le \theta \le T$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:*

$$v(t,\mu) \;=\; \inf_{\alpha\in\mathcal{A}} \Big[\int_t^\theta \hat f(s, \mathbb{P}^{t,\mu}_s, \tilde\alpha(s,.,\mathbb{P}^{t,\mu}_s))ds \;+\; v(\theta, \mathbb{P}^{t,\mu}_\theta)\Big]. \tag{0.1.10}$$

In order to obtain a PDE from the DPP (0.1.10), we follow Lion's approach to the differential calculus on the Wasserstein space first introduced in [Lio12], then detailed in [Car12] and [CD18, Vol I, Chapter 5], and Itô's formula for functions of deterministic measure-valued processes in [BLPR17], [CCD15] and [CD18, Vol I, Chapter 5]. The Bellman equation associated to (0.1.10) is given by

$$\begin{cases} \partial_t v \;+\; \inf\limits_{\tilde\alpha\in L(\mathbb{R}^d;A)} \Big[\hat f(t,\mu,\tilde\alpha) + \; < \mathcal{L}^{\tilde\alpha}_t v(t,\mu), \mu > \Big] \;=\; 0, \;\text{ on }\; [0,T) \times \mathcal{P}_2(\mathbb{R}^d), \\ \qquad\qquad\qquad\qquad v(T,\mu) \;=\; \hat g(\mu), \;\text{ on }\; \mathcal{P}_2(\mathbb{R}^d) \end{cases} \tag{0.1.11}$$

where for $\tilde\alpha \in L(\mathbb{R}^d;A)$, $\varphi \in \mathcal{C}^2_b(\mathcal{P}_2(\mathbb{R}^d))$ and $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, $\mathcal{L}^{\tilde\alpha}_t$ is the function: $\mathbb{R}^d \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{L}^{\tilde\alpha}_t \varphi(\mu)(x) \;:=&\; \partial_\mu\varphi(\mu)(x).b(t,x,\tilde\alpha(x), Id\tilde\alpha\star\mu) \\ &+ \frac{1}{2}\mathrm{tr}\big(\partial_x\partial_\mu\varphi(\mu)(x)\sigma\sigma^\intercal(t,x,\tilde\alpha(x), Id\tilde\alpha\star\mu)\big). \end{aligned}$$

To complete the DPP, we turn to the verification argument when the Bellman equation (0.1.11) has a smooth solution. This is a quite straightforward consequence of the chain rule for functions on the Wasserstein space. We then apply the verification theorem to one important class of LQ McKean-Vlasov control problem and obtain the explicit solution, which is reduced to the resolution of Riccati differential equation system. In particular, both mean-variance portfolio selection model and interbank system risk model fall into the LQ framework and are solved explicitly.

However, except the class of LQ McKean-Vlasov control problem, in general the value function v does not have good regularity enough to interpret the fully nonlinear PDE of second order (0.1.11), and thus a notion of viscosity solutions is required. Indeed, the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ is infinite dimensional locally non compact space. There are few tools developed for viscosity solutions on Wasserstein space. To circumvent this difficulty, we work in Hilbert space $L^2(\mathcal{F}_0;\mathbb{R}^d)$ instead of Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ via the lifting identification, and identify the value function v and its lifted version. By abuse of notation $v(t,\mathcal{L}(\xi)) = v(t,\xi)$, the lifted Bellman equation in $[0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d)$ is given by:

$$\begin{cases} -\dfrac{\partial v}{\partial t} + \mathcal{H}(t,\xi,Dv(t,\xi),D^2v(t,\xi)) & = & 0 \quad \text{on } [0,T) \times L^2(\mathcal{F}_0;\mathbb{R}^d), \\ v(T,\xi) & = & \hat{G}(\xi) \ := \ \mathbb{E}[g(\xi,\mathbb{P}_\xi)], \quad \xi \in L^2(\mathcal{F}_0;\mathbb{R}^d), \end{cases} \tag{0.1.12}$$

where $\mathcal{H} : [0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d) \times L^2(\mathcal{F}_0;\mathbb{R}^d) \times S(L^2(\mathcal{F}_0;\mathbb{R}^d)) \to \mathbb{R}$ is defined by

$$\begin{aligned} \mathcal{H}(t,\xi,P,Q) \quad = \quad & -\inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \Big\{ \mathbb{E}\Big[f(t,\xi,\tilde\alpha(\xi), Id\tilde\alpha \star \mathbb{P}_\xi) + P.b(t,\xi,\tilde\alpha(\xi), Id\tilde\alpha \star \mathbb{P}_\xi) \\ & + \frac{1}{2} Q\big(\sigma(t,\xi,\tilde\alpha(\xi), Id\tilde\alpha \star \mathbb{P}_\xi)N\big).\big(\sigma(t,\xi,\tilde\alpha(\xi), Id\tilde\alpha \star \mathbb{P}_\xi)N\big)\Big]\Big\}, \end{aligned}$$

with $N \in L^2(\mathcal{F}_0;\mathbb{R}^n)$ of zero mean, unit variance, and independent of ξ. We choose all the twice continuously Fréchet differentiable functions on $L^2(\mathcal{F}_0;\mathbb{R}^d)$ as test functions and then naturally define the viscosity subsolution (resp. supersolution) to the Bellman equation (0.1.11). After making stronger assumptions on the coefficients and exploiting the lifted form of DPP, we show that the value function v is a viscosity solution to the Bellman equation (0.1.11) satisfying the quadratic growth condition. Assume further that the σ-algebra $\mathcal{F}_0$ is countably generated upto null sets, which guarantees that the Hilbert space $L^2(\mathcal{F}_0;\mathbb{R}^d)$ is separable. By [FGS15, Theorem 3.5], we prove a comparison principle (hence uniqueness result) for the lifted Bellman equation (0.1.12). Our value function v in (0.1.9) is thus characterized as the unique solution to Bellman equation (0.1.11) in the viscosity sense.

In the last section of this chapter, we drop the closed-loop restriction on the admissible controls, and discuss a bit about more general open-loop controls when the coefficients depend only upon the mean-field component through the law of the state variable. In the closed-loop case, the reformulation (0.1.9) of the value function is an important step towards the DPP, however, in the open-loop case, the reformulation requires more mathematical tools and measurability issues. Anyway, HJB equation is proposed and its solution coincides with case of the closed-loop form when the solution is smooth, see [CCD15] for the existence of smooth solution under some coercive assumptions. We also refer to more recently [BCP18] for the case of open-loop controls with the randomization method.

0.1.4 Dynamic programming for continuous time conditional McKean-Vlasov control problem

In this chapter, we deal with the McKean-Vlasov control problem with common noise. This is a generalization of the previous chapters without common noise. While the DPP in the case without common noise is quite straightforward, the presence of common noise makes it more intricate because of measurability issues, which requires more care.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space in the form $(\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$, Ω^0 the canonical space $C(\mathbb{R}_+,\mathbb{R}^m)$, W^0 the canonical process, $\mathbb{P}^0$ the Wiener measure, $\mathcal{F}^0$ the $\mathbb{P}^0$-completion of the natural filtration generated by W^0, B a n-dimensional Brownian motion on $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$, $\mathcal{G}$ a sub-σ-algebra of

$\mathcal{F}^1$ independent of B, $\mathcal{F}$ the natural filtration generated by W^0 and B, augmented with the independent $\mathcal{G}$. We denote by $\mathbb{E}^0$ (resp. $\mathbb{E}^1$) the expectation under $\mathbb{P}^0$ (resp. $\mathbb{P}^1$). We assume that $\mathcal{G}$ is rich enough in the sense that $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi : \xi \in L^2(\mathcal{G};\mathbb{R}^d)\}$. Then, the controlled stochastic McKean-Vlasov dynamics starting from $\xi \in L^2(\mathcal{F}_t, \mathbb{R}^d)$ at time $t \in [0,T]$ follows the stochastic McKean-Vlasov equation:

$$dX_s^{t,\xi} \;=\; b(X_s^{t,\xi}, \mathbb{P}^{W^0}_{X_s^{t,\xi}}, \alpha_s)ds + \sigma(X_s^{t,\xi}, \mathbb{P}^{W^0}_{X_s^{t,\xi}}, \alpha_s)dB_s + \sigma_0(X_s^{t,\xi}, \mathbb{P}^{W^0}_{X_s^{t,\xi}}, \alpha_s)dW_s^0, \qquad (0.1.13)$$

where $\mathbb{P}^{W^0}_{X_s^{t,\xi}}$ denotes the regular conditional distribution of $X_s^{t,\xi}$ given W^0 or equivalently $\mathcal{F}^0$, and the control α is $\mathbb{F}^0$-progressive process valued in some polish space $\mathbf{A}$ (typically a closed subset of the space $C(\mathbb{R}^d, A)$). We denote by $\mathcal{A}$ the set of admissible control processes. It's worth mentioning that unlike the first two chapters, the admissible control α is allowed to be in a more general form, see Remark 4.2.1 for details. As usual, the linear growth and lipschitz conditions on the coefficients b, σ guarantee the existence and uniqueness of a square integrable solution to (0.1.13). The dynamic cost functional associated to the stochastic McKean-Vlasov equation (0.1.13) is

$$J(t,\xi,\alpha) \;:=\; \mathbb{E}\Big[\int_t^T f\big(X_s^{t,\xi}, \mathbb{P}^{W^0}_{X_s^{t,\xi}}, \alpha_s\big)ds + g\big(X_T^{t,\xi}, \mathbb{P}^{W^0}_{X_T^{t,\xi}}\big)\Big],$$

and the objective is to minimize over all admissible control processes $\alpha \in \mathcal{A}$ the cost functional

$$v(t,\xi) \;=\; \inf_{\alpha\in\mathcal{A}} J(t,\xi,\alpha), \quad (t,\xi) \in [0,T] \times L^2(\mathcal{G};\mathbb{R}^d). \qquad (0.1.14)$$

Note that by the law of iterated conditional expectations, and as $\alpha \in \mathcal{A}$ is $\mathbb{F}^0$-progressive, we rewrite the cost functional in terms of the conditional law of the state

$$J(t,\xi,\alpha) \;=\; \mathbb{E}[\int_t^T \hat{f}(\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \alpha_s)ds + \hat{g}(\mathbb{P}^{W^0}_{X_T^{t,\xi,\alpha}})],$$

where the functions $\hat{f}$: $\mathcal{P}_2(\mathbb{R}^d) \times \mathbf{A} \to \mathbb{R}$, and $\hat{g}$: $\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are defined by

$$\hat{f}(\mu,a) \;:=\; \mu\big(f(\cdot,\mu,a)\big), \quad \hat{g}(\mu) \;:=\; \mu\big(g(\cdot,\mu)\big).$$

Since $\mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}$ depends on the ξ only through its conditional law $\mathbb{P}^{W_0}_\xi$, and $\mathcal{G} \subset \mathcal{F}_t$ is rich enough and independent of W^0, we define for any $t \in [0,T]$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$,

$$\rho_s^{t,\mu,\alpha} \;:=\; \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}, \quad t \le s \le T, \;\text{ for }\; \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d) \;\text{ such that } \mathbb{P}^{W^0}_\xi = \mu,$$

a square integrable $\mathbb{F}^0$-progressive continuous process in $\mathcal{P}_2(\mathbb{R}^d)$. From the pathwise uniqueness of $X_s^{t,\xi,\alpha}(\omega^0,\cdot)$ on $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$, $\mathbb{P}^1$-a.s., $\omega^0 \in \Omega^0$, we obtain a flow property on the controlled conditional distribution $\{\rho_s^{t,\mu,\alpha}, t \le s \le T\}$

$$\rho_s^{t,\mu,\alpha}(\omega^0) \;=\; \rho_s^{\theta(\omega^0),\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0),\alpha^{\theta(\omega^0),\omega^0}}(\omega^0), \quad s \in [\theta,T], \; \mathbb{P}^0(d\omega^0) - a.s. \qquad (0.1.15)$$

for all $\theta \in \mathcal{T}^0_{t,T}$, the set of $\mathbb{F}^0$-stopping times valued in $[t,T]$, where $\alpha^{\theta(\omega^0),\omega^0}$ is a shifted control process by concatenation, independent of $\mathcal{F}^0_{\theta(\omega^0)}$ introduced in [CTT16].

Notice that the cost functional $J(t,\xi,\alpha)$ depends on ξ only through its law $\mu = \mathcal{L}(\xi)$, and since $\{\rho_s^{t,\mu}, t \le s \le T\}$ and the control α are both $\mathbb{F}^0$-progressive. Therefore, the expectation is taken under $\mathbb{P}^0$ and the value function defined on $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ by abuse of notation

$$v(t,\mu) \;=\; \inf_{\alpha\in\mathcal{A}} \mathbb{E}^0[\int_t^T \hat{f}(\rho_s^{t,\mu,\alpha}, \alpha_s)ds + \hat{g}(\rho_T^{t,\mu,\alpha})].$$

As usual in the proof of the DPP for standard stochastic control problem, we use the flow property (0.1.15) and measurability selection arguments. The difference is that the process $\{\rho_s^{t,\mu,\alpha}, t \leq s \leq T\}$ here is measure-valued. We obtain the DPP for the conditional McKean-Vlasov control problem

$$\begin{aligned} v(t,\mu) &= \inf_{\alpha\in\mathcal{A}} \inf_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big] \\ &= \inf_{\alpha\in\mathcal{A}} \sup_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big]. \end{aligned}$$

Next, by relying on the notion of differentiability with respect to probability measures introduced by P.L. Lions [Lio12] and the chain rule along a flow of probability measures [BLM17], [CCD15], we derive the Bellman equation associated to the value function

$$\begin{cases} -\partial_t v - \inf_{a\in\mathbf{A}} \Big[\hat{f}(\mu,a) + \mu\big(\text{Ł}^a v(t,\mu)\big) + \mu\otimes\mu\big(\mathbb{M}^a v(t,\mu)\big)\Big] &= 0, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ v(T,\mu) &= \hat{g}(\mu), \quad \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{0.1.16}$$

where for $\phi\in\mathcal{C}_c^2(\mathcal{P}_2(\mathbb{R}^d))$, $a\in\mathbf{A}$, and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $\text{Ł}^a\phi(\mu)\in L^2_\mu(\mathbb{R})$ is the function $\mathbb{R}^d\to\mathbb{R}$ defined by

$$\text{Ł}^a\phi(\mu)(x) \ := \ \partial_\mu\phi(\mu)(x).b(x,\mu,a) + \frac{1}{2}\text{tr}\big(\partial_x\partial_\mu\phi(\mu)(x)(\sigma\sigma^\intercal + \sigma_0\sigma_0^\intercal)(x,\mu,a)\big),$$

and $\mathbb{M}^a\phi(\mu)\in L^2_{\mu\otimes\mu}(\mathbb{R})$ is the function $\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined by

$$\mathbb{M}^a\phi(\mu)(x,x') \ := \ \frac{1}{2}\text{tr}\big(\partial_\mu^2\phi(\mu)(x,x')\sigma_0(x,\mu,a)\sigma_0^\intercal(x',\mu,a)\big).$$

In general, there is no classical solution to the Bellman equation (0.1.16). We turn to the viscosity characterization of the value function to the Bellman equation (0.1.16). As it is difficult to obtain comparison principle for viscosity solutions in the Wasserstein space which is a locally non compact space, we instead work in the Hilbert space $L^2(\mathcal{G};\mathbb{R}^d)$ by viewing the value function as a function on $[0,T]\times L^2(\mathcal{G},\mathbb{R}^d)$ via the lifting identification. With the same notation $v(t,\xi) = v(t,\mathcal{L}(\xi))$, the lifted Bellman equation is then written as

$$\begin{cases} -\partial_t v - H\big(\xi, Dv(t,\xi), D^2v(t,\xi)\big) &= 0, \quad (t,\xi)\in[0,T)\times L^2(\mathcal{G};\mathbb{R}^d), \\ v(T,\xi) &= \tilde{\mathbb{E}}^1\big[g(\xi,\mathcal{L}(\xi))\big], \quad \xi\in L^2(\mathcal{G};\mathbb{R}^d), \end{cases} \tag{0.1.17}$$

where $H: L^2(\mathcal{G};\mathbb{R}^d)\times L^2(\mathcal{G};\mathbb{R}^d)\times S(L^2(\mathcal{G};\mathbb{R}^d))\to\mathbb{R}$ is defined by

$$\begin{aligned} H(\xi,P,Q) \ = \ \inf_{a\in\mathbf{A}} \mathbb{E}^1\Big[& f(\xi,\mathcal{L}(\xi),a) + P.b(\xi,\mathcal{L}(\xi),a) \\ &+ \frac{1}{2}Q(\sigma_0(\xi,\mathcal{L}(\xi),a)).\sigma_0(\xi,\mathcal{L}(\xi),a) + \frac{1}{2}Q(\sigma(\xi,\mathcal{L}(\xi),a)N).\sigma(\xi,\mathcal{L}(\xi),a)N\Big], \end{aligned}$$

with $N\in L^2(\mathcal{G};\mathbb{R}^n)$ of zero mean, and unit variance, and independent of ξ. We start with defining the viscosity subsolution (resp. supersolution) to the lifted Bellman equation (0.1.17). It is further assumed that the σ-algebra $\mathcal{G}$ is countably generated upto null sets, hence $L^2(\mathcal{G};\mathbb{R}^d)$ is separable Hilbert space. From the DPP and [FGS15, Theorem 3.5], we can show that v is the unique viscosity solution to the Bellman equation (0.1.16). We also state some verification results, which allow us to get an analytic feedback of the optimal control when there is a smooth function to the Bellman equation (0.1.16). The proof of the verification theorem is based on the Itô's formula along a flow of conditional probability measures and also require an additional linear growth condition on $\partial_\mu v(t,\mu)(x)$ to guarantee the vanishing of the martingale part of Itô's formula.

Finally, we illustrate our results on a class of LQ stochastic McKean-Vlasov type, for which explicit computations are possible. In particular, interbank systemic risk model fits into the LQ framework.

0.2 Part II: Robust mean-variance problem under model uncertainty

As in standard stochastic control problem, one interesting perspective of McKean-Vlasov control problem is to study it under G-expectation introduced by Peng [Pen10], called robust McKean-Vlasov control problem. This is a new and challenging problem in that it involves both the possible dependence of the coefficients upon the marginal distributions of the solutions and nonlinear expectation. In this part of this thesis, we consider one of its applications in finance: a robust mean-variance portfolio selection problem where model uncertainty carries on both rate of return and correlation matrix of the multiply assets.

Because of different kinds of reasons, model risk,see e.g. [Tal09], occurs in optimal portfolio allocation. In the literature on the robust portfolio selection problem, model uncertainty is usually parametrized by a prior set Θ, in a general setting, a family of tuples (b,σ) describing the uncertainty about drift and volatility. In a probability setup, $\mathcal{P}^\Theta$ consists of all semimartingale laws $\mathbb{P}^\theta$ such that the associated differential characteristics (b_t,σ_t) take values in Θ, see [NN18]. In the study of drift uncertainty, many authors introduced a dominated set of probability measures which are absolutely continuous with respect to a reference probability measure. While, in the framework of volatility uncertainty, the set of probability measures becomes non-dominated, see [MPZ15].

The majority of works on the robust portfolio selection problem focus on utility maximization. The mean-variance criterion has received little attention in the continuous-time setting, however, more recently [IP17] considered the robust mean-variance portfolio selection under covariance matrix ambiguity, in particular correlation ambiguity, and solved it by a McKean-Vlasov dynamic programming approach. Their method, however, can not be extended to tackle drift uncertainty. One key assumption in [IP17] is that one can aggregate a family of processes. In the case of drift uncertainty, this condition does not hold anymore. As mentioned in the first part of introduction, the nonlinear dependence of the cost functional on the law of the state makes the robust mean-variance portfolio under drift uncertainty into a non standard stochastic differential game. The objective of this part is to study the portfolio diversification under model uncertainty on drift and correlation of multi-aasets with a dynamic robust mean-variance approach.

Let us describe the model in detail. We consider a financial market which consists of one risk-free asset and $d \geq 2$ risky assets on a finite investment horizon $T > 0$. Let Ω be a canonical state space with the uniform norm and Borel σ-field $\mathcal{F}$, $B = (B_t)_{t\in[0,T]}$ the canonical space, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the canonical filtration. We assume that the investor knows the marginal volatility of each asset σ_i, $1 \leq i \leq d$, but is diffident about the rate of return b and correlation ρ, assumed to be valued in a nonempty convex set $\Theta \subset \mathbb{R}^d \times \mathbb{C}^d_{>+}$ where $\mathbb{C}^d_{>+}$ is the subset of all elements $\rho = (\rho_{ij})_{1\leq i<j\leq d} \in [-1,1]^{d(d-1)/2}$ s.t. correlation matrix $C(\rho)$ of the assets belongs to $\mathbb{S}^d_{>+}$. We consider the following cases for the parametrization of the ambiguity set Θ:

(HΘ)

(i) *Product set*: $\Theta = \Delta \times \Gamma$, where Δ is a compact convex set of $\mathbb{R}^d$ and Γ a convex set of $\mathbb{C}^d_{>+}$.

(ii) *Ellipsoidal set*: $\Theta = \{(b,\rho) \in \mathbb{R}^d \times \Gamma : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \leq \delta\}$, for some convex set Γ of $\mathbb{C}^d_{>+}$, where $\hat{b}$ is a known vector and $\delta > 0$ radius of ambiguity. For each given realization of ρ, the drift is allowed to vary in the ellipsoidal set.

This parametrization **(H**Θ**)** includes large classes of model uncertainty. When there is no correlation ambiguity, i.e. Γ is singleton, the case of different levels of drift uncertainty on different asset subclass, see [GUW06], is included in **(H**Θ**)**(i). However, it is difficult to study this case in the structure of correlation ambiguity. When there is one subclass for all assets, it's an ellipsoidal set described in **H**Θ(ii). When there is d subclass for all assets, it's a rectangular set included in **H**Θ(i). The other subclass m, $1 < m < d$ with ambiguity on correlation is left for future studies. We denote by $\Sigma(\rho)$ the prior covariance matrix of the assets and introduce the prior risk premium

$$R(b,\rho) \quad = \quad b^\intercal \Sigma(\rho)^{-1} b \quad \text{for } \theta = (b,\rho) \in \Theta.$$

We consider all $\mathbb{F}$-progressively measurable processes $\theta = (\theta_t) = (b_t, \rho_t)$ taking value in Θ, denoted by $\mathcal{V}_\Theta$, and then build a correspondence between elements in $\mathcal{V}_\Theta$ and probability measures on $(\Omega, \mathcal{F})$, namely,

$$\mathcal{P}^\Theta \quad = \quad \{\mathbb{P}^\theta : \theta \in \mathcal{V}_\Theta\},$$

where under each probability measure $\mathbb{P}^\theta$, there exists a Brownian motion W^θ_t such that $dB_t = b_t dt + \sigma(\rho_t) dW^\theta_t$. It is worth mentioning that in this modelling the drift and correlation is allowed to be random processes.

Now given an admissible portfolio strategy $\alpha \in \mathcal{A}$, the dynamics of the wealth process X^α_t with initial value at $x_0 \in \mathbb{R}$ evolves according to

$$\begin{aligned} dX^\alpha_t &= \alpha_t^\intercal \mathrm{diag}(S_t)^{-1} dS_t \;=\; \alpha_t^\intercal dB_t, \quad 0 \leq t \leq T,\ \mathcal{P}^\Theta - q.s. \\ &= \alpha_t^\intercal \big(b_t dt + \sigma(\rho_t) dW^\theta_t\big), \quad 0 \leq t \leq T,\ \mathbb{P}^\theta - a.s. \end{aligned}$$

In our framework, the problem of mean-variance minimization problem is formulated as a McKean-Vlasov differential game,

$$V_0 \quad := \quad \sup_{\alpha \in \mathcal{A}} \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} J(\alpha, \theta) \;=\; \sup_{\alpha \in \mathcal{A}} \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} \big\{\mathbb{E}_\theta[X^\alpha_T] - \lambda \mathrm{Var}_\theta(X^\alpha_T)\big\}. \tag{0.2.1}$$

In the classical mean-variance minimization problem, $\mathcal{P}^\Theta$ contains only one probability measure, which corresponds to a singleton $\Theta = \{(b^o, \rho^o)\}$. This means that the investor knows the "historical" probability measure denoted by $\mathbb{P}^o$ that describes the dynamics of the underlying assets. In this case, the optimal portfolio strategy is explicitly given by, see [LZ00] and Section 3.4.1 in Chapter 3

$$\alpha^*_t \quad = \quad \Big[x_0 + \frac{e^{R^o T}}{2\lambda} - X^*_t\Big](\Sigma^o)^{-1} b^o, \quad 0 \leq t \leq T,\ \mathbb{P}^o - a.s. \tag{0.2.2}$$

where X^*_t is the state process associated to α^*_t. We see from (0.2.2) that the investor diversifies her portfolio among all the available assets according to (up to a scalar term) the vector $(\Sigma^o)^{-1} b^o$ and in general all the available risky assets will be held, called well-diversification.

To find the initial value function V_0 and an optimal portfolio strategy α^*_t for (0.2.1), we follow the idea of the general martingale optimality principle approach. With the nonlinear dependence on the law of the state process via the variance term in the mean-variance criterion, we have to adapt it to our framework.

Lemma 0.2.1. *Let $\{V_t^{\alpha,\theta}, t \in [0,T], \alpha \in \mathcal{A}, \theta \in \mathcal{V}_\Theta\}$ be a family of real-valued processes in the form*

$$V_t^{\alpha,\theta} : \;=\; v_t(X_t^\alpha, \mathbb{E}_\theta[X_t^\alpha]),$$

for some measurable functions v_t on $\mathbb{R} \times \mathbb{R}$, $t \in [0,T]$, such that :

(i) $v_T(x, \bar{x}) = x - \lambda(x - \bar{x})^2$, *for all* $x, \bar{x} \in \mathbb{R}$,

(ii) the function $t \in [0,T] \mapsto \mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}]$ *is nonincreasing for all* $\alpha \in \mathcal{A}$ *and some* $\theta^* \in \mathcal{V}_\Theta$,

(iii) $\mathbb{E}_\theta[V_T^{\alpha^*,\theta}] \geq V_0^{\alpha^*,\theta} = v_0(x_0, x_0)$, *for some* $\alpha^* \in \mathcal{A}$ *and all* $\theta \in \mathcal{V}_\Theta$.

Then, α^* is an optimal portfolio strategy for the robust mean-variance problem (5.2.4) *with a worst-case scenario θ^*, and*

$$V_0 \;=\; \sup_{\alpha \in \mathcal{A}} \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} J(\alpha, \theta) \;=\; \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta) \;=\; v_0(x_0, x_0) \;=\; J(\alpha^*, \theta^*).$$

According to the above lemma, we have to work on an enlarged space of both state and the expectation of the state. So we construct a function $v_t(x, \bar{x})$, $t \in [0,T]$, x, $\bar{x} \in \mathbb{R}$ in the following form

$$v_t(x, \bar{x}) \;=\; K_t(x - \bar{x})^2 + Y_t x + \chi_t,$$

and choose (if exist) $\theta^*: = (b^*, \rho^*) = \arg\min_\Theta R(\theta) \in \Theta \subset \mathcal{V}_\Theta$ and α_t^*, the optimal portfolio strategy in the classical mean-variance problem in the Black-Schole model with parameter $(b^*, \sigma(\rho^*))$ under $\mathbb{P}^{\theta^*}$. We check that such a pair (θ^*, α^*) satisfies weak optimality principle, which yields the main result called separation principle.

Theorem 0.2.1. *Let us consider a parametric set Θ for model uncertainty as in* **(HΘ)**. *Suppose that there exists $\theta^* = (b^*, \rho^*) \in \Theta$ solution to $\arg\min_{\theta \in \Theta} R(\theta)$. Then the robust mean-variance problem* (5.2.4) *admits an optimal portfolio strategy given by*

$$\alpha_t^* \;=\; (x_0 + \frac{e^{R(\theta^*)T}}{2\lambda} - X_t^*)\Sigma(\rho^*)^{-1} b^*, \quad 0 \leq t \leq T, \;\; \mathcal{P}^\Theta - q.s., \tag{0.2.3}$$

where X^* is the state process associated to α_t^*. Moreover, the corresponding initial value function is

$$V_0 \;=\; x_0 + \frac{1}{4\lambda}\big[e^{R(\theta^*)T} - 1\big].$$

In the last section of this chapter, we provide some examples for explicit computation of the minimal risk premium function $\theta \in \Theta \mapsto R(\theta)$ in Theorem 0.2.1 and implication for the optimal robust portfolio strategy and the portfolio diversification. According to drift uncertainty in **(HΘ)**, we distinguish the case of ellipsoidal set $\Theta = \{(b, \rho) \in \mathbb{R}^d \times \Gamma : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \leq \delta\}$ and rectangular set $\Theta = \prod_{i=1}^d [\underline{b}_i, \bar{b}_i] \times \Gamma$. Our first observation is that whenever $\delta > \inf_{\rho \in \Gamma} \|\sigma(\rho)^{-1}\hat{b}\|_2$ in the ellipsoidal set and $\underline{b}_i \leq 0 \leq \bar{b}_i$ for each $1 \leq i \leq d$ in the rectangular set, $\alpha_t^* \equiv 0$. In other words, when the investor is poorly confident about the estimation on the expected rate of return, then she doesn't make risky investment at all. In what follows, we assume that $\delta < \inf_{\rho \in \Gamma} \|\sigma(\rho)^{-1}\hat{b}\|_2$ in the ellipsoidal set and $\underline{b}_i > 0$ for each $1 \leq i \leq d$, in the rectangular set. Our second observation is that if the investor is completely ambiguous about correlation, i.e. $\Gamma = \mathbb{C}_{>+}^d$, then the investor only holds the asset with the highest instantaneous (absolute value) Sharpe ratio. Next, we state our observation for two-asset model with drift uncertainty and partial correlation ambiguity, i.e. $\Gamma = [\underline{\rho}, \bar{\rho}] \subset (-1, 1)$. Whenever ellipsoidal set or rectangular set, there are three possible cases depending on the relation between Sharpe ratio "proximity" and the correlation bounds: directional trading, i.e. long or short in both assets, spread trading, i.e. long in one asset and short in the other one, and anti-diversification, i.e. only one asset with the highest Sharpe ratio is held in the portfolio. At the end, we give the result for three-asset model with drift uncertainty and partial correlation ambiguity, i.e. $\Gamma = \prod_{j=1}^{3} \prod_{i=1}^{j-1} [\underline{\rho}_{ij}, \bar{\rho}_{ij}] \subset \mathbb{C}^3_{>+}$. In this model, there are roughly five possible cases depending on the Sharpe ratio "proximity" and correlation bounds: anti-diversification, under-diversification in the sense that two assets are held in the optimal portfolio strategy, and well-diversification. One contribution in this part is to unify and extend, to some extent, the results obtained by [GUW06], [BGUW12] and [LZ17] with one type of model uncertainty in one period.

Part I

The optimal control of McKean-Vlasov dynamics

Chapter 1

Some differential calculus on Wasserstein space

Abstract: In this chapter, we present some useful elements of differential calculus and stochastic analysis such as differentiability and Itô's formula on Wasserstein space derived from the investigation of MFG. We mention [Lio12], [Car12], [BLPR17], [CCD15], [WZ17], [GT18], from which are quoted most of the results recalled without proof. Such a calculus plays an important role in the following chapters when we drive the Hamiltonian-Jacobi-Bellman (HJB) equation of stochastic control of McKean-Vlasov dynamics.

Keywords: Wasserstein space, differentiability, Itô's formula

1.1 Wasserstein space

Assume that (E,d) is a complete separable metric space. Let us consider a probability space $(\Omega,\mathcal{G},\mathbb{P})$ (we have to adapt the notation of probability space to the need of the chapters below) which is rich enough in the sense that for every $\mu\in\mathcal{P}_2(E)$, there is a random variable $\xi\in L^2(\mathcal{G};E)$ such that $\mathbb{P}_\xi=\mu$. For instance, Ω is Polish space, $\mathcal{G}$ its Borel σ-filed, $\mathbb{P}$ is atomless probability measure on $(\Omega,\mathcal{G})$ (Since Ω is Polish, then $\mathbb{P}$ is atomless if and only if every singleton has zero probability measure). The probability measure space $\mathcal{P}_2(E)$ is endowed with the 2-Wasserstein distance defined by

$$\begin{aligned}\mathcal{W}_2(\mu,\mu') &:= \inf\Big\{\Big(\int_{E\times E} d(x,y)^2\pi(dx,dy)\Big)^{\frac12} : \pi\in\mathcal{P}_2(E\times E) \text{ with marginals } \mu \text{ and } \mu'\Big\}\\ &= \inf\Big\{\Big(\mathbb{E}[d(\xi,\xi')^2]\Big)^{\frac12} :\ \xi,\xi'\in L^2(\mathcal{G};E) \text{ with } \mathbb{P}_\xi=\mu,\ \mathbb{P}_{\xi'}=\mu'\Big\},\end{aligned}$$

Then, it's shown that $(\mathcal{P}_2(E),\mathcal{W}_2)$ is a also complete separable metric space (see e.g. Theorem 6.18 in [Vil08]). We shall equip $\mathcal{P}_2(E)$ with the corresponding Borel σ-field $\mathcal{B}(\mathcal{P}_2(E))$. We next recall some useful topological properties on this Borel σ-field when $E=\mathbb{R}^d$. This is implicitly used throughout the manuscript.

Proposition 1.1.1. *We denote by $\mathscr{C}_2(\mathbb{R}^d)$ the set of continuous functions on $\mathbb{R}^d$ with quadratic growth, and for any $\varphi\in\mathscr{C}_2(\mathbb{R}^d)$, define the map $\Lambda_\varphi:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ by $\Lambda_\varphi\mu := \mu(\varphi)=\int_{\mathbb{R}^d}\varphi(x)\mu(dx)$, for $\mu\in\mathcal{P}_2(\mathbb{R}^d)$. Then*

(i) for $(\mu_n)_n$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, we have that $\mathcal{W}_2(\mu_n,\mu)\to 0$ if and only if, for every $\varphi\in\mathscr{C}_2(\mathbb{R}^d)$, $\Lambda_\varphi\mu_n\to\Lambda_\varphi\mu$.

(ii) given a measurable space $(O,\mathcal{O})$ and a map $\rho:O\to\mathcal{P}_2(\mathbb{R}^d)$, ρ is measurable if and only if the map $\Lambda_\varphi\circ\rho=\rho(\varphi):O\to\mathbb{R}$ is measurable, for any $\varphi\in\mathscr{C}_2(\mathbb{R}^d)$.

Notice that $\mathcal{B}(\mathcal{P}_2(\mathbb{R}^d))$ is generated by the family of functions $(\mu\in\mathcal{P}_2(\mathbb{R}^d)\mapsto\mu(D))_{D\in\mathcal{B}(\mathbb{R}^d)}$. Alternatively, it coincides with the cylindrical σ-algebra $\sigma(\Lambda_\varphi,\ \varphi\in\mathscr{C}_2(\mathbb{R}^d))$ and that the map Λ_φ is $\mathcal{B}(\mathcal{P}_2(\mathbb{R}^d))$-measurable, for any measurable function φ with quadratic growth condition, by using a monotone class argument since it holds true whenever $\varphi\in\mathscr{C}_2(\mathbb{R}^d)$.

1.2 Differentiability on Wassersetin space

There are various notions of differentiability in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. For example, the notion of Wasserstein derivative have been discussed in the theory of optimal transport. Briefly speaking, based on the geometric theory of Wasserstein space, Wasserstein derivative is defined in terms of sub- and super-differentials. An alternative notion has been introduced by Lions in his lectures at Collège de France [Lio12], and then detailed in [Car12] and [CD18, Vol I , Chapter 5]. Lions' notion is based on the *lifting* of the function $u:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ into a function $\tilde u$ defined on the so-called "flat" space $L^2(\mathcal{G};\mathbb{R}^d)$ by $\tilde u(\xi)=u(\mathbb{P}_\xi)$. Conversely, given a function $\tilde u$ defined on $L^2(\mathcal{G};\mathbb{R}^d)$, we call inverse-lifted function of $\tilde u$ defined on $\mathcal{P}_2(\mathbb{R}^d)$ by $u(\mu)=\tilde u(\xi)$ for $\mu=\mathcal{L}(\xi)$, and we notice that such u exists iff $\tilde u(\xi)$ depends only on the distribution of ξ for any $\xi\in L^2(\mathcal{G};\mathbb{R}^d)$. In this case, we shall often identify the function u and its lifted version $\tilde u$ by using the same notation $u=\tilde u$.

Definition 1.2.1. *We say that u is L-differentiable (resp. $\mathcal{C}^1$) on $\mathcal{P}_2(\mathbb{R}^d)$ if the lifting $\tilde u$ is Fréchet differentiable (resp. Fréchet differentiable with continuous derivatives) on $L^2(\mathcal{G};\mathbb{R}^d)$.*

In this case, the Fréchet derivative $[D\tilde{u}](\xi)$ is viewed as an element $D\tilde{u}(\xi)$ of $L^2(\mathcal{G};\mathbb{R}^d)$ by Riesz' theorem: $[D\tilde{u}](\xi)(Y) = \mathbb{E}[D\tilde{u}(\xi).Y]$ for any $Y \in L^2(\mathcal{G};\mathbb{R}^d)$. In [Car12] and [CD18, Vol I, Chapter 5], it has been shown that the law of the random variable $D\tilde{u}(\xi)$ does not depend upon the particular choice of ξ satisfying $\mathbb{P}_\xi = \mu$. Moreover, if u is $\mathcal{C}^1$, then there exists a deterministic measurable function $h : \mathbb{R}^d \to \mathbb{R}^d$ such that $D\tilde{u}(\xi)$ can be represented as

$$D\tilde{u}(\xi) \;=\; h(\xi), \quad \mu \;=\; \mathbb{P}_\xi - a.e. \tag{1.2.1}$$

We also refer to [WZ17] for an alternative proof of this structure of Lions derivative. Therefore, we denote by $\partial_\mu u(\mathbb{P}_\xi)(\xi) := h(\xi)$, which is called L-derivative of u at $\mu = \mathbb{P}_\xi$. From a geometric analysis perspective, Lions L-differentiability may be not as intuitive as differentiability in the sense of Wasserstein, actually, one can reconcile the notion of Lions L-derivative and Wasserstein derivative, that is, u is continuously L-differentiable iff u is Wasserstein differentiable, and $\partial_\mu u(\mu)$ equals Wasserstein derivative, see [GT18] for a deep connection between these two notions of derivatives.

Next, we consider the structure of second-order derivatives.

Definition 1.2.2. *We say that u is partially $\mathcal{C}^2$ if it is $\mathcal{C}^1$, and one can find, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, a continuous version of the mapping $x \in \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$, such that the mapping $(\mu, x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$ is jointly continuous at any point (μ, x) such that $x \in Supp(\mu)$, and if for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the mapping $x \in \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$ is differentiable, its derivative being jointly continuous at any point (μ, x) such that $x \in Supp(\mu)$. The gradient is then denoted by $\partial_x\partial_\mu u(\mu)(x) \in \mathbb{S}^d$.*

Definition 1.2.3. *We say that u is fully $\mathcal{C}^2$ if it is $\mathcal{C}^1$, and one can find, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, a continuous version of the mapping $x \in \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$, such that the mapping $(\mu, x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$ is jointly continuous at any point (μ, x) such that $x \in Supp(\mu)$, and*

(i) for each fixed $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the mapping $x \in \mathbb{R}^d \mapsto \partial_\mu u(\mu)(x)$ is differentiable in the standard (classical) sense, with a gradient denoted by $\partial_x\partial_\mu u(\mu)(x) \in \mathbb{R}^{d\times d}$, and s.t. the mapping $(\mu, x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \partial_x\partial_\mu u(\mu)(x)$ is jointly continuous

(ii) for each fixed $x \in \mathbb{R}^d$, the mapping $\mu \in \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_\mu u(\mu)(x)$ is differentiable in the above lifted sense. Its derivative, interpreted thus as a mapping $x' \in \mathbb{R}^d \mapsto \partial_\mu\big[\partial_\mu u(\mu)(x)\big](x') \in \mathbb{R}^{d\times d}$ in $L^2_\mu(\mathbb{R}^{d\times d})$, is denoted by $x' \in \mathbb{R}^d \mapsto \partial^2_\mu u(\mu)(x, x')$, and s.t. the mapping $(\mu, x, x') \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \partial^2_\mu u(\mu)(x, x')$ is continuous.

Definition 1.2.4. *We say that u is $\mathcal{C}^2_b(\mathcal{P}_2(\mathbb{R}^d))$ if it is partially $\mathcal{C}^2$, $\partial_x\partial_\mu u(\mu) \in L^\infty_\mu(\mathbb{R}^{d\times d})$, and for any compact set $\mathcal{K}$ of $\mathcal{P}_2(\mathbb{R}^d)$, we have*

$$\sup_{\mu\in\mathcal{K}} \Big[\int_{\mathbb{R}^d} \big|\partial_\mu u(\mu)(x)\big|^2 \mu(dx) + \big\|\partial_x\partial_\mu u(\mu)\big\|_\infty \Big] \;<\; \infty.$$

As shown in [CCD15], if the *lifting* $\tilde{u} \in \mathcal{C}^2_c(L^2(\mathcal{G};\mathbb{R}^d))$, the set of twice continuously Fréchet differentiable on $L^2(\mathcal{G};\mathbb{R}^d)$ with Lipschitz Fréchet derivative, then u lies in $\mathcal{C}^2_b(\mathcal{P}_2(\mathbb{R}^d))$. In this case, the second Fréchet derivative $D^2\tilde{u}(\xi)$ is identified indifferently by Riesz' theorem as a bilinear form on $L^2(\mathcal{G};\mathbb{R}^d)$ or as a symmetric operator (hence bounded) on $L^2(\mathcal{G};\mathbb{R}^d)$, denoted by $D^2\tilde{u}(\xi) \in S(L^2(\mathcal{G};\mathbb{R}^d))$, and we have the relation (see Appendix A.2 in [CD14]):

$$D^2\tilde{u}(\xi)[YN, YN] \;=\; \mathbb{E}\Big[D^2\tilde{u}(\xi)(YN).YN\Big] \;\;=\;\; \mathbb{E}\Big[\mathrm{tr}\big(\partial_x\partial_\mu u(\mathbb{P}_\xi)(\xi)YY^\intercal\big)\Big], \tag{1.2.2}$$

for any $\xi \in L^2(\mathcal{G};\mathbb{R}^d)$, $Y \in L^2(\mathcal{G};\mathbb{R}^{d\times q})$, and where $N \in L^2(\mathcal{G};\mathbb{R}^q)$ is independent of (ξ, Y) with zero mean and unit variance.

Definition 1.2.5. *We say that u is fully $\mathcal{C}_c^2$ if it is fully $\mathcal{C}^2$, $\partial_x\partial_\mu u(\mu) \in L^\infty_\mu(\mathbb{R}^{d\times d})$, $\partial_\mu^2 u(\mu) \in L^\infty_{\mu\otimes\mu}(\mathbb{R}^{d\times d})$ for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and for any compact set $\mathcal{K}$ of $\mathcal{P}_2(\mathbb{R}^d)$, we have*

$$\sup_{\mu\in\mathcal{K}}\Big[\int_{\mathbb{R}^d}\big|\partial_\mu u(\mu)(x)\big|^2\mu(dx) + \big\|\partial_x\partial_\mu u(\mu)\big\|_\infty + \big\|\partial_\mu^2 u(\mu)\big\|_\infty\Big] \quad < \quad \infty. \tag{1.2.3}$$

We illustrate the notion of differentiability with some fundamental examples, some of which are from [CD18].

Example 1.2.1. *(i) The function u is of the first order form:*

$$u(\mu) \ = <\varphi,\mu>: = \int_{R^d}\varphi(x)\mu(dx),$$

for some continuously differentiable function φ defined on $\mathbb{R}^d$, whose derivative is at most of linear growth. In this case, the lifted function of u is given by $\tilde{u}(\xi) = \mathbb{E}[\varphi(\xi)]$ and

$$\begin{aligned}
\tilde{u}(\xi+Y) &= \mathbb{E}[\varphi(\xi+Y)] \\
&= \mathbb{E}[\varphi(\xi)] + \int_0^1 \mathbb{E}[\nabla\varphi(\xi+hY).Y]dh \\
&= \mathbb{E}[\varphi(\xi)] + \mathbb{E}[\nabla\varphi(\xi).Y] + \mathbb{E}\int_0^1[(\nabla\varphi(\xi+hY)-\nabla\varphi(\xi)).Y]dh.
\end{aligned}$$

It is easy to check that the last term in the r.h.s is $o(\|Y\|_2^2)$, and thus the Fréchet derivative of $\tilde{u}$ at ξ is given by $\nabla\varphi$. Consequently, we have $\partial_\mu u(\mu) = \nabla\varphi$.

(ii) For any $\Lambda \in \mathbb{S}^d$, we set

$$\bar{\mu} : = \int_{\mathbb{R}^d} x\mu(dx), \ \ \bar{\mu}_2(\Lambda) : = \int_{\mathbb{R}^d} x^\intercal\Lambda x\mu(dx), \ \ \mathrm{Var}(\mu)(\Lambda) : = \ \bar{\mu}_2(\Lambda) - \bar{\mu}^\intercal\Lambda\bar{\mu},$$

which corresponds to $\varphi(x) = x$, $\varphi(x) = x^\intercal\Lambda x$, and $\varphi(x) = (x-\mu)^\intercal\Lambda(x-\mu)$ in (i) respectively. Therefore, we have

$$\partial_\mu\bar{\mu} \ = \ I_{d\times d}, \ \ \partial_\mu\bar{\mu}_2(\Lambda) \ = \ 2\Lambda x, \ \ \partial_\mu\mathrm{Var}(\mu)(\Lambda) = 2\Lambda(x-\bar{\mu}),$$

which shall be used in the class of LQ McKean-Vlasov problem.

(iii) The function u is in the cylindrical form:

$$u(\mu) \quad = \quad F(<\varphi_1,\mu>,\ldots,<\varphi_d,\mu>)$$

for some continuously differentiable functions F, φ_1, ..., φ_d defined on $\mathbb{R}^d$, whose derivatives are at most of linear growth. In this case, the lifted function of u is given by $\tilde{u}(\xi) = F(\mathbb{E}[\varphi_1(\xi)],\ldots,\mathbb{E}[\varphi_d(\xi)])$ and

$$\begin{aligned}
\tilde{u}(\xi+Y) &= F(\mathbb{E}[\varphi_1(\xi+Y)],\ldots,\mathbb{E}[\varphi_d(\xi+Y)]) \\
&= F(\mathbb{E}[\varphi_1(\xi)],\ldots,\mathbb{E}[\varphi_d(\xi)]) + \sum_{i=1}^d \partial_{x_i}F(\mathbb{E}[\varphi_1(\xi)],\ldots,\mathbb{E}[\varphi_d(\xi)])E[\nabla\varphi_i(\xi).Y] \\
&\quad + \int_0^1\sum_{i=1}^d\big[\partial_{x_i}F(\mathbb{E}[\varphi_1(\xi+hY)],\ldots,\mathbb{E}[\varphi_d(\xi+hY)])\mathbb{E}[\nabla\varphi_i(\xi+hY).Y] \\
&\quad - \partial_{x_i}F(\mathbb{E}[\varphi_1(\xi)],\ldots,\mathbb{E}[\varphi_d(\xi)])\mathbb{E}[\nabla\varphi_i(\xi).Y]\big]dh
\end{aligned}$$

We check that the last term in the r.h.s is $o(\|Y\|_2^2)$*, and thus the Fréchete derivative of* $\tilde{u}$ *is given by* $\sum_{i=1}^{d} \partial_{x_i} F(\mathbb{E}[\varphi_1(\xi)], \ldots, \mathbb{E}[\varphi_d(\xi)]) \nabla \varphi_i(\xi)$*. Therefore, we have*

$$\partial_\mu \varphi(\mu) \quad = \quad \sum_{i=1}^{d} \partial_{x_i} F(< \varphi_1, \mu >, \ldots, < \varphi_d, \mu >) \nabla \varphi_i.$$

1.3 Itô's formula on Wasserstein space

1.3.1 Itô's formula along a flow of deterministic measures

In this thesis, we shall use a chain rule (or Itô's formula) along a flow of deterministic probability measures and recall it here. Let us consider an $\mathbb{R}^d$-valued Itô process

$$X_t \quad = \quad X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s, \quad 0 \leq t \leq T, \tag{1.3.1}$$

where (b_t) and (σ_t) are progressively measurable processes with respect to the filtration generated by the n-dimensional Brownian motion B, valued respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d \times n}$, and satisfying the integrability condition:

$$\mathbb{E}\Big[\int_0^T |b_t|^2 + |\sigma_t|^2 dt \Big] \quad < \quad \infty. \tag{1.3.2}$$

We then claim:

Proposition 1.3.1. *Let* u *be* $\mathcal{C}_b^2(\mathcal{P}_2(\mathbb{R}^d))$*. Then, under condition* (1.3.2)*, for all* $t \in [0, T]$*,*

$$u(\mathbb{P}_{X_t}) \quad = \quad u(\mathbb{P}_{X_0}) \; + \; \int_0^t \mathbb{E}\big[\partial_\mu u(\mathbb{P}_{X_s})(X_s).b_s + \frac{1}{2} \mathrm{tr}\big(\partial_x \partial_\mu u(\mathbb{P}_{X_s})(X_s) \sigma_s \sigma_s^\intercal \big) \big] ds. \tag{1.3.3}$$

This proposition is proved independently in [BLPR17] and [CCD15], see also the Appendix in [CD14].

It will be useful to reformulate the Itô's formula for the lifted function $\tilde{u}$ on $L^2(\mathcal{G}; \mathbb{R}^d)$ $(= L^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^d))$. Now, denoting by $(\tilde{B}_t)_{0 \leq t \leq T}$, $(\tilde{b}_t)_{0 \leq t \leq T}$, $(\tilde{\sigma}_t)_{0 \leq t \leq T}$ the copies of $(B_t)_{0 \leq t \leq T}$, $(b_t)_{0 \leq t \leq T}$ and $(\sigma_t)_{0 \leq t \leq T}$ on the space $(\Omega, \mathcal{G}, \mathbb{P})$, we then have an Itô process of the form on $(\Omega, \mathcal{G}, \mathbb{P})$

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \tilde{b}_t dt + \int_0^t \tilde{\sigma}_t d\tilde{B}_t, \quad 0 \leq t \leq T,$$

which is then the copy of X in (1.3.1). Thus, when the lifted function $\tilde{u} \in \mathcal{C}_2(L^2(\mathcal{G}; \mathbb{R}^d))$, we obtain from (1.3.7) and (1.2.2) an Itô's formula on the lifted space $L^2(\mathcal{G}; \mathbb{R}^d)$,

$$u(\tilde{X}_t) = u(\tilde{X}_0) + \int_0^t \mathbb{E}[D\tilde{u}(\tilde{X}_s).\tilde{b}_s + \frac{1}{2} D^2 \tilde{u}(\tilde{X}_s)(\tilde{\sigma}_s N).\tilde{\sigma}_s N], \quad 0 \leq t \leq T, \tag{1.3.4}$$

where $N \in L^2(\mathcal{G}; \mathbb{R}^d)$, is independent of $(\tilde{X}_0, \tilde{B})$, with zero mean, and unit variance.

1.3.2 Itô's formula along a flow of random measures

We next give an Itô's formula along a flow of conditional measures proved in [CCD15] (see also [CDLL15] and [CD15]). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space of the form $(\Omega,\mathcal{F},\mathbb{P})=(\Omega^0\times\Omega^1,\mathcal{F}^0\otimes\mathcal{F}^1,\mathbb{P}^0\otimes\mathbb{P}^1)$, where $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0)$ supports W^0 and $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ supports B. Let us consider an Itô process in $\mathbb{R}^d$ of the form:

$$X_t \;=\; X_0+\int_0^t b_s ds+\int_0^t \sigma_s dB_s+\int_0^t \sigma_s^0 dW_s^0, \quad 0\leq t\leq T, \tag{1.3.5}$$

where X_0 is independent of (B,W^0), and the coefficients b, σ and σ^0 are progressively measurable processes with respect to the natural filtration $\mathbb{F}$ generated by (X^0,B,W^0), and satisfying the square integrability condition:

$$\mathbb{E}\Big[\int_0^T |b_t|^2+|\sigma_t|^2+|\sigma_t^0|^2dt\Big] \;<\; \infty. \tag{1.3.6}$$

Denote by $\mathbb{P}^{W^0}_{X_t}$ the conditional law of X_t, $t\in[0,T]$, given the σ-algebra $\mathcal{F}^0$ generated by the whole filtration of W^0, and by $\mathbb{E}_{W^0}=\mathbb{E}^1$ the conditional expectation w.r.t. $\mathcal{F}^0$.

Proposition 1.3.2. *Let $u\in\mathcal{C}^2_c(\mathcal{P}_2(\mathbb{R}^d))$. Then, under condition (1.3.6), for all $t\in[0,T]$, we have:*

$$\begin{aligned} u(\mathbb{P}^{W^0}_{X_t}) \;=\; & u(\mathbb{P}_{X_0})+\int_0^t \mathbb{E}_{W^0}\Big[\partial_\mu u(\mathbb{P}^{W^0}_{X_s})(X_s).b_s+\frac{1}{2}\mathrm{tr}\big(\partial_x\partial_\mu u(\mathbb{P}^{W^0}_{X_s})(X_s)(\sigma_s\sigma_s^{\intercal}+\sigma_s^0(\sigma_s^0)^{\intercal})\big)\Big]\\ & +\mathbb{E}_{W^0}\Big[\mathbb{E}'_{W^0}\big[\frac{1}{2}\mathrm{tr}\big(\partial_\mu^2 u(\mathbb{P}^{W^0}_{X_s})(X_s,X'_s)\sigma_s^0({\sigma'}_s^0)^{\intercal}\big)\big]\Big]ds\\ & +\int_0^t \mathbb{E}_{W^0}\Big[\partial_\mu u(\mathbb{P}^{W^0}_{X_s})(X_s)^{\intercal}\sigma_s^0\Big]dW_s^0, \end{aligned} \tag{1.3.7}$$

where X' and ${\sigma'}^0$ are copies of X and σ^0 on another probability space $(\Omega^0\times{\Omega'}^1,\mathcal{F}^0\otimes{\mathcal{F}'}^1,\mathbb{P}^0\times{\mathbb{P}'}^1)$, with $({\Omega'}^1,{\mathcal{F}'}^1,{\mathbb{P}'}^1)$ supporting B' a copy of B, and $\mathbb{E}'_{W^0}={\mathbb{E}'}^1$.

Assume further that $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ is in the form $\Omega^1=\tilde\Omega^1\times{\Omega'}^1$, $\mathcal{F}^1=\mathcal{G}\otimes{\mathcal{F}'}^1$, $\mathbb{P}^1=\tilde{\mathbb{P}}^1\otimes{\mathbb{P}'}^1$, where $\tilde\Omega^1$ is a Polish space, $\mathcal{G}$ its Borel σ-algebra, $\tilde{\mathbb{P}}^1$ an atomless probability measure on $(\tilde\Omega^1,\mathcal{G})$, while $({\Omega'}^1,{\mathcal{F}'}^1,{\mathbb{P}'}^1)$ supports B. We denote by $\mathbb{E}^1$ (resp. $\tilde{\mathbb{E}}^1$) the expectation under $\mathbb{P}^1$ (resp. $\tilde{\mathbb{P}}^1$). It will be useful to formulate Itô's formula for the lifted function $\tilde u$ on $L^2(\mathcal{G};\mathbb{R}^d)$ $(=L^2(\tilde\Omega^1,\mathcal{G},\tilde{\mathbb{P}}^1;\mathbb{R}^d))$. Notice, however, that even if $u\in\mathcal{C}^2_c(\mathcal{P}_2(\mathbb{R}^d))$, then its lifted function $\tilde u$ may not be in general twice continuously Fréchet differentiable on $L^2(\mathcal{G};\mathbb{R}^d)$, as discussed in Example 2.1 in [BLPR17]. Under the extra-assumption that the lift $\tilde u\in\mathcal{C}^2(L^2(\mathcal{G};\mathbb{R}^d))$ (the set of real-valued twice continuously Fréchet differentiable on $L^2(\mathcal{G};\mathbb{R}^d)$), the second Fréchet derivative $D^2\tilde u(\xi)$ is identified indifferently by Riesz' theorem as a bilinear form on $L^2(\mathcal{G};\mathbb{R}^d)$ or as a self-adjoint operator (hence bounded) on $L^2(\mathcal{G};\mathbb{R}^d)$, denoted by $D^2\tilde u(\xi)\in S(L^2(\mathcal{G};\mathbb{R}^d))$, and we have the relation (see Appendix A.2 in [CD14]):

$$\left\{\begin{aligned} D^2\tilde u(\xi)[Y,Y] \;&=\; \tilde{\mathbb{E}}^1\Big[D^2\tilde u(\xi)(Y).Y\Big] \;=\; \tilde{\mathbb{E}}^1\Big[\tilde{\mathbb{E}}'^1\big[\mathrm{tr}\big(\partial_\mu^2 u(\mathcal{L}(\xi))(\xi,\xi')Y(Y')^{\intercal}\big)\big]\Big]\\ &\qquad\qquad\qquad\qquad\qquad\quad +\;\tilde{\mathbb{E}}^1\Big[\mathrm{tr}\big(\partial_x\partial_\mu u(\mathcal{L}(\xi))(\xi)YY^{\intercal}\big)\Big],\\ D^2\tilde u(\xi)[ZN,ZN] \;&=\; \tilde{\mathbb{E}}^1\Big[D^2\tilde u(\xi)(ZN).ZN\Big] \;=\; \tilde{\mathbb{E}}^1\Big[\mathrm{tr}\big(\partial_x\partial_\mu u(\mathcal{L}(\xi))(\xi)ZZ^{\intercal}\big)\Big], \end{aligned}\right. \tag{1.3.8}$$

for any $\xi\in L^2(\mathcal{G};\mathbb{R}^d)$, $Y\in L^2(\mathcal{G};\mathbb{R}^d)$, $Z\in L^2(\mathcal{G};\mathbb{R}^{d\times q})$, and where (ξ',Y') is a copy of (ξ,Y) on another Polish and atomless probability space $(\tilde\Omega'^1,\mathcal{G}',\tilde{\mathbb{P}}'^1)$, $N\in L^2(\mathcal{G};\mathbb{R}^q)$ is independent of (ξ,Z) with zero mean, and unit variance. Now, let us consider a copy $\tilde{B}$ of B on the probability space $(\tilde{\Omega}^1, \mathcal{G}, \tilde{\mathbb{P}}^1)$, denote by $\tilde{X}_0$, $\tilde{b}$, $\tilde{\sigma}$, $\tilde{\sigma}_0$ copies of X_0, b, σ, σ_0 on $(\tilde{\Omega} = \Omega^0 \times \tilde{\Omega}^1, \tilde{\mathcal{F}} = \mathcal{F}^0 \otimes \mathcal{G}, \tilde{\mathbb{P}} = \mathbb{P}^0 \otimes \tilde{\mathbb{P}}^1)$, and consider the Itô process $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the form

$$\tilde{X}_t \quad = \quad \tilde{X}_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s d\tilde{B}_s + \int_0^t \tilde{\sigma}_s^0 dW_s^0, \quad 0 \leq t \leq T,$$

which is then a copy of X in (1.3.5). The process $\check{X}$ defined by $\check{X}_t(\omega^0) = \tilde{X}_t(\omega^0, .)$, $0 \leq t \leq T$, is $\mathbb{F}^0$-progressive, and valued in $L^2(\mathcal{G}; \mathbb{R}^d)$. Similarly, the processes defined by $\check{b}_t(\omega^0) = \tilde{b}_t(\omega^0, .)$, $\check{\sigma}_t(\omega^0) = \tilde{\sigma}_t(\omega^0, .)$, $\check{\sigma}_t^0(\omega^0) = \tilde{\sigma}_t^0(\omega^0, .)$, $0 \leq t \leq T$, are valued in $L^2(\mathcal{G}; \mathbb{R}^d)$, $\mathbb{P}^0$-a.s. Thus, when the lifted function $\tilde{u} \in \mathcal{C}^2(L^2(\mathcal{G}; \mathbb{R}^d))$, we obtain from (1.3.7) and relation (1.2.1)-(1.3.8) an Itô's formula on the lifted space $L^2(\mathcal{G}; \mathbb{R}^d)$:

$$\begin{aligned}\tilde{u}(\check{X}_t) \quad &= \quad \tilde{u}(\check{X}_0) + \int_0^t \tilde{\mathbb{E}}^1 \Big[D\tilde{u}(\check{X}_s).\check{b}_s + \frac{1}{2} D^2\tilde{u}(\check{X}_s)(\check{\sigma}_s N).\check{\sigma}_s N + \frac{1}{2} D^2\tilde{u}(\check{X}_s)(\check{\sigma}_s^0).\check{\sigma}_s^0 \Big] ds \\ &\quad + \int_0^t \tilde{\mathbb{E}}^1 \big[D\tilde{u}(\check{X}_s)^{\intercal} \check{\sigma}_s^0 \big] dW_s^0, \qquad 0 \leq t \leq T, \ \mathbb{P}^0 - a.s. \end{aligned} \tag{1.3.9}$$

where $N \in L^2(\mathcal{G}; \mathbb{R}^d)$ is independent of $(\tilde{B}, \tilde{X}_0)$, with zero mean, and unit variance.

Remark 1.3.1. Itô's formula (1.3.9) is proved in Proposition 6.3 in [CD15], and holds true for any function $\tilde{u}$ which is twice continuously Fréchet differentiable on $L^2(\mathcal{G}; \mathbb{R}^d)$. The fact that $\tilde{u}$ has a lifted structure plays no role, and is used only to derive from (1.2.1)-(1.3.8) Itô's formula (1.3.7) on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. Recall however that Itô's formula (1.3.7) holds even if the lift is not twice continuously Fréchet differentiable as shown in [CCD15] (see also [CDLL15]). □

Chapter 2

Dynamic programming for discrete time McKean-Vlasov control problem[a]

Abstract: We consider the stochastic optimal control problem of nonlinear mean-field systems in discrete time. We reformulate the problem into a deterministic control problem with marginal distribution as controlled state variable, and prove that dynamic programming principle holds in its general form. We apply our method for solving explicitly the mean-variance portfolio selection and the multivariate linear-quadratic McKean-Vlasov control problem.

Keywords: McKean-Vlasov equation, dynamic programming, calculus of variations.

a. This chapter is based on a paper in collaboration with Huyên Pham [PW16]. This paper is published in *Applied Mathematics and Optimization*, **74**(3), 2016.

2.1 Introduction

The problem studied in this paper concerns the optimal control of nonlinear stochastic dynamical systems in discrete time of McKean-Vlasov type. Such topic is related to the modeling of collective behaviors for a large number of players with mutual interactions, which has led to the theory of MFGs, introduced in [LL07] and [HMC06].

Since the emergence of MFG theory, the optimal control of mean-field dynamical systems has attracted a lot of interest in the literature, mostly in continuous time. It has been first studied in [AD01] by functional analysis method with a value function expressed in terms of the Nisio semigroup of operators. More recently, several papers have adopted the stochastic maximum principle for characterizing solutions to the controlled McKean-Vlasov systems in terms of adjoint backward stochastic differential equations (BSDEs), see [AD10], [BDL11], [CD15]. We also refer to the paper [Yon13] which focused on the LQ case where the BSDE from the maximum principle leads to a Riccati equation system. It is mentioned in these papers that due to the non-markovian nature of the McKean-Vlasov systems, dynamic programming (also called Bellman optimality) principle does not hold and the problem is time inconsistent in general. Indeed, the standard Markov property of the state process, say X, is ruled out, however, as noticed in [BFY13], [BFY15], this can be restored by working with the marginal distribution of X. The dynamic programming has then been applied in [LP14] for a specific control problem where the objective function depends upon statistics of X like its mean value but with no mean-field interaction on the dynamics of X, and by assuming the existence at all times of a density function for the marginal distribution of X.

The purpose of this paper is to provide a detailed analysis of the dynamic programming method for the optimal control of nonlinear mean-field systems in discrete time, where the coefficients may depend both upon the marginal distributions of the state and of the control. The case of continuous time McKean-Vlasov equations requires more technicalities and mathematical tools, and is under current investigation. The discrete time framework has been also considered in [ELN13] for LQ problem, and arises naturally in situations where signal values are available only at certain times. On the other hand, it can also be viewed as the discrete time version or approximation of the optimal control of continuous time McKean-Vlasov stochastic differential equations. Our methodology is the following. By using closed-loop (also called feedback) controls, we first convert the stochastic optimal control problem into a deterministic control problem involving only the marginal distribution of the state process. We then derive the deterministic evolution of the controlled marginal distribution, and prove in its general form the DPP. This gives sufficient conditions for optimality in terms of calculus of variations in the space of feedback control functions. Classical DPP for stochastic control problem without mean-field interaction falls within our approach. We finally apply our method for solving explicitly the mean-variance portfolio selection problem and the multivariate LQ mean-field control problem, and retrieve in particular the results obtained in [ELN13] by four different approaches.

The outline of the paper is as follows. The next section formulates the McKean-Vlasov control problem in discrete time. In Section 2.3, we develop the dynamic programming method in this framework. Section 2.4 is devoted to applications of the DPP with explicit solutions in the LQ case including the mean-variance problem.

2.2 McKean-Vlasov control problem

We consider a general class of optimal control of mean-field type in discrete time. We are given two measurable spaces $(E, \mathcal{B}(E))$ and $(A, \mathcal{B}(A))$ representing respectively the state space, and the control space. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a controlled stochastic dynamics of McKean-Vlasov type:

$$X^\alpha_{k+1} \;=\; F_{k+1}(X^\alpha_k, \mathbb{P}_{X^\alpha_k}, \alpha_k, \mathbb{P}_{\alpha_k}, \varepsilon_{k+1}), \quad k \in \mathbb{N}, \quad X^\alpha_0 \;=\; \xi, \tag{2.2.1}$$

for some measurable functions F_k defined from $E \times \mathcal{P}(E) \times A \times \mathcal{P}(A) \times \mathbb{R}^d$ into E, where $(\varepsilon_k)_k$ is a sequence of i.i.d. random variables, independent of the initial random value ξ, and we denote by $\mathbb{F} = (\mathcal{F}_k)_k$ the filtration with $\mathcal{F}_k$ the σ-algebra generated by $\{\xi, \varepsilon_1, \ldots, \varepsilon_k\}$. Here, $(X^\alpha_k)_k$ is the state process valued in E controlled by the $\mathbb{F}$-adapted process $(\alpha_k)_k$ valued in A, Thus, the dynamics of (X^α_k) depends at any time k of its marginal distribution, but also of the marginal distribution of the control, which represents an additional mean-field feature with respect to classical McKean Vlasov equations, and also considered recently in [ELN13].

Let us now precise the assumptions on the McKean-Vlasov equation. We shall assume that $(E, |.|)$ is a normed space (most often $\mathbb{R}^d$), $(A, |.|)$ is also a normed space (typically a subset of $\mathbb{R}^m$), and we recall from Notations that $\mathcal{P}_2(E)$ the space of square integrable probability measures over E , i.e. $\mu \in \mathcal{P}(E)$ s.t. $\|\mu\|_2^2 := \int_E |x|^2 \mu(dx) < \infty$, and similarly for $\mathcal{P}_2(A)$. For any $(x, \mu, a, \lambda) \in E \times \mathcal{P}(E) \times A \times \mathcal{P}(A)$, and $k \in \mathbb{N}$, we denote by $P_{k+1}(x, \mu, a, \lambda, dx')$ the probability distribution of the E-valued random variable $F_{k+1}(x, \mu, a, \lambda, \varepsilon_{k+1})$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and we assume

(H1) For any $k \in \mathbb{N}$, there exists some positive constant $C_{k,F}$ s.t. for all $(x, a, \mu, \lambda) \in E \times A \times \mathcal{P}(E) \times \mathcal{P}(A)$:

$$\begin{aligned} \int_E |x'|^2 P_{k+1}(x, \mu, a, \lambda, dx') \;&=\; \mathbb{E}\Big[\big|F_{k+1}(x, \mu, a, \lambda, \varepsilon_{k+1})\big|^2\Big] \\ &\leq\; C_{k,F}(1 + |x|^2 + |a|^2 + \|\mu\|_2^2 + \|\lambda\|_2^2). \end{aligned}$$

Assuming that the initial random value ξ is square integrable, and considering admissible controls α which are square integrable, i.e. $\mathbb{E}|\alpha_k|^2 < \infty$, for any k, it is then clear under **(H1)** that $\mathbb{E}|X^\alpha_k|^2 < \infty$, i.e. $\mathbb{P}_{X^\alpha_k} \in \mathcal{P}_2(E)$, and there exists some positive constant C_k s.t.

$$\mathbb{E}|X^\alpha_k|^2 \;\leq\; C_k\big(1 + \mathbb{E}|\xi|^2 + \sum_{j=0}^{k-1} \mathbb{E}|\alpha_j|^2\big). \tag{2.2.2}$$

The cost functional associated to the system (2.2.1) over a finite horizon $n \in \mathbb{N} \setminus \{0\}$ is:

$$J(\alpha) \;:=\; \mathbb{E}\Big[\sum_{k=0}^{n-1} f_k(X^\alpha_k, \mathbb{P}_{X^\alpha_k}, \alpha_k, \mathbb{P}_{\alpha_k}) \;+\; g(X^\alpha_n, \mathbb{P}_{X^\alpha_n})\Big], \tag{2.2.3}$$

for any square integrable $\mathbb{F}$-adapted processes α valued in A, where the running cost functions f_k, $k = 0, \ldots, n-1$, are measurable real-valued functions on $E \times \mathcal{P}_2(E) \times A \times \mathcal{P}_2(A)$, and the terminal cost function g is a real-valued measurable function on $E \times \mathcal{P}_2(E)$. We shall assume

(H2) There exist some positive constant C_g and for any $k = 0, \ldots, n-1$, some positive constant $C_{k,f}$ s.t. for all $(x, a, \mu, \lambda) \in E \times A \times \mathcal{P}_2(E) \times \mathcal{P}_2(A)$:

$$\begin{aligned} \big|f_k(x, \mu, a, \lambda)\big| \;&\leq\; C_{k,f}(1 + |x|^2 + |a|^2 + \|\mu\|_2^2 + \|\lambda\|_2^2), \\ \big|g(x, \mu)\big| \;&\leq\; C_g(1 + |x|^2 + \|\mu\|_2^2). \end{aligned}$$

Under **(H1)-(H2)**, the cost functional $J(\alpha)$ is well-defined and finite for any admissible control, and the objective is to minimize over all admissible controls the cost functional, i.e. by solving

$$V_0 \quad := \quad \inf_{\alpha} J(\alpha), \tag{2.2.4}$$

and when $V_0 > -\infty$, find an optimal control α^* i.e. achieving the minimum in (2.2.4) if it exists.

Problem (2.2.1)-(2.2.4) arises in the study of collective behaviors of a large number of players (particles) resulting from mean-field interactions: typically, the controlled dynamics of a system of N symmetric particles are given by

$$X_{k+1}^{i,\alpha^i} \quad = \quad F_{k+1}(X_k^{i,\alpha^i}, \frac{1}{N}\sum_{j=1}^N \delta_{X_k^{j,\alpha^i}}, \alpha_k^i, \frac{1}{N}\sum_{j=1}^N \delta_{\alpha_k^j}, \varepsilon_{k+1}^i), \quad i = 1, \ldots, N,$$

by assuming that a center decides of the general same policy $\alpha^i = \alpha$ for all players with same running and terminal gain functions, the propagation of chaos argument from McKean-Vlasov theory (see [Szn91]) states that when the number of players N goes to infinity, the problem of each agent is asymptotically reduced to the problem of a single agent with controlled dynamics (2.2.1) and objective (2.2.4). We refer to [CDL13] for a detailed discussion about optimal control of McKean-Vlasov equations and connection with equilibrium of large populations of individuals with mean-field interactions.

2.3 Dynamic programming

In this section, we make the standing assumptions **(H1)-(H2)**, and our purpose is to show that dynamic programming principle holds for problem (2.2.4), which we would like to combine with some Markov property of the controlled state process. However, notice that the McKean-Vlasov type dependence on the dynamics of the state process rules out the standard Markov property of the controlled process (X_k^α). Actually, this Markov property can be restored by considering its probability law $(\mathbb{P}_{X_k^\alpha})_k$. To be more precise and for the sake of definiteness, we shall restrict ourselves to controls $\alpha = (\alpha_k)_k$ given in closed-loop (or feedback) form:

$$\alpha_k \quad = \quad \tilde{\alpha}_k(X_k^\alpha), \qquad k = 0, \ldots, n-1, \tag{2.3.1}$$

for some deterministic measurable functions $\tilde{\alpha}_k$ of the state. Notice that the feedback control may also depend on the (deterministic) marginal distribution, and it will be indeed the case for the optimal one, but to alleviate notation, we omit this dependence which is implicit through the deterministic function $\tilde{\alpha}_k$. We denote by A^E the set of measurable functions on E valued in A, which satisfy a linear growth condition, and by $\mathcal{A}$ the set of admissible controls α in closed loop form (2.3.1) with $\tilde{\alpha}_k$ in A^E, $k \in \mathbb{N}$. We shall often identify $\alpha \in \mathcal{A}$ with the sequence $(\tilde{\alpha}_k)_k$ in A^E via (2.3.1). Notice that any $\alpha \in \mathcal{A}$ satisfies the square integrability condition, i.e. $\mathbb{E}|\alpha_k|^2 < \infty$, for all k. Indeed from the linear growth condition on $\tilde{\alpha}_k$ in A^E, we have $\mathbb{E}|\alpha_k|^2 \leq C_\alpha(1 + \mathbb{E}|X_k^\alpha|^2)$ for some constant C_α (depending on α), which gives the square integrability condition by (2.2.2).

Next, we show that the initial stochastic control problem can be reduced to a deterministic control problem. Indeed, the key point is to observe by definition of $\mathbb{P}_{X_k^\alpha}$ and noting that $\mathbb{P}_{\alpha_k}$ is the image by $\tilde{\alpha}_k$ of $\mathbb{P}_{X_k^\alpha}$ for a feedback control $\alpha \in \mathcal{A}$, that the gain functional in (2.3.2) can be rewritten as:

$$J(\alpha) \quad = \quad \sum_{k=0}^{n-1} \hat{f}_k(\mathbb{P}_{X_k^\alpha}, \tilde{\alpha}_k) + \hat{g}(\mathbb{P}_{X_n^\alpha}), \tag{2.3.2}$$

where $\hat{f}_k$, $k = 0, \ldots, n-1$, are defined on $\mathcal{P}_2(E) \times A^E$, $\hat{g}$ is defined on $\mathcal{P}_2(E)$ by:

$$\hat{f}_k(\mu, \tilde{\alpha}) := \int_E f_k(x, \mu, \tilde{\alpha}(x), \tilde{\alpha} \star \mu)\mu(dx), \qquad \hat{g}(\mu) := \int_E g(x, \mu)\mu(dx), \tag{2.3.3}$$

and $\tilde{\alpha} \star \mu \in \mathcal{P}_2(A)$ is the

$$\big(\tilde{\alpha} \star \mu\big)(B) = \mu\big(\tilde{\alpha}^{-1}(B)\big), \quad \forall B \in \mathcal{B}(A).$$

Hence, the original problem (2.2.4) is transformed into a deterministic control problem involving the infinite dimensional marginal distribution process. Let us then define the dynamic version for problem (2.2.4):

$$V_k^\alpha := \inf_{\beta \in \mathcal{A}_k(\alpha)} \sum_{j=k}^{n-1} \hat{f}_j(\mathbb{P}_{X_j^\beta}, \tilde{\beta}_j) + \hat{g}(\mathbb{P}_{X_n^\beta}), \quad k = 0, \ldots, n, \tag{2.3.4}$$

for $\alpha \in \mathcal{A}$, where $\mathcal{A}_k(\alpha) = \{\beta \in \mathcal{A} : \beta_j = \alpha_j, j = 0, \ldots, k-1\}$, with the convention that $\mathcal{A}_0(\alpha) = \mathcal{A}$, so that $V_0 = \inf_{\alpha \in \mathcal{A}} J(\alpha)$ is equal to V_0^α. It is clear that $V_k^\alpha < \infty$, and we shall assume that

$$V_k^\alpha > -\infty, \qquad k = 0, \ldots, n, \ \alpha \in \mathcal{A}. \tag{2.3.5}$$

Remark 2.3.1. The finiteness condition (2.3.5) can be checked a priori directly from the assumptions on the model. For example, when f_k, g, hence $\hat{f}_k$, g, $k = 0, \ldots, n-1$, are lower-bounded functions, condition (2.3.5) clearly holds. Another example is the case when $f_k(x, \mu, a, \lambda)$, $k = 0, \ldots, n-1$, and g are lower bounded by a quadratic function in x, μ, and λ, so that by the linear growth condition on $\tilde{\alpha}$,

$$\hat{f}_k(\mu, \tilde{\alpha}) + \hat{g}(x, \mu) \geq -C_k\big(1 + \|\mu\|_2\big), \quad \forall\, \mu \in \mathcal{P}_2(E), \tilde{\alpha} \in A^E,$$

and we are able to derive moment estimates on X_k^α, uniformly in α: $\big\|\mathbb{P}_{X_k^\alpha}\big\|_2^2 = \mathbb{E}[|X_k^\alpha|^2] \leq C_k$, which arises typically when A is bounded from (2.2.2). Then, it is clear that (2.3.5) holds true. Otherwise, this finiteness condition can be checked a posteriori from a verification theorem, see Theorem 2.3.2. □

The DPP for the deterministic control problem (2.3.4) takes the following formulation:

Lemma 2.3.1. *(Dynamic Programming Principle)*

Under (2.3.5)*, we have*

$$\begin{cases} V_n^\alpha = \hat{g}(\mathbb{P}_{X_n^\alpha}) \\ V_k^\alpha = \inf_{\beta \in \mathcal{A}_k(\alpha)} \hat{f}_k(\mathbb{P}_{X_k^\beta}, \tilde{\beta}_k) + V_{k+1}^\beta, \quad k = 0, \ldots, n-1. \end{cases} \tag{2.3.6}$$

Proof. In the context of deterministic control problem, the proof of the DPP is standard and does not require any measurable selection arguments. For sake of completeness and since it is quite elementary, we give it. Denote by $J_k(\alpha)$ the cost functional at time k, i.e.

$$J_k(\alpha) := \sum_{j=k}^{n-1} \hat{f}_k(\mathbb{P}_{X_k^\alpha}, \tilde{\alpha}_k) + \hat{g}(\mathbb{P}_{X_n^\alpha}), \quad k = 0, \ldots, n,$$

so that $V_k^\alpha = \inf_{\beta\in\mathcal{A}_k(\alpha)} J_k(\beta)$, and by W_k^α the r.h.s. of (2.3.6). Then,

$$\begin{aligned} W_k^\alpha &= \inf_{\beta\in\mathcal{A}_k(\alpha)} \big[\hat f_k(\mathbb{P}_{X_k^\beta},\tilde\beta_k) + \inf_{\gamma\in\mathcal{A}_{k+1}(\beta)} J_{k+1}(\gamma)\big] \\ &= \inf_{\beta\in\mathcal{A}_k(\alpha)} \inf_{\gamma\in\mathcal{A}_{k+1}(\beta)} \big[\hat f_k(\mathbb{P}_{X_k^\beta},\tilde\beta_k) + J_{k+1}(\gamma)\big] \\ &= \inf_{\beta\in\mathcal{A}_k(\alpha)} \inf_{\gamma\in\mathcal{A}_{k+1}(\beta)} \big[\hat f_k(\mathbb{P}_{X_k^\gamma},\tilde\gamma_k) + J_{k+1}(\gamma)\big] \\ &= \inf_{\gamma\in\{\mathcal{A}_{k+1}(\beta):\beta\in\mathcal{A}_k(\alpha)\}} J_k(\gamma), \end{aligned}$$

where we used in the third equality the fact that $X_k^\beta = X_k^\gamma$, $\beta_k=\gamma_k$ for $\gamma\in\mathcal{A}_{k+1}(\beta)$. Finally, we notice that $\{\mathcal{A}_{k+1}(\beta):\beta\in\mathcal{A}_k(\alpha)\} = \mathcal{A}_k(\alpha)$. Indeed, the inclusion $\subset$ is clear while for the converse inclusion, it suffices to observe that any γ in $\mathcal{A}_k(\alpha)$ satisfies obviously $\gamma\in\mathcal{A}_{k+1}(\gamma)$. This proves the required equality: $W_k^\alpha = V_k^\alpha$. □

Let us now show how one can simplify the DPP by exploiting the flow property of $(\mathbb{P}_{X_k^\alpha})_k$ for any admissible control α in feedback form $\in\mathcal{A}$. Actually, we can derive the evolution of the controlled deterministic process $(\mathbb{P}_{X_k^\alpha})_k$.

Lemma 2.3.2. *For any admissible control in closed-loop form $\alpha\in\mathcal{A}$, we have*

$$\mathbb{P}_{X_{k+1}^\alpha} = \Phi_{k+1}\big(\mathbb{P}_{X_k^\alpha},\tilde\alpha_k\big), \quad k\in\mathbb{N}, \;\; \mathbb{P}_{X_0^\alpha} = \mathbb{P}_\xi \tag{2.3.7}$$

where Φ_{k+1} is the measurable function defined from $\mathcal{P}_2(E)\times A^E$ into $\mathcal{P}_2(E)$ by:

$$\Phi_{k+1}(\mu,\tilde\alpha)(dx') = \int_E \mu(dx)P_{k+1}(x,\mu,\tilde\alpha(x),\tilde\alpha\star\mu,dx'). \tag{2.3.8}$$

Proof. Fix $\alpha\in\mathcal{A}$. Recall from the definition of the transition probability $P_{k+1}(x,\mu,a,\lambda,dx')$ associated to (2.2.1) that

$$\mathbb{P}\big[X_{k+1}^\alpha\in dx'\big|\mathcal{F}_k\big] = P_{k+1}(X_k^\alpha,\mathbb{P}_{X_k^\alpha},\alpha_k,\mathbb{P}_{\alpha_k},dx'), \quad k\in\mathbb{N}. \tag{2.3.9}$$

For any bounded measurable function φ on E, we have by the law of iterated conditional expectation and (2.3.9):

$$\begin{aligned} \mathbb{E}\Big[\varphi(X_{k+1}^\alpha)\Big] &= \mathbb{E}\Big[\mathbb{E}\big[\varphi(X_{k+1}^\alpha)\big|\mathcal{F}_k\big]\Big] \\ &= \mathbb{E}\Big[\int_E \varphi(x')P_{k+1}(X_k^\alpha,\mathbb{P}_{X_k^\alpha},\alpha_k,\mathbb{P}_{\alpha_k},dx')\Big] \\ &= \mathbb{E}\Big[\int_{E\times E}\varphi(x')P_{k+1}(x,\mathbb{P}_{X_k^\alpha},\tilde\alpha_k(x),\tilde\alpha_k\star\mathbb{P}_{X_k^\alpha},dx')\mathbb{P}_{X_k^\alpha}(dx)\Big] \end{aligned}$$

where we used in the last equality the fact that $\alpha_k=\tilde\alpha_k(X_k^\alpha)$ is in closed loop form, the definition of $\mathbb{P}_{X_k^\alpha}$, and noting that $\mathbb{P}_{\alpha_k} = \tilde\alpha_k\star\mathbb{P}_{X_k^\alpha}$. This shows the required inductive relation for $\mathbb{P}_{X_k^\alpha}$. □

Remark 2.3.2. Relation (2.3.7) is the Fokker-Planck equation in discrete time for the marginal distribution of the controlled process (X_k^α). In absence of control and McKean-Vlasov type dependence, i.e. $P_{k+1}(x,dx')$ does not depend on (μ,a,λ), we retrieve the standard Fokker-Planck equation with a linear function $\Phi_{k+1}(\mu)=\mu P_{k+1}$. In our McKean-Vlasov control context, the function $\Phi_{k+1}(\mu,\tilde\alpha)$ is nonlinear in μ. □

By exploiting the inductive relation (2.3.7) on the controlled process $(\mathbb{P}_{X_k^\alpha})_k$, the calculation of the value processes V_k^α can be reduced to the recursive computation of deterministic functions (called value functions) on $\mathcal{P}(E)$.

Theorem 2.3.1. *(Dynamic programming and value functions)*

Under (2.3.5), we have for any $\alpha \in \mathcal{A}$, $V_k^\alpha = v_k(\mathbb{P}_{X_k^\alpha})$, $k = 0, \dots, n$, where $(v_k)_k$ is the sequence of value functions defined recursively on $\mathcal{P}_2(E)$ by:

$$\begin{cases} v_n(\mu) & = & \hat{g}(\mu) \\ v_k(\mu) & = & \inf_{\tilde{\alpha} \in A^E} \Big[\hat{f}_k(\mu, \tilde{\alpha}) + v_{k+1}\big(\Phi_{k+1}(\mu, \tilde{\alpha})\big) \Big] \end{cases} \tag{2.3.10}$$

for $k = 0, \dots, n-1$, $\mu \in \mathcal{P}_2(E)$.

Proof. First observe that for any $\beta \in \mathcal{A}_k(\alpha)$, $X_k^\beta = X_k^\alpha$, $k = 0, \dots, n$. Let us prove the result by backward induction. For $k = n$, the result clearly holds since $V_n^\alpha = \hat{g}(\mathbb{P}_{X_n^\alpha})$. Suppose now that at time $k+1$, $V_{k+1}^\alpha = v_{k+1}(\mathbb{P}_{X_{k+1}^\alpha})$ for some deterministic function v_{k+1} and any $\alpha \in \mathcal{A}$. Then, from the DPP (2.3.6) and Lemma 2.3.2, we get

$$\begin{aligned} V_k^\alpha & = \inf_{\beta \in \mathcal{A}_k(\alpha)} \hat{f}_k(\mathbb{P}_{X_k^\alpha}, \tilde{\beta}_k) + v_{k+1}(\mathbb{P}_{X_{k+1}^\beta}) \\ & = \inf_{\beta \in \mathcal{A}_k(\alpha)} w_k(\mathbb{P}_{X_k^\alpha}, \tilde{\beta}_k) \end{aligned} \tag{2.3.11}$$

where

$$w_k(\mu, \tilde{\alpha}) \quad := \quad \hat{f}_k(\mu, \tilde{\alpha}) + v_{k+1}\big(\Phi_{k+1}(\mu, \tilde{\alpha})\big).$$

Now, for any $\beta \in \mathcal{A}_k(\alpha)$, and since $\tilde{\beta}_k$ is valued in A^E, we clearly have: $w_k(\mu, \beta_k) \geq \inf_{\tilde{\alpha} \in A^E} w_k(\mu, \tilde{\alpha})$, and so $\inf_{\beta \in \mathcal{A}_k(\alpha)} w_k(\mu, \tilde{\beta}_k) \geq \inf_{\tilde{\alpha} \in A^E} w_k(\mu, \tilde{\alpha})$. Conversely, for any $\tilde{\alpha} \in A^E$, the control β defined by $\beta_j = \alpha_j$, $j \leq k-1$, and $\tilde{\beta}_j = \tilde{\alpha}$ for $j \geq k$, lies in $\mathcal{A}_k(\alpha)$, so: $w_k(\mu, \tilde{\alpha}) \geq \inf_{\beta \in \mathcal{A}_k(\alpha)} w_k(\mu, \tilde{\beta}_k)$, and thus $\inf_{\beta \in \mathcal{A}_k(\alpha)} w_k(\mu, \tilde{\beta}_k) = \inf_{\tilde{\alpha} \in A^E} w_k(\mu, \tilde{\alpha})$. We conclude from (2.3.11) that: $V_k^\alpha = v_k(\mathbb{P}_{X_k^\alpha})$ with $v_k(\mu) = \inf\limits_{\tilde{\alpha} \in A^E} w_k(\mu, \tilde{\alpha})$, i.e. given by (2.3.10). □

Remark 2.3.3. Problem (2.2.4) includes the case where the cost functional in (2.3.2) is a nonlinear function of the expected value of the state process, i.e. the running cost functions and the terminal gain function are in the form: $f_k(X_k^\alpha, \mathbb{P}_{X_k^\alpha}, \alpha_k) = \bar{f}_k(X_k^\alpha, \mathbb{E}[X_k^\alpha], \alpha_k)$, $k = 0, \dots, n-1$, $g(X_n^\alpha, \mathbb{P}_{X_n^\alpha}) = \bar{g}(X_n^\alpha, \mathbb{E}[X_n^\alpha])$, which arise for example in mean-variance problem (see Section 2.4). It is claimed in [BM14] and [Yon13] that Bellman optimality principle does not hold, and therefore the problem is time-inconsistent. This is true when one takes into account only the state process X^α (that is its realization), since it is not Markovian, but as shown in this section, dynamic programming principle holds whenever we consider the marginal distribution as state variable. This gives more information and the price to paid is the infinite-dimensional feature of the marginal distribution state variable. □

We complete the above Bellman's optimality principle with a verification theorem, which gives a sufficient condition for finding an optimal control.

Theorem 2.3.2. *(Verification theorem)* *(i) Suppose we can find a sequence of real-valued functions w_k, $k = 0, \ldots, n$, defined on $\mathcal{P}_2(E)$ and satisfying the dynamic programming relation:*

$$\begin{cases} w_n(\mu) & = & \hat{g}(\mu) \\ w_k(\mu) & = & \inf_{\tilde{\alpha} \in A^E} \Big[\hat{f}_k(\mu, \tilde{\alpha}) + w_{k+1}\big(\Phi_{k+1}(\mu, \tilde{\alpha})\big) \Big] \end{cases} \tag{2.3.12}$$

for $k = 0, \ldots, n-1$, $\mu \in \mathcal{P}_2(E)$. Then $V_k^\alpha = w_k(\mathbb{P}_{X_k^\alpha})$, for all $k = 0, \ldots, n$, $\alpha \in \mathcal{A}$, and thus $w_k = v_k$.

(ii) Moreover, suppose that at any time k and $\mu \in \mathcal{P}(E)$, the infimum in (2.3.12) for $w_k(\mu)$ is attained, by some $\tilde{\alpha}_k^*(., \mu)$ in A^E. Then, by defining by induction the control α^* in feedback form by $\alpha_k^* = \tilde{\alpha}_k^*(X_k^{\alpha^*}, \mathbb{P}_{X_k^{\alpha^*}})$, $k = 0, \ldots, n-1$, we have

$$V_0 \;=\; J(\alpha^*),$$

which means that $\alpha^* \in \mathcal{A}$ is an optimal control.

Proof. (i) Fix some $\alpha \in \mathcal{A}$, and arbitrary $\beta \in \mathcal{A}$ associated to a feedback sequence $(\tilde{\beta}_k)_k$ in A^E. Then, from the dynamic programming relation (2.3.12) for w_k, and recalling the evolution (2.3.7) of the controlled marginal distribution $\mathbb{P}_{X_k^\beta}$, we have

$$w_k(\mathbb{P}_{X_k^\beta}) \;\leq\; \hat{f}_k(\mathbb{P}_{X_k^\beta}, \tilde{\beta}_k) + v_{k+1}(\mathbb{P}_{X_{k+1}^\beta}), \quad k = 0, \ldots, n-1.$$

By induction and since $w_n = \hat{g}$, this gives

$$w_k(\mathbb{P}_{X_k^\beta}) \;\leq\; \sum_{j=k}^{n-1} \hat{f}_j(\mathbb{P}_{X_j^\beta}, \tilde{\beta}_j) + \hat{g}(\mathbb{P}_{X_n^\beta}).$$

By noting that $\mathbb{P}_{X_k^\alpha} = \mathbb{P}_{X_k^\beta}$, when $\beta \in \mathcal{A}_k(\alpha)$, and since β is arbitrary, this proves that $w_k(\mathbb{P}_{X_k^\alpha}) \leq V_k^\alpha$. In particular, $V_k^\alpha > -\infty$, i.e. relation (2.3.5) holds, and then by Theorem 2.3.1, V_k^α is characterized by the sequence of value functions $(v_k)_k$ defined by the DP (2.3.10). This DP obviously defines by backward induction a unique sequence of functions on $\mathcal{P}_2(E)$, hence $w_k = v_k$, $k = 0, \ldots, n$, and therefore $V_k^\alpha = w_k(\mathbb{P}_{X_k^\alpha})$.

(ii) By definition of $\tilde{\alpha}_k^*$ which attains the infimum in (2.3.12), we have

$$w_k\big(\mathbb{P}_{X_k^{\alpha^*}}\big) \;=\; \hat{f}_k(\mathbb{P}_{X_k^{\alpha^*}}, \tilde{\alpha}_k^*(., \mathbb{P}_{X_k^{\alpha^*}})) + w_{k+1}\big(\mathbb{P}_{X_{k+1}^{\alpha^*}}\big), \quad k = 0, \ldots, n-1.$$

By induction this implies that

$$V_0 \;=\; w_0(\mathbb{P}_\xi) \;=\; \sum_{k=0}^{n-1} \hat{f}_k\big(\mathbb{P}_{X_k^{\alpha^*}}, \tilde{\alpha}_k^*(., \mathbb{P}_{X_k^{\alpha^*}})\big) \;+\; \hat{g}(\mathbb{P}_{X_n^{\alpha^*}}) \;=\; J(\alpha^*),$$

which shows that α^* is an optimal control. □

The above verification theorem, which consists in solving the dynamic programming relation (2.3.12), is useful to check a posteriori the finiteness condition (2.3.5), and can be applied in practice fo find explicit solutions to some McKean-Vlasov control problems, as investigated in the next section.

2.4 Applications

2.4.1 Special cases

We consider some particular cases, and provide the special forms of the DPP.

2.4.1.1 No mean-field interaction

We first consider the standard control case where there is no mean-field interaction in the dynamics of the state process, i.e. $F_{k+1}(x, a, \varepsilon_{k+1})$, hence $P_{k+1}(x, a, dx')$ do not depend on μ, λ, as well as in the cost functions $f_k(x, a)$ and $g(x)$. For simplicity, we assume that A is a bounded set, which ensures the finiteness condition (2.3.5). In this case, we can see that the value functions v_k are in the form

$$v_k(\mu) \quad = \quad \int_E \tilde{v}_k(x)\mu(dx), \quad k = 0, \ldots, n, \tag{2.4.1}$$

where the functions $\tilde{v}_k$ defined on E satisfy the classical dynamic programming principle:

$$\begin{cases} \tilde{v}_n(x) & = & g(x) \\ \tilde{v}_k(x) & = & \inf_{a \in A} \Big[f_k(x, a) + \mathbb{E}\big[\tilde{v}_{k+1}(X_{k+1}^\alpha)\big|X_k^\alpha = x, \alpha_k = a\big]\Big], \end{cases} \tag{2.4.2}$$

for $k = 0, \ldots, n-1$. Let us check this result by backward induction. This holds true for $k = n$ since $v_n(\mu)$ $= \hat{g}(\mu) = \int_E g(x)\mu(dx)$. Suppose that (2.4.1) holds true at time $k+1$. Then, from the DPP (2.3.10), (2.3.8) and Fubini's theorem, we have

$$\begin{aligned} v_k(\mu) &= \inf_{\tilde{\alpha} \in A^E} \Big[\int_E f_k(x, \tilde{\alpha}(x))\mu(dx) + \int_E \tilde{v}_{k+1}(x')\Phi_{k+1}(\mu, \tilde{\alpha})(dx')\Big] \\ &= \inf_{\tilde{\alpha} \in A^E} \Big[\int_E \big[f_k(x, \tilde{\alpha}(x)) + \int_E \tilde{v}_{k+1}(x')P_{k+1}(x, \tilde{\alpha}(x), dx')\big]\mu(dx)\Big] \\ &= \inf_{\tilde{\alpha} \in A^E} \int_E \tilde{w}_k(x, \tilde{\alpha}(x))\mu(dx) \end{aligned}$$

where we set $\tilde{w}_k(x, a) = f_k(x, a) + \int_E \tilde{v}_{k+1}(x')P_{k+1}(x, a, dx')$. Now, we observe that

$$\inf_{\tilde{\alpha} \in A^E} \int_E \tilde{w}_k(x, \tilde{\alpha}(x))\mu(dx) \quad = \quad \int_E \inf_{a \in A} \tilde{w}_k(x, a)\mu(dx). \tag{2.4.3}$$

Indeed, since for any $\tilde{\alpha} \in A^E$, the value $\tilde{\alpha}(x)$ is valued in A for any $x \in E$, it is clear that the inequality $\geq$ in (2.4.3) holds true. Conversely, for any $\varepsilon > 0$, and $x \in E$, one can find $\tilde{\alpha}^\varepsilon(x)$ in A such that

$$\tilde{w}_k(x, \tilde{\alpha}^\varepsilon(x)) \quad \leq \quad \inf_{a \in A} \tilde{w}_k(x, a) + \varepsilon.$$

By a measurable selection theorem, the map $x \mapsto \tilde{\alpha}^\varepsilon(x)$ can be chosen measurable, and since A is bounded, the function $\tilde{\alpha}^\varepsilon$ lies in A^E. It follows that

$$\inf_{\tilde{\alpha} \in A^E} \int_E \tilde{w}_k(x, \tilde{\alpha}(x))\mu(dx) \;\leq\; \int_E \tilde{w}_k(x, \tilde{\alpha}^\varepsilon(x))\mu(dx) \;\leq\; \int_E \inf_{a \in A} \tilde{w}_k(x, a)\mu(dx) + \varepsilon,$$

which shows (2.4.3) since ε is arbitrary. Therefore, we have $v_k(\mu) = \int_E \tilde{v}_k(x)\mu(dx)$ with

$$\begin{aligned} \tilde{v}_k(x) &= \inf_{a \in A} \tilde{w}_k(x, a) \\ &= \inf_{a \in A} \big[f_k(x, a) + \int_E \tilde{v}_{k+1}(x')P_{k+1}(x, a, dx')\big], \end{aligned}$$

which is the relation (2.4.2) at time k from the definition of the transition probability P_{k+1}.

2.4.1.2 First order interactions

We consider the case of first order interactions, i.e. the dependence of the model coefficients upon the probability measures is linear in the sense that for any $(x, \mu, a) \in E \times \mathcal{P}_2(E) \times A$, $\tilde{\alpha} \in A^E$,

$$\begin{aligned} P_{k+1}(x, \mu, a, \tilde{\alpha} \star \mu, dx') &= \int_E \tilde{P}_{k+1}(x, y, a, \tilde{\alpha}(y), dx')\mu(dy), \\ f_k(x, \mu, a, \tilde{\alpha} \star \mu) &= \int_E \tilde{f}_k(x, y, a, \tilde{\alpha}(y))\mu(dy), \quad g(x, \mu) = \int_E \tilde{g}(x, y)\mu(dy), \end{aligned}$$

for some transition probability kernels $\tilde{P}_{k+1}$ from $E \times E \times A \times A$ into E, measurable functions $\tilde{f}_k$ defined on $E \times E \times A \times A$, $k = 0, \ldots, n-1$, and $\tilde{g}$ defined on $E \times E$. In this case, the value functions v_k are in the reduced form

$$v_k(\mu) = \int_{E^{2^{n-k+1}}} \tilde{v}_k(\mathbf{x}_{2^{n-k+1}})\mu(d\mathbf{x}_{2^{n-k+1}}), \quad k = 0, \ldots, n, \tag{2.4.4}$$

where we denote by $\mathbf{x}_p$ the p-tuple $(x_1, \ldots, x_p) \in E^p$, by $\mu(d\mathbf{x}_p)$ the product measure $\mu(dx_1) \otimes \ldots \otimes \mu(dx_p)$, and the functions $\tilde{v}_k$ are defined recursively on $E^{2^{n-k+1}}$ by

$$\begin{cases} \tilde{v}_n(x, y) = \tilde{g}(x, y) \\ \tilde{v}_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}) = \inf_{\tilde{\alpha} \in A^E} \Big[\tilde{f}_k(x_1, y_1, \tilde{\alpha}(x_1), \tilde{\alpha}(y_1)) \\ \qquad + \int_{E^{2^{n-k}}} \tilde{v}_{k+1}(\mathbf{x}'_{2^{n-k}}) \mathbf{\tilde{P}_{k+1}}(\mathbf{x_{2^{n-k}}}, \mathbf{y_{2^{n-k}}}, \tilde{\alpha}(\mathbf{x_{2^{n-k}}}), \tilde{\alpha}(\mathbf{y_{2^{n-k}}}), \mathbf{dx'_{2^{n-k}}}) \Big], \end{cases}$$

where we set

$$\begin{aligned} & \mathbf{\tilde{P}_{k+1}}(\mathbf{x_p}, \mathbf{y_p}, \tilde{\alpha}(\mathbf{x_p}), \tilde{\alpha}(\mathbf{y_p}), \mathbf{dx'_p}) \\ = \; & \tilde{P}_{k+1}(x_1, y_1, \tilde{\alpha}(x_1), \tilde{\alpha}(y_1), dx'_1) \otimes \ldots \otimes \tilde{P}_{k+1}(x_p, y_p, \tilde{\alpha}(x_p), \tilde{\alpha}(y_p), dx'_p). \end{aligned}$$

Let us check this result by backward induction. This holds true for $k = n$ since $\hat{v}_n(\mu) = \int_{E^2} \tilde{g}(x, y)\mu(dx)\mu(dy)$. Suppose that (2.4.4) holds true at time $k+1$. Then, from the DPP (2.3.10), (2.3.8) and Fubini's theorem, we have

$$\begin{aligned} v_k(\mu) &= \inf_{\tilde{\alpha} \in A^E} \{ \int_{E^2} \tilde{f}_k(x_1, y_1, \tilde{\alpha}(x_1), \tilde{\alpha}(y_1))\mu(dx_1)\mu(dy_1) \\ &\quad + \int_{E^{3*2^{n-k}}} \tilde{v}_{k+1}(\mathbf{x}'_{2^{n-k}}) \mathbf{\tilde{P}_{k+1}}(\mathbf{x_{2^{n-k}}}, \mathbf{y_{2^{n-k}}}, \tilde{\alpha}(\mathbf{x_{2^{n-k}}}), \tilde{\alpha}(\mathbf{y_{2^{n-k}}}), \mathbf{dx'_{2^{n-k}}})\mu(d\mathbf{x}_{2^{n-k}})\mu(d\mathbf{y}_{2^{n-k}}) \} \\ &= \inf_{\tilde{\alpha} \in A^E} \int_{E^{2^{n-k+1}}} \tilde{w}_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}, \tilde{\alpha}(\mathbf{x}_{2^{n-k}}), \tilde{\alpha}(\mathbf{y}_{2^{n-k}}))\mu(d\mathbf{x}_{2^{n-k}})\mu(d\mathbf{y}_{2^{n-k}}) \end{aligned}$$

where we set

$$\begin{aligned} \tilde{w}_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}, \tilde{\alpha}(\mathbf{x}_{2^{n-k}}), \tilde{\alpha}(\mathbf{y}_{2^{n-k}})) &= \tilde{f}_k(x_1, y_1, \tilde{\alpha}(x_1), \tilde{\alpha}(y_1)) \\ &\quad + \int_{E^{2^{n-k}}} \tilde{v}_{k+1}(\mathbf{x}'_{2^{n-k}}) \mathbf{\tilde{P}_{k+1}}(\mathbf{x_{2^{n-k}}}, \mathbf{y_{2^{n-k}}}, \tilde{\alpha}(\mathbf{x_{2^{n-k}}}), \tilde{\alpha}(\mathbf{y_{2^{n-k}}}), \mathbf{dx'_{2^{n-k}}}). \end{aligned}$$

Now, proceeding the similar argument as no mean-field interaction case, we observe that

$$\begin{aligned} & \inf_{\tilde{\alpha} \in A^E} \int_{E^{2^{n-k+1}}} \tilde{w}_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}, \tilde{\alpha}(\mathbf{x}_{2^{n-k}}), \tilde{\alpha}(\mathbf{y}_{2^{n-k}}))\mu(d\mathbf{x}_{2^{n-k}})\mu(d\mathbf{y}_{2^{n-k}}) \\ = \; & \int_{E^{2^{n-k+1}}} \inf_{\tilde{\alpha} \in A^E} \tilde{w}_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}, \tilde{\alpha}(\mathbf{x}_{2^{n-k}}), \tilde{\alpha}(\mathbf{y}_{2^{n-k}}))\mu(d\mathbf{x}_{2^{n-k}})\mu(d\mathbf{y}_{2^{n-k}}). \end{aligned}$$

Therefore, we have $v_k(\mu) = \int_{E^{2^{n-k+1}}} v_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}})\mu(d\mathbf{x}_{2^{n-k}})\mu(d\mathbf{y}_{2^{n-k}})$ with

$$\begin{aligned} v_k(\mathbf{x}_{2^{n-k}}, \mathbf{y}_{2^{n-k}}) &= \inf_{\tilde\alpha \in A^E} \Big[\tilde f_k(x_1, y_1, \tilde\alpha(x_1), \tilde\alpha(y_1)) \\ &\quad + \int_{E^{2^{n-k}}} \tilde v_{k+1}(\mathbf{x}'_{2^{n-k}}) \tilde{\mathbf{P}}_{\mathbf{k+1}}(\mathbf{x}_{\mathbf{2^{n-k}}}, \mathbf{y}_{\mathbf{2^{n-k}}}, \tilde\alpha(\mathbf{x}_{\mathbf{2^{n-k}}}), \tilde\alpha(\mathbf{y}_{\mathbf{2^{n-k}}}), \mathbf{dx}'_{\mathbf{2^{n-k}}}) \Big], \end{aligned}$$

2.4.2 Linear-quadratic McKean-Vlasov control problem

We consider a general multivariate linear McKean-Vlasov dynamics in $E = \mathbb{R}^d$ with control valued in $A = \mathbb{R}^m$:

$$\begin{aligned} X^\alpha_{k+1} &= \big(B_k X^\alpha_k + \bar B_k \mathbb{E}[X^\alpha_k] + C_k \alpha_k + \bar C_k \mathbb{E}[\alpha_k]\big) \\ &\quad + \big(D_k X^\alpha_k + \bar D_k \mathbb{E}[X^\alpha_k] + H_k \alpha_k + \bar H_k \mathbb{E}[\alpha_k]\big)\varepsilon_{k+1}, \quad k = 0, \ldots, n-1, \end{aligned} \tag{2.4.5}$$

starting from $X^\alpha_0 = \xi$, where B_k, $\bar B_k$, D_k, $\bar D_k$ are constant matrices in $\mathbb{R}^{d\times d}$, C_k, $\bar C_k$, H_k, $\bar H_k$ are constant matrices in $\mathbb{R}^{d\times m}$, and (ε_k) is a sequence of i.i.d. random variables with distribution $\mathcal{N}(0,1)$, independent of ξ. The quadratic cost functional to be minimized is given by

$$\begin{aligned} J(\alpha) &= \mathbb{E}\Big[\sum_{k=0}^{n-1} \big[(X^\alpha_k)^\intercal Q_k X^\alpha_k + \big(\mathbb{E}[X^\alpha_k]\big)^\intercal \bar Q_k \mathbb{E}[X^\alpha_k] + L_k^\intercal X^\alpha_k + \bar L_k^\intercal \mathbb{E}[X^\alpha_k] \\ &\qquad + \alpha_k^\intercal R_k \alpha_k + \big(\mathbb{E}[\alpha_k]\big)^\intercal \bar R_k \mathbb{E}[\alpha_k]\big] \\ &\qquad + \big(X^\alpha_n\big)^\intercal Q X^\alpha_n + \big(\mathbb{E}[X^\alpha_n]\big)^\intercal \bar Q \mathbb{E}[X^\alpha_n] + L^\intercal X^\alpha_n + \bar L^\intercal \mathbb{E}[X^\alpha_k]\Big], \end{aligned} \tag{2.4.6}$$

for some constants matrices Q_k, $\bar Q_k$, Q, $\bar Q$, in $\mathbb{R}^{d\times d}$, R_k, $\bar R_k$ in $\mathbb{R}^{m\times m}$, and vectors L_k, $\bar L_k$, L, $\bar L \in \mathbb{R}^d$, $k = 0, \ldots, n-1$. Since the cost functions are real-valued, we may assume w.l.o.g. that all these matrices Q_k, $\bar Q_k$, Q, $\bar Q$, R_k and $\bar R_k$ are symmetric. This model is in the form (2.2.1) and associated to a transition probability satisfying:

$$\begin{aligned} P_{k+1}(x, \mu, a, \lambda, dx') &\rightsquigarrow \mathcal{N}\Big(M_k(x,\mu,a,\lambda); \Sigma_k(x,\mu,a,\lambda)\Sigma_k(x,\mu,a,\lambda)^\intercal\Big) \\ M_k(x,\mu,a,\lambda) &= B_k x + \bar B_k \bar\mu + C_k a + \bar C_k \bar\lambda \\ \Sigma_k(x,\mu,a,\lambda) &= D_k x + \bar D_k \bar\mu + H_k a + \bar H_k \bar\lambda \end{aligned} \tag{2.4.7}$$

where we set for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ (resp. $\mathcal{P}_2(\mathbb{R}^m)$) symmetric matrix $\Lambda \in \mathbb{R}^{d\times d}$ (resp. in $\mathbb{R}^{m\times m}$):

$$\bar\mu := \int x\mu(dx), \quad \bar\mu_2(\Lambda) := \int x^\intercal \Lambda x \mu(dx), \quad \mathrm{Var}(\mu)(\Lambda) := \bar\mu_2(\Lambda) - \bar\mu^\intercal \Lambda \bar\mu,$$

and in the form (2.3.2), hence (2.3.2) for feedback controls, with

$$\begin{aligned} \hat f_k(\mu, \tilde\alpha) &= \mathrm{Var}(\mu)(Q_k) + \bar\mu^\intercal (Q_k + \bar Q_k)\bar\mu + (L_k + \bar L_k)^\intercal \bar\mu \\ &\quad + \mathrm{Var}(\tilde\alpha \star \mu)(R_k) + \overline{\tilde\alpha \star \mu}^\intercal (R_k + \bar R_k)\overline{\tilde\alpha \star \mu} \\ \hat g(\mu) &= \mathrm{Var}(\mu)(Q) \ + \ \bar\mu^\intercal (Q + \bar Q)\bar\mu \ + \ (L + \bar L)^\intercal \bar\mu. \end{aligned}$$

We look for candidate w_k, $k = 0, \ldots, n$, of values functions satisfying the dynamic programming principle (2.3.10), in the quadratic form:

$$w_k(\mu) = \mathrm{Var}(\mu)(\Lambda_k) + \bar\mu^\intercal \Gamma_k \bar\mu + \rho_k^\intercal \bar\mu + \chi_k, \tag{2.4.8}$$

for some constant symmetric matrices Λ_k and Γ_k in $\mathbb{R}^{d\times d}$, vector $\rho_k \in \mathbb{R}^d$ and real χ_k to be determined below. We proceed by backward induction. For $k = n$, we see that $w_k = \hat{g}$ $(= v_k)$ iff

$$\Lambda_n \;=\; Q, \qquad \Gamma_n \;=\; Q + \bar{Q}, \;\; \rho_n \;=\; L + \bar{L}, \; \chi_n \;=\; 0. \tag{2.4.9}$$

Now, suppose that the form (2.4.8) holds true at time $k+1$, and observe from (2.3.8) and (2.4.7) that for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\tilde{\alpha} \in A^E$, $\Lambda \in \mathbb{R}^{d\times d}$, we have by Fubini's theorem:

$$\begin{aligned}
\overline{\Phi_{k+1}(\mu,\tilde{\alpha})} &= \int_{\mathbb{R}^d} \mathbb{E}\big[Y(x,\mu,\tilde{\alpha})\big]\mu(dx) \\
\overline{\Phi_{k+1}(\mu,\tilde{\alpha})}_2(\Lambda) &= \int_{\mathbb{R}^d} \mathbb{E}\big[Y(x,\mu,\tilde{\alpha})^\intercal \Lambda Y(x,\mu,\tilde{\alpha})\big]\mu(dx),
\end{aligned}$$

where $Y(x,\mu,\tilde{\alpha}) \rightsquigarrow \mathcal{N}\Big(M_k(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu); \Sigma_k(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu)\Sigma_k(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu)^\intercal\Big)$. Therefore,

$$\overline{\Phi_{k+1}(\mu,\tilde{\alpha})} \;=\; (B_k + \bar{B}_k)\bar{\mu} + (C_k + \bar{C}_k)\overline{\tilde{\alpha}\star\mu},$$

and after some tedious but straightforward calculation:

$$\begin{aligned}
\mathrm{Var}(\Phi_{k+1}(\mu,\tilde{\alpha}))(\Lambda) &= \overline{\Phi_{k+1}(\mu,\tilde{\alpha})}_2(\Lambda) - \overline{\Phi_{k+1}(\mu,\tilde{\alpha})}^\intercal \Lambda \overline{\Phi_{k+1}(\mu,\tilde{\alpha})} \\
&= \int_{\mathbb{R}^d} \big[\Sigma_k(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu)^\intercal \Lambda \Sigma(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu) \\
&\qquad + \; M_k(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu)^\intercal \Lambda M(x,\mu,\tilde{\alpha}(x),\tilde{\alpha}\star\mu)\big]\mu(dx) \\
&\quad - \Big((B_k + \bar{B}_k)\bar{\mu} + (C_k + \bar{C}_k)\overline{\tilde{\alpha}\star\mu}\Big)^\intercal \Lambda \Big((B_k + \bar{B}_k)\bar{\mu} + (C_k + \bar{C}_k)\overline{\tilde{\alpha}\star\mu}\Big) \\
&= \mathrm{Var}(\mu)(B_k^\intercal \Lambda B_k + D_k^\intercal \Lambda D_k) + \bar{\mu}^\intercal (D_k + \bar{D}_k)^\intercal \Lambda (D_k + \bar{D}_k)\bar{\mu} \\
&\quad + \mathrm{Var}(\tilde{\alpha} * \mu)(H_k^\intercal \Lambda H_k + C_k^\intercal \Lambda C_k) \\
&\quad + \overline{\tilde{\alpha}\star\mu}^\intercal (H_k + \bar{H}_k)^\intercal \Lambda (H_k + \bar{H}_k)\overline{\tilde{\alpha}\star\mu} \\
&\quad + \; 2\int_{\mathbb{R}^d} (x - \bar{\mu})^\intercal (D_k^\intercal \Lambda H_k + B_k^\intercal \Lambda C_k)\tilde{\alpha}(x)\mu(dx) \\
&\quad + \; 2\bar{\mu}^\intercal (D_k + \bar{D}_k)^\intercal \Lambda (H_k + \bar{H}_k) \int_{\mathbb{R}^d} \overline{\tilde{\alpha}\star\mu}.
\end{aligned}$$

Then, w_k satisfies the DPP (2.3.10) iff

$$\begin{aligned}
w_k(\mu) &= \inf_{\tilde{\alpha}\in A^E} \Big[\hat{f}_k(\mu,\tilde{\alpha}) + \mathrm{Var}(\Phi_{k+1}(\mu,\tilde{\alpha}))(\Lambda_{k+1}) + \overline{\Phi_{k+1}(\mu,\tilde{\alpha})}^\intercal \Gamma_{k+1} \overline{\Phi_{k+1}(\mu,\tilde{\alpha})}\Big] && (2.4.10) \\
&= \mathrm{Var}(\mu)(Q_k + B_k^\intercal \Lambda_{k+1} B_k + D_k^\intercal \Lambda_{k+1} D_k) + \inf_{\tilde{\alpha}\in A^E} G^\mu_{k+1}(\tilde{\alpha}) \\
&\quad + \bar{\mu}^\intercal \big(Q_k + \bar{Q}_k + (D_k + \bar{D}_k)^\intercal \Lambda_{k+1}(D_k + \bar{D}_k) + (B_k + \bar{B}_k)^\intercal \Gamma_{k+1}(B_k + \bar{B}_k)\big)\bar{\mu} \\
&\quad + \big(L_k + \bar{L}_k + (B_k + \bar{B}_k)^\intercal \rho_{k+1}\big)^\intercal \bar{\mu} + \chi_{k+1}, && (2.4.11)
\end{aligned}$$

where we define the function $G^\mu_{k+1} : L^2(\mu; A) \mapsto \mathbb{R}$ by

$$\begin{aligned}
G^\mu_{k+1}(\tilde{\alpha}) &= \mathrm{Var}(\tilde{\alpha}\star\mu)(V_k) \;+\; \overline{\tilde{\alpha}\star\mu}^\intercal W_k \overline{\tilde{\alpha}\star\mu} \;+\; 2\int_{\mathbb{R}^d} (x - \bar{\mu})^\intercal S_k \tilde{\alpha}(x)\mu(dx) \\
&\quad + 2\bar{\mu}^\intercal T_k \overline{\tilde{\alpha}\star\mu} \;+\; \rho_{k+1}^\intercal (C_k + \bar{C}_k)\overline{\tilde{\alpha}\star\mu},
\end{aligned} \tag{2.4.12}$$

and we set $V_k = V_k(\Lambda_{k+1})$, $W_k = W_k(\Lambda_{k+1},\Gamma_{k+1})$, $S_k = S_k(\Lambda_{k+1})$, $T_k = T_k(\Lambda_{k+1},\Gamma_{k+1})$, with

$$\left\{ \begin{array}{ccl} V_k(\Lambda_{k+1}) & = & R_k + H_k^\intercal \Lambda_{k+1} H_k + C_k^\intercal \Lambda_{k+1} C_k; \\ W_k(\Lambda_{k+1},\Gamma_{k+1}) & = & R_k + \bar R_k + (C_k+\bar C_k)^\intercal \Gamma_{k+1}(C_k+\bar C_k) + (H_k+\bar H_k)^\intercal \Lambda_{k+1}(H_k+\bar H_k) \\ S_k(\Lambda_{k+1}) & = & D_k^\intercal \Lambda_{k+1} H_k + B_k^\intercal \Lambda_{k+1} C_k; \\ T_k(\Lambda_{k+1},\Gamma_{k+1}) & = & (D_k+\bar D_k)^\intercal \Lambda_{k+1}(H_k+\bar H_k) + (B_k+\bar B_k))^\intercal \Gamma_{k+1}(C_k+\bar C_k). \end{array} \right. \tag{2.4.13}$$

Here, $L^2(\mu;A) \supset A^E$ is the Hilbert space of measurable functions on $E=\mathbb{R}^d$ valued in $A=\mathbb{R}^m$ and square integrable w.r.t. $\mu \in \mathcal{P}_2(E)$.

We now search for the infimum of the function G^μ_{k+1}, and shall make the following assumptions on the symmetric matrices of the quadratic cost functional and on the coefficients of the state dynamics:

(c0)

$$\left\{ \begin{array}{lll} Q \;\geq\; 0,\; Q+\bar Q \;\geq\; 0, & Q_k \;\geq\; 0,\; Q_k+\bar Q_k \;\geq\; 0, & \\ & R_k \;\geq\; 0,\; R_k+\bar R_k \;\geq\; 0, & k=0,\ldots,n-1, \end{array} \right.$$

and for all $k=0,\ldots,n-1$,

(c1) $R_k > 0$ or $[C_k$ of full rank, $Q_{k+1} > 0]$, or $[H_k$ of full rank, $Q_{k+1} > 0]$,

(c2) $R_k+\bar R_k > 0$ or $[C_k+\bar C_k$ of full rank, $Q_{k+1}+\bar Q_{k+1} > 0]$, or $[H_k+\bar H_k$ of full rank, $Q_{k+1} > 0]$.

Conditions **(c0)-(c1)-(c2)** is slightly weaker than the condition in [ELN13] (see their Theorem 3.1), where the condition **(c0)** is strengthened to $R_k > 0$ and $R_k+\bar R_k > 0$ for all $k=0,\ldots,n-1$, for ensuring the existence of an optimal control. We relax this positivity condition with the conditions **(c1)-(c2)** in order to include the case of mean-variance problem (see the example at the end of this section). Actually, as we shall see in Remark 2.4.1, these conditions will guarantee that for Λ_k, Γ_k to be determined below, the function G^μ_{k+1} is convex and coercive on $L^2(\mu;A)$ for any $k=0,\ldots,n-1$. For the moment, we derive after some straightforward calculation the Gateaux derivative of G^μ_{k+1} at $\tilde\alpha$ in the direction $\beta \in L^2(\mu;A)$, which is given by:

$$DG^\mu_{k+1}(\tilde\alpha;\beta) \;:=\; \lim_{\varepsilon\to 0} \frac{G^\mu_{k+1}(\tilde\alpha+\varepsilon\beta) - G^\mu_{k+1}(\tilde\alpha)}{\varepsilon} \;=\; \int_{\mathbb{R}^d} g_{k+1}(x,\tilde\alpha)\beta(x)\mu(dx)$$

with

$$\begin{array}{rcl} g_{k+1}(x,\tilde\alpha) & = & 2\tilde\alpha(x)^\intercal V_k + 2\overline{\tilde\alpha\star\mu}^\intercal\big(W_k - V_k\big) \\ & & + 2(x-\mu)^\intercal S_k + 2\bar\mu^\intercal T_k + \rho_{k+1}^\intercal (C_k+\bar C_k). \end{array}$$

We shall check later in Remark 2.4.1 that V_k and W_k are positive symmetric matrices, hence invertible. We thus see that $DG^\mu_{k+1}(\tilde\alpha;.)$ vanishes for $\tilde\alpha = \tilde\alpha_k^*(.,\mu)$ s.t. $g_{k+1}(x,\tilde\alpha_k^*(.,\mu)) = 0$ for all $x\in\mathbb{R}^d$, which gives:

$$\tilde\alpha_k^*(x,\mu) \;=\; -V_k^{-1}S_k^\intercal(x-\bar\mu) \;-\; W_k^{-1}T_k^\intercal\bar\mu \;-\; \frac{1}{2}W_k^{-1}(C_k+\bar C_k)^\intercal\rho_{k+1} \tag{2.4.14}$$

and then after some straightforward caculation:

$$\begin{array}{rcl} G^\mu_{k+1}(\tilde\alpha_k^*(.,\mu)) & = & -\mathrm{Var}(\mu)\big(S_k V_k^{-1} S_k^\intercal\big) \;-\; \bar\mu^\intercal\big(T_k W_k^{-1} T_k^\intercal\big)\bar\mu \;-\; \bar\mu^\intercal T_k W_k^{-1}(C_k+\bar C_k)^\intercal\rho_{k+1} \\ & & -\; \frac{1}{4}\rho_{k+1}^\intercal(C_k+\bar C_k)W_k^{-1}(C_k+\bar C_k)^\intercal\rho_{k+1}. \end{array}$$

Assuming for the moment that $\tilde{\alpha}^*_k(.,\mu)$ attains the infimum of G^μ_{k+1} (this is a consequence of the convexity and coercivity of G^μ_{k+1} shown in Remark 2.4.1), and plugging the above expression in (2.4.10), we see that w_k is like the function $\mu \mapsto G^\mu_{k+1}(\tilde{\alpha}^*_k(.,\mu))$, a linear combination of terms in $\mathrm{Var}(\mu)(.)$, $\bar{\mu}^\intercal(.)\bar{\mu}$, and by identification with the form (2.4.8), we obtain an inductive relation for Λ_k, Γ_k, ρ_k, χ_k:

$$\left\{\begin{array}{rcl} \Lambda_k &=& Q_k + B_k^\intercal \Lambda_{k+1} B_k + D_k^\intercal \Lambda_{k+1} D_k - S_k(\Lambda_{k+1}) V_k^{-1}(\Lambda_{k+1}) S_k^\intercal(\Lambda_{k+1}) \\ \Gamma_k &=& (Q_k + \bar{Q}_k) + (B_k + \bar{B}_k)^\intercal \Gamma_{k+1}(B_k + \bar{B}_k) + (D_k + \bar{D}_k)^\intercal \Lambda_{k+1}(D_k + \bar{D}_k) \\ && - T_k(\Lambda_{k+1},\Gamma_{k+1}) W_k^{-1}(\Lambda_{k+1},\Gamma_{k+1}) T_k^\intercal(\Lambda_{k+1},\Gamma_{k+1}) \\ \rho_k &=& L_k + \bar{L}_k + \big[(B_k + \bar{B}_k) - (C_k + \bar{C}_k) W_k^{-1}(\Lambda_{k+1},\Gamma_{k+1}) T_k^\intercal(\Lambda_{k+1},\Gamma_{k+1})\big]\rho_{k+1} \\ \chi_k &=& \chi_{k+1} - \frac{1}{4}\rho_{k+1}^\intercal (C_k + \bar{C}_k) W_k^{-1}(\Lambda_{k+1},\Gamma_{k+1})(C_k + \bar{C}_k)^\intercal \rho_{k+1}. \end{array}\right. \tag{2.4.15}$$

for all $k = 0, \ldots, n-1$, starting from the terminal condition (2.4.9). The relations for (Λ_k, Γ_k) in (2.4.15) are two algebraic Riccati difference equations, while the equations for ρ_k and χ_k are linear equations once (Λ_k, Γ_k) are determined. This system (2.4.15) is the same as the one obtained in [ELN13]. In the particular mean-variance problem considered at the end of this section, we can obtain explicit closed-form expressions for the solutions $(\Lambda_k, \Gamma_k, \rho_k, \chi_k)$ to this Riccati system. However, in general, there are no closed-form formulae, and these quantities are simply computed by induction.

In the following remark, we check the issues that have left open up to now.

Remark 2.4.1. Let conditions **(c0)-(c1)-(c2)** hold. We prove by backward induction that for all $k = 1, \ldots, n$, the matrices $V_{k-1} = V_{k-1}(\Lambda_k)$, $W_{k-1} = W_{k-1}(\Lambda_k, \Gamma_k)$ are symmetric positive, hence invertible, with (Λ_k, Γ_k) given by (2.4.15), together with the nonnegativity of the symmetric matrices Λ_k, Γ_k, which will immediately gives the convexity and coercivity of the function G^μ_k in (4.5.7) for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

At time $k = n$, we have $\Lambda_n = Q \geq 0$, $\Gamma_n = Q + \bar{Q} \geq 0$, and thus from (2.4.13), $V_{n-1} = V_{n-1}(\Lambda_n)$, $W_{n-1} = W_{n-1}(\Lambda_n, \Gamma_n)$ are symmetric positive under **(c0)-(c1)-(c2)**. Now, suppose that the assertion is true at time $k+1$, i.e. V_k, W_k are symmetric positive, and Λ_{k+1}, Γ_{k+1} are symmetric nonnegative. Then, it is clear from (2.4.15) that Λ_k and Γ_k are symmetric, and noting that they can be rewritten from the expression of V_k, W_k, S_k, T_k in (2.4.13) as

$$\left\{\begin{array}{rcl} \Lambda_k &=& Q_k + S_k V_k^{-1} R_k \big(S_k V_k^{-1}\big)^\intercal \;+\; \Big(B_k - C_k(S_k V_k^{-1})^\intercal\big]^\intercal \Lambda_{k+1}\big[B_k - C_k(S_k V_k^{-1})^\intercal\Big) \\ && + \Big(D_k - H_k(S_k V_k^{-1})^\intercal\Big)^\intercal \Lambda_{k+1}\Big(D_k - H_k(S_k V_k^{-1})^\intercal\Big) \\ \Gamma_k &=& Q_k + \bar{Q}_k \;+\; T_k W_k^{-1}(R_k + \bar{R}_k)\big(T_k W_k^{-1}\big)^\intercal \\ && + \Big(B_k + \bar{B}_k - (C_k + \bar{C}_k)(T_k W_k^{-1})^\intercal\Big)^\intercal \Gamma_{k+1}\Big(B_k + \bar{B}_k - (C_k + \bar{C}_k)(T_k W_k^{-1})^\intercal\Big) \\ && + \Big(D_k + \bar{D}_k - (H_k + \bar{H}_k)(T_k W_k^{-1})^\intercal\Big)^\intercal \Gamma_{k+1}\Big(D_k + \bar{D}_k - (H_k + \bar{H}_k)(T_k W_k^{-1})^\intercal\Big), \end{array}\right.$$

it is also clear that they are nonnegative under **(c0)**. Finally from the expression (2.4.13) at time $k-1$, we see that $V_{k-1} = V_{k-1}(\Lambda_k)$ and $W_{k-1} = W_{k-1}(\Lambda_k, \Gamma_k)$ are symmetric positive under **(c0)-(c1)-(c2)**, which shows the required assertion. □

In view of the above derivation and Remark 2.4.1, it follows that the functions w_k, $k = 0, \ldots, n$, given in the quadratic form (2.4.8) with $(\Lambda_k, \Gamma_k, \rho_k, \chi_k)$ as in (2.4.15), satisfy by construction the DPP (2.3.10), and by the verification theorem, this implies that the value functions are given by $v_k = w_k$, while the optimal control is given in feedback form from (2.4.14) by:

$$\alpha^*_k \;=\; \tilde{\alpha}_k(X^*_k, \mathbb{P}_{X^*_k}) \;=\; -V_k^{-1} S_k^\intercal\big(X^*_k - \mathbb{E}[X^*_k]\big) \;-\; W_k^{-1} T_k^\intercal \mathbb{E}[X^*_k], \tag{2.4.16}$$

where $X^*_k = X^{\alpha^*}_k$ is the optimal wealth process with the feedback control α^*. We retrieve the expression obtained in [ELN13] by four different methods (see e.g. their Theorem 3.1). We can push further our calculations to get an explicit form of the optimal control expressed only in terms of the state process (and not on its mean). Indeed, from the linear dynamics (2.4.5), we have

$$\begin{aligned}\mathbb{E}[X^*_{k+1}] &= (B_k+\bar B_k)\mathbb{E}[X^*_k] + (C_k+\bar C_k)\mathbb{E}[\alpha^*_k] \\ &= (B_k+\bar B_k)\mathbb{E}[X^*_k] - (C_k+\bar C_k)\big(W_k^\intercal\big)^{-1}T_k^\intercal\mathbb{E}[X^*_k] \;=\; N_k\mathbb{E}[X^*_k],\end{aligned}$$

with $N_k = B_k + \bar B_k - (C_k+\bar C_k)W_k^{-1}T_k$, for $k=0,\ldots,n-1$, and so by induction:

$$\mathbb{E}[X^*_k] \;=\; N_{k-1}\ldots N_0\mathbb{E}[\xi].$$

Plugging into (2.4.16), this gives the explicit form of the optimal control as

$$\alpha^*_k \;=\; -V_k^{-1}S_k^\intercal X^*_k \;+\; \big(V_k^{-1}S_k^\intercal - W_k^{-1}T_k^\intercal\big)N_{k-1}\ldots N_0\mathbb{E}[\xi], \quad k=0,\ldots,n-1. \tag{2.4.17}$$

We observe that the optimal control at any time k does not only depend on the current state X^*_k but also on its the initial state ξ (via its mean).

Example: Mean-variance portfolio selection

The mean-variance discrete-time problem consists in minimizing the cost functional:

$$\begin{aligned}J(\alpha) &= \frac{\gamma}{2}\mathrm{Var}(X^\alpha_n) - \mathbb{E}[X^\alpha_n] \\ &= \mathbb{E}\Big[\frac{\gamma}{2}\big(X^\alpha_n\big)^2 - X^\alpha_n\Big] \;-\; \frac{\gamma}{2}\Big(\mathbb{E}[X^\alpha_n]\Big)^2,\end{aligned}$$

for some $\gamma > 0$, with a dynamics for the wealth process (X^α_k) valued in $E=\mathbb{R}$ controlled by the amount α_k valued in $A=\mathbb{R}$ invested in the stock at time k (we assume zero interest rate):

$$X^\alpha_{k+1} \;=\; X^\alpha_k + \alpha_k(b\Delta + \sigma\sqrt{\Delta}\varepsilon_{k+1}), \quad k=0,\ldots,n-1,\; X^\alpha_0 = x_0. \tag{2.4.18}$$

Here $x_0\in\mathbb{R}$ is the initial capital, b, $\sigma > 0$ are some constants, representing respectively the rate of return and volatility of the stock, $\Delta > 0$ is a parameter, e.g. $\Delta = T/n$, arising when considering a time discretization of a continuous-time model over $[0,T]$, and (ε_k) is a sequence of i.i.d. random variables with distribution $\mathcal{N}(0,1)$. This univariate model fits into the LQ framework (2.4.5)-(2.4.6) with:

$$\begin{aligned}&B_k=1,\; \bar B_k = 0,\; C_k = b\Delta,\; \bar C_k = 0,\; D_k = \bar D_k = 0,\; H_k = \sigma\sqrt{\Delta},\; \bar H_k = 0,\\ &Q_k = \bar Q_k = L_k = \bar L_k = R_k = \bar R_k = 0,\; Q = \frac{\gamma}{2},\; \bar Q = -\frac{\gamma}{2},\; L=0,\; \bar L = -1.\end{aligned}$$

Conditions **(c0)-(c1)-(c2)** are clearly satisfied, and the Riccati system (2.4.15) for $(\Lambda_k,\Gamma_k,\rho_k,\chi_k)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ is written in this case as:

$$\begin{cases}\Lambda_k &= \Lambda_{k+1}\frac{\sigma^2}{\sigma^2+b^2\Delta}\\ \Gamma_k &= \frac{\sigma^2\Lambda_{k+1}}{\sigma^2\Lambda_{k+1}+b^2\Delta\Gamma_{k+1}}\Gamma_{k+1}\\ \rho_k &= \frac{\sigma^2\Lambda_{k+1}}{b^2\Delta\Gamma_{k+1}+\sigma^2\Lambda_{k+1}}\rho_{k+1}\\ \chi_k &= \chi_{k+1} - \frac{1}{4}\frac{b^2\Delta\rho^2_{k+1}}{\sigma^2\Lambda_{k+1}+b^2\Delta\Gamma_{k+1}},\end{cases}$$

together with the terminal condition $\Lambda_n = \frac{\gamma}{2}$, $\Gamma_n = 0$, $\rho_n = -1$, $\chi_n = 0$. This leads by induction to the explicit form for $(\Lambda_k,\Gamma_k,\rho_k,\chi_k)$:

$$\begin{cases}\Lambda_k &= \frac{\gamma}{2}\Big(\frac{\sigma^2}{\sigma^2+b^2\Delta}\Big)^{n-k},\\ \Gamma_k &= 0,\quad \rho_k \;=\; -1\\ \chi_k &= -\frac{1}{2\gamma}\Big(\Big(\frac{\sigma^2+b^2\Delta}{\sigma^2}\Big)^{n-k} - 1\Big).\end{cases} \tag{2.4.19}$$

The value functions are then explicitly given by

$$v_k(\mu) \;=\; \frac{\gamma}{2}\Big(\frac{\sigma^2}{\sigma^2+b^2\Delta}\Big)^{n-k}\text{Var}(\mu) - \bar{\mu} - \frac{1}{2\gamma}\Big(\Big(\frac{\sigma^2+b^2\Delta}{\sigma^2}\Big)^{n-k} - 1\Big),$$

for all $k = 0,\ldots,n$, $\mu \in \mathcal{P}_2(\mathbb{R})$. Moreover, the optimal control is given in feedback form from (2.4.16) by:

$$\alpha_k^* \;=\; \tilde{\alpha}_k(X_k^*,\mathbb{P}_{X_k^*}) \;=\; -\frac{b}{\sigma^2+b^2\Delta}\big(X_k^* - \mathbb{E}[X_k^*]\big) \;+\; \frac{b}{\sigma^2\gamma}\Big(\frac{\sigma^2+b^2\Delta}{\sigma^2}\Big)^{n-k-1},$$

where $X_k^* = X_k^{\alpha^*}$ is the optimal wealth process with the feedback control α^*. It is then explicitly written from (2.4.17) by

$$\alpha_k^* \;=\; -\frac{b}{\sigma^2+b^2\Delta}\Big[X_k^* - x_0 - \frac{1}{\gamma}\Big(1+\frac{b^2}{\sigma^2}\Delta\Big)^n\Big]. \tag{2.4.20}$$

We then observe that the optimal control at any time k does not only depend on the current wealth X_k^* but also on the initial wealth x_0. This expression (2.4.20) of the optimal control is the discrete time analog of the continuous time optimal control obtained in [LZ00] or [AD10]. Actually, if we view (2.4.18) as a time discretization (with a time step $\Delta = T/n$) of a continuous time Black-Scholes model for the stock price over $[0,T]$, with a controlled wealth dynamics

$$dX_t^\alpha \;=\; \alpha_t(bdt + \sigma dW_t), \quad X_0^\alpha = x_0,$$

then by sending n to infinity (hence Δ to zero) into (2.4.20), we retrieve the closed-form expression of the optimal control in [LZ00] or [AD10]:

$$\alpha_t^* \;=\; -\frac{b}{\sigma^2}\Big[X_t^{\alpha^*} - x_0 - \frac{1}{\gamma}\exp\big(\frac{b^2}{\sigma^2}T\big)\Big].$$

Chapter 3

Bellman equation and viscosity solutions for continuous time McKean-Vlasov control problem [a]

Abstract: We consider the stochastic optimal control problem of McKean-Vlasov stochastic differential equation where the coefficients may depend upon the joint marginal law of the state and control. By using feedback controls, we reformulate the problem into a deterministic control problem with only the marginal distribution of the process as controlled state variable, and prove that dynamic programming principle holds in its general form. Then, by relying on the notion of differentiability with respect to probability measures recently introduced by P.L. Lions in [Lio12], and a special Itô formula for flows of probability measures, we derive the (dynamic programming) Bellman equation for mean-field stochastic control problem, and prove a verification theorem in our McKean-Vlasov framework. We give explicit solutions to the Bellman equation for the linear quadratic mean-field control problem, with applications to the mean-variance portfolio selection and a systemic risk model. We also consider a notion of lifted viscosity solutions for the Bellman equation, and show the viscosity property and uniqueness of the value function to the McKean-Vlasov control problem. Finally, we discuss the case of McKean-Vlasov control problem with open-loop controls and compare the associated dynamic programming equation with the case of feedback controls.

Keywords: McKean-Vlasov SDEs, dynamic programming, Bellman equation, Wasserstein space, viscosity solutions.

a. This chapter is based on a paper written in collaboration with Pham Huyên [PW18]. This paper is published in ESAIM: *Control, Optimisation, Calculus of Variations*, **24**(1), 2018.

3.1 Introduction

The problem studied in this paper concerns the optimal control of mean-field SDEs, also known as McKean-Vlasov equations. This topic is closely related to the mean-field game (MFG) problem as originally formulated by Lasry and Lions in [LL07] and simultaneously by Huang, Caines and Malhamé in [HMC06]. It aims at describing equilibrium states of large population of symmetric players (particles) with mutual interactions of mean-field type, and we refer to [CDL13] for a discussion pointing out the subtle differences between the notions of equilibrium in MFG and optimal control of McKean-Vlasov dynamics.

While the analysis of McKean-Vlasov SDEs has a long history with the pioneering works by Kac [Kac56] and H. McKean [McK67], and later on with papers in the general framework of propagation of chaos, see e.g. [Szn91], [JMW08], the optimal control of McKean-Vlasov dynamics is a rather new problem, which attracts an increasing interest since the emergence of the MFG theory and its numerous applications in several areas outside physics, like economics and finance, biology, social interactions, networks. Actually, it has been first studied in [AD01] by functional analysis method with a value function expressed in terms of the Nisio semigroup of operators. More recently, several papers have adopted the stochastic maximum (also called Pontryagin) principle for characterizing solutions to the controlled McKean-Vlasov systems in terms of an adjoint backward stochastic differential equation (BSDE) coupled with a forward SDE: see [AD10], [BDL11], [Yon13] with a state dynamics depending upon moments of the distribution, and [CD15] for a deep investigation in a more general setting. Alternatively, and although the dynamics of mean-field SDEs is non-Markovian, it is tempting to use DP method (also called Bellman principle), which is known to be a powerful tool for standard Markovian stochastic control problem, see e.g. [FS06], [Pha09], and does not require any convexity assumption usually imposed in Pontryagin principle. Indeed, mean-field type control problem was tackled by DP in [LP14] and [BFY15] for specific McKean-Vlasov SDE and cost functional, typically depending only upon statistics like its mean value or with uncontrolled diffusion coefficient, and especially by assuming the existence at all times of a density for the marginal distribution of the state process. The key idea in both papers [LP14] and [BFY15] is to reformulate the stochastic control problem with feedback strategy as a deterministic control problem involving the density of the marginal distribution, and then to derive a dynamic programming equation in the space of density functions.

Inspired by the works [BFY15] and [LP14], the objective of this paper is to analyze in detail the dynamic programming method for the optimal control of mean-field SDEs where the drift, diffusion coefficients and running costs may depend both upon the joint marginal distribution of the state and of the control. This additional dependence related to the mean-field interaction on control is natural in the context of McKean-Vlasov control problem, but has been few considered in the literature, see however [Yon13] for a dependence only through the moments of the control. Our paper can be viewed as the continuous time version of the discrete time problem studied recently in [PW16]. By using closed-loop (also called feedback) controls, we first convert the stochastic optimal control problem into a deterministic control problem where the marginal distribution is the sole controlled state variable, and we prove that dynamic programming holds in its general form. The next step for exploiting the DP is to differentiate functions defined on the space of probability measures. There are various notions of derivatives with respect to measures which have been developed in connection with the theory of optimal transport and using Wasserstein metric on the space of probability measures, see e.g. the monographs [AGS08], [Vil08]. For our purpose, we shall use the notion of differentiability introduced by P.L. Lions in his lectures at the Collège de France [Lio12], see also the helpful redacted notes [Car12]. This notion of derivative is based on the lifting of functions defined on the space of square integrable probability measures into functions defined on the Hilbert space of square integrable random variables distributed according to the "lifted" probability measure. It has been used in [CD15] for differentiating the Hamiltonian function appearing in stochastic Pontryagin principle for controlled McKean-Vlasov dynamics. As usual in continuous time control problem, we need a dynamic differential calculus for deriving the infinitesimal version of the DP, and shall rely on a special Itô's chain rule for flows of probability measures as recently developed in [BLPR17] and [CCD15], and used in [CD14] for deriving the so-called Master equation in MFG. We are then able to derive the dynamic programming Bellman equation for mean-field stochastic control problem. This infinite dimensional fully nonlinear partial differential equation (PDE) of second order in the Wassertein space of probability measures extends previous results in the literature [BFY15], [CD14], [LP14]: it reduces in particular to the Bellman equation in the space of density functions derived by Bensoussan, Frehse and Yam [BFY17] when the marginal distribution admits a density, and on the other hand, we notice that it differs from the Master equation for McKean-Vlasov control problem obtained by Carmona and Delarue in [CD14] where the value function is a function of both the state and its marginal distribution, and so with associated PDE in the state space comprising probability measures but also Euclidian vectors. Following the traditional approach for stochastic control problem, we prove a verification theorem for the Bellman equation of the McKean-Vlasov control problem, which reduces to the classical Bellman equation in the case of no mean-field interaction. We apply our verification theorem to the important class of LQ McKean-Vlasov control problems, addressed e.g. in [Yon13] and [BSYY16] by maximum principle and adjoint equations, and that we solve by a different approach where it turns out that derivations in the space of probability measures are quite tractable and lead to explicit classical solutions for the Bellman equation. We illustrate these results with two examples arising from finance: the mean-variance portfolio selection and an interbank systemic risk model, and retrieve the results obtained in [LZ00], [FL16] and [CFS15] by different methods.

In general, there are no classical solutions to the Bellman equation, and we thus introduce a notion of viscosity solutions for the Bellman equation in the Wasserstein space of probability measures. There are several definitions of viscosity solutions for Hamilton Jacobi equations of first order in Wasserstein space and more generally in metric spaces, see e.g. [AGS08], [GNT08], [FK09] or [GŚ15b]. We adopt the approach in [Lio12], and detailed in [Car12], which consists, after the lifting identification between measures and random variables, in working in the Hilbert space of square integrable random variables instead of working in the Wasserstein space of probability measures, in order to use the various tools developed for viscosity solutions in separable Hilbert spaces, in particular in our context, for second order Hamilton-Jacobi equations, see [Lio88], [Lio89a], [Lio89b], and the recent monograph [FGS15]. We then prove the viscosity property of the value function and a comparison principle, hence uniqueness result, for our Bellman equation associated to the McKean-Vlasov control problem.

Finally, we consider the more general class of open-loop controls instead of (Lipschitz) closed-loop controls, hence allowing a priori bang-bang controls, which is useful in the applications. We derive the corresponding dynamic programming equation, and compare with the Bellman equation arising from McKean-Vlasov control problem with feedback controls.

The rest of the paper is organized as follows. Section 2 describes the McKean-Vlasov control problem and fix the standing assumptions. In Section 3, we state the dynamic programming principle after the reformulation into a deterministic control problem, and derive the Bellman equation together with the proof of the verification theorem. We present in Section 4 the applications to the LQ framework where explicit solutions are provided with two examples arising from financial models. Section 5 deals with viscosity solutions for the Bellman equation, and the last section considers the case of open-loop controls.

3.2 McKean-Vlasov control problem

Let us fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a n-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ its natural filtration, augmented with an independent σ-algebra $\mathcal{F}_0 \subset \mathcal{F}$. Given a normed space $(E, |.|)$, recall that $\mathcal{P}_2(E)$ is Wasserstein space and $L^2(\mathcal{F}_0; E)$ $(= L^2(\Omega, \mathcal{F}_0, \mathbb{P}; E))$ the set of square integrable E-valued random variables on $(\Omega, \mathcal{F}_0, \mathbb{P})$. In the sequel, E will be either $\mathbb{R}^d$, the state space, or A, the control space, a subset of $\mathbb{R}^m$, or the product space $\mathbb{R}^d \times A$. We shall assume without loss of generality (see Remark 3.2.1 below) that $\mathcal{F}_0$ is rich enough to carry E-valued random variables with any arbitrary square integrable distribution, i.e. $\mathcal{P}_2(E) = \{\mathbb{P}_\xi, \xi \in L^2(\mathcal{F}_0; E)\}$.

Remark 3.2.1. A possible construction of a probability space, which is rich enough to satisfy the above conditions is the following. We consider a Polish space Ω_0, its Borel σ-algebra $\mathcal{F}_0$ and let $\mathbb{P}_0$ be an atomless probability measure on $(\Omega_0, \mathcal{F}_0)$. We consider another probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ supporting a n-dimensional Brownian motion B and denote by $\mathbb{F}^B = (\mathcal{F}_t^B)$ its natural filtration. By defining $\Omega = \Omega_0 \times \Omega_1$, $\mathcal{F} = \mathcal{F}_0 \vee \mathcal{F}_1$, $\mathbb{P} = \mathbb{P}_0 \otimes \mathbb{P}_1$, and $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_0$, $0 \leq t \leq T$, we then obtain that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the required condition in the above framework. □

We also denote by $\mathcal{W}_2$ the 2-Wasserstein distance defined on $\mathcal{P}_2(E)$ by

$$\begin{aligned}
\mathcal{W}_2(\mu, \mu') &:= \inf\Big\{ \Big(\int_{E \times E} |x - y|^2 \pi(dx, dy) \Big)^{\frac{1}{2}} : \pi \in \mathcal{P}_2(E \times E) \text{ with marginals } \mu \text{ and } \mu' \Big\} \\
&= \inf\Big\{ \Big(\mathbb{E}|\xi - \xi'|^2 \Big)^{\frac{1}{2}} : \ \xi, \xi' \in L^2(\mathcal{F}_0; E) \text{ with } \mathbb{P}_\xi = \mu, \ \mathbb{P}_{\xi'} = \mu' \Big\}.
\end{aligned}$$

We consider a controlled stochastic dynamics of McKean-Vlasov type for the process $X^\alpha = (X_t^\alpha)_{0 \leq t \leq T}$ valued in $\mathbb{R}^d$:

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t, \mathbb{P}_{(X_t^\alpha, \alpha_t)})dt + \sigma(t, X_t^\alpha, \alpha_t, \mathbb{P}_{(X_t^\alpha, \alpha_t)})dB_t, \ \ X_0^\alpha = X_0, \tag{3.2.1}$$

where $X_0 \in L^2(\mathcal{F}_0; \mathbb{R}^d)$, and the control process $\alpha = (\alpha_t)_{0 \leq t \leq T}$ is progressively measurable with values in a subset A of $\mathbb{R}^m$. The coefficients b and σ are deterministic measurable functions from $[0, T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d \times A)$ into $\mathbb{R}^d$ and $\mathbb{R}^{d \times n}$ respectively. Notice here that the drift and diffusion coefficients b, σ of the controlled state process do not depend only on the marginal distribution of the state process X_t at time t but more generally on the joint marginal distribution of the state/control (X_t, α_t) at time t, which represents an additional mean-field feature with respect to classical McKean-Vlasov equations. We make the following assumption:

(H1) There exists some constant $C_{b,\sigma} > 0$ s.t. for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $a, a' \in A$, $\lambda, \lambda' \in \mathcal{P}_2(\mathbb{R}^d \times A)$,

$$\begin{aligned}
& |b(t, x, a, \lambda) - b(t, x', a', \lambda')| + |\sigma(t, x, a, \lambda) - \sigma(t, x', a', \lambda')| \\
\leq \ & C_{b,\sigma}\big[|x - x'| + |a - a'| + \mathcal{W}_2(\lambda, \lambda')\big],
\end{aligned}$$

and

$$\int_0^T |b(t, 0, 0, \delta_{(0,0)})|^2 + |\sigma(t, 0, 0, \delta_{(0,0)})|^2 dt < \infty.$$

Condition **(H1)** ensures that for any control process α, which is square integrable, i.e. $\mathbb{E}[\int_0^T |\alpha_t|^2 dt] < \infty$, there exists a unique solution X^α to (3.2.1), and moreover this solution satisfies (see e.g. [Szn91] or [JMW08]):

$$\mathbb{E}\big[\sup_{0\le t\le T}|X_t^\alpha|^2\big] \;\le\; C\Big(1+\mathbb{E}|X_0|^2+\mathbb{E}\big[\int_0^T|\alpha_t|^2dt\big]\Big) \;<\; \infty. \tag{3.2.2}$$

In the sequel of the paper, we stress the dependence of X^α on α if needed, but most often, we shall omit this dependence and simply write $X = X^\alpha$ when there is no ambiguity.

The cost functional associated to the McKean-Vlasov equation (3.2.1) is

$$J(\alpha) \;:=\; \mathbb{E}\Big[\int_0^T f(t,X_t,\alpha_t,\mathbb{P}_{(X_t,\alpha_t)})dt + g(X_T,\mathbb{P}_{X_T})\Big] \tag{3.2.3}$$

for a square integrable control process α. The running cost function f is a deterministic real-valued function on $[0,T]\times\mathbb{R}^d\times A\times\mathcal{P}_2(\mathbb{R}^d\times A)$ and the terminal gain function g is a deterministic real-valued function on $\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$. We shall assume the following quadratic condition on f, g:

(H2) There exists some constant $C_{f,g}>0$ s.t. for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $a\in A$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $\lambda\in\mathcal{P}_2(\mathbb{R}^d\times A)$,

$$|f(t,x,a,\lambda)|+|g(x,\mu)| \;\le\; C_{f,g}\big(1+|x|^2+|a|^2+\|\mu\|_2^2+\|\lambda\|_2^2\big).$$

Under Condition **(H2)**, and from (3.2.2), we see that $J(\alpha)$ is well-defined and finite for any square integrable control process α. The stochastic control problem of interest in this paper is to minimize the cost functional:

$$V_0 \;:=\; \inf_{\alpha\in\mathcal{A}} J(\alpha), \tag{3.2.4}$$

over a set of admissible controls $\mathcal{A}$ to be precised later.

3.3 Dynamic programming and Bellman equation

3.3.1 Dynamic programming principle

In this paragraph, we make the standing assumptions **(H1)-(H2)**, and our purpose is to show that dynamic programming principle holds for problem (3.2.4), which we would like to combine with some Markov property of the controlled state process. However, notice that the McKean-Vlasov type dependence on the dynamics of the state process rules out the standard Markov property of the controlled process $(X_t)_t$. Actually, this Markov property can be restored by considering its probability law $(\mathbb{P}_{X_t})_t$. To be more precise and for the sake of definiteness, we shall restrict ourselves to controls $\alpha=(\alpha_t)_{0\le t\le T}$ given in closed loop (or feedback) form:

$$\alpha_t \;=\; \tilde\alpha(t,X_t,\mathbb{P}_{X_t}), \qquad 0\le t\le T, \tag{3.3.1}$$

for some deterministic measurable function $\tilde\alpha(t,x,\mu)$ defined on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$. We shall discuss in the last section how one deal more generally with open-loop controls. We denote by $Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$ the set of deterministic measurable functions $\tilde\alpha$ on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, valued in A, which are Lipschitz in (x,μ), and satisfy a linear growth condition on (x,μ), uniformly on $t \in [0,T]$, i.e. there exists some positive constant $C_{\tilde\alpha}$ s.t. for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} |\tilde\alpha(t,x,\mu) - \tilde\alpha(t,x',\mu')| &\leq C_{\tilde\alpha}\big(|x-x'| + \mathcal{W}_2(\mu,\mu')\big), \\ \int_0^T |\tilde\alpha(t,0,\delta_0)|^2 dt &< \infty. \end{aligned}$$

Notice that for any $\tilde\alpha \in Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$, and under the Lipschitz condition in **(H1)**, there exists a unique solution to the SDE:

$$\begin{aligned} dX_t &= b(t, X_t, \tilde\alpha(t,X_t,\mathbb{P}_{X_t}), \mathbb{P}_{(X_t,\tilde\alpha(t,X_t,\mathbb{P}_{X_t}))})dt \\ &\quad + \sigma(t, X_t, \tilde\alpha(t,X_t,\mathbb{P}_{X_t}), \mathbb{P}_{(X_t,\tilde\alpha(t,X_t,\mathbb{P}_{X_t}))})dB_t, \end{aligned} \tag{3.3.2}$$

starting from some square integrable random variable, and this solution satisfies the square integrability condition (3.2.2). The set $\mathcal{A}$ of so-called admissible controls α is then defined as the set of control processes α of feedback form (3.3.1) with $\tilde\alpha \in Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$. We shall often identify $\alpha \in \mathcal{A}$ with $\tilde\alpha$ in $Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$ via (3.3.1), and we see that any α in $\mathcal{A}$ is square-integrable: $\mathbb{E}[\int_0^T |\alpha_t|^2 dt] < \infty$, by (3.2.2) and Gronwall's lemma.

Let us now check the flow property of the marginal distribution process $\mathbb{P}_{X_t} = \mathbb{P}_{X_t^\alpha}$ for any admissible control α in $\mathcal{A}$. For any $\tilde\alpha \in L(\mathbb{R}^d;A)$, the set of Lipschitz functions from $\mathbb{R}^d$ into A, we denote by $Id\tilde\alpha$ the function

$$\begin{aligned} Id\tilde\alpha : \mathbb{R}^d &\to \mathbb{R}^d \times A \\ x &\mapsto (x, \tilde\alpha(x)). \end{aligned}$$

We observe that the joint distribution $\mathbb{P}_{(X_t,\alpha_t)}$ associated to a feedback control $\alpha \in \mathcal{A}$ is equal to the image by $Id\tilde\alpha(t,.,\mathbb{P}_{X_t})$ of the marginal distribution $\mathbb{P}_{X_t}$ of the controlled state process X, i.e. $\mathbb{P}_{(X_t,\alpha_t)} = Id\tilde\alpha(t,.,\mathbb{P}_{X_t}) \star \mathbb{P}_{X_t}$, recall that $\star$ denotes the standard pushforward of measures: for any $\tilde\alpha \in L(\mathbb{R}^d;A)$, and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$(Id\tilde\alpha \star \mu)(B) = \mu\big(Id\tilde\alpha^{-1}(B)\big), \quad \forall B \in \mathcal{B}(\mathbb{R}^d \times A).$$

We consider the dynamic version of (3.3.2) starting at time $t \in [0,T]$ from $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, which is then written as:

$$\begin{aligned} X_s^{t,\xi} &= \xi + \int_t^s b(r, X_r^{t,\xi}, \tilde\alpha(r, X_r^{t,\xi}, \mathbb{P}_{X_r^{t,\xi}}), Id\tilde\alpha(r,.,\mathbb{P}_{X_r^{t,\xi}}) \star \mathbb{P}_{X_r^{t,\xi}})dr \\ &\quad + \int_t^s \sigma(r, X_r^{t,\xi}, \tilde\alpha(r, X_r^{t,\xi}, Id\tilde\alpha(r,.,\mathbb{P}_{X_r^{t,\xi}}) \star \mathbb{P}_{X_r^{t,\xi}})dB_r, \quad t \leq s \leq T. \end{aligned} \tag{3.3.3}$$

Existence and uniqueness of a solution to (3.5.9) implies the flow property:

$$X_s^{t,\xi} = X_s^{\theta, X_\theta^{t,\xi}}, \quad \forall\, 0 \leq t \leq \theta \leq s \leq T,\ \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d). \tag{3.3.4}$$

Moreover, as pointed out in Remark 3.1 in [BLPR17] (see also the remark following (2.3) in [CCD15]), the solution to (3.5.9) is also unique in law from which it follows that the law of $X^{t,\xi}$ depends on ξ only through its law $\mathbb{P}_\xi$. Therefore, we can define

$$\mathbb{P}_s^{t,\mu} := \mathbb{P}_{X_s^{t,\xi}}, \quad \text{for} \quad 0 \leq t \leq s \leq T,\ \mu = \mathbb{P}_\xi \in \mathcal{P}_2(\mathbb{R}^d), \tag{3.3.5}$$

As a consequence of the flow property (3.3.4), and recalling that $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi, \xi \in L^2(\mathcal{F}_0;\mathbb{R}^d)\}$, it is clear that we also get the flow property for the marginal distribution process:

$$\mathbb{P}_s^{t,\mu} \;=\; \mathbb{P}_s^{\theta,\mathbb{P}_\theta^{t,\mu}}, \quad \forall\, 0 \le t \le \theta \le s \le T,\ \mu \in \mathcal{P}_2(\mathbb{R}^d). \tag{3.3.6}$$

Recall that the process $X^{t,\xi}$, hence also the law process $\mathbb{P}^{t,\mu}$ depends on the feedback control $\alpha \in \mathcal{A}$, and if needed, we shall stress the dependence on α by writing $\mathbb{P}^{t,\mu,\alpha}$.

We next show that the initial stochastic control problem can be reduced to a deterministic control problem. Indeed, by definition of the marginal distribution $\mathbb{P}_{X_t}$, recalling that $\mathbb{P}_{(X_t,\alpha_t)} = Id\tilde{\alpha}(t,.,\mathbb{P}_{X_t}) \star \mathbb{P}_{X_t}$, and Fubini's theorem, we see that the cost functional can be written for any admissible control $\alpha \in \mathcal{A}$ as:

$$J(\alpha) \;=\; \int_0^T \hat{f}(t,\mathbb{P}_{X_t},\tilde{\alpha}(t,.,\mathbb{P}_{X_t}))dt + \hat{g}(\mathbb{P}_{X_T}),$$

where the function $\hat{f}$ is defined on $[0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times L(\mathbb{R}^d;A)$ and $\hat{g}$ is defined on $\mathcal{P}_2(\mathbb{R}^d)$ by

$$\hat{f}(t,\mu,\tilde{\alpha}) \;:=\, < f(t,.,\tilde{\alpha}(.), Id\tilde{\alpha} \star \mu), \mu >, \qquad \hat{g}(\mu) \;:=\, < g(.,\mu), \mu > . \tag{3.3.7}$$

We have thus transformed the initial control problem (3.2.4) into a deterministic control problem involving the infinite dimensional controlled marginal distribution process valued in $\mathcal{P}_2(\mathbb{R}^d)$. In view of the flow property (3.3.6), it is then natural to define the value function

$$v(t,\mu) \;:=\; \inf_{\alpha\in\mathcal{A}} \Big[\int_t^T \hat{f}(s,\mathbb{P}_s^{t,\mu},\tilde{\alpha}(s,.,\mathbb{P}_s^{t,\mu}))ds + \hat{g}(\mathbb{P}_T^{t,\mu}) \Big],\ \ t \in [0,T],\ \mu \in \mathcal{P}_2(\mathbb{R}^d), \tag{3.3.8}$$

so that the initial control problem in (3.2.4) is given by: $V_0 = v(0,\mathbb{P}_{X_0})$. It is clear that $v(t,\mu) < \infty$, and we shall assume that

$$v(t,\mu) \;>\; -\infty, \quad \forall\, t \in [0,T],\ \mu \in \mathcal{P}_2(\mathbb{R}^d). \tag{3.3.9}$$

Remark 3.3.1. The finiteness condition (3.3.9) can be checked a priori directly from the assumptions on the model. For example, when f, g, hence $\hat{f}$, $\hat{g}$, are lower-bounded functions, condition (3.3.9) clearly holds. Another example is the case when $f(t,x,a,\lambda)$, and $g(x,\mu)$ are lower bounded by a quadratic function in x, μ, and λ (uniformly in (t,a)) so that

$$\hat{f}(t,\mu,\tilde{\alpha}) + \hat{g}(x,\mu) \;\ge\; -C\big(1 + \|\mu\|_2\big), \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d),\ \tilde{\alpha} \in L(\mathbb{R}^d;A),$$

and we are able to derive moment estimates on the controlled process X, uniformly in α: $\big\|\mathbb{P}_s^{t,\mu}\big\|_2^2 = \mathbb{E}[|X_s^{t,\xi}|^2] \le C(1 + \|\mu\|_2^2)$, (for $\mu = \mathbb{P}_\xi$) which arises typically from (3.2.2) when A is bounded. Then, it is clear that (3.3.9) holds true. Otherwise, this finiteness condition can be checked a posteriori from a verification theorem, see Theorem 3.3.1. □

The DPP for the deterministic control problem (3.3.8) takes the following formulation:

Theorem 3.3.1. *(Dynamic Programming Principle)*

Under (3.3.9), *we have for all* $0 \le t \le \theta \le T$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$v(t,\mu) \;=\; \inf_{\alpha\in\mathcal{A}} \Big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu},\tilde{\alpha}(s,.,\mathbb{P}_s^{t,\mu}))ds \;+\; v(\theta,\mathbb{P}_\theta^{t,\mu}) \Big]. \tag{3.3.10}$$

Proof. In the context of deterministic control problem, the proof of the DPP is elementary and does not require any measurable selection arguments. For sake of completeness, we provide it. Denote by $J(t,\mu,\alpha)$ the cost functional:

$$J(t,\mu,\alpha) \quad := \quad \int_t^T \hat{f}(s,\mathbb{P}_s^{t,\mu,\alpha},\tilde{\alpha}(s,.)) + \hat{g}(\mathbb{P}_T^{t,\mu,\alpha}), \quad 0\le t\le T,\ \mu\in\mathcal{P}_2(\mathbb{R}^d), \alpha\in\mathcal{A},$$

so that $v(t,\mu) = \inf_{\alpha\in\mathcal{A}} J(t,\mu,\alpha)$, and by $w(t,\mu)$ the r.h.s. of (3.3.10) (here we stress the dependence of the controlled marginal distribution process $\mathbb{P}^{t,\mu,\alpha}$ on α). Then,

$$\begin{aligned} w(t,\mu) &= \inf_{\alpha\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu,\alpha},\tilde{\alpha}(s,.,\mathbb{P}_s^{t,\mu,\alpha}))ds + \inf_{\beta\in\mathcal{A}} J(\theta,\mathbb{P}_\theta^{t,\mu,\beta},\beta)\big] \\ &= \inf_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu,\alpha},\tilde{\alpha}(s,.,\mathbb{P}_s^{t,\mu,\alpha}))ds + J(\theta,\mathbb{P}_\theta^{t,\mu,\beta},\beta)\big] \\ &= \inf_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu,\gamma[\alpha,\beta]},\tilde{\gamma}[\alpha,\beta](s,.,\mathbb{P}_s^{t,\mu,\gamma[\alpha,\beta]}))ds + J(\theta,P_\theta^{t,\mu,\gamma[\alpha,\beta]},\gamma[\alpha,\beta])\big] \end{aligned}$$

where we define $\gamma[\alpha,\beta]\in\mathcal{A}$ by: $\tilde{\gamma}[\alpha,\beta](s,.) = \tilde{\alpha}(s,.)1_{0\le s\le\theta} + \tilde{\beta}(s,.)1_{\theta<s\le T}$. Now, it is clear that when α,β run over $\mathcal{A}$, then $\gamma[\alpha,\beta]$ also runs over $\mathcal{A}$, and so:

$$\begin{aligned} w(t,\mu) &= \inf_{\gamma\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu,\gamma},\tilde{\gamma}(s,.,\mathbb{P}_s^{t,\mu,\gamma}))ds + J(\theta,P_\theta^{t,\mu,\gamma},\gamma)\big] \\ &= \inf_{\gamma\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu},\tilde{\gamma}(s,.,\mathbb{P}_s^{t,\mu}))ds + \int_\theta^T \hat{f}(s,\mathbb{P}_s^{\theta,\mathbb{P}_\theta^{t,\mu}},\tilde{\gamma}(s,.,\mathbb{P}_s^{\theta,\mathbb{P}_\theta^{t,\mu}})) + \hat{g}(\mathbb{P}_T^{\theta,\mathbb{P}_\theta^{t,\mu}})\big] \\ &= \inf_{\gamma\in\mathcal{A}}\big[\int_t^\theta \hat{f}(s,\mathbb{P}_s^{t,\mu},\tilde{\gamma}(s,.,\mathbb{P}_s^{t,\mu}))ds + \int_\theta^T \hat{f}(s,\mathbb{P}_s^{t,\mu},\tilde{\gamma}(s,.,\mathbb{P}_s^{t,\mu})) + \hat{g}(\mathbb{P}_T^{t,\mu})\big], \end{aligned}$$

by the flow property (3.3.6) (here we have omitted in the second and third line the dependence of $\mathbb{P}_s$ in γ). This proves the required equality: $w(t,\mu) = v(t,\mu)$. □

Remark 3.3.2. Problem (3.2.4) includes the case where the cost functional in (3.2.3) is a nonlinear function of the expected value of the state process, i.e. the running cost functions and the terminal gain function are in the form: $f(t,X_t,\alpha_t,\mathbb{P}_{(X_t,\alpha_t)}) = \bar{f}(t,X_t,\mathbb{E}[X_t],\alpha_t)$, $t\in[0,T]$, $g(X_T,\mathbb{P}_{X_T}) = \bar{g}(X_T,\mathbb{E}[X_T])$, which arises for example in mean-variance problem (see Section 3.4). It is claimed in [BM14] and [Yon13] that Bellman optimality principle does not hold, and therefore the problem is time-inconsistent. This is correct when one takes into account only the state process X (that is its realization), since it is not Markovian, but as shown in this section, dynamic programming principle holds true whenever we consider the marginal distribution as state variable. This gives more information and the price to paid is the infinite-dimensional feature of the marginal distribution state variable. □

3.3.2 Bellman equation

The purpose of this paragraph is to derive from the dynamic programming principle (3.3.10), a partial differential equation (PDE) for the value function $v(t,\mu)$, called Bellman equation.

Notice that under condition **(H1)** on the coefficients b and σ, we have

$$\begin{aligned} \mathbb{E}\Big[\int_t^T &\big|b(s,X_s^{t,\xi},\tilde{\alpha}(s,X_s^{t,\xi},\mathbb{P}_{X_s^{t,\xi}}),Id\tilde{\alpha}(s,.,\mathbb{P}_{X_s^{t,\xi}})\star\mathbb{P}_{X_s^{t,\xi}})\big|^2 \\ &+ \big|\sigma(s,X_s^{t,\xi},\tilde{\alpha}(s,X_s^{t,\xi},\mathbb{P}_{X_s^{t,\xi}}),Id\tilde{\alpha}(s,.,\mathbb{P}_{X_s^{t,\xi}})\star\mathbb{P}_{X_s^{t,\xi}})\big|^2 ds\Big] \ < \ \infty. \end{aligned}$$

Therefore, by relying on Itô's formula along a flow of probability measures in Proposition 1.3.7, we derive the Bellman equation associated to the DPP (3.3.10), which turns out to be in the following form:

$$\begin{cases} \partial_t v \; + \inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \Big[\hat f(t,\mu,\tilde\alpha) + \, < \mathcal{L}_t^{\tilde\alpha} v(t,\mu), \mu > \Big] & = \; 0, \;\text{ on } [0,T) \times \mathcal{P}_2(\mathbb{R}^d), \\ v(T,.) & = \; \hat g, \;\text{ on } \mathcal{P}_2(\mathbb{R}^d) \end{cases} \tag{3.3.11}$$

where for $\tilde\alpha \in L(\mathbb{R}^d;A)$, $\varphi \in \mathcal{C}_b^2(\mathcal{P}_2(\mathbb{R}^d))$ and $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, $\mathcal{L}_t^{\tilde\alpha}\varphi(\mu) \in L^2_\mu(\mathbb{R})$ is the function: $\mathbb{R}^d \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{L}_t^{\tilde\alpha}\varphi(\mu)(x) \;\; := \;\; & \partial_\mu\varphi(\mu)(x).b(t,x,\tilde\alpha(x), Id\tilde\alpha \star \mu) \\ & + \frac{1}{2}\mathrm{tr}\big(\partial_x\partial_\mu\varphi(\mu)(x)\sigma\sigma^\intercal(t,x,\tilde\alpha(x), Id\tilde\alpha\star\mu)\big). \end{aligned} \tag{3.3.12}$$

In the spirit of classical verification theorem for stochastic control of diffusion processes, we prove the following result in our McKean-Vlasov control framework, which is a consequence of the Itô's formula for functions defined on the Wasserstein space.

Proposition 3.3.1. *(Verification theorem)*

Let $w : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a function in $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, i.e. w is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $w(t,.) \in \mathcal{C}_b^2(\mathcal{P}_2(\mathbb{R}^d))$, for all $t \in [0,T]$, and $w(.,\mu) \in \mathcal{C}^1([0,T))$. Suppose that w is solution to (3.3.11), and there exists for all $(t,\mu) \in [0,T)\times\mathcal{P}_2(\mathbb{R}^d)$ an element $\tilde\alpha^*(t,.,\mu) \in L(\mathbb{R}^d;A)$ attaining the infimum in (3.3.11) s.t. the mapping $(t,x,\mu) \mapsto \tilde\alpha^*(t,x,\mu) \in Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$. Then, $w = v$, and the feedback control $\alpha^* \in \mathcal{A}$ defined by

$$\alpha_t^* \;\; = \;\; \tilde\alpha^*(t, X_t, \mathbb{P}_{X_t}), \quad 0 \le t < T,$$

is an optimal control, i.e. $V_0 = J(\alpha^*)$.

Proof. Fix $(t,\mu = \mathbb{P}_\xi) \in [0,T)\times\mathcal{P}_2(\mathbb{R}^d)$, and consider some arbitrary feedback control $\alpha \in \mathcal{A}$ associated to $X^{t,\xi}$ the solution to the controlled SDE (3.5.9). Under condition **(H1)**, we have the standard estimate

$$\mathbb{E}\big[\sup_{t\le s\le T}|X_s^{t,\xi}|^2\big] \;\; \le \;\; C\big(1+\mathbb{E}|\xi|^2\big) \; < \; \infty,$$

which implies that

$$\begin{aligned} \mathbb{E}\Big[\int_t^T & \big|b(s,X_s^{t,\xi},\tilde\alpha(s,X_s^{t,\xi},\mathbb{P}_{X_s^{t,\xi}}), Id\tilde\alpha(s,.,\mathbb{P}_{X_s^{t,\xi}})\star\mathbb{P}_{X_s^{t,\xi}})\big|^2 \\ & + \big|\sigma(s,X_s^{t,\xi},\tilde\alpha(s,X_s^{t,\xi},\mathbb{P}_{X_s^{t,\xi}}), Id\tilde\alpha(s,.,\mathbb{P}_{X_s^{t,\xi}})\star\mathbb{P}_{X_s^{t,\xi}})\big|^2 ds\Big] \; < \; \infty. \end{aligned}$$

One can then apply the Itô's formula (1.3.7) to $w(s,\mathbb{P}_{X_s^{t,\xi}}) = w(s,\mathbb{P}_s^{t,\mu})$ (with the definition (3.3.5)) between $s = t$ and $s = T$, and obtain

$$\begin{aligned} w(T,\mathbb{P}_T^{t,\mu}) \;\; = \;\; & w(t,\mu) \; + \int_t^T \frac{\partial w}{\partial t}(s,\mathbb{P}_s^{t,\mu}) + \\ & \mathbb{E}\Big[\partial_\mu w(s,\mathbb{P}_s^{t,\mu})(X_s^{t,\xi}).b(s,X_s^{t,\xi},\tilde\alpha(s,X_s^{t,\xi},\mathbb{P}_s^{t,\mu}), Id\tilde\alpha(s,.,\mathbb{P}_s^{t,\mu})\star\mathbb{P}_s^{t,\mu}) \\ & + \frac{1}{2}\mathrm{tr}\big(\partial_x\partial_\mu w(s,\mathbb{P}_s^{t,\mu})(X_s^{t,\xi})\sigma_s\sigma_s^\intercal(s,X_s^{t,\xi},\tilde\alpha(s,X_s^{t,\xi},\mathbb{P}_s^{t,\mu}), Id\tilde\alpha(s,.,\mathbb{P}_s^{t,\mu})\star\mathbb{P}_s^{t,\mu})\big)\Big] ds \\ = \;\; & w(t,\mu) \; + \int_t^T \frac{\partial w}{\partial t}(s,\mathbb{P}_s^{t,\mu}) + \, < \mathcal{L}_s^{\tilde\alpha(s,.,\mathbb{P}_s^{t,\mu})} w(s,\mathbb{P}_s^{t,\mu}), \mathbb{P}_s^{t,\mu} > ds, \end{aligned} \tag{3.3.13}$$

where we used in the second equality the fact that $\mathbb{P}_s^{t,\mu}$ is the distribution of $X_s^{t,\xi}$ for $s \in [t,T]$. Since $x \mapsto \tilde\alpha(s,.,\mathbb{P}_s^{t,\mu}) \in L(\mathbb{R}^d;A)$ for $s \in [t,T]$, we deduce from the Bellman equation satisfied by w and (3.3.13) that

$$\hat g(\mathbb{P}_T^{t,\mu}) \quad \geq \quad w(t,\mu) - \int_t^T \hat f(s,\mathbb{P}_s^{t,\mu},\tilde\alpha(s,.,\mathbb{P}_s^{t,\mu}))ds.$$

Since α is arbitrary in $\mathcal{A}$, this shows that $w(t,\mu) \leq v(t,\mu)$.

In the final step, let us apply the same Itô's argument (3.3.13) with the feedback control $\alpha^* \in \mathcal{A}$ associated with the fonction $\tilde\alpha^* \in Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$. Since $\tilde\alpha$ attains the infimum in (3.3.11), we thus get

$$\hat g(\mathbb{P}_T^{t,\mu}) \quad = \quad w(t,\mu) - \int_t^T \hat f(s,\mathbb{P}_s^{t,\mu},\tilde\alpha^*(s,.,\mathbb{P}_s^{t,\mu}))ds,$$

which shows that $w(t,\mu) = J(t,\mu,\alpha^*)$ $(\geq v(t,\mu))$, and therefore gives the required result: $v(t,\mu) = w(t,\mu)$ $= J(t,\mu,\alpha^*)$. □

We shall apply the verification theorem in the next section, where we can derive explicit (smooth) solutions to the Bellman equation (3.3.11) in some class of examples, but first discuss below the case when there are no mean-field interaction, and the structure of the optimal control (when it exists).

Remark 3.3.3. (No mean-field interaction)

We consider the classical case of stochastic control where there is no mean-field interaction in the dynamics of the state process, i.e. $b(x,a)$ and $\sigma(x,a)$ do not depend on μ,λ, as well as in the cost functions $f(x,a)$ and $g(x)$. In this special case, let us show how the verification Theorem 3.3.1 is reduced to the classical verification result for smooth functions on $[0,T]\times\mathbb{R}^d$, see e.g. [FS06] or [Pha09].

Suppose that there exists a function u in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ solution to the standard HJB equation

$$\begin{cases} \partial_t u + \inf_{a\in A}\big[f(t,x,a) + L_t^a u(t,x)\big] & = \quad 0, \quad \text{on} \quad [0,T)\times\mathbb{R}^d, \\ \qquad\qquad\qquad\qquad u(T,.) & = \quad g \quad \text{on} \quad \mathbb{R}^d. \end{cases} \tag{3.3.14}$$

where L_t^a is the second-order differential operator

$$L_t^a u(t,x) \quad = \quad \partial_x u(t,x).b(t,x,a) + \frac{1}{2}\mathrm{tr}\big(\partial_{xx}^2 u(t,x)\sigma\sigma^{\intercal}(t,x,a)\big),$$

and that for all $(t,x)\in[0,T)\times\mathbb{R}^d$, there exists $\hat a(t,x)$ attaining the argmin in (3.3.14), s.t. the map $x \mapsto \hat a(t,x)$ is Lipschitz on $\mathbb{R}^d$.

Let us then consider the function defined on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ by

$$w(t,\mu) \quad = \quad < u(t,.),\mu > = \int_{\mathbb{R}^d} u(t,x)\mu(dx).$$

The lifted function of w is thus equal to $\mathcal{W}(t,X) = \mathbb{E}[u(t,X)]$ with Fréchet derivative (with respect to $X \in L^2(\mathcal{F}_0,\mathbb{P})$): $[D\mathcal{W}](t,X)(Y) = \mathbb{E}[\partial_x u(t,X).Y]$. Assuming that the time derivative of u w.r.t. t satisfies a quadratic growth condition in x, the first derivative of u w.r.t. x satisfies a linear growth condition, and the second derivative of u w.r.t. x is bounded, this shows that w lies in $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ with

$$\partial_t w(t,\mu) \;=\; < \partial_t u(t,.),\mu >, \qquad \partial_\mu w(t,\mu) \;=\; \partial_x u(t,.), \quad \partial_x\partial_\mu v(t,\mu) \;=\; \partial_{xx}^2 u(t,.).$$

Recalling the definition (3.3.12) of $\mathcal{L}_t^{\tilde\alpha} w(t,\mu)$, we then get for any fixed $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$:

$$\begin{aligned}
& \partial_t w(t,\mu) \ + \inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \Big[\hat f(t,\mu,\tilde\alpha) + \ < \mathcal{L}_t^{\tilde\alpha} w(t,\mu), \mu > \Big] \\
= & \inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \int_{\mathbb{R}^d} \big[\partial_t u(t,x) + f(t,x,\tilde\alpha(x)) + L_t^{\tilde\alpha(x)} u(t,x) \big] \mu(dx) \\
= & \int_{\mathbb{R}^d} \inf_{a \in A} \big[\partial_t u(t,x) + f(t,x,a) + L_t^a u(t,x) \big] \mu(dx). \qquad (3.3.15)
\end{aligned}$$

Indeed, the inequality $\geq$ in (3.3.15) is clear since $\tilde\alpha(x)$ lies in A for all $x \in \mathbb{R}^d$, and $\tilde\alpha \in L(\mathbb{R}^d;A)$. Conversely, by taking $\hat a(t,x)$ which attains the infimum in (3.3.14), and since the map $x \in \mathbb{R}^d \mapsto \hat a(t,x)$ is Lipschitz, we then have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \inf_{a \in A} \big[\partial_t u(t,x) + f(t,x,a) + L_t^a u(t,x) \big] \mu(dx) \\
= & \int_{\mathbb{R}^d} \big[\partial_t u(t,x) + f(t,x,\hat a(t,x)) + L_t^{\hat a(t,x)} u(t,x) \big] \mu(dx) \\
\geq & \inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \int_{\mathbb{R}^d} \big[\partial_t u(t,x) + f(t,x,\tilde\alpha(x)) + L_t^{\tilde\alpha(x)} u(t,x) \big] \mu(dx),
\end{aligned}$$

which thus shows the equality (3.3.15). Since u is solution to (3.3.14), this proves that w is solution to the Bellman equation (3.3.11), $\tilde\alpha^*(t,x) = \hat a(t,x)$ is an optimal feedback control, and therefore, the value function is equal to $v(t,\mu) = \ < u(t,.), \mu >$. □

Remark 3.3.4. (Form of the optimal control)

Consider the case where the coefficients of the McKean-Vlasov SDE and of the running costs do not depend upon the law of the control, hence in the form: $b(t, X_t, \alpha_t, \mathbb{P}_{X_t})$, $\sigma(t, X_t, \alpha_t, \mathbb{P}_{X_t})$, $f(t, X_t, \alpha_t, \mathbb{P}_{X_t})$, and denote by

$$\mathbb{H}(t,x,a,\mu,q,M) \quad = \quad f(t,x,a,\mu) + q.b(t,x,a,\mu) + \frac{1}{2}\mathrm{tr}\big(M\sigma\sigma^{\intercal}(t,x,a,\mu)\big) \qquad (3.3.16)$$

for $(t,x,a,\mu,q,M) \in [0,T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{S}^d$, the Hamiltonian function related to the Bellman equation (3.3.11) rewritten as:

$$\partial_t w(t,\mu) \ + \inf_{\tilde\alpha \in L(\mathbb{R}^d;A)} \int_{\mathbb{R}^d} \mathbb{H}\big(t,x,\tilde\alpha(x),\mu,\partial_\mu w(t,\mu)(x), \partial_x\partial_\mu w(t,\mu)(x)\big)\mu(dx) \quad = \quad 0, \qquad (3.3.17)$$

for $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$. Under suitable convexity conditions on the function $a \in A \mapsto \mathbb{H}(t,x,a,\mu,q,M)$, there exists a minimizer, say $\hat a(t,x,\mu,q,M)$, to $\inf_{a\in A} \mathbb{H}(t,x,a,\mu,q,M)$. Then, an optimal control $\tilde\alpha^*$ in the statement of the verification theorem 3.3.1, obtained from the minimization of the (infinite dimensional) Hamiltonian in (3.3.17), is written merely as $\tilde\alpha^*(t,x,\mu) = \hat a(t,x,\mu,\partial_\mu w(\mu)(x), \partial_x\partial_\mu w(\mu)(x))$, which extends the form discuss in Remark 3.3.3, and says that it depends locally upon the derivatives of the value function. In the more general case when the coefficients depend upon the law of the control, we shall see how one can derive the form of the optimal control for the linear-quadratic problem. □

3.4 Application: linear-quadratic McKean-Vlasov control problem

We consider a multivariate linear McKean-Vlasov controlled dynamics with coefficients given by

$$\begin{aligned} b(t,x,\mu,a,\lambda) &= b_0(t) + B(t)x + \bar{B}(t)\bar{\mu} + C(t)a + \bar{C}(t)\bar{\lambda}, \\ \sigma(t,x,\mu,a,\lambda) &= \sigma_0(t) + D(t)x + \bar{D}(t)\bar{\mu} + F(t)a + \bar{F}(t)\bar{\lambda}, \end{aligned} \tag{3.4.1}$$

for $(t,x,\mu,a,\lambda) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m)$, where we set

$$\bar{\mu} := \int_{\mathbb{R}^d} x\mu(dx), \qquad \bar{\lambda} := \int_{\mathbb{R}^m} a\lambda(da).$$

Here B, $\bar{B}$, D, $\bar{D}$ are deterministic continuous functions valued in $\mathbb{R}^{d\times d}$, and C, $\bar{C}$, F, $\bar{F}$ are deterministic continuous functions valued in $\mathbb{R}^{d\times m}$, and b_0, σ_0 are deterministic continuous function valued in $\mathbb{R}^d$. The quadratic cost functions are given by

$$\begin{aligned} f(t,x,\mu,a,\lambda) &= x^\intercal Q_2(t)x + \bar{\mu}^\intercal \bar{Q}_2(t)\bar{\mu} + a^\intercal R_2(t)a + \bar{\lambda}^\intercal \bar{R}_2(t)\bar{\lambda} + 2x^\intercal M_2(t)a \\ &\quad + 2\bar{\mu}^\intercal \bar{M}_2(t)\bar{\lambda} + q_1(t).x + \bar{q}_1(t).\bar{\mu} + r_1(t).a + \bar{r}_1(t).\bar{\lambda}, \\ g(x,\mu) &= x^\intercal P_2 x + \bar{\mu}^\intercal \bar{P}_2\bar{\mu} + p_1.x + \bar{p}_1.\bar{\mu}, \end{aligned} \tag{3.4.2}$$

where Q_2, $\bar{Q}_2$ are deterministic continuous functions, P_2, $\bar{P}_2$ are constants valued in $\mathbb{R}^{d\times d}$, R_2, $\bar{R}_2$ are deterministic continuous functions valued in $\mathbb{R}^{m\times m}$, M_2, $\bar{M}_2$ are deterministic continuous functions valued in $\mathbb{R}^{d\times m}$, q_1, $\bar{q}_1$ are deterministic continuous functions, p_1, $\bar{p}_1$ are constants valued in $\mathbb{R}^d$, and r_1, $\bar{r}_1$ are deterministic continuous functions valued in $\mathbb{R}^m$. Since f and g are real-valued, we may assume w.l.o.g. that all the matrices Q_2, $\bar{Q}_2$, R_2, $\bar{R}_2$, P_2, $\bar{P}_2$ are symmetric. This linear quadratic (LQ) framework is similar to the one in [Yon13], and extends the one considered in [BSYY16] where there is no dependence on the law of the control, and the diffusion coefficient is deterministic.

The functions $\hat{f}$ and $\hat{g}$ defined in (3.3.7) are then given by

$$\left\{ \begin{aligned} \hat{f}(t,\mu,\tilde{\alpha}) &= \mathrm{Var}(\mu)(Q_2(t)) + \bar{\mu}^\intercal(Q_2(t) + \bar{Q}_2(t))\bar{\mu} \\ &\quad + \mathrm{Var}(\tilde{\alpha}\star\mu)(R_2(t)) + \overline{\tilde{\alpha}\star\mu}^\intercal(R_2(t) + \bar{R}_2(t))\overline{\tilde{\alpha}\star\mu} \\ &\quad + 2\bar{\mu}^\intercal(M_2(t) + \bar{M}_2(t))\overline{\tilde{\alpha}\star\mu} + 2\int_{\mathbb{R}^d}(x-\bar{\mu})^\intercal M_2(t)\tilde{\alpha}(x)\mu(dx) \\ &\quad + \big(q_1(t) + \bar{q}_1(t)\big).\bar{\mu} + \big(r_1(t) + \bar{r}_1(t)\big).\overline{\tilde{\alpha}\star\mu} \\ \hat{g}(\mu) &= \mathrm{Var}(\mu)(P_2) + \bar{\mu}^\intercal(P_2 + \bar{P}_2)\bar{\mu} + (p_1 + \bar{p}_1).\bar{\mu}, \end{aligned} \right. \tag{3.4.3}$$

for any $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$, $\tilde{\alpha} \in L(\mathbb{R}^d;A)$ (here with $A = \mathbb{R}^m$), where we set for any Λ in $\mathbb{S}^d$ (resp. in $\mathbb{S}^m$), and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ (resp. $\mathcal{P}_2(\mathbb{R}^m)$):

$$\bar{\mu}_2(\Lambda) := \int x^\intercal \Lambda x\mu(dx), \qquad \mathrm{Var}(\mu)(\Lambda) := \bar{\mu}_2(\Lambda) - \bar{\mu}^\intercal\Lambda\bar{\mu}.$$

We look for a value function solution to the Bellman equation (3.3.11) in the form

$$w(t,\mu) = \mathrm{Var}(\mu)(\Lambda(t)) + \bar{\mu}^\intercal\Gamma(t)\bar{\mu} + \gamma(t).\bar{\mu} + \chi(t), \tag{3.4.4}$$

for some functions Λ, $\Gamma \in \mathcal{C}^1([0,T];\mathbb{S}^d)$, $\gamma \in \mathcal{C}^1([0,T];\mathbb{R}^d)$, and $\chi \in \mathcal{C}^1([0,T];\mathbb{R})$. The lifted function of w in (3.4.4) is given by

$$\mathcal{W}(t,X) = \mathbb{E}[X^\intercal\Lambda(t)X] + \mathbb{E}[X]^\intercal(\Gamma(t) - \Lambda(t))\mathbb{E}[X] + \gamma(t).\mathbb{E}[X] + \chi(t),$$

for $X \in L^2(\mathcal{F}_0;\mathbb{R}^d)$. By computing for all $Y \in L^2(\mathcal{F}_0;\mathbb{R}^d)$ the difference

$$\mathcal{W}(t,X+Y) - \mathcal{W}(t,X) \;=\; \mathbb{E}\Big[\big(2X^\intercal\Lambda(t) + 2\mathbb{E}[X]^\intercal(\Gamma(t)-\Lambda(t)) + \gamma(t)\big).Y\Big] \;+\; o(\|Y\|_{L^2}),$$

we see that $\mathcal{W}$ is Fréchet differentiable (w.r.t. X) with $[D\mathcal{W}](t,X)(Y) = \mathbb{E}\big[\big(2X^\intercal\Lambda(t) + 2\mathbb{E}[X]^\intercal(\Gamma(t) - \Lambda(t)) + \gamma(t)\big).Y\big]$. This shows that w lies in $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ with

$$\begin{aligned}
\partial_t w(t,\mu) &= \mathrm{Var}(\mu)(\Lambda'(t)) + \bar\mu^\intercal\Gamma'(t)\bar\mu + \gamma'(t)\bar\mu + \chi'(t),\\
\partial_\mu w(t,\mu)(x) &= 2x^\intercal\Lambda(t) + 2\bar\mu^\intercal(\Gamma(t)-\Lambda(t)) + \gamma(t),\\
\partial_x\partial_\mu w(t,\mu)(x) &= 2\Lambda(t).
\end{aligned}$$

Together with the quadratic expression (3.4.3) of $\hat f$, $\hat g$, we then see that w satisfies the Bellman equation (3.3.11) iff

$$\begin{aligned}
& \mathrm{Var}(\mu)(\Lambda(T)) + \bar\mu^\intercal\Gamma(T)\bar\mu + \gamma(T).\bar\mu + \chi(T)\\
=\; & \mathrm{Var}(\mu)(P_2) + \bar\mu^\intercal(P_2+\bar P_2)\bar\mu + (p_1+\bar p_1).\bar\mu,
\end{aligned} \tag{3.4.5}$$

holds for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and

$$\begin{aligned}
& \mathrm{Var}(\mu)\big(\Lambda'(t) + Q_2(t) + D(t)^\intercal\Lambda(t)D(t) + \Lambda(t)B(t) + B(t)^\intercal\Lambda(t)\big) + \inf_{\tilde\alpha\in L(\mathbb{R}^d,A)} G_t^\mu(\tilde\alpha)\\
& + \bar\mu^\intercal\Big(\Gamma'(t) + Q_2(t) + \bar Q_2(t) + (D(t)+\bar D(t))^\intercal\Lambda(t)(D(t)+\bar D(t))\\
& \qquad + \Gamma(t)(B(t)+\bar B(t)) + (B(t) \;+\; \bar B(t))^\intercal\Gamma(t)\Big)\bar\mu\\
& + \big(q_1(t) + \bar q_1(t) + \gamma(t)(B(t)+\bar B(t)) + 2\sigma_0^\intercal\Lambda(t)(D(t)+\bar D(t)) + 2b_0(t)^\intercal\Gamma(t)\big)\bar\mu\\
& + \chi'(t) \;+\; \gamma(t).b_0(t) + \sigma_0(t)^\intercal\Lambda(t)b_0(t)\\
=\; & 0,
\end{aligned} \tag{3.4.6}$$

holds for all $t \in [0,T)$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, where the function $G_t^\mu : L_\mu^2(A) \supset L(\mathbb{R}^d;A) \to \mathbb{R}$ is defined by

$$\begin{aligned}
G_t^\mu(\tilde\alpha) \;=\; & \mathrm{Var}(\tilde\alpha\star\mu)(U_t) \;+\; \overline{\tilde\alpha\star\mu}^\intercal V_t\overline{\tilde\alpha\star\mu} \;+\; 2\int_{\mathbb{R}^d}(x-\bar\mu)^\intercal S_t\tilde\alpha(x)\mu(dx)\\
& + 2\bar\mu^\intercal Z_t\overline{\tilde\alpha\star\mu} \;+\; Y_t.\overline{\tilde\alpha\star\mu},
\end{aligned} \tag{3.4.7}$$

and we set $U_t = U(t,\Lambda(t))$, $V_t = V(t,\Lambda(t))$, $S_t = S(t,\Lambda(t))$, $Z_t = Z(t,\Lambda(t),\Gamma(t))$, $Y_t = Y(t,\Gamma(t),\gamma(t))$ with

$$\left\{\begin{aligned}
U(t,\Lambda(t)) &= F(t)^\intercal\Lambda(t)F(t) + R_2(t),\\
V(t,\Lambda(t)) &= (F(t)+\bar F(t))^\intercal\Lambda(t)(F(t)+\bar F(t)) + R_2(t) + \bar R_2(t),\\
S(t,\Lambda(t)) &= D(t)^\intercal\Lambda(t)F(t) + \Lambda(t)C(t) + M_2(t),\\
Z(t,\Lambda(t),\Gamma(t)) &= (D(t)+\bar D(t))^\intercal\Lambda(t)(F(t)+\bar F(t)) + \Gamma(t)(C(t)+\bar C(t)) + M_2(t) + \bar M_2(t)\\
Y(t,\Gamma(t),\gamma(t)) &= \big(C(t)+\bar C(t)\big)^\intercal\gamma(t) + r_1(t) + \bar r_1(t) + 2\big(F(t)+\bar F(t)\big)^\intercal\Lambda(t)\sigma_0(t).
\end{aligned}\right. \tag{3.4.8}$$

We now search for the infimum of the function G_t^μ. After some straightforward calculation, we derive the Gateaux derivative of G_t^μ at $\tilde\alpha$ in the direction $\beta \in L_\mu^2(A)$, which is given by:

$$DG_t^\mu(\tilde\alpha,\beta) \;:=\; \lim_{\varepsilon\to 0}\frac{G_t^\mu(\tilde\alpha+\varepsilon\beta) - G_t^\mu(\tilde\alpha)}{\varepsilon} \;=\; \int_{\mathbb{R}^d}\dot g_t^\mu(x,\tilde\alpha).\beta(x)\mu(dx)$$

with

$$\dot{g}_t^\mu(x,\tilde{\alpha}) \;=\; 2U_t\tilde{\alpha} + 2(V_t - U_t)\overline{\tilde{\alpha}\star\mu} + 2S_t^\intercal(x-\bar{\mu}) + 2Z_t^\intercal\bar{\mu} + Y_t.$$

Suppose that the symmetric matrices U_t and V_t in (3.4.8) are positive, hence invertible (this will be discussed later on). Then, the function G_t^μ is convex and coercive on the Hilbert space $L^2_\mu(A)$, and attains its infimum at some $\tilde{\alpha} = \tilde{\alpha}^*(t,.,\mu)$ s.t. $DG_t^\mu(\tilde{\alpha};.)$ vanishes, i.e. $\dot{g}_t^\mu(x,\tilde{\alpha}^*(t,.,\mu)) = 0$ for all $x \in \mathbb{R}^d$, which gives:

$$\tilde{\alpha}^*(t,x,\mu) \;=\; -U_t^{-1}S_t^\intercal(x-\bar{\mu}) \;-\; V_t^{-1}Z_t^\intercal\bar{\mu} \;-\; \frac{1}{2}V_t^{-1}Y_t. \tag{3.4.9}$$

It is clear that $\tilde{\alpha}^*(t,.,\mu)$ lies in $L(\mathbb{R}^d;A)$, and so after some straightforward caculation:

$$\begin{aligned}\inf_{\tilde{\alpha}\in L(R^d,A)} G_t^\mu(\tilde{\alpha}) \;=\; G_t^\mu(\tilde{\alpha}^*(t,.,\mu)) \;&=\; -\mathrm{Var}(\mu)\big(S_tU_t^{-1}S_t^\intercal\big) \;-\; \bar{\mu}^\intercal\big(Z_tV_t^{-1}Z_t^\intercal\big)\bar{\mu}\\ &\quad -\; Y_t^\intercal V_t^{-1}Z_t^\intercal\bar{\mu} \;-\; \frac{1}{4}Y_t^\intercal V_t^{-1}Y_t.\end{aligned}$$

Plugging the above expression in (3.4.6), we observe that the relation (3.4.5)-(3.4.6), hence the Bellman equation, is satisfied by identifying the terms in $\mathrm{Var}(\mu)(.)$, $\bar{\mu}^\intercal(.)\bar{\mu}$, $\bar{\mu}$, which leads to the system of ordinary differential equations (ODEs) for $(\Lambda,\Gamma,\gamma,\chi)$:

$$\left\{\begin{aligned}\Lambda'(t) + Q_2(t) + D(t)^\intercal\Lambda(t)D(t) + \Lambda(t)B(t) + B(t)^\intercal\Lambda(t)&\\ -S(t,\Lambda(t))U(t,\Lambda(t))^{-1}S(t,\Lambda(t))^\intercal &= 0,\\ \Lambda(T) &= P_2,\end{aligned}\right. \tag{3.4.10}$$

$$\left\{\begin{aligned}\Gamma'(t) + Q_2(t) + \bar{Q}_2(t) + (D(t)+\bar{D}(t))^\intercal\Lambda(t)(D(t)+\bar{D}(t)) + \Gamma(t)(B(t)+\bar{B}(t))&\\ +\,(B(t)+\bar{B}(t))^\intercal\Gamma(t) - Z(t,\Lambda(t),\Gamma(t))V(t,\Lambda(t))^{-1}Z(t,\Lambda(t),\Gamma(t))^\intercal &= 0,\\ \Gamma(T) &= P_2 + \bar{P}_2,\end{aligned}\right. \tag{3.4.11}$$

$$\left\{\begin{aligned}\gamma'(t) + \big(B(t)+\bar{B}(t)\big)^\intercal\gamma(t) - Z(t,\Lambda(t),\Gamma(t))V(t,\Lambda(t))^{-1}Y(t,\Gamma(t),\gamma(t))&\\ +\, q_1(t) + \bar{q}_1(t) + 2\big(D(t)+\bar{D}(t)\big)^\intercal\Lambda(t)\sigma_0(t) + 2\Gamma(t)b_0(t) &= 0,\\ \gamma(T) &= p_1 + \bar{p}_1\end{aligned}\right. \tag{3.4.12}$$

$$\left\{\begin{aligned}\chi'(t) - \tfrac{1}{4}Y(t,\Gamma(t),\gamma(t))^\intercal V(t,\Lambda(t))^{-1}Y(t,\Gamma(t),\gamma(t))&\\ +\,\gamma(t).b_0(t) + \sigma_0(t)^\intercal\Lambda(t)\sigma_0(t) &= 0,\\ \chi(T) &= 0.\end{aligned}\right. \tag{3.4.13}$$

Therefore, the resolution of the Bellman equation in the LQ framework is reduced to the resolution of the Riccati equations (3.4.10) and (3.4.11) for Λ and Γ, and then given (Λ,Γ), to the resolution of the linear ODEs (3.4.12) and (3.4.13) for γ and χ. Suppose that there exists a solution $(\Lambda,\Gamma) \in \mathcal{C}^1([0,T];\mathbb{S}^d)\times\mathcal{C}^1([0,T];\mathbb{S}^d)$ to (3.4.10)-(3.4.11) s.t. (U_t,V_t) in (3.4.8) lies in $\mathbb{S}^m_{>+}\times\mathbb{S}^m_{>+}$ for all $t\in[0,T]$ (see Remark 3.4.1). Then, the above calculations are justified a posteriori, and by noting also that the mapping $(t,x,\mu)\mapsto\tilde{\alpha}^*(t,x,\mu)\in Lip([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d);A)$, we deduce by the verification theorem that the value function v is equal to w in (3.4.4) with $(\Lambda,\Gamma,\gamma,\chi)$ solution to (3.4.10)-(3.4.11)-(3.4.12)-(3.4.13). Moreover, the optimal control is given in feedback form from (3.4.9) by

$$\alpha_t^* \;=\; \tilde{\alpha}^*(t,X_t^*,\mathbb{P}_{X_t^*}) = -U_t^{-1}S_t^\intercal(X_t^* - \mathbb{E}[X_t^*]) \;-\; V_t^{-1}Z_t^\intercal\mathbb{E}[X_t^*] \;-\; \frac{1}{2}V_t^{-1}Y_t, \tag{3.4.14}$$

where X^* is the state process controlled by α^*.

Remark 3.4.1. In the case where $M_2 = \bar{M}_2 = 0$ (i.e. no crossing term between the state and the control in the quadratic cost function f), it is shown in Proposition 3.1 and 3.2 in [Yon13] that under the condition

$$P_2 \ \geq\ 0,\ P_2 + \bar{P}_2 \ \geq\ 0, \qquad \begin{array}{l} Q_2(t) \ \geq\ 0,\ Q_2(t) + \bar{Q}_2(t) \ \geq\ 0, \\ R_2(t) \ \geq \delta I_m,\ R_2(t) + \bar{R}_2(t) \ \geq\ \delta I_m \end{array} \tag{3.4.15}$$

for some $\delta > 0$, the Riccati equations (3.4.10)-(3.4.11) admit unique solutions $(\Lambda, \Gamma) \in \mathcal{C}^1([0,T]; \mathbb{S}^d_+) \times \mathcal{C}^1([0,T]; \mathbb{S}^d_+)$, and then U_t, V_t in (3.4.8) are symmetric positive definite matrices, i.e. lie in $\mathbb{S}^m_{>+}$ for all $t \in [0,T]$. In this case, we retrieve the expressions (3.4.14) of the optimal control in feedback form obtained in [Yon13].

We shall see in the next two paragraphs, some other examples arising from finance with explicit solutions where condition (3.4.15) is not satisfied. □

3.4.1 Mean-variance portfolio selection

The mean-variance problem consists in minimizing a cost functional of the form:

$$\begin{aligned} J(\alpha) &= \frac{\eta}{2}\mathrm{Var}(X_T) - \mathbb{E}[X_T] \\ &= \mathbb{E}\big[\frac{\eta}{2}\big(X_T\big)^2 - X_T\big] \ - \ \frac{\eta}{2}\Big(\mathbb{E}[X_T]\Big)^2 \end{aligned}$$

for some $\eta > 0$, with a dynamics for the wealth process $X = (X^\alpha)$ controlled by the amount α_t valued in $A = \mathbb{R}^d$ invested in d risky stocks at time t:

$$dX_t \ = \ r(t)X_t dt + \alpha_t^\intercal\big(\rho(t)dt + \vartheta(t)dB_t\big), \quad X_0 \ = x_0 \ \in \ \mathbb{R},$$

where r is the interest rate, ρ and $\vartheta > 0$ are the excess rate of return (w.r.t. the interest rate) valued in $\mathbb{R}^d$ and volatility of the stock price valued in $R^{d\times d}$, and we denote by $\Sigma(t) = \vartheta(t)\vartheta(t)^\intercal$, called the covariance matrix of the stock price, and these deterministic functions are assumed to be continuous. This model fits into the LQ framework (3.4.1)-(3.4.2) of the McKean-Vlasov problem, with a linear controlled dynamics that does not have mean-field interaction:

$$\begin{aligned} b_0 = 0,\ B(t) = r(t),\ \bar{B} = 0,\ C(t) = \rho(t),\ \bar{C} = 0, \\ \sigma_0 = D = \bar{D} = 0,\ F(t) = \vartheta(t),\ \bar{F} = 0, \\ Q_2 = \bar{Q}_2 = M_2 = \bar{M}_2 = R_2 = \bar{R}_2 = 0, \\ q_1 = \bar{q}_1 = r_1 = \bar{r}_1 = 0,\ P_2 = \frac{\eta}{2},\ \bar{P}_2 = -\frac{\eta}{2},\ p_1 = 0,\ \bar{p}_1 = -1. \end{aligned}$$

The Riccati system (3.4.10)-(3.4.11)-(3.4.12)-(3.4.13) for $(\Lambda(t), \Gamma(t), \gamma(t), \chi(t))$ is written in this case as

$$\left\{ \begin{array}{rclrcl} \Lambda'(t) - \big(\rho(t)^\intercal\Sigma(t)^{-1}\rho(t) - 2r(t)\big)\Lambda(t) &=& 0, & \Lambda(T) &=& \frac{\eta}{2}, \\ \Gamma'(t) - \rho(t)^\intercal\Sigma(t)^{-1}\rho(t)\frac{\Gamma^2(t)}{\Lambda(t)} + 2r(t)\Gamma(t) &=& 0, & \Gamma(T) &=& 0, \\ \gamma'(t) + r(t)\gamma(t) - \rho(t)^\intercal\Sigma(t)^{-1}\rho(t)\frac{\Gamma(t)}{\Lambda(t)} &=& 0, & \gamma(T) &=& -1, \\ \chi'(t) - \rho(t)^\intercal\Sigma(t)^{-1}\rho(t)\frac{\gamma^2(t)}{4\Lambda(t)} &=& 0, & \chi(T) &=& 0, \end{array} \right. \tag{3.4.16}$$

whose explicit solution is given by

$$\left\{ \begin{array}{rcl} \Lambda(t) &=& \frac{\eta}{2}\exp\big(\int_t^T 2r(s) - \rho(s)^\intercal\Sigma(s)^{-1}\rho(s)\ ds\big), \\ \Gamma(t) &=& 0, \\ \gamma(t) &=& -\exp\big(\int_t^T r(s)ds\big) \\ \chi(t) &=& \frac{1}{4}\exp\big(\int_t^T \rho(s)^\intercal\Sigma(s)^{-1}\rho(s)ds\big) - \frac{1}{4}. \end{array} \right. \tag{3.4.17}$$

Although the condition (3.4.15) is not satisfied, we see that (U_t, V_t) in (3.4.8), which are here explicitly given by $U_t = V_t = \Sigma(t)\Lambda(t)$, are positive definite, and this validates our calculations for the verification theorem. Notice also that the functions (Z_t, Y_t) in (3.4.8) are explicitly given by $Z_t = 0$, $Y_t = \rho(t)\gamma(t)$. Therefore, the optimal control is given in feedback form from (3.4.14) by

$$\begin{aligned} \alpha_t^* &= \tilde{\alpha}^*(t, X_t^*, \mathbb{P}_{X_t^*}) \\ &= -\Sigma(t)^{-1}\rho(t)(X_t^* - \mathbb{E}[X_t^*]) \; + \; \frac{1}{\eta}\Sigma(t)^{-1}\rho(t)\exp\Big(\int_t^T \rho(s)^\intercal\Sigma(s)^{-1}\rho(s) - r(s)\, ds\Big), \end{aligned} \tag{3.4.18}$$

where X^* is the optimal wealth process with portfolio strategy α^*, hence with mean process governed by

$$d\mathbb{E}[X_t^*] \;=\; r(t)\mathbb{E}[X_t^*]dt \; + \; \frac{1}{\eta}\rho(t)^\intercal\Sigma(t)^{-1}\rho(t)\exp\Big(\int_t^T \rho(s)^\intercal\Sigma(s)^{-1}\rho(s) - r(s)\, ds\Big)dt,$$

and explicitly given by

$$\mathbb{E}[X_t^*] \;=\; x_0 e^{\int_0^t r(s)ds} \; + \; \frac{1}{\eta}\exp\Big(\int_t^T \rho(s)^\intercal\Sigma(s)^{-1}\rho(s) - r(s)\, ds\Big)\Big(\exp\Big(\int_0^t \rho(s)^\intercal\Sigma(s)^{-1}\rho(s)ds\Big) - 1\Big).$$

Plugging into (3.4.18), we get the optimal control for the mean-variance portfolio problem

$$\alpha_t^* \;=\; \Sigma(t)^{-1}\rho(t)\Big[x_0 e^{\int_0^t r(s)ds} + \frac{1}{\eta}\exp\Big(\int_0^T \rho(s)^\intercal\Sigma(s)^{-1}\rho(s)ds - \int_t^T r(s)ds\Big) - X_t^*\Big],$$

and retrieve the closed-form expression of the optimal control found in [LZ00], [AD10] or [FL16] by different approaches.

3.4.2 Interbank systemic risk model

We consider a model of interbank borrowing and lending studied in [CFS15] where the log-monetary reserve of each bank in the asymptotics when the number of banks tend to infinity, is governed by the McKean-Vlasov equation:

$$dX_t \;=\; [\kappa(\mathbb{E}[X_t] - X_t) + \alpha_t]dt + \sigma dB_t, \qquad X_0 \;=\; x_0 \in \mathbb{R}. \tag{3.4.19}$$

Here, $\kappa \geq 0$ is the rate of mean-reversion in the interaction from borrowing and lending between the banks, and $\sigma > 0$ is the volatility coefficient of the bank reserve, assumed to be constant. Moreover, all banks can control their rate of borrowing/lending to a central bank with the same policy α in order to minimize a cost functional of the form

$$J(\alpha) \;=\; \mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}\alpha_t^2 - q\alpha_t(\mathbb{E}[X_t] - X_t) + \frac{\eta}{2}(\mathbb{E}[X_t] - X_t)^2\Big)dt + \frac{c}{2}(\mathbb{E}[X_T] - X_T)^2\Big],$$

where $q > 0$ is a positive parameter for the incentive to borrowing ($\alpha_t > 0$) or lending ($\alpha_t < 0$), and $\eta > 0$, $c > 0$ are positive parameters for penalizing departure from the average. This model fits into the LQ McKean-Vlasov framework (3.4.1)-(3.4.2) with $d = m = 1$ and

$$\begin{aligned} b_0 = 0,\ B = -\kappa,\ \bar{B} = \kappa,\ C = 1,\ \bar{C} = 0, \\ \sigma_0 = \sigma,\ D = \bar{D} = F = \bar{F} = 0, \\ Q_2 = \frac{\eta}{2},\ \bar{Q}_2 = -\frac{\eta}{2},\ R_2 = \frac{1}{2},\ \bar{R}_2 = 0,\ M_2 = \frac{q}{2},\ \bar{M}_2 = -\frac{q}{2}, \\ q_1 = \bar{q}_1 = r_1 = \bar{r}_1 = 0, \quad P_2 = \frac{c}{2},\ \bar{P}_2 = -\frac{c}{2},\ p_1 = \bar{p}_1 = 0. \end{aligned}$$

The Riccati system (3.4.10)-(3.4.11)-(3.4.12)-(3.4.13) for $(\Lambda(t), \Gamma(t), \gamma(t), \chi(t))$ is written in this case as

$$\left\{ \begin{array}{rclrcl} \Lambda'(t) - 2(\kappa+q)\Lambda(t) - 2\Lambda^2(t) - \frac{1}{2}(q^2-\eta) & = & 0, & \Lambda(T) & = & \frac{c}{2}, \\ \Gamma'(t) - 2\Gamma^2(t) & = & 0, & \Gamma(T) & = & 0, \\ \gamma'(t) - 2\gamma(t)\Gamma(t) & = & 0, & \gamma(T) & = & 0, \\ \chi'(t) + \sigma^2\Lambda(t) - \frac{1}{2}\gamma^2(t) & = & 0, & \chi(T) & = & 0, \end{array} \right. \tag{3.4.20}$$

whose explicit solution is given by $\Gamma = \gamma = 0$, and

$$\begin{aligned} \chi(t) &= \sigma^2 \int_t^T \Lambda(s)ds, \\ \Lambda(t) &= \frac{1}{2} \frac{(q-\eta^2)\big(e^{(\delta^+-\delta^-)(T-t)} - 1\big) - c\big(\delta^+ e^{(\delta^+-\delta^-)(T-t)} - \delta^-\big)}{\delta^- e^{(\delta^+-\delta^-)(T-t)} - \delta^+\big) - ce^{(\delta^+-\delta^-)(T-t)} - 1}, \end{aligned}$$

where we set

$$\delta^{\pm} = -(\kappa+q) \pm \sqrt{(\kappa+q)^2 + (\eta - q^2)}.$$

Moreover, the functions (U_t, V_t, Z_t, Y_t) in (3.4.8) are explicitly given by: $U_t = V_t = \frac{1}{2}$ (hence > 0), $S_t = \Lambda(t) + \frac{q}{2}$, $Z_t = \Gamma(t) = 0$, $Y_t = \gamma(t) = 0$. Therefore, the optimal control is given in feedback form from (3.4.14) by

$$\alpha_t^* = \tilde{\alpha}^*(t, X_t^*, \mathbb{P}_{X_t^*}) = -(2\Lambda(t) + q)(X_t^* - \mathbb{E}[X_t^*]), \tag{3.4.21}$$

where X^* is the optimal log-monetary reserve controlled by the rate of borrowing/lending α^*. We then retrieve the expression found in [CFS15] by sending the number of banks N to infinity in their formula for the optimal control. Actually, from (3.4.19), we have $d\mathbb{E}[X_t^*] = \mathbb{E}[\alpha_t^*]dt$, while $\mathbb{E}[\alpha_t^*] = 0$ from (3.4.21). We conclude that the optimal rate of borrowing/lending is equal to

$$\alpha_t^* = -(2\Lambda(t) + q)(X_t^* - x_0), \quad 0 \le t \le T.$$

3.5 Viscosity solutions

In general, there are no smooth solutions to the HJB equation, and in the spirit of HJB equation for standard stochastic control, we shall introduce in this section a notion of viscosity solutions for the Bellman equation (3.3.11) in the Wasserstein space of probability measures $\mathcal{P}_2(\mathbb{R}^d)$. We adopt the approach in [Lio12], and detailed in [Car12], which consists, after the lifting identification between measures and random variables, in working in the Hilbert space $L^2(\mathcal{F}_0; \mathbb{R}^d)$ instead of working in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$, in order to use the various tools developed for viscosity solutions in Hilbert spaces, in particular in our context, for second order Hamilton-Jacobi equations.

Let us rewrite the the Bellman equation (3.3.11) in the "Hamiltonian" form:

$$\left\{ \begin{array}{rcll} -\dfrac{\partial v}{\partial t} + H(t, \mu, \partial_\mu v(t,\mu), \partial_x \partial_\mu v(t,\mu)) & = & 0 & \text{on } [0,T) \times \mathcal{P}_2(\mathbb{R}^d), \\ v(T,.) & = & \hat{g} & \text{on } \mathcal{P}_2(\mathbb{R}^d) \end{array} \right. \tag{3.5.1}$$

where H is the function defined by

$$\begin{aligned} H(t,\mu,p,\Gamma) = & - \inf_{\tilde{\alpha} \in L(\mathbb{R}^d; A)} \Big[< f(t,.,\tilde{\alpha}(.), Id\tilde{\alpha} \star \mu) + p(.).b(t,.,\tilde{\alpha}(.), Id\tilde{\alpha} \star \mu) \\ & + \frac{1}{2} \mathrm{tr}\big(\Gamma(.)\sigma\sigma^\intercal(t,.,\tilde{\alpha}(.), Id\tilde{\alpha} \star \mu)\big), \mu > \Big], \end{aligned} \tag{3.5.2}$$

for $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $(p,\Gamma) \in L^2_\mu(\mathbb{R}^d)\times L^\infty_\mu(\mathbb{S}^d)$.

We identify v and its lifted version by using the same notation $v(t,\mathcal{L}(\xi)) = v(t,\xi)$, and then consider the "lifted" Bellman equation in $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$:

$$\begin{cases} -\dfrac{\partial v}{\partial t} + \mathcal{H}(t,\xi,Dv(t,\xi),D^2v(t,\xi)) & = & 0 \quad \text{on} \;\; [0,T)\times L^2(\mathcal{F}_0;\mathbb{R}^d), \\ v(T,\xi) & = & \hat{G}(\xi) \; := \; \mathbb{E}[g(\xi,\mathbb{P}_\xi)], \quad \xi \in L^2(\mathcal{F}_0;\mathbb{R}^d), \end{cases} \tag{3.5.3}$$

where $\mathcal{H} : [0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)\times L^2(\mathcal{F}_0;\mathbb{R}^d)\times S(L^2(\mathcal{F}_0;\mathbb{R}^d)) \to \mathbb{R}$ is defined by

$$\begin{aligned} \mathcal{H}(t,\xi,P,Q) \;\; = \;\; & -\inf_{\tilde{\alpha}\in L(\mathbb{R}^d;A)} \Big\{ \mathbb{E}\Big[f(t,\xi,\tilde{\alpha}(\xi), Id\tilde{\alpha}\star\mathbb{P}_\xi) + P.b(t,\xi,\tilde{\alpha}(\xi),Id\tilde{\alpha}\star\mathbb{P}_\xi) \\ & + \frac{1}{2}Q\big(\sigma(t,\xi,\tilde{\alpha}(\xi),Id\tilde{\alpha}\star\mathbb{P}_\xi)N\big).\big(\sigma(t,\xi,\tilde{\alpha}(\xi),Id\tilde{\alpha}\star\mathbb{P}_\xi)N\big)\Big]\Big\}, \end{aligned} \tag{3.5.4}$$

with $N \in L^2(\mathcal{F}_0;\mathbb{R}^n)$ of zero mean, unit variance, and independent of ξ. Observe that when $v(t,\mu)$ and $v(t,\xi)$ are smooth functions respectively in $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ and $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$, linked by the lifting relation $v(t,\xi) = v(t,\mathbb{P}_\xi)$, then from the relations (1.2.1)-(1.2.2), $v(t,\mu)$ is solution to the "lifted" Bellman equation (3.5.1) iff $v(t,\xi)$ with $\mathcal{L}(\xi)=\mu$ is solution to the Bellman equation (3.5.3). Let us mention that the lifted Bellman equation was also derived in [BFY15] in the case where $\sigma = \sigma(x)$ is not controlled and does not depend on the distribution of the state process, and there is no dependence on the marginal distribution of the control process on the coefficients b and f.

It is then natural to define viscosity solutions for the Bellman equation (3.5.1) (hence (3.3.11)) from viscosity solutions to (3.5.3). As usual, we say that a function u (resp. $\tilde{u}$) is locally bounded in on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ (resp. $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$) if it is bounded on bounded subsets of $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ (resp. $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$), and we denote by u^* (resp. $\tilde{u}^*$) its upper semicontinuous envelope, and by u_*(resp. $\tilde{u}_*$) its lower semicontinuous envelope. Similarly as in [GNT08], we define the set $\mathcal{C}^2_\ell([0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d))$ of test functions for the lifted Bellman equation, as the set of real-valued continuous functions φ on $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$ which are continuously differentiable in $t\in[0,T)$, twice continuously Fréchet differentiable on $L^2(\mathcal{F}_0;\mathbb{R}^d)$.

Definition 3.5.1. *We say that a locally bounded function $u : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a viscosity (sub, super) solution to* (3.5.1) *if the lifted function $\tilde{u} : [0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d) \to \mathbb{R}$ defined by*

$$\tilde{u}(t,\xi) \;\; = \;\; u(t,\mathbb{P}_\xi), \qquad (t,\xi) \in [0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d),$$

is a viscosity (sub, super) solution to the lifted Bellman equation (3.5.3)*, that is:*

(i) $\tilde{u}^*(T,.) \le \hat{G}$, and for any test function $\varphi \in \mathcal{C}^2_\ell([0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d))$ such that $\tilde{u}^*-\varphi$ has a maximum at $(t_0,\xi_0) \in [0,T)\times L^2(\mathcal{F}_0;\mathbb{R}^d)$, one has

$$-\partial_t\varphi(t_0,\xi_0) + \mathcal{H}(t_0,\xi_0,D\varphi(t_0,\xi_0),D^2\varphi(t_0,\xi_0)) \;\; \le \;\; 0.$$

(ii) $\tilde{u}_*(T,.) \ge \hat{G}$, and for any test function $\varphi \in \mathcal{C}^2_\ell([0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d))$ such that $\tilde{u}_*-\varphi$ has a minimum at $(t_0,\xi_0) \in [0,T)\times L^2(\mathcal{F}_0;\mathbb{R}^d)$, one has

$$-\partial_t\varphi(t_0,\xi_0) + \mathcal{H}(t_0,\xi_0,D\varphi(t_0,\xi_0),D^2\varphi(t_0,\xi_0)) \;\; \ge \;\; 0.$$

The main goal of this section is to prove the viscosity characterization of the value function v in (3.3.8) to the Bellman equation (3.3.11), hence equivalently the viscosity characterization of the lifted value function $v : [0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$ via the lifted identification

$$v(t,\xi) \;\; = \;\; v(t,\mathbb{P}_\xi), \quad \xi \in L^2(\mathcal{F}_0;\mathbb{R}^d),$$

to the lifted Bellman equation (3.5.3). We shall strenghten condition **(H1)** by assuming in addition that b, σ are uniformly continuous in t, and bounded in (a, λ):

(H1') There exists some constant $C_{b,\sigma} > 0$ s.t. for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^d$, $a, a' \in A$, $\lambda, \lambda' \in \mathcal{P}_2(\mathbb{R}^d \times A)$,

$$\begin{aligned} & |b(t,x,a,\lambda) - b(t',x',a',\lambda')| + |\sigma(t,x,a,\lambda) - \sigma(t',x',a',\lambda')| \\ \leq \quad & C_{b,\sigma}\big[m_{b,\sigma}(|t-t'|) + |x-x'| + |a-a'| + \mathcal{W}_2(\lambda,\lambda')\big], \end{aligned}$$

for some modulus $m_{b,\sigma}$ (i.e. $m_{b,\sigma}(\tau) \to 0$ when τ goes to zero) and

$$|b(t,0,a,\delta_{(0,0)})| + |\sigma(t,0,a,\delta_{(0,0)})| \quad \leq \quad C_{b,\sigma}.$$

We also strenghten condition **(H2)** by making additional (uniform) continuity assumptions on the running and terminal cost functions, and boundedness conditions in (a, λ):

(H2') (i) g is continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ and there exists some constant $C_g > 0$ s.t. for all $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|g(x,\mu)| \quad \leq \quad C_g\big(1 + |x|^2 + \|\mu\|_2^2\big).$$

(ii) There exists some constant $C_f > 0$ s.t. for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $a \in A$, $\lambda \in \mathcal{P}_2(\mathbb{R}^d \times A)$,

$$|f(t,x,a,\lambda)| \quad \leq \quad C_f\big(1 + |x|^2 + \|\lambda\|_2^2\big),$$

and some modulus m_f (i.e. $m_f(\tau) \to 0$ when τ goes to zero) s.t. for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^d$, $a \in A$, $\lambda, \lambda' \in \mathcal{P}_2(\mathbb{R}^d \times A)$,

$$|f(t,x,a,\lambda) - f(t',x',a,\lambda')| \quad \leq \quad m_f\big(|t-t'| + |x-x'| + \mathcal{W}_2(\lambda,\lambda')\big).$$

The boundedness condition in **(H1')-(H2')** of b, σ, f w.r.t. $(a,\lambda) \in A \times \mathcal{P}_2(A)$ is typically satisfied when A is bounded. Under **(H1')**, we get by standard arguments

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\big[\sup_{t \leq s \leq T} |X_s^{t,\xi}|^2\big] \quad < \quad \infty,$$

for any $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, which shows under the quadratic growth condition of g and f in **(H2')** (uniformly in a) that v also satisfy a quadratic growth condition: there exists some positive constant C s.t.

$$\begin{cases} |v(t,\mu)| & \leq \quad C\big(1 + \|\mu\|_2^2\big), \quad (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d), \\ |v(t,\xi)| & \leq \quad C\big(1 + \mathbb{E}|\xi|^2\big), \quad (t,\xi) \in [0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d), \end{cases} \tag{3.5.5}$$

and are thus in particular locally bounded.

We first state a flow continuity property of the marginal distribution of the controlled state process, and the continuity property of the value function. Indeed, from standard estimates on the state process under **(H1')** one easily checks (see also Lemma 3.1 in [BLPR17]) that there exists some positive constant C (independent of t, μ, α), such that for all $\alpha \in \mathcal{A}$, $t, t' \in [0,T]$, $t \leq s \leq T, t' \leq s' \leq T$, $\mu = \mathbb{P}_\xi, \mu' = \mathbb{P}_{\xi'}$ $\in \mathcal{P}_2(\mathbb{R}^d)$:

$$\mathbb{E}\big|X_s^{t,\xi} - X_{s'}^{t',\xi'}\big|^2 \quad \leq \quad C\big(1 + \mathbb{E}|\xi|^2 + \mathbb{E}|\xi'|^2\big)\big(|t-t'| + |s-s'| + \mathbb{E}|\xi-\xi'|^2\big), \tag{3.5.6}$$

and so from the definition of the 2-Wasserstein distance

$$\begin{aligned}\|\mathbb{P}^{t,\mu}_s\|_2 &\leq C(1+\|\mu\|_2) &(3.5.7)\\ \mathcal{W}_2(\mathbb{P}^{t,\mu}_s,\mathbb{P}^{t',\mu'}_{s'}) &\leq C\big(1+\|\mu\|_2+\|\mu'\|_2\big)\big(|t-t'|^{\frac12}+|s-s'|^{\frac12}+\mathcal{W}_2(\mu,\mu')\big) &(3.5.8)\end{aligned}$$

Now from the definition of $v(t,\mu)$ in (3.3.8), together with (3.5.7), (3.5.8) and the growth condition on f in **(H2')**, we have for all $0\leq t\leq t'\leq T$, μ, $\mu'\in\mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}|v(t,\mu)-v(t',\mu')| &\leq \sup_{\alpha\in\mathcal{A}}|J(t,\mu,\alpha)-J(t',\mu',\alpha)|\\
&\leq \sup_{\alpha\in\mathcal{A}}\Big\{\int_t^{t'}|\hat f(s,\mathbb{P}^{t,\mu}_s,\tilde\alpha(s,\cdot,\mathbb{P}^{t,\mu}_s))|ds\\
&\quad+\int_{t'}^T|\hat f(s,\mathbb{P}^{t,\mu}_s,\tilde\alpha(s,\cdot,\mathbb{P}^{t,\mu}_s))-\hat f(s,\mathbb{P}^{t',\mu'}_s,\tilde\alpha(s,\cdot,\mathbb{P}^{t',\mu'}_s))|ds+|\hat g(\mathbb{P}^{t,\mu}_T)-\hat g(\mathbb{P}^{t',\mu'}_T)|\Big\}\\
&\leq C(t'-t)(1+\sup_{t\leq s\leq t'}\|\mathbb{P}^{t,\mu}_s\|_2)\\
&\quad+Cm_f\big(\sup_{t'\leq s\leq T}\mathcal{W}_2(\mathbb{P}^{t,\mu}_s,\mathbb{P}^{t',\mu'}_s)\big)+|\hat g(\mathbb{P}^{t,\mu}_T)-\hat g(\mathbb{P}^{t',\mu'}_T)|\\
&\leq C(t'-t)(1+\|\mu\|_2)+Cm_f\big((1+\|\mu\|_2+\|\mu'\|_2)(|t-t'|^{\frac12}+\mathcal{W}_2(\mu,\mu'))\big)\\
&\quad+|\hat g(\mathbb{P}^{t,\mu}_T)-\hat g(\mathbb{P}^{t',\mu'}_T)|.\end{aligned}$$

By the continuity assumption on g together with the growth condition on g in (**H2′**), which allows to use the dominated converge theorem, we deduce from (3.5.8) that $\hat g(\mathbb{P}^{t,\mu}_s)$ converges to $\hat g(\mathbb{P}^{t',\mu'}_s)$ when $t\nearrow t'$ uniformly in $\alpha\in\mathcal{A}$. Then we get the desired continuity property of the value function $v(t,\mu)$, i.e. $v^*(t,\mu)=v_*(t,\mu)=v(t,\mu)$, and equivalently $v^*(T,\xi)=v_*(T,\xi)=v(T,\xi)$.

The next result states the viscosity property of the value function to the Bellman equation as a consequence of the dynamic programming principle (3.3.10).

Proposition 3.5.1. *The value function v is a viscosity solution to the Bellman equation* (3.3.11).

Proof. Let us first reformulate the dynamic programming principle (DPP) of Theorem 3.3.1 for the value function viewed as a function on $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$. For this, we take a copy $\tilde B$ of B on the probability space $(\Omega,\mathcal{F}_0,\mathbb{P})$, and given $(t,\xi)\in[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$, $\alpha\in\mathcal{A}$, we consider the solution $\tilde X^{t,\xi,\alpha}_s$, $t\leq s\leq T$, to the McKean-Vlasov equation

$$\begin{aligned}\tilde X^{t,\xi}_s &= \xi+\int_t^s b(r,\tilde X^{t,\xi}_r,\tilde\alpha(r,\tilde X^{t,\xi}_r,\mathbb{P}_{\tilde X^{t,\xi}_r}),Id\tilde\alpha(r,.,\mathbb{P}_{\tilde X^{t,\xi}_r})\star\mathbb{P}_{\tilde X^{t,\xi}_r})dr\\
&\quad+\int_t^s\sigma(r,\tilde X^{t,\xi}_r,\tilde\alpha(r,\tilde X^{t,\xi}_r,\mathbb{P}_{\tilde X^{t,\xi}_r}),Id\tilde\alpha(r,.,\mathbb{P}_{\tilde X^{t,\xi}_r})\star\mathbb{P}_{\tilde X^{t,\xi}_r})d\tilde B_r,\quad t\leq s\leq T.\end{aligned}$$

Therefore, $\tilde X^{t,\xi}$ is a copy of $X^{t,\xi}$ on $(\Omega,\mathcal{F}_0,\mathbb{P})$. The lifted value function on $[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$ identified with the value function on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ satisfies $v(s,\tilde X^{t,\xi}_s)=v(s,\rho^{t,\mu}_s)$. By noting that $\hat f(s,\mathbb{P}^{t,\mu}_s,\tilde\alpha(s,\cdot,\mathbb{P}^{t,\mu}_s))=\mathbb{E}[f(s,\tilde X^{t,\xi}_s,\tilde\alpha(s,\tilde X^{t,\xi}_s,\mathbb{P}^{t,\mu}_s),Id\tilde\alpha(s,\tilde X^{t,\xi}_s,\mathbb{P}^{t,\mu}_s)\star\mathbb{P}^{t,\mu}_s)]$, we obtain from Theorem 3.3.1 the lifted DPP: for all $(t,\xi)\in[0,T]\times L^2(\mathcal{F}_0;\mathbb{R}^d)$,

$$v(t,\xi)=\inf_{\alpha\in\mathcal{A}}\int_t^\theta\mathbb{E}[f(s,\tilde X^{t,\xi}_s,\tilde\alpha(s,\tilde X^{t,\xi}_s,\mathbb{P}^{t,\mu}_s),Id\tilde\alpha(s,\tilde X^{t,\xi}_s,\mathbb{P}^{t,\mu}_s)\star\mathbb{P}^{t,\mu}_s)]ds+v(\theta,\tilde X^{t,\xi}_\theta)\tag{3.5.9}$$

We already know that v is continuous on $[0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d)$, hence in particular at T, so that $v(T,\xi) = \mathbb{E}[g(\xi,\mathbb{P}_\xi)]$, and it remains to derive the viscosity property for the value function in $[0,T) \times L^2(\mathcal{F}_0;\mathbb{R}^d)$ by following standard arguments that we adapt in our context.

(i) Subsolution property. Let us now prove the viscosity subsolution property of v on $[0,T) \times L^2(\mathcal{F}_0;\mathbb{R}^d)$. Fix $(t_0,\xi_0) \in [0,T) \times L^2(\mathcal{F}_0;\mathbb{R}^d)$, and consider some test function $\varphi \in \mathcal{C}^2_\ell([0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d))$ such that $v-\varphi$ has a maximum at (t_0,ξ_0), and w.l.o.g. $v(t_0,\xi_0) = \varphi(t_0,\xi_0)$, so that $v \leq \varphi$. and let h be a strictly positive s.t. $h \to 0$, let $\tilde\alpha$ be an arbitrary element in $L(\mathbb{R}^d;A)$, and consider the time-independent feedback control $\alpha \in \mathcal{A}$ associated with $\tilde\alpha$. From the DPP (3.5.9) applied to $v(t_0,\xi_0)$, we have

$$v(t_0,\xi_0) \quad \leq \quad \int_{t_0}^{t_0+h} \mathbb{E}[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})]ds \;+\; v(t_0+h,\tilde X_{t_0+h}^{t_0,\xi_0}).$$

Since $v(t,\xi) \leq v^*(t,\xi) \leq \varphi(t,\xi)$ for all $(t,\xi) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, this implies

$$0 \quad \leq \quad \frac{1}{h}\int_{t_0}^{t_0+h} \mathbb{E}[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})]ds \;+\; \frac{\varphi(t_0+h,\tilde X_{t_0+h}^{t_0,\xi_0}) - \varphi(t_0,\xi_0)}{h}$$

Applying Itô's formula (1.3.4) to $\varphi(s,\tilde X_s^{t,\xi})$ between t_0 and t_0+h, we get

$$\begin{aligned}0 \quad \leq \quad & \frac{1}{h}\int_{t_0}^{t_0+h} \Big[\frac{\partial\varphi}{\partial t}(s,\tilde X_s^{t_0,\xi_0}) \;+\; \mathbb{E}\Big[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0}) \\ & + D\varphi(s,\tilde X_s^{t_0,\xi_0}).b(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0}) \\ & + \frac{1}{2}D^2\varphi(s,\tilde X_s^{t_0,\xi_0})(\sigma(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})N).(\sigma(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})N)\Big]ds\end{aligned}$$

with $N \in L^2(\mathcal{F}_0;\mathbb{R}^n)$ of zero mean, and unit variance, independent of ξ. By the continuity of b,σ,f,φ on their respective domains, and the flow continuity property (3.5.6), we then obtain by sending h to zero in the above inequality:

$$\begin{aligned}&-\frac{\partial\varphi}{\partial t}(t_0,\xi_0) - \mathbb{E}\Big[f(t_0,\xi_0,\tilde\alpha(\xi_0),Id\tilde\alpha \star \mathbb{P}_{\xi_0}) + D\varphi(t_0,\xi_0).b(t_0,\xi_0,\tilde\alpha(\xi_0),Id\tilde\alpha \star \mathbb{P}_{\xi_0})) \\ &\qquad -\frac{1}{2}D^2\varphi(t_0,\xi_0)(\sigma(t_0,\xi_0,\tilde\alpha(\xi_0),Id\tilde\alpha \star \mathbb{P}_{\xi_0})N)(\sigma(t_0,\xi_0,\tilde\alpha(\xi_0),Id\tilde\alpha \star \mathbb{P}_{\xi_0})N)\Big] \quad \leq \quad 0.\end{aligned}$$

Since $\tilde\alpha$ is arbitrary in $L(\mathbb{R}^d;A)$, this shows

$$-\frac{\partial\varphi}{\partial t}(t_0,\xi_0) + \mathcal{H}(t_0,\xi_0,D\varphi(t_0,\xi_0),D^2\varphi(t_0,\xi_0)) \quad \leq \quad 0,$$

which is the required viscosity subsolution property.

(ii) Supersolution property. We proceed finally with the viscosity supersolution property. Fix $(t_0,\xi_0) \in [0,T) \times L^2(\mathcal{F}_0;\mathbb{R}^d)$, and consider some test function $\varphi \in \mathcal{C}^2_\ell([0,T] \times L^2(\mathcal{F}_0;\mathbb{R}^d))$ such that $v-\varphi$ has a minimum at (t_0,ξ_0), and w.l.o.g. $v(t_0,\xi_0) = \varphi(t_0,\xi_0)$, so that $v \geq \varphi$. From the lifted DPP (3.5.9), for small $h \in [0,T-t]$, there exists $\alpha \in \mathcal{A}$ associated to a feedback control $\tilde\alpha \in Lip([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d);A)$ s.t.

$$v(t_0,\xi_0) + h^2 \quad \geq \quad \int_{t_0}^{t_0+h} \mathbb{E}[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})]ds \;+\; v(t_0+h,\tilde X_{t_0+h}^{t_0,\xi_0}).$$

Since $v(t,\xi) \geq v_*(t,\xi) \geq \varphi(t,\xi)$ for all $(t,\xi) \in [0,T] \times L^2(\mathcal{G};\mathbb{R}^d)$, this implies

$$h \quad \geq \quad \frac{1}{h}\int_{t_0}^{t_0+h} \mathbb{E}[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha \star \mathbb{P}_s^{t_0,\mu_0})]ds \;+\; \frac{\varphi(t_0+h,\tilde X_{t_0+h}^{t_0,\xi_0}) - \varphi(t_0,\xi_0)}{h}.$$

Applying again Itô's formula (1.3.7) to $\varphi(s,\tilde X_s^{t,\xi})$, we then get

$$
\begin{aligned}
h \;\geq\; & \frac{1}{h}\int_{t_0}^{t_0+h} \big[\frac{\partial\varphi}{\partial t}(s,\tilde X_s^{t_0,\xi_0}) \;+\; \mathbb{E}\Big[f(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha\star\mathbb{P}_s^{t_0,\mu_0})\\
& + D\varphi(s,\tilde X_s^{t_0,\xi_0}).b(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha\star\mathbb{P}_s^{t_0,\mu_0})\\
& + \frac{1}{2}D^2\varphi(s,\tilde X_s^{t_0,\xi_0})(\sigma(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha\star\mathbb{P}_s^{t_0,\mu_0})N).(\sigma(s,\tilde X_s^{t_0,\xi_0},\tilde\alpha(\tilde X_s^{t_0,\xi_0}),Id\tilde\alpha\star\mathbb{P}_s^{t_0,\mu_0})N)\big]\\
\geq\; & \frac{1}{h}\int_{t_0}^{t_0+h} \big[\frac{\partial\varphi}{\partial t}(s,\tilde X_s^{t_0,\xi_0}) + \mathcal{H}(s,\tilde X_s^{t_0,\xi_0},D\varphi(\tilde X_s^{t_0,\xi_0}),D^2\varphi(\tilde X_s^{t_0,\xi_0}))\big]
\end{aligned}
$$

where N is independent of (ξ_0,B), with zero mean and unit variance. By sending h to zero together with the continuity assumption in **(H1')-(H2')** of b,σ,f,φ, uniformly in $a\in A$, and the flow continuity property (3.5.6), we get

$$
-\frac{\partial\varphi}{\partial t}(t_0,\xi_0)+\mathcal{H}(t_0,\xi_0,D\varphi(t_0,\xi_0),D^2\varphi(t_0,\xi_0)) \;\;\geq\;\; 0,
$$

which gives the required viscosity supersolution property of v, and ends the proof. □

We finally turn to comparison principle (hence uniqueness result) for the Bellman equation (3.3.11) (or (3.5.1)), hence equivalently for the lifted Bellman equation (3.5.3), which shall follow from comparison results for second order Hamilton-Jacobi equations in separable Hilbert space stated in [Lio89b], see also [FGS15]. We shall assume that the σ-algebra $\mathcal{F}_0$ is countably generated upto null sets, which ensures that the Hilbert space $L^2(\mathcal{F}_0;\mathbb{R}^d)$ is separable, see [Doo94], p. 92. This is satisfied for example when $\mathcal{F}_0$ is the Borel σ-algebra of a canonical space Ω_0 of continuous functions on $\mathbb{R}_+$, in which case, $\mathcal{F}_0=\vee_{s\geq 0}\mathcal{F}_s^{B^0}$, where $(\mathcal{F}_s^{B^0})$ is the canonical filtration on Ω_0, and it is then known that $\mathcal{F}_0$ is countably generated, see for instance Exercise 4.21 in Chapter 1 of [RY99].

Proposition 3.5.2. *Let u and w be two functions defined on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ satisfying a quadratic growth condition such that u (resp. w) is an upper (resp. lower) semicontinuous viscosity subsolution (resp. supersolution) to* (3.3.11)*. Then $u\leq w$. Consequently, the value function v is the unique viscosity solution to the Bellman equation* (3.3.11) *satisfying a quadratic growth condition* (3.5.5)*.*

Proof. In view of our definition 3.5.1 of viscosity solution, we have to show a comparison principle for viscosity solutions to the lifted Bellman equation (3.5.3). We use the comparison principle proved in Theorem 3.50 in [FGS15] and only need to check that the hypotheses of this theorem are satisfied in our context for the lifted Hamiltonian $\mathcal{H}$ defined in (3.5.4). Notice that the lifted Bellman equation (3.5.3) is a bounded equation in the terminology of [FGS15] (see their section 3.3.1) meaning that there is no linear dissipative operator on $L^2(\mathcal{F}_0;\mathbb{R}^d)$ in the equation. Therefore, the notion of B-continuity reduces to the standard notion of continuity in $L^2(\mathcal{F}_0;\mathbb{R}^d)$ since one can take for B the identity operator. Their Hypothesis 3.44 follows from the uniform continuity of b, σ, and f in **(H1')-(H2')**. Hypothesis 3.45 is immediately satisfied since there is no discount factor in our equation, i.e. $\mathcal{H}$ does not depend on v but only on its derivatives. The monotonicity condition in $Q\in S(L^2(\mathcal{F}_0;\mathbb{R}^d))$ of $\mathcal{H}$ in Hypothesis 3.46 is clearly satisfied. Hypothesis 3.47 holds directly when dealing with bounded equations. Hypothesis 3.48 is obtained from the Lipschitz condition of b,σ in **(H1')**, and the uniform continuity condition on f in **(H2')**, while Hypothesis 3.49 follows from the quadratic growth condition of σ in **(H1')**. One can then apply Theorem 3.50 in [FGS15] and conclude that comparison principle holds for the Bellman equation (3.5.3), hence for the Bellman equation (3.3.11). □

3.6 The case of open-loop controls

In this section, we discuss how one can consider more generally open-loop controls instead of (Lipschitz) closed-loop controls as imposed in the previous sections, hence allowing a priori in particular bang-bang controls, which is useful in the applications. We shall restrict our framework to usual controlled McKean-Vlasov SDE with coefficients that not depend on the law of the control but only on the law of the state process, hence in the form

$$dX_s = b(s, X_s, \alpha_s, \mathbb{P}_{X_s})ds + \sigma(s, X_s, \alpha_s, \mathbb{P}_{X_s})dB_s, \tag{3.6.1}$$

where b, σ are measurable functions from $[0,T] \times \mathbb{R}^d \times A \times \mathcal{P}_2(\mathbb{R}^d)$ into $\mathbb{R}^d$, respectively $\mathbb{R}^{d\times n}$, satisfying a Lipschitz condition: for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $a \in A$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} & |b(t,x,a,\mu) - b(t,x',a,\mu')| + |\sigma(t,x,a,\mu) - \sigma(t,x',a,\mu')| \\ \leq \;& C\big[|x - x'| + \mathcal{W}_2(\mu, \mu')\big], \end{aligned} \tag{3.6.2}$$

for some positive constant C. We denote by $\mathcal{A}_o$ the set of $\mathbb{F}$-progressive processes α valued in A, assumed for simplicity here to be a compact space of $\mathbb{R}^m$, and consider the McKean-Vlasov control problem with open-loop controls with running cost not depending on the law of the control:

$$\mathcal{V}_0 := \inf_{\alpha \in \mathcal{A}_o} \mathbb{E}\Big[\int_0^T f(t, X_t, \alpha_t, \mathbb{P}_{X_t})dt + g(X_T, \mathbb{P}_{X_T})\Big].$$

Under (3.6.2), and given $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, $\alpha \in \mathcal{A}_o$, there exists a unique (pathwise and in law) solution $X_s^{t,\xi} = X_s^{t,\xi,\alpha}$, $t \leq s \leq T$, solution to (3.6.1) starting from ξ at time t, satisfying

$$\mathbb{E}\big[\sup_{t \leq s \leq T} |X_s^{t,\xi}|^2\big] \leq C\big(1 + \mathbb{E}|\xi|^2\big),$$

for some positive constant C independent of $\alpha \in \mathcal{A}_o$. As in (3.3.5), one can then define the flow $\mathbb{P}_s^{t,\mu} = \mathbb{P}_s^{t,\mu,\alpha}$, $t \leq s \leq T$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}_o$, of the law of $X_s^{t,\xi}$, for $\mu = \mathbb{P}_\xi$, and they satisfy the flow properties (3.3.4) and (3.3.6). We then define the value function in the Wasserstein space

$$v_o(t,\mu) := \inf_{\alpha \in \mathcal{A}_o} \mathbb{E}\Big[\int_t^T f(s, X_s^{t,\xi}, \alpha_s, \mathbb{P}_s^{t,\mu})ds + g(X_T^{t,\xi}, \mathbb{P}_T^{t,\mu})\Big], \;\; t \in [0,T], \;\; \mu = \mathbb{P}_\xi \in \mathcal{P}_2(\mathbb{R}^d),$$

so that $\mathcal{V}_0 = v_o(0, \mathbb{P}_{X_0})$. Since the set of open-loop controls is larger than the set of feedback controls, it is clear that v_o is smaller than v, the value function to the McKean-Vlasov control problem with feedback controls considered in the previous sections.

Notice that when $f \equiv 0$, one can reformulate the value function v_o as a deterministic control problem as in the case of feedback controls: $v_o(t,\mu) = \inf_{\alpha \in \mathcal{A}_o} \hat{g}(\mathbb{P}_T^{t,\mu})$, but in general this is not possible. Anyway, by similar arguments as in Theorem 3.3.1, but more technical and requiring some measurable selection arguments due to the stochastic control formulation of the value function v_o, one could show the DPP for the value function with open-loop controls, namely:

$$v_o(t,\mu) = \inf_{\alpha \in \mathcal{A}_o} \mathbb{E}\Big[\int_t^\theta f(s, X_s^{t,\xi}, \alpha_s, \mathbb{P}_s^{t,\mu})ds + v_o(\theta, \mathbb{P}_\theta^{t,\mu})\Big]$$

for all $0 \leq t \leq \theta \leq T$, $\mu = \mathbb{P}_\xi \in \mathcal{P}_2(\mathbb{R}^d)$. From Itô's formula (1.3.7), the infinitesimal version of the above DPP leads to the dynamic programming Bellman equation:

$$\left\{ \begin{aligned} -\partial_t v_o(t,\mu) + H_o\big(t, \mu, \partial_\mu v_o(t,\mu), \partial_x \partial_\mu v_o(t,\mu)\big) &= 0, \quad \text{on } [0,T) \times \mathcal{P}_2(\mathbb{R}^d), \\ v_o(T,.) &= \hat{g}, \quad \text{on } \mathcal{P}_2(\mathbb{R}^d) \end{aligned} \right. \tag{3.6.3}$$

where H_o is the function defined by

$$H_o(t,\mu,p,\Gamma) \quad := \quad -\inf_{\alpha\in\mathcal{A}_o} \mathbb{E}\big[f(t,\xi,\alpha_t,\mu) + p(\xi).b(t,\xi,\alpha_t,\mu) + \frac{1}{2}\mathrm{tr}\big(\Gamma(\xi)\sigma\sigma^{\intercal}(t,\xi,\alpha_t,\mu)\big)\big],$$

for $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $(p,\Gamma)\in L^2_\mu(\mathbb{R}^d)\times L^\infty_\mu(\mathbb{S}^d)$, and with $\mathbb{P}_\xi=\mu$. Similarly as in Propositions 3.3.1 and 3.5.2, one can show a verification theorem for v_o and prove that v_o is the unique viscosity solution to (3.6.3).

For any $\tilde\alpha\in L(\mathbb{R}^d;A)$, it is clear that the control α defined by $\alpha_s=\tilde\alpha(\xi)$, $t\le s\le T$, lies in $\mathcal{A}_o$, so that

$$\begin{aligned} H_o(t,\mu,p,\Gamma) \quad \geq \quad & -\inf_{\tilde\alpha\in L(\mathbb{R}^d;A)} \mathbb{E}\big[f(t,\xi,\tilde\alpha(\xi),\mu) \\ & + p(\xi).b(t,\xi,\tilde\alpha(\xi),\mu) + \frac{1}{2}\mathrm{tr}\big(\Gamma(\xi)\sigma\sigma^{\intercal}(t,\xi,\tilde\alpha(\xi),\mu)\big)\big] \;=\; H(t,\mu,p,\Gamma),\end{aligned}$$

with H the Hamiltonian in (3.5.2) for the McKean-Vlasov control problem with feedback control. This inequality $H_o\ge H$ combined with comparison principle for the Bellman equation (3.6.3) is consistent with the inequality $v\ge v_o$. If we could prove that H_o is equal to H (which is not trivial in general), then this would show that v_o is equal to v, i.e. the value functions to the McKean-Vlasov control problems with open-loop and feedback controls coincide. Actually, we notice that the minimization over the infinite dimensional space $\mathcal{A}_o$ in the Hamiltonian H_o can be reduced into a minimization over the finite dimensional space A, namely:

$$H_o(t,\mu,p,\Gamma) \quad = \quad \tilde H_o(t,\mu,p,\Gamma) := -<\inf_{a\in A}\mathbb{H}(t,x,a,\mu,p(.),\Gamma(.)),\mu> \tag{3.6.4}$$

where $\mathbb{H}(t,x,a,\mu,q,M)$ is defined by (3.3.16) in Remark 3.3.4.
Indeed, it is clear that $H_o\le\tilde H_o$. Conversely, by continuity of the coefficients b, σ, f w.r.t. the argument a lying the compact space A, and invoking a measurable selection theorem, one can find for any (t,μ,p,Γ) $\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times L^2_\mu(\mathbb{R}^d)\times L^\infty_\mu(\mathbb{S}^d)$, some measurable function $x\in\mathbb{R}^d\mapsto\hat a(t,x,\mu,p(x),\Gamma(x))=\hat\alpha(x)$ s.t. for all $x\in\mathbb{R}^d$,

$$\begin{aligned} & \inf_{a\in A}\big[f(t,x,a,\mu)+p(x).b(t,x,a,\mu)+\frac{1}{2}\mathrm{tr}\big(\Gamma(x)\sigma\sigma^{\intercal}(t,x,a,\mu)\big)\big] \\ = \quad & f(t,x,\hat\alpha(x),\mu)+p(x).b(t,x,\hat\alpha(x),\mu)+\frac{1}{2}\mathrm{tr}\big(\Gamma(x)\sigma\sigma^{\intercal}(t,x,\hat\alpha(x),\mu)\big).\end{aligned}$$

By integrating w.r.t. μ, we then get

$$\begin{aligned} \tilde H_o(t,\mu,p,\Gamma) \quad & = \quad -\mathbb{E}\big[f(t,\xi,\hat\alpha(\xi),\mu)+p(\xi).b(t,\xi,\hat\alpha(\xi),\mu)+\frac{1}{2}\mathrm{tr}\big(\Gamma(\xi)\sigma\sigma^{\intercal}(t,\xi,\hat\alpha(\xi),\mu)\big)\big] \\ & \le \quad H_o(t,\mu,p,\Gamma),\end{aligned}$$

which shows the equality (3.6.4). Suppose now that there exists some smooth solution w on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ to the equation:

$$\begin{cases} -\partial_t w(t,\mu)+\tilde H_o\big(t,\mu,\partial_\mu w(t,\mu),\partial_x\partial_\mu w(t,\mu)\big) \;=\; 0, & \text{on } [0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ w(T,.) \;=\; \hat g, & \text{on } \mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{3.6.5}$$

such that for all $(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$, the element $x\mapsto\hat a(t,x,\mu,\partial_\mu w(t,\mu)(x),\partial_x\partial_\mu w(t,\mu)(x))$ achieving the infimum in the definition of $\tilde H_o\big(t,\mu,\partial_\mu w(t,\mu),\partial_x\partial_\mu w(t,\mu)\big)$, is Lipschitz, i.e. lies in $L(\mathbb{R}^d;A)$, then (recall also Remark 3.3.2)

$$\tilde H_o\big(t,\mu,\partial_\mu w(t,\mu),\partial_x\partial_\mu w(t,\mu)\big) \quad = \quad H\big(t,\mu,\partial_\mu w(t,\mu),\partial_x\partial_\mu w(t,\mu)\big),$$

which shows with (3.6.4) that w solves both the Bellman equations (3.6.3) and (3.5.1). By comparison principle, we conclude that $w = v = v_o$, which means in this case that the value functions to the McKean-Vlasov control problems with open-loop and feedback controls coincide. Such condition was satisfied for example in the case of the mean-variance portfolio selection problem studied in paragraph 3.4.1.

Remark 3.6.1. (Existence of smooth solution)

Let us mention that the existence of classical solution to HJB equation (3.6.3) or (3.6.5) was derived in [CCD15] in the case where σ is uncontrolled. Briefly speaking, under coercive conditions on the coefficients, $\hat{\alpha}(x)$ is the minimizer of $\mathbb{H}(t, x, \partial_\mu v_o(t,\mu), \partial_x\partial_\mu v_o(t,\mu))$, and $V(t,x,\mu)$ on the enlarged space $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ defined by

$$V(t,x,\mu) \quad = \quad \mathbb{E}\Big[\int_t^T f(s, X_s^{t,x,\mu}, \mathbb{P}_{X_s^{t,\mu}}, \hat{\alpha}(X_s^{t,x,\mu}))ds + g(X_T^{t,x,\mu}, \mathbb{P}_{X_T^{t,\mu}})\Big],$$

where $X_s^{t,x,\mu}$ is the solution of the SDE

$$dX_s^{t,x,\mu} = b(s, X_s^{t,x,\mu}, \mathbb{P}_{X_s^{t,\mu}}, \hat{\alpha}(X_s^{t,x,\mu}))ds + \sigma(s, X_s^{t,x,\mu}, \mathbb{P}_{X_s^{t,\mu}})dB_s, \quad t \le s \le T,$$

with the initial condition x at time t. Then $V(t,x,\mu)$ is the unique classical solution to the following master equation

$$\begin{aligned}
&\partial_t V(t,x,\mu) + b(t,x,\mu,\hat{\alpha}(x)).\partial_x V(t,x,\mu) + f(t,x,\mu,\hat{\alpha}(x)) \\
&\quad + \frac{1}{2}\mathrm{tr}(\partial^2_{xx} V(t,x,\mu)\sigma\sigma^\intercal(t,x,\mu)) + \int_{\mathbb{R}^d} \big[b(t,x',\mu,\hat{\alpha}(x')).\partial_\mu V(t,x,\mu)(x') \\
&\quad + \frac{1}{2}\mathrm{tr}(\partial_x\partial_\mu V(t,x,\mu)(x')\sigma\sigma^\intercal(t,x,\mu))\big]\mu(dx') = 0.
\end{aligned}$$

Moreover, the identification of value function $v_o(t,\mu)$ in terms of $V(t,x,\mu)$ is given by $v_o(t,\mu) = v(t,\mu)$ $= \int_{\mathbb{R}^d} V(t,x,\mu)\mu(dx)$, or equivalently reads as $\partial_\mu v_o(t,\mu)(x) = \partial_x V(t,x,\mu) + \int_{\mathbb{R}^d} \partial_\mu V(t,x',\mu)(x)\mu(dx')$. Therefore, we conclude that $v_o(t,\mu)$ is the unique classical solution to HJB equation. $\square$

Chapter 4

Dynamic programming for continuous time conditional McKean-Vlasov control problem [a]

Abstract: We study the optimal control of general stochastic McKean-Vlasov equation. Such problem is motivated originally from the asymptotic formulation of cooperative equilibrium for a large population of particles (players) in mean-field interaction under common noise. Our first main result is to state a dynamic programming principle for the value function in the Wasserstein space of probability measures, which is proved from a flow property of the conditional law of the controlled state process. Next, by relying on the notion of differentiability with respect to probability measures due to P.L. Lions [Lio12], and Itô's formula along a flow of conditional measures, we derive the dynamic programming Hamilton-Jacobi-Bellman equation, and prove the viscosity property together with a uniqueness result for the value function. Finally, we solve explicitly the linear-quadratic stochastic McKean-Vlasov control problem and give an application to an interbank systemic risk model with common noise.

Keywords: Stochastic McKean-Vlasov SDEs, dynamic programming principle, Bellman equation, Wasserstein space, viscosity solutions.

a. This chapter is based on a paper in collaboration with Pham Huyên [PW17]. This paper is published in *SIAM Journal on Control and Optimization*, **55**(2), 2017.

4.1 Introduction

Let us consider the controlled McKean-Vlasov dynamics in $\mathbb{R}^d$ given by

$$dX_t = b(X_t, \mathbb{P}^{W^0}_{X_t}, \alpha_t)dt + \sigma(X_t, \mathbb{P}^{W^0}_{X_t}, \alpha_t)dB_t + \sigma_0(X_t, \mathbb{P}^{W^0}_{X_t}, \alpha_t)dW^0_t, \tag{4.1.1}$$

where B, W^0 are two independent Brownian motions on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{P}^{W^0}_{X_t}$ denotes the conditional distribution of X_t given W^0 (or equivalently given $\mathcal{F}^0_t$ where $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t\geq 0}$ is the natural filtration generated by W^0), valued in $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$, and the control α is an $\mathbb{F}^0$-progressive process valued in some Polish space $\mathbf{A}$. When there is no control, the dynamics (4.1.1) is sometimes called stochastic McKean-Vlasov equation (see [DV95]), where the term “stochastic" refers to the presence of the random noise caused by the Brownian motion W^0 w.r.t. a McKean-Vlasov equation when $\sigma_0 = 0$, and for which coefficients depend on the (deterministic) marginal distribution $\mathbb{P}_{X_t}$. One also uses the terminology conditional mean-field stochastic differential equation (CMFSDE) to emphasize the dependence of the coefficients on the conditional law with respect to the random noise, and such CMFSDE was studied in [CZ16], and more generally in [BLM17]. In this context, the control problem is to minimize over α a cost functional of the form:

$$J(\alpha) = \mathbb{E}\Big[\int_0^T f(X_t, \mathbb{P}^{W^0}_{X_t}, \alpha_t)dt + g(X_T, \mathbb{P}^{W^0}_{X_T})\Big]. \tag{4.1.2}$$

The motivation and applications for the study of such stochastic control problem, referred to alternatively as control of stochastic McKean-Vlasov dynamics, or stochastic control of conditional McKean-Vlasov equation, comes mainly from the *McKean-Vlasov control problem with common noise*, that we briefly describe now: we consider a system of controlled individuals (referred also to as particles or players) in mutual interaction, where the dynamics of the state process X^i of player $i \in \{1, \dots, N\}$ is governed by

$$dX^i_t = \tilde{b}(X^i_t, \bar{\rho}^N_t, \tilde{\alpha}^i_t)dt + \tilde{\sigma}(X^i_t, \bar{\rho}^N_t, \tilde{\alpha}^i_t)dB^i_t + \tilde{\sigma}_0(X^i_t, \bar{\rho}^N_t, \tilde{\alpha}^i_t)dW^0_t.$$

Here, the Wiener process W^0 accounts for the common random environment in which all the individuals evolve, called common noise, and $B^1, \dots, B^N$ are independent Brownian motions, independent of W^0, called idiosyncratic noises. The particles are in interaction of mean-field type in the sense that at any time t, the coefficients $\tilde{b}$, $\tilde{\sigma}$, $\tilde{\sigma}_0$ of their state process depend on the empirical distribution of all individual states

$$\bar{\rho}^N_t = \frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}.$$

The processes $(\tilde{\alpha}^i_t)_{t\geq 0}$, $i = 1, \dots, N$, are in general progressively measurable w.r.t. the filtration generated by $B^1, \dots, B^N, W^0$, valued in some subset A of a Euclidian space, and represent the control processes of the players with cost functionals:

$$J^i(\tilde{\alpha}^1, \dots, \tilde{\alpha}^n) = \mathbb{E}\Big[\int_0^T \tilde{f}(X^i_t, \bar{\rho}^N_t, \tilde{\alpha}^i_t)dt + g(X^i_T, \bar{\rho}^N_T)\Big].$$

For this N-player stochastic differential game, one looks for equilibriums, and different notions may be considered. Classically, the search for a consensus among the players leads to the concept of Nash equilibrium where each player minimizes its own cost functional, and the goal is to find a N-tuple control strategy for which there is no interest for any player to leave from this consensus state. The asymptotic formulation of this Nash equilibrium when the number of players N goes to infinity leads to the (now well-known) theory of MFG pioneered in the works by Lasry and Lions [LL07], and Huang, Malhamé and Caines [HMC06]. In this framework, the analysis is reduced to the problem of a single representative player in interaction with the theoretical distribution of the whole population by the propagation of chaos phenomenon, who first solves a control problem by freezing a probability law in the coefficients of her/his state process and cost function, and then has to find a fixed point probability measure that matches the distribution of her/his optimal state process. The case of MFG with common noise has been recently studied in [Ahu16] and [CDL16]. Alternatively, one may take the point of view of a center of decision (or social planner), which decides the strategies for all players, with the goal of minimizing the global cost to the collectivity. This leads to the concept of Pareto or cooperative equilibrium whose asymptotic formulation is reduced to the optimal control of McKean-Vlasov dynamics for a representative player. More precisely, given the symmetry of the set-up, when the social planner chooses the same control policy for all the players in feedback form: $\tilde\alpha_t^i = \tilde\alpha(t, X_t^i, \bar\rho_t^N)$, $i = 1, \ldots, N$, for some deterministic function $\tilde\alpha$ depending upon time, private state of player, and the empirical distribution of all players, then the theory of propagation of chaos implies that, in the limit $N \to \infty$, the particles X^i become asymptotically independent conditionally on the random environment W^0, and the empirical measure $\bar\rho_t^N$ converge to the distribution $\mathbb{P}_{X_t}^{W^0}$ of X_t given W^0, and X is governed by the (stochastic) McKean-Vlasov equation:

$$\begin{aligned} dX_t &= \tilde b(X_t, \mathbb{P}_{X_t}^{W^0}, \tilde\alpha(t, X_t, \mathbb{P}_{X_t}^{W^0}))dt + \tilde\sigma(X_t, \mathbb{P}_{X_t}^{W^0}, \tilde\alpha(t, X_t, \mathbb{P}_{X_t}^{W^0}))dB_t \\ &\quad + \tilde\sigma_0(X_t, \mathbb{P}_{X_t}^{W^0}, \tilde\alpha(t, X_t, \mathbb{P}_{X_t}^{W^0}))dW_t^0, \end{aligned}$$

for some Brownian motion B independent of W^0. The objective of the representative player for the Pareto equilibrium becomes the minimization of the functional

$$J(\tilde\alpha) = \mathbb{E}\Big[\int_0^T \tilde f(X_t, \mathbb{P}_{X_t}^{W^0}, \tilde\alpha(t, X_t, \mathbb{P}_{X_t}^{W^0}))dt + g(X_T, \mathbb{P}_{X_T}^{W^0}) \Big]$$

over the class of feedback controls $\tilde\alpha$. We refer to [CDL13] for a detailed discussion of the differences between the nature and solutions to the MFG and optimal control of McKean-Vlasov dynamics related respectively to the notions of Nash and Pareto equilibrium. Notice that in this McKean-Vlasov control formulation, the control $\tilde\alpha$ is of feedback (also called closed-loop) form both w.r.t. the state process X_t, and its conditional law process $\mathbb{P}_{X_t}^{W^0}$, which is $\mathbb{F}^0$-adapted. More generally, we can consider semi-feedback control $\alpha(t, x, \omega^0)$, in the sense that it is of closed-loop form w.r.t. the state process X_t, but of open-loop form w.r.t. the common noise W^0. In other words, one can consider random field control $\mathbb{F}^0$-progressive control process $\alpha = \{\alpha_t(x), x \in \mathbb{R}^d\}$, which may be viewed equivalently as processes valued in some functional space $\mathbf{A}$ on $\mathbb{R}^d$, typically a closed subset of the Polish space $C(\mathbb{R}^d, A)$, of continuous functions from $\mathbb{R}^d$ into some Euclidian space A. In this case, we are in the framework (4.1.1)-(4.1.2) with $b(x, \mu, a) = \tilde b(x, \mu, a(x))$, $\sigma(x, \mu, a) = \tilde\sigma(x, \mu, a(x))$, $\sigma_0(x, \mu, a) = \tilde\sigma_0(x, \mu, a(x))$, $f(x, \mu, a) = \tilde f(x, \mu, a(x))$, for $(x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbf{A}$.

We also mention that partial observation control problem arises as a particular case of our stochastic control framework (4.1.1)-(4.1.2): Indeed, let us consider a controlled process with dynamics

$$d\bar X_t = \bar b(\bar X_t, \alpha_t)dt + \bar\sigma(\bar X_t, \alpha_t)dB_t + \bar\sigma_0(\bar X_t, \alpha_t)dB_t^0,$$

where B, B^0 are two independent Brownian motions on some physical probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, and the signal control process can only be observed through W^0 given by

$$dW_t^0 = h(\bar X_t)dt + dB_t^0.$$

The control process α is progressively measurable w.r.t. the observation filtration $\mathbb{F}^0$ generated by W^0, valued typically in some Euclidian space A, and the cost functional to minimize over α is

$$J(\alpha) \;=\; \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \bar f(\bar X_t, \alpha_t)dt + \bar g(\bar X_T)\Big].$$

By considering the process Z via

$$Z_t^{-1} \;=\; \exp\big(-\int_0^t h(\bar X_s)dB_s^0 - \frac{1}{2}\int_0^t |h(\bar X_s)|^2 ds\big),\; 0 \leq t \leq T,$$

the process Z^{-1} is (under suitable integrability conditions on h) a martingale under $\mathbb{Q}$, and by Girsanov's theorem, this defines a probability measure $\mathbb{P}(d\omega) = Z_T^{-1}(\omega)\mathbb{Q}(d\omega)$, called reference probability measure, under which the pair (B, W^0) is a Brownian motion. We then see that the partial observation control problem can be recast into the framework (4.1.1)-(4.1.2) of a particular stochastic McKean-Vlasov control problem with $X = (\bar X, Z)$ governed by

$$\begin{aligned} d\bar X_t &= \big(\bar b(\bar X_t, \alpha_t) - \bar\sigma_0(\bar X_t, \alpha_t)h(\bar X_t)\big)dt + \bar\sigma(\bar X_t, \alpha_t)dB_t + \bar\sigma_0(\bar X_t, \alpha_t)dW_t^0, \\ dZ_t &= Z_t h(\bar X_t)dW_t^0, \end{aligned}$$

and a cost functional rewritten under the reference probability measure from Bayes formula as

$$J(\alpha) \;=\; \mathbb{E}\Big[\int_0^T Z_t \bar f(\bar X_t, \alpha_t)dt + Z_T \bar g(\bar X_T)\Big].$$

The optimal control of McKean-Vlasov dynamics is a rather new problem with an increasing interest in the field of stochastic control problem. It has been studied by maximum principle methods in [AD10], [BDL11], [CD15] for state dynamics depending upon marginal distribution, and in [CZ16], [BLM17] for conditional McKean-Vlasov dynamics. This leads to a characterization of the solution in terms of an adjoint BSDE coupled with a forward SDE, and we refer to [CCD15] for a theory of BSDE of McKean-Vlasov type. Alternatively, dynamic programming approach for the control of McKean-Vlasov dynamics has been considered in [BFY15], [BFY17], [LP14] for specific McKean-Vlasov dynamics and under a density assumption on the probability law of the state process, and then analyzed in a general framework in [PW18] (without noise W^0), where the problem is reformulated into a deterministic control problem involving the marginal distribution process.

The aim of this paper is to develop the dynamic programming method for stochastic McKean-Vlasov equation in a general setting. For this purpose, a key step is to show the flow property of the conditional distribution $\mathbb{P}^{W^0}_{X_t}$ of the controlled state process X_t given the noise W^0. Then, by reformulating the original control problem into a stochastic control problem where the conditional law $\mathbb{P}^{W^0}_{X_t}$ is the sole controlled state variable driven by the random noise W^0, and by showing the continuity of the value function in the Wasserstein space of probability measures, we are able to prove a dynamic programming principle (DPP) for our stochastic McKean-Vlasov control problem. Next, for exploiting the DPP, we use a notion of differentiability with respect to probability measures introduced by P.L. Lions in his lectures at the Collège de France [Lio12], and detailed in the notes [Car12]. This notion of derivative is based on the lifting of functions defined on the Hilbert space of square integrable random variables distributed according to the "lifted" probability measure. By combining with a special Itô's chain rule for flows of conditional distributions, we derive the dynamic programming Bellman equation for stochastic McKean-Vlasov control problem, which is a fully nonlinear second order PDE in the infinite dimensional Wasserstein space of probability measures. By adapting standard arguments to our context, we prove the viscosity property of the value function to the Bellman equation from the dynamic programming principle. To complete our PDE characterization of the value function with a uniqueness result, it is convenient to work in the lifted Hilbert space of square integrable random variables instead of the Wasserstein metric space of probability measures, in order to rely on the general results for viscosity solutions of second order Hamilton-Jacobi-Bellman equations in separable Hilbert spaces, see [Lio88], [Lio89b], [FGS15]. We also state a verification theorem which is useful for getting an analytic feedback form of the optimal control when there is a smooth solution to the Bellman equation. Finally, we apply our results to the class of linear-quadratic (LQ) stochastic McKean-Vlasov control problem for which one can obtain explicit solutions, and we illustrate with an example arising from an interbank systemic risk model.

The outline of the paper is organized as follows. Section 2 formulates the stochastic McKean-Vlasov control problem, and fix the standing assumptions. Section 3 is devoted to the proof and statement of the dynamic programming principle. We prove in Section 4 the viscosity characterization of the value function to the Bellman equation, and the last Section 5 presents the application to the LQ framework with explicit solutions.

4.2 Conditional McKean-Vlasov control problem

Let us fix some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ assumed of the form $(\Omega^0\times\Omega^1,\mathcal{F}^0\otimes\mathcal{F}^1,\mathbb{P}^0\otimes\mathbb{P}^1)$, where $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0)$ supports a m-dimensional Brownian motion W^0, and $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ supports a n-dimensional Brownian motion B. So an element $\omega\in\Omega$ is written as $\omega=(\omega^0,\omega^1)\in\Omega^0\times\Omega^1$, and we extend canonically W^0 and B on Ω by setting $W^0(\omega^0,\omega^1):=W^0(\omega^0)$, $B(\omega^0,\omega^1):=B(\omega^1)$, and extend similarly on Ω any random variable on Ω^0 or Ω^1. We assume that $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ is in the form $\Omega^1=\tilde\Omega^1\times\Omega'^1$, $\mathcal{F}^1=\mathcal{G}\otimes\mathcal{F}'^1$, $\mathbb{P}^1=\tilde{\mathbb{P}}^1\otimes\mathbb{P}'^1$, where $\tilde\Omega^1$ is a Polish space, $\mathcal{G}$ its Borel σ-algebra, $\tilde{\mathbb{P}}^1$ an atomless probability measure on $(\tilde\Omega^1,\mathcal{G})$, while $(\Omega'^1,\mathcal{F}'^1,\mathbb{P}'^1)$ supports B. We denote by $\mathbb{E}^0$ (resp. $\mathbb{E}^1$ and $\tilde{\mathbb{E}}^1$) the expectation under $\mathbb{P}^0$ (resp. $\mathbb{P}^1$ and $\tilde{\mathbb{P}}^1$), by $\mathbb{F}^0=(\mathcal{F}^0_t)_{t\geq 0}$ the $\mathbb{P}^0$-completion of the natural filtration generated by W^0 (and w.l.o.g. we assume that $\mathcal{F}^0=\mathcal{F}^0_\infty$), and by $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by W^0,B, augmented with the independent σ-algebra $\mathcal{G}$. We often omit some arguments and write $L^2(\tilde\Omega^1,\mathcal{G},\tilde{\mathbb{P}}^1;\mathbb{R}^d)$ (resp. $L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$) as $L^2(\mathcal{G};\mathbb{R}^d)$ (resp. $L^2(\mathcal{F}_t;\mathbb{R}^d)$). We know that $\mathcal{P}_2(\mathbb{R}^d)$ $=\{\mathbb{P}_\xi=\tilde{\mathbb{P}}^1_\xi:\xi\in L^2(\mathcal{G};\mathbb{R}^d)\}$ since $(\tilde\Omega^1,\mathcal{G},\tilde{\mathbb{P}}^1)$ is Polish and atomless (we say that $\mathcal{G}$ is rich enough). We often write $\mathcal{L}(\xi)=\mathbb{P}_\xi=\tilde{\mathbb{P}}^1_\xi$ for the law of $\xi\in L^2(\mathcal{G};\mathbb{R}^d)$.

• *Admissible controls.* We are given a Polish set $\mathbf{A}$ equipped with the distance d_A, satisfying w.l.o.g. d_A <1, representing the control set, and we denote by $\mathcal{A}$ the set of $\mathbb{F}^0$-progressive processes α valued in $\mathbf{A}$. Notice that $\mathcal{A}$ is a separable metric space endowed with the Krylov distance $\Delta(\alpha,\beta)=\mathbb{E}^0[\int_0^T d_A(\alpha_t,\beta_t)dt]$. We denote by $\mathcal{B}_\mathcal{A}$ the Borel σ-algebra of $\mathcal{A}$.

• *Controlled stochastic McKean-Vlasov dynamics.* For $(t,\xi)\in[0,T]\times L^2(\mathcal{F}_t;\mathbb{R}^d)$, and given $\alpha\in\mathcal{A}$, we consider the stochastic McKean-Vlasov equation:

$$\left\{\begin{array}{rcl} dX_s &=& b(X_s,\mathbb{P}^{W^0}_{X_s},\alpha_s)ds+\sigma(X_s,\mathbb{P}^{W^0}_{X_s},\alpha_s)dB_s \\ & & \quad +\,\sigma_0(X_s,\mathbb{P}^{W^0}_{X_s},\alpha_s)dW^0_s, \quad t\leq s\leq T, \\ X_t &=& \xi. \end{array}\right. \tag{4.2.1}$$

Here, $\mathbb{P}^{W^0}_{X_s}$ denotes the regular conditional distribution of X_s given $\mathcal{F}^0$, and its realization at some ω^0 $\in\Omega^0$ also reads as the law under $\mathbb{P}^1$ of the random variable $X_s(\omega^0,.)$ on $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$, i.e. $\mathbb{P}^{W^0}_{X_s}(\omega^0)=$ $\mathbb{P}^1_{X_s(\omega^0,.)}$. The coefficients b, σ, σ_0 are measurable functions from $\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbf{A}$ into $\mathbb{R}^d$, respectively

$\mathbb{R}^{d\times n}$, $\mathbb{R}^{d\times m}$, and satisfy the condition:

(H1)

(i) There exists some positive constant C s.t. for all $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \times\mathcal{P}_2(\mathbb{R}^d)$, and $a \in \mathbf{A}$,

$$\begin{aligned} & |b(x,\mu,a) - b(x',\mu',a)| + |\sigma(x,\mu,a) - \sigma(x',\mu',a)| + |\sigma_0(x,\mu,a) - \sigma_0(x',\mu',a)| \\ \leq \;\; & C\Big(|x-x'| + \mathcal{W}_2(\mu,\mu')\Big), \end{aligned}$$

and

$$|b(0,\delta_0,a)| + |\sigma(0,\delta_0,a)| + |\sigma_0(0,\delta_0,a)| \quad \leq \quad C.$$

(ii) For all $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the functions $a \mapsto b(x,\mu,a)$, $\sigma(x,\mu,a)$, $\sigma_0(x,\mu,a)$ are continuous on $\mathbf{A}$.

Remark 4.2.1. We have chosen a control formulation where the process α is required to be progressively measurable w.r.t. the filtration $\mathbb{F}^0$ of the sole common noise. This form is used for rewriting the cost functional in terms of the conditional law as sole state variable, see (4.3.3), which is then convenient for deriving the dynamic programming principle. In the case where $\mathbf{A}$ is a functional space on the state space $\mathbb{R}^d$, meaning that α is a semi closed-loop control, and when the coefficients are in the form: $b(x,\mu,a)$ $= \tilde{b}(x,\mu,a(x))$, $\sigma(x,\mu,a) = \tilde{\sigma}(x,\mu,a(x))$, $\sigma_0(x,\mu,a) = \tilde{\sigma}_0(x,\mu,a(x))$ (see discussion in the introduction), the Lipschitz condition in **(H1)**(i) requires that $a \in \mathbf{A}$ is Lipschitz continuous with a prescribed Lipschitz constant, which is somewhat a restrictive condition. The more general case where the control α is allowed to be measurable with respect to the filtration $\mathbb{F}$ of both noises, i.e., α of open-loop form, is certainly an important extension, and left for future work. In this case, one should consider as state variables the pair composed of the process X_t and its conditional law $\mathbb{P}^{W^0}_{X_t}$, see the recent paper [BCP18] where a dynamic programming principle is stated when the control is allowed to be of open-loop form in the case without common noise. □

Under **(H1)**(i), there exists a unique solution to (4.2.1) (see e.g. [KX99]), denoted by $\{X_s^{t,\xi,\alpha}, t \leq s \leq T\}$, which is $\mathbb{F}$-adapted, and satisfies the square-integrability condition:

$$\mathbb{E}\Big[\sup_{t\leq s\leq T} |X_s^{t,\xi,\alpha}|^2\Big] \quad \leq \quad C\Big(1 + \mathbb{E}|\xi|^2\Big) \; < \; \infty, \tag{4.2.2}$$

for some positive constant C independent of α. We shall sometimes omit the dependence of $X^{t,\xi} = X^{t,\xi,\alpha}$ on α when there is no ambiguity. Since $\{X_s^{t,\xi}, t \leq s \leq T\}$ is $\mathbb{F}$-adapted, and W^0 is a $(\mathbb{P},\mathbb{F})$-Wiener process, we notice that $\mathbb{P}^{W^0}_{X_s^{t,\xi}}(dx) = \mathbb{P}[X_s^{t,\xi} \in dx|\mathcal{F}^0] = \mathbb{P}[X_s^{t,\xi} \in dx|\mathcal{F}_s^0]$. We thus have for any $\varphi \in \mathscr{C}_2(\mathbb{R}^d)$:

$$\mathbb{P}^{W^0}_{X_s^{t,\xi}}(\varphi) \quad = \quad \mathbb{E}\Big[\varphi(X_s^{t,\xi})\big|\mathcal{F}^0\Big] \; = \; \mathbb{E}\Big[\varphi(X_s^{t,\xi})\big|\mathcal{F}_s^0\Big], \quad t \leq s \leq T, \tag{4.2.3}$$

which shows that $\mathbb{P}^{W^0}_{X_s^{t,\xi}}(\varphi)$ is $\mathcal{F}_s^0$-measurable, and therefore, in view of the measurability property in Property 1.1.1, that $\{\mathbb{P}^{W^0}_{X_s^{t,\xi}}, t \leq s \leq T\}$ is $(\mathcal{F}_s^0)_{t\leq s\leq T}$-adapted. Moreover, since $\{\mathbb{P}^{W^0}_{X_s^{t,\xi}}, t \leq s \leq T\}$ is valued in $\mathcal{P}_2(C([t,T];\mathbb{R}^d))$, the set of square integrable probability measures on the space $C([t,T];\mathbb{R}^d)$ of continuous functions from $[t,T]$ into $\mathbb{R}^d$, it also has continuous trajectories, and is then $\mathbb{F}^0$-progressively measurable (actually even $\mathbb{F}^0$-predictable).

• *Cost functional and value function.* We are given a running cost function f defined on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbf{A}$, and a terminal cost function g defined on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, assumed to satisfy the condition

(H2)

(i) There exists some positive constant C s.t. for all $(x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbf{A}$,

$$|f(x, \mu, a)| + |g(x, \mu)| \quad \leq \quad C\big(1 + |x|^2 + \|\mu\|_2^2\big).$$

(ii) The functions f, g are continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbf{A}$, resp. on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and satisfy the local Lipschitz condition, uniformly w.r.t. $\mathbf{A}$: there exists some positive constant C s.t. for all $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, $a \in \mathbf{A}$,

$$\begin{aligned} & |f(x, \mu, a) - f(x', \mu', a)| + |g(x, \mu) - g(x', \mu')| \\ \leq \quad & C(1 + |x| + |x'| + \|\mu\|_2 + \|\mu'\|_2)\big(|x - x'| + \mathcal{W}_2(\mu, \mu')\big). \end{aligned}$$

We then consider the cost functional:

$$J(t, \xi, \alpha) \quad := \quad \mathbb{E}\Big[\int_t^T f\big(X_s^{t,\xi}, \mathbb{P}^{W^0}_{X_s^{t,\xi}}, \alpha_s\big) ds + g\big(X_T^{t,\xi}, \mathbb{P}^{W^0}_{X_T^{t,\xi}}\big)\Big],$$

which is well-defined and finite for all $(t, \xi, \alpha) \in [0, T] \times L^2(\mathcal{G}; \mathbb{R}^d) \times \mathcal{A}$, and we define the value function of the conditional McKean-Vlasov control problem as

$$v(t, \xi) \quad := \quad \inf_{\alpha \in \mathcal{A}} J(t, \xi, \alpha), \qquad (t, \xi) \in [0, T] \times L^2(\mathcal{G}; \mathbb{R}^d). \tag{4.2.4}$$

From the estimate (4.2.2) and the growth condition in **(H2)**(i), it is clear that v also satisfies a quadratic growth condition:

$$|v(t, \xi)| \quad \leq \quad C\big(1 + \mathbb{E}|\xi|^2\big), \quad \forall \xi \in L^2(\mathcal{G}; \mathbb{R}^d). \tag{4.2.5}$$

Our goal is to characterize the value function v as solution of a partial differential equation by means of a dynamic programming approach.

4.3 Dynamic programming

The aim of this section is to prove the dynamic programming principle (DPP) for the value function v in (4.2.4) of the conditional McKean-Vlasov control problem.

4.3.1 Flow properties

We shall assume that $(\Omega^0, W^0, \mathbb{P}^0)$ is the canonical space, i.e. $\Omega^0 = C(\mathbb{R}_+, \mathbb{R}^m)$, the set of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^m$, W^0 is the canonical process, and $\mathbb{P}^0$ the Wiener measure. Following [CTT16], we introduce the class of shifted control processes constructed by concatenation of paths: for $\alpha \in \mathcal{A}$, $(t, \bar\omega^0) \in [0, T] \times \Omega^0$, we set

$$\alpha_s^{t,\bar\omega^0}(\omega^0) \quad := \quad \alpha_s(\bar\omega^0 \otimes_t \omega^0), \quad (s, \omega^0) \in [0, T] \times \Omega^0,$$

where $\bar\omega^0 \otimes_t \omega^0$ is the element in Ω^0 defined by

$$\bar\omega^0 \otimes_t \omega^0(s) \quad := \quad \bar\omega^0(s)1_{s<t} + \big(\bar\omega^0(t) + \omega^0(s) - \omega^0(t)\big)1_{s\geq t}.$$

We notice that for fixed $(t,\bar\omega^0)$, the process $\alpha^{t,\bar\omega^0}$ lies in $\mathcal{A}_t$, the set of elements in $\mathcal{A}$ which are independent of $\mathcal{F}^0_t$ under $\mathbb{P}^0$. For any $\alpha \in \mathcal{A}$, and $\mathbb{F}^0$-stopping time θ, we denote by α^θ the map

$$\begin{array}{rcl}\alpha^\theta : (\Omega^0, \mathcal{F}^0_\theta) & \rightarrow & (\mathcal{A}, \mathcal{B}_\mathcal{A}) \\ \omega^0 & \mapsto & \alpha^{\theta(\omega^0),\omega^0}.\end{array}$$

The key step in the proof of the DPP is to obtain a flow property on the controlled conditional distribution $\mathbb{F}^0$-progressively measurable process $\{\mathbb{P}^{W^0}_{X^{t,\xi}_s}, t \leq s \leq T\}$, for $(t,\xi) \in [0,T] \times L^2(\mathcal{F}_t;\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$.

Lemma 4.3.1. *For any $t \in [0,T]$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, the relation given by*

$$\rho^{t,\mu,\alpha}_s \quad := \quad \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s}, \quad t \leq s \leq T, \quad \textit{for } \ \xi \in L^2(\mathcal{F}_t;\mathbb{R}^d) \ \textit{ s.t. } \ \mathbb{P}^{W^0}_\xi = \mu, \tag{4.3.1}$$

defines a square integrable $\mathbb{F}^0$-progressive continuous process in $\mathcal{P}_2(\mathbb{R}^d)$. Moreover, the map $(s,t,\omega^0,\mu,\alpha) \in [0,T] \times [0,T] \times \Omega^0 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A} \rightarrow \rho^{t,\mu,\alpha}_s(\omega^0) \in \mathcal{P}_2(\mathbb{R}^d)$ (with the convention that $\rho^{t,\mu,\alpha}_s = \mu$ for $s \leq t$) is measurable, and satisfies the flow property: $\rho^{t,\mu,\alpha}_s = \rho^{\theta,\rho^{t,\mu,\alpha}_\theta,\alpha^\theta}_s$, $\mathbb{P}^0$-a.s., i.e.

$$\rho^{t,\mu,\alpha}_s(\omega^0) \quad = \quad \rho_s^{\theta(\omega^0),\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0),\alpha^{\theta(\omega^0),\omega^0}}(\omega^0), \quad s \in [\theta,T], \ \mathbb{P}^0(d\omega^0) - a.s \tag{4.3.2}$$

for all $\theta \in \mathcal{T}^0_{t,T}$, the set of $\mathbb{F}^0$-stopping times valued in $[t,T]$.

Proof. 1. First observe that for any $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, we have: $\mathbb{E}^0[\|\mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s}\|_2^2] = \mathbb{E}[|X^{t,\xi,\alpha}_s|^2] < \infty$, which means that the process $\{\mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s}, t \leq s \leq T\}$ is square integrable, and we recall (see the discussion after (4.2.3)) that it is $\mathbb{F}^0$-progressively measurable.

(i) Notice that for $\mathbb{P}^0$-a.s $\omega^0 \in \Omega^0$, the law of the solution $\{X^{t,\xi,\alpha}_s(\omega^0,.), t \leq s \leq T\}$ to (4.2.1) on $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ is unique in law, which implies that $\mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_s}(\omega^0) = \mathbb{P}^1_{X^{t,\xi,\alpha}_s(\omega^0,.)}$, $t \leq s \leq T$, depends on ξ only through $\mathbb{P}^{W^0}_\xi(\omega^0) = \mathbb{P}^1_{\xi(\omega^0,.)}$. In other words, for any ξ_1, $\xi_2 \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ s.t. $\mathbb{P}^{W^0}_{\xi_1} = \mathbb{P}^{W^0}_{\xi_2}$, the processes $\{\mathbb{P}^{W^0}_{X^{t,\xi_1,\alpha}_s}, t \leq s \leq T\}$ and $\{\mathbb{P}^{W^0}_{X^{t,\xi_2,\alpha}_s}, t \leq s \leq T\}$ are indistinguishable.

(ii) Let us now check that for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, one can find $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$ s.t. $\mathbb{P}^{W^0}_\xi = \mu$. Indeed, recalling that $\mathcal{G}$ is rich enough, one can find $\xi \in L^2(\mathcal{G};\mathbb{R}^d) \subset L^2(\mathcal{F}_t;\mathbb{R}^d)$ s.t. $\mathcal{L}(\xi) = \mu$. Since $\mathcal{G}$ is independent of W^0, this also means that $\mathbb{P}^{W^0}_\xi = \mu$.

In view of the uniqueness result in (i), and the representation result in (ii), one can define the process $\{\rho^{t,\mu,\alpha}_s, t \leq s \leq T\}$ by the relation (4.3.1), and this process is a square integrable $\mathbb{F}^0$-progressively measurable process in $\mathcal{P}_2(\mathbb{R}^d)$.

2. Let us now prove the joint measurability of $\rho^{t,\mu,\alpha}_s(\omega^0)$ in $(t,s,\omega^0,\mu,\alpha) \in [0,T]\times[0,T]\times\Omega^0\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{A}$. Given $t \in [0,T]$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, let $\xi \in L^2(\mathcal{G};\mathbb{R}^d)$ s.t. $\mathcal{L}(\xi) = \mu$. We construct $X^{t,\xi,\alpha}$ using Picard's iteration by defining recursively a sequence of processes $(X^{(m),t,\xi,\alpha})_m$ as follows: we start from $X^{(0),t,\xi,\alpha} \equiv 0$, and define $\rho^{(0),t,\mu,\alpha}$ by formula (4.3.1) with $X^{(0),t,\xi,\alpha}$ instead of $X^{t,\xi,\alpha}$, and see that $\rho^{(0),t,\mu,\alpha} = \delta_0$.

- The process $X^{(1),t,\xi,\alpha}$ is given by

$$X_s^{(1),t,\xi,\alpha} \quad = \quad \xi + \int_t^s b(0,\delta_0,\alpha_r)dr + \int_t^s \sigma(0,\delta_0,\alpha_r)dB_r + \int_t^s \sigma_0(0,\delta_0,\alpha_r)dW_r^0,$$

for $0 \leq t \leq s \leq T$ (and $X_s^{(1),t,\xi,\alpha} = \xi$ when $s < t$), and we notice that the map $X^{(1),t,\xi,\alpha}$: $([t,T]\times\Omega, \mathcal{B}([t,T])\otimes\mathcal{F}) \to (\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ is measurable, up to indistinguishability. We then define $\rho^{(1),t,\mu,\alpha}$ by formula (4.3.1) with $X^{(1),t,\xi,\alpha}$ instead of $X^{t,\xi,\alpha}$, so that

$$\rho_s^{(1),t,\mu,\alpha}(\omega^0)(\varphi) \quad = \quad \mathbb{E}^1\Big[\varphi\big(X_s^{(1),t,\xi,\alpha}(\omega^0,.)\big)\Big] \;=\; \int_{\mathbb{R}^d}\Phi^{(1)}(x,t,s,\omega^0,\alpha)\mu(dx),$$

for any $\varphi \in \mathscr{C}_2(\mathbb{R}^n)$, where $\Phi^{(1)} : \mathbb{R}^d\times[0,T]\times[0,T]\times\Omega^0\times\mathcal{A}\to\mathbb{R}$ is measurable with quadratic growth condition in x, uniformly in (t,s,ω^0,α), and given by:

$$\begin{aligned}\Phi^{(1)}(x,t,s,\omega^0,\alpha) \quad = \quad & \mathbb{E}^1\Big[\varphi\big(x + \int_t^s b(0,\delta_0,\alpha_r(\omega^0))dr + \int_t^s \sigma(0,\delta_0,\alpha_r(\omega^0))dB_r \\ & \qquad + \int_t^s \sigma_0(0,\delta_0,\alpha_r(\omega^0))dW_r^0(\omega^0)\big)\Big],\ t\leq s\leq T,\end{aligned}$$

and $\Phi^{(1)}(x,t,s,\omega^0,\alpha) = \varphi(x)$ when $s < t$. By a monotone class argument (first considering the case when $\Phi^{(1)}(x,t,s,\omega^0,\alpha)$ is expressed as a product $h(x)\ell(t,s,\omega^0,\alpha)$ for some measurable and bounded functions h,ℓ), we deduce that $\rho_s^{(1),t,\mu,\alpha}(\omega^0)(\varphi)$ is jointly measurable in $(t,s,\omega^0,\mu,\alpha)$. By Proposition 1.1.1, this means that the map $(t,s,\omega^0,\mu,\alpha) \in [0,T]\times[0,T]\times\Omega^0\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{A}\mapsto \rho_s^{(1),t,\mu,\alpha}(\omega^0)\in\mathcal{P}_2(\mathbb{R}^d)$ is measurable.

- We define recursively $X^{(m+1),t,\xi,\alpha}$ assuming that $X^{(m),t,\xi,\alpha}$ has been already defined. We assume that the map $X^{(m),t,\xi,\alpha} : ([t,T]\times\Omega,\mathcal{B}([t,T])\otimes\mathcal{F})\to(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ is measurable (up to indistinguishability), and we define $\rho_s^{(m),t,\mu,\alpha}(\omega^0)$ given by formula (4.3.1) with $X^{(m),t,\xi,\alpha}$ instead of $X^{t,\xi,\alpha}$. Moreover, we suppose that $\rho_s^{(m),t,\mu,\alpha}(\omega^0)$ is jointly measurable in $(t,s,\omega^0,\mu,\alpha)$. Then, we define the process $X^{(m+1),t,\xi,\alpha}$ as follows:

$$\begin{aligned}X_s^{(m+1),t,\xi,\alpha} \quad = \quad & \xi + \int_t^s b(X_r^{(m),t,\xi,\alpha},\rho_r^{(m),t,\mu,\alpha},\alpha_r)dr + \int_t^s \sigma(X_r^{(m),t,\xi,\alpha},\rho_r^{(m),t,\mu,\alpha},\alpha_r)dB_r \\ & + \int_t^s \sigma_0(X_r^{(m),t,\xi,\alpha},\rho_r^{(m),t,\mu,\alpha},\alpha_r)dW_r^0,\end{aligned}$$

for $0\leq t\leq s\leq T$ (and $X_s^{(m+1),t,\xi,\alpha} = \xi$ when $s<t$), and notice by construction that the map $X^{(m+1),t,\xi,\alpha} : [t,T]\times\Omega,\mathcal{B}([t,T])\otimes\mathcal{F})\to(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ is measurable, up to indistinguishability. We can then define $\rho^{(m+1),t,\mu,\alpha}$ by formula (4.3.1) with $X^{(m+1),t,\xi,\alpha}$ instead of $X^{t,\xi,\alpha}$, namely

$$\rho_s^{(m+1),t,\mu,\alpha}(\omega^0)(\varphi) \quad = \quad \mathbb{E}^1\Big[\varphi\big(X_s^{(m+1),t,\xi,\alpha}(\omega^0,.)\big)\Big],$$

for any $\varphi\in\mathscr{C}_2(\mathbb{R}^n)$, $\omega^0\in\Omega^0$. From the (iterated) dependence of $X^{(m+1),t,\xi,\alpha}$ on ξ, and by Fubini's theorem (recalling the product structure of the probability space Ω^1 on which are defined the random variable ξ of law μ and the Brownian motion B), we then have

$$\rho_s^{(m+1),t,\mu,\alpha}(\omega^0)(\varphi) \quad = \quad \int_{\mathbb{R}^d}\Phi^{(m+1)}(x,t,s,\omega^0,\mu,\alpha)\mu(dx),$$

where $\Phi^{(m+1)} : \mathbb{R}^d \times [0,T] \times [0,T] \times \Omega^0 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A} \to \mathbb{R}$ is measurable with quadratic growth condition uniformly in (t, s, ω^0, α), and given by

$$\begin{aligned}\Phi^{(m+1)}(x,t,s,\omega^0,\mu,\alpha) \quad = \quad & \mathbb{E}^1\Big[\varphi\Big(x + \int_t^s b(x+\ldots,\rho_r^{(m),t,\mu,\alpha},\alpha_r)dr \\ & + \int_t^s \sigma(x+\ldots,\rho_r^{(m),t,\mu,\alpha},\alpha_r)dB_r \\ & + \int_t^s \sigma_0(x+\ldots,\rho_r^{(m),t,\mu,\alpha},\alpha_r)dW_r(\omega^0)\Big)\Big], \; t \le s \le T,\end{aligned}$$

and $\Phi^{(m+1)}(x,t,s,\omega^0,\alpha) = \varphi(x)$ when $s < t$. We then see that $\rho_s^{(m+1),t,\mu,\alpha}(\omega^0)(\varphi)$ is jointly measurable in $(t,s,\omega^0,\mu,\alpha)$ (using again a monotone class argument), and deduce by Proposition 1.1.1 that the map $(t,s,\omega^0,\mu,\alpha) \in [0,T] \times [0,T] \times \Omega^0 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A} \mapsto \rho_s^{(m+1),t,\mu,\alpha}(\omega^0) \in \mathcal{P}_2(\mathbb{R}^d)$ is measurable.

Now that we have constructed the sequence $(X^{(m),t,\xi,\alpha})_m$, one can show by proceeding along the same lines as in the proof of Theorem IX.2.1 in [RY99] or Theorem V.8 in [Pro05] that

$$\sup_{t \le s \le T} |X_s^{(m),t,\xi,\alpha} - X_s^{t,\xi,\alpha}| \quad \xrightarrow[m\to\infty]{\mathbb{P}} \quad 0,$$

where the convergence holds in probability. Then, by the same arguments as in the proof of Lemma 3.2 in [BCP18] (see their Appendix B), this implies that the following convergence holds in probability:

$$\mathcal{W}_2\big(\rho_s^{(m),t,\mu,\alpha}, \rho_s^{t,\mu,\alpha}\big) \quad \xrightarrow[m\to\infty]{\mathbb{P}^0} \quad 0,$$

for all $s \in [t,T]$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and $\alpha \in \mathcal{A}$. Since for any $m \in \mathbb{N}$, $\rho_s^{(m),t,\mu,\alpha}(\omega^0)$ is jointly measurable in $(t,s,\omega^0,\mu,\alpha)$, we deduce by proceeding for instance as in the first item of Exercise IV.5.17 in [RY99], and recalling that $\mathcal{F}^0$ is assumed to be a complete σ-field, that the map $(t,s,\omega^0,\mu,\alpha) \in [0,T] \times [0,T] \times \Omega^0 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A} \mapsto \rho_s^{t,\mu,\alpha}(\omega^0) \in \mathcal{P}_2(\mathbb{R}^d)$ is measurable.

3. Let us finally check the flow property (4.3.2). From pathwise uniqueness of the solution $\{X_s(\omega^0,.), t \le s \le T\}$ to (4.2.1) on $(\Omega, \mathcal{F}^1, \mathbb{P}^1)$ for $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, and recalling the definition of the shifted control process, we have the flow property: for $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, and $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$,

$$X_s^{t,\xi,\alpha}(\omega^0,.) \quad = \quad X_s^{\theta(\omega^0), X^{t,\xi,\alpha}_{\theta(\omega^0)}(\omega^0,.), \alpha^{\theta(\omega^0),\omega^0}}(\omega^0,.), \quad \mathbb{P}^1 - \text{a.s.}$$

for all $\mathbb{F}^0$-stopping time θ valued in $[t,T]$. It follows that for any Borel-measurable bounded function φ on $\mathbb{R}^d$, and for $\mathbb{P}^0$-a.s $\omega^0 \in \Omega^0$,

$$\begin{aligned}\rho_s^{t,\mu,\alpha}(\omega^0)(\varphi) \;=\; \mathbb{E}^1\Big[\varphi\big(X_s^{t,\xi,\alpha}(\omega^0,.)\big)\Big] \quad & = \quad \mathbb{E}^1\Big[\varphi\big(X_s^{\theta(\omega^0), X^{t,\xi,\alpha}_{\theta(\omega^0)}(\omega^0,.), \alpha^{\theta(\omega^0),\omega^0}}(\omega^0,.)\big)\Big] \\ & = \quad \rho_s^{\theta(\omega^0), \rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0), \alpha^{\theta(\omega^0),\omega^0}}(\omega^0)(\varphi),\end{aligned}$$

where the last equality is obtained by noting that $\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0) = \mathbb{P}^{W^0}_{X^{t,\xi,\alpha}_{\theta(\omega^0)}(\omega^0,.)}$, and the definition of $\rho_s^{t,\mu,\alpha}$. This shows the required flow property (4.3.2). □

Now, by the law of iterated conditional expectations, from (4.2.3), (4.3.1), and recalling that $\alpha \in \mathcal{A}$

is $\mathbb{F}^0$-progressive, we can rewrite the cost functional as

$$\begin{aligned} J(t,\xi,\alpha) &= \mathbb{E}\Big[\int_t^T \mathbb{E}\big[f\big(X_s^{t,\xi},\mathbb{P}^{W^0}_{X_s^{t,\xi}},\alpha_s\big)\big|\mathcal{F}_s^0\big]ds + \mathbb{E}\big[g\big(X_T^{t,\xi},\mathbb{P}^{W^0}_{X_T^{t,\xi}}\big)\big|\mathcal{F}_T^0\big]\Big] \\ &= \mathbb{E}\Big[\int_t^T \rho_s^{t,\mu}\big(f(.,\rho_s^{t,\mu},\alpha_s)\big)ds + \rho_T^{t,\mu}\big(g(.,\rho_T^{t,\mu})\big)\Big] \\ &= \mathbb{E}\Big[\int_t^T \hat{f}(\rho_s^{t,\mu},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu})\Big], \end{aligned} \tag{4.3.3}$$

for $t \in [0,T]$, $\xi \in L^2(\mathcal{G};\mathbb{R}^d)$ with law $\mu = \mathcal{L}(\xi) = \mathbb{P}^{W^0}_\xi \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, and with the functions $\hat{f}$: $\mathcal{P}_2(\mathbb{R}^d) \times \mathbf{A} \to \mathbb{R}$, and $\hat{g} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, defined by

$$\begin{cases} \hat{f}(\mu,a) := \mu\big(f(.,\mu,a)\big) = \int_{\mathbb{R}^d} f(x,\mu,a)\mu(dx) \\ \hat{g}(\mu) := \mu\big(g(.,\mu)\big) = \int_{\mathbb{R}^d} g(x,\mu)\mu(dx). \end{cases} \tag{4.3.4}$$

(To alleviate notations, we have omitted here the dependence of $\rho_s^{t,\mu} = \rho_s^{t,\mu,\alpha}$ on α). Relation (4.3.3) means that the cost functional depends on ξ only through its distribution $\mu = \mathcal{L}(\xi)$, and by misuse of notation, we set:

$$J(t,\mu,\alpha) := J(t,\xi,\alpha) = \mathbb{E}^0\Big[\int_t^T \hat{f}(\rho_s^{t,\mu},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu})\Big],$$

for $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, $\xi \in L^2(\mathcal{G};\mathbb{R}^d)$ with $\mathcal{L}(\xi) = \mu$, and the expectation is taken under $\mathbb{P}^0$ since $\{\rho_s^{t,\mu}, t \le s \le T\}$ is $\mathbb{F}^0$-progressive, and the control $\alpha \in \mathcal{A}$ is an $\mathbb{F}^0$-progressive process. Therefore, the value function can be identified with a function defined on $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$, equal to (we keep the same notation $v(t,\mu) = v(t,\xi)$):

$$v(t,\mu) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}^0\Big[\int_t^T \hat{f}(\rho_s^{t,\mu},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu})\Big],$$

and satisfying from (4.2.5) the quadratic growth condition

$$|v(t,\mu)| \le C(1 + \|\mu\|_2^2), \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d). \tag{4.3.5}$$

As a consequence of the flow property in Lemma 4.3.1, we obtain the following conditioning lemma, also called pseudo-Markov property in the terminology of [CTT16], for the controlled conditional distribution $\mathbb{F}^0$-progressive process $\{\rho_s^{t,\mu,\alpha}, t \le s \le T\}$.

Lemma 4.3.2. *For any $(t,\mu,\alpha) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{A}$, and $\theta \in \mathcal{T}^0_{t,T}$, we have*

$$J(\theta,\rho_\theta^{t,\mu,\alpha},\alpha^\theta) = \mathbb{E}^0\Big[\int_\theta^T \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu,\alpha})\big|\mathcal{F}_\theta^0\Big], \quad \mathbb{P}^0 - a.s \tag{4.3.6}$$

Proof. By the joint measurability property of $\rho_s^{t,\mu,\alpha}$ in $(t,s,\omega^0,\mu,\alpha)$ in Lemma 4.3.1, the flow property (4.3.2), and since $\rho_\theta^{t,\mu,\alpha}$ is $\mathcal{F}_\theta^0$-measurable for θ $\mathbb{F}^0$-stopping time, we have for $\mathbb{P}^0$-a.s $\omega^0 \in \Omega^0$,

$$\begin{aligned} &\mathbb{E}^0\Big[\int_\theta^T \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu,\alpha})\big|\mathcal{F}_\theta^0\Big](\omega^0) \\ =\ & \mathbb{E}^0\Big[\int_r^T \hat{f}(\rho_s^{r,\pi,\beta},\beta_s) + \hat{g}(\rho_T^{r,\pi,\beta})\big|\mathcal{F}_r^0\Big](\omega^0)\Big|_{r=\theta(\omega^0),\pi=\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0),\beta=\alpha^{r,\omega^0}} \\ =\ & \mathbb{E}^0\Big[\int_r^T \hat{f}(\rho_s^{r,\pi,\beta},\beta_s) + \hat{g}(\rho_T^{r,\pi,\beta})\Big]\Big|_{r=\theta(\omega^0),\pi=\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0),\beta=\alpha^{r,\omega^0}}, \end{aligned}$$

where we used in the second equality the fact that for fixed ω^0, $r \in [t,T]$, $\pi \in \mathcal{P}_2(\mathbb{R}^d)$ represented by $\eta \in L^2(\mathcal{G};\mathbb{R}^d)$ s.t. $\mathcal{L}(\xi) = \pi$, the process α^{r,ω^0} lies in $\mathcal{A}_r$, hence is independent of $\mathcal{F}^0_r$, which implies that $X_s^{r,\eta,\alpha^{r,\omega^0}}$ is independent of $\mathcal{F}_r$, and thus $\rho_s^{r,\pi,\alpha^{r,\omega^0}}$ is also independent of $\mathcal{F}^0_r$ for $r \leq s$. This shows the conditioning relation (4.3.6). □

4.3.2 Continuity of the value function and dynamic programming principle

In this paragraph, we show the continuity of the value function, which is helpful for proving next the dynamic programming principle. We mainly follow arguments from [Kry08] for the continuity result that we extend to our McKean-Vlasov framework.

Lemma 4.3.3. *The function $(t,\mu) \mapsto J(t,\mu,\alpha)$ is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, uniformly with respect to $\alpha \in \mathcal{A}$, and the function $\alpha \mapsto J(t,x,\alpha)$ is continuous on $\mathcal{A}$ for any $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. Consequently, the cost functional J is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{A}$, and the value function v is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$.*

Proof. *(1)* For any $0 \leq t \leq s \leq T$, $\mu,\pi \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, recall that $\mathbb{P}^0$-a.s. $\omega^0 \in \Omega^0$, we have $\mathbb{P}^1_{X_r^{t,\xi,\alpha}(\omega^0,.)} = \rho_r^{t,\mu,\alpha}(\omega^0)$, $\mathbb{P}^1_{X_r^{s,\zeta,\alpha}(\omega^0,.)} = \rho_r^{s,\pi,\alpha}(\omega^0)$ for $r \in [s,T]$, and any $\xi,\zeta \in L^2(\mathcal{G};\mathbb{R}^d)$ s.t. $\mathcal{L}(\xi) = \mu$, $\mathcal{L}(\zeta) = \pi$. By definition of $\|.\|_2$ and the Wasserstein distance in $\mathcal{P}_2(\mathbb{R}^d)$, we then have: $\|\rho_r^{t,\mu,\alpha}(\omega^0)\|_2 = \mathbb{E}^1|X_r^{t,\xi,\alpha}(\omega^0,.)|^2$, and $\mathcal{W}_2^2\big(\rho_r^{t,\mu,\alpha}(\omega^0),\rho_r^{s,\pi,\alpha}(\omega^0)\big) \leq \mathbb{E}^1|X_r^{t,\xi,\alpha}(\omega^0,.) - X_r^{s,\zeta,\alpha}(\omega^0,.)|^2$, so that

$$\mathbb{E}^0\Big[\sup_{s\leq r\leq T}\|\rho_r^{t,\mu,\alpha}\|_2^2\Big] \leq \mathbb{E}\Big[\sup_{s\leq r\leq T}|X_r^{t,\xi,\alpha}|^2\Big], \tag{4.3.7}$$

$$\mathbb{E}^0\Big[\sup_{s\leq r\leq T}\mathcal{W}_2^2(\rho_r^{t,\mu,\alpha},\rho_r^{s,\pi,\alpha})\Big] \leq \mathbb{E}\Big[\sup_{s\leq r\leq T}|X_r^{t,\xi,\alpha} - X_r^{s,\zeta,\alpha}|^2\Big]. \tag{4.3.8}$$

From the state equation (4.2.1), and using standard arguments involving Burkholder-Davis-Gundy inequalities, (4.3.7), (4.3.8), and Gronwall lemma, under the Lipschitz condition in **(H1)**(i), we obtain the following estimates similar to the ones for controlled diffusion processes (see [Kry08], Chap.2, Thm.5.9, Cor.5.10): there exists some positive constant C s.t. for all $t \in [0,T]$, $\xi,\zeta \in L^2(\mathcal{G};\mathbb{R}^d)$, $\alpha \in \mathcal{A}$, $h \in [0,T-t]$,

$$\mathbb{E}\big[\sup_{t\leq s\leq t+h}|X_s^{t,\xi,\alpha} - \xi|^2\big] \leq C(1+\mathbb{E}|\xi|^2)h,$$

$$\mathbb{E}\big[\sup_{t\leq s\leq T}|X_s^{t,\xi,\alpha} - X_s^{t,\zeta,\alpha}|^2\big] \leq C\mathbb{E}[|\xi-\zeta|^2],$$

from which we easily deduce that for all $0 \leq t \leq s \leq T$, $\xi,\zeta \in L^2(\mathcal{G};\mathbb{R}^d)$, $\alpha \in \mathcal{A}$

$$\mathbb{E}\Big[\sup_{s\leq r\leq T}|X_r^{t,\xi,\alpha} - X_r^{s,\zeta,\alpha}|^2\Big] \leq C\big(\mathbb{E}|\xi-\zeta|^2 + (1+\mathbb{E}|\xi|^2+\mathbb{E}|\zeta|^2)|s-t|\big). \tag{4.3.9}$$

Together with the estimates (4.2.2), and by definition of $\mathcal{W}_2(\mu,\pi)$, $\|\mu\|_2$, $\|\pi\|_2$, we then get from (4.3.7), (4.3.8):

$$\mathbb{E}^0\Big[\sup_{s\leq r\leq T}\|\rho_r^{t,\mu,\alpha}\|_2^2\Big] \leq C(1+\|\mu\|_2^2), \tag{4.3.10}$$

$$\mathbb{E}^0\Big[\sup_{s\leq r\leq T}\mathcal{W}_2^2(\rho_r^{t,\mu,\alpha},\rho_r^{s,\pi,\alpha})\Big] \leq C\big(\mathcal{W}_2^2(\mu,\pi) + (1+\|\mu\|_2^2+\|\pi\|_2^2)|s-t|\big). \tag{4.3.11}$$

(2) Let us now show the continuity of the cost functional J in (t,μ), uniformly w.r.t. $\alpha \in \mathcal{A}$. First, we notice from the growth condition in **(H2)**(i) and the local Lipschitz condition in **(H2)**(ii) that there

exists some positive constant C s.t. for all $\mu, \pi \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$,

$$\begin{aligned} |\hat{f}(\mu,\alpha)| &\leq C\big(1+\|\mu\|_2^2\big), \\ |\hat{f}(\mu,\alpha)-\hat{f}(\pi,\alpha)|+|\hat{g}(\mu)-\hat{g}(\pi)| &\leq C(1+\|\mu\|_2+\|\pi\|_2)\mathcal{W}_2(\mu,\pi). \end{aligned}$$

Then, we have for all $0 \leq t \leq s \leq T$, $\mu, \pi \in \mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$

$$\begin{aligned} \big|J(t,\mu,\alpha)-J(s,\pi,\alpha)\big| &\leq \mathbb{E}^0\Big[\int_t^s |\hat{f}(\rho_r^{t,\mu,\alpha})|dr\Big] \\ &\quad + \mathbb{E}^0\Big[\int_s^T \big|\hat{f}(\rho_r^{t,\mu,\alpha},\alpha_r)-\hat{f}(\rho_r^{s,\pi,\alpha},\alpha_r)\big|dr + \big|\hat{g}(\rho_T^{t,\mu,\alpha})-\hat{g}(\rho_T^{s,\pi,\alpha})\big|\Big] \\ &\leq C\mathbb{E}^0\Big[(1+\sup_{t\leq r\leq s}(\|\rho_r^{t,\mu,\alpha}\|_2)|s-t|\Big] \\ &\quad + C\mathbb{E}^0\Big[\big(1+\sup_{s\leq r\leq T}(\|\rho_r^{t,\mu,\alpha}\|_2+\|\rho_r^{s,\pi,\alpha}\|_2)\big)\sup_{s\leq r\leq T}\mathcal{W}_2(\rho_r^{t,\mu,\alpha},\rho_r^{s,\pi,\alpha})\Big] \\ &\leq C(1+\|\mu\|_2)|s-t| \\ &\quad + C(1+\|\mu\|_2+\|\pi\|_2)\big(\mathcal{W}_2(\mu,\pi)+(1+\|\mu\|_2+\|\pi\|_2)|s-t|^{\frac{1}{2}}\big), \end{aligned}$$

by Cauchy Schwarz inequality and (4.3.10)-(4.3.11), which shows the desired continuity result.

(3) Let us show the continuity of the cost functional with respect to the control. Fix $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, and consider $\alpha \in \mathcal{A}$, a sequence $(\alpha^n)_n$ in $\mathcal{A}$ s.t. $\Delta(\alpha^n,\alpha) \to 0$, i.e. $d_A(\alpha_t^n,\alpha_t) \to 0$ in $dt\otimes d\mathbb{P}^0$-measure, as n goes to infinity. Denote by $\rho^n = \rho^{t,\mu,\alpha^n}$, $\rho = \rho^{t,\mu,\alpha}$, $X^n = X^{t,\xi,\alpha^n}$, $X = X^{t,\xi,\alpha}$ for $\xi \in L^2(\mathcal{G};\mathbb{R}^d)$ s.t. $\mathcal{L}(\xi) = \mu$. By the same arguments as in (4.3.8), we have

$$\mathbb{E}^0\Big[\sup_{t\leq s\leq T}\mathcal{W}_2^2(\rho_s^n,\rho_s)\Big] \leq \mathbb{E}\Big[\sup_{t\leq s\leq T}|X_s^n-X_s|^2\Big]. \tag{4.3.12}$$

Next, starting from the state equation (4.2.1), using standard arguments involving Burkholder-Davis-Gundy inequalities, (4.3.12), and Gronwall lemma, under the Lipschitz condition in **(H1)**(i), we arrive at:

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\leq s\leq T}|X_s^n-X_s|^2\Big] \leq C\Big\{\mathbb{E}\Big[&\int_t^T |b(X_s,\rho_s,\alpha_s)-b(X_s,\rho_s,\alpha_s^n)|^2 ds \\ &+ \int_t^T |\sigma(X_s,\rho_s,\alpha_s)-\sigma(X_s,\rho_s,\alpha_s^n)|^2 ds \\ &+ \int_t^T |\sigma_0(X_s,\rho_s,\alpha_s)-\sigma_0(X_s,\rho_s,\alpha_s^n)|^2 ds\Big]\Big\}, \end{aligned}$$

for some positive constant C independent of n. Recalling the bound (4.2.2), and (4.3.7), we deduce by the dominated convergence theorem under the linear growth condition in **(H1)**(i), and the continuity assumption in **(H1)**(ii) that $\mathbb{E}\big[\sup_{t\leq s\leq T}|X_s^n-X_s|^2\big] \to 0$, and thus by (4.3.12)

$$\mathbb{E}^0\Big[\sup_{t\leq s\leq T}\mathcal{W}_2^2(\rho_s^n,\rho_s)\Big] \to 0, \quad \text{as } n\to\infty. \tag{4.3.13}$$

Now, by writing

$$\begin{aligned} &\big|J(t,\mu,\alpha^n)-J(t,\mu,\alpha)\big| \\ \leq\ & \mathbb{E}^0\Big[\int_t^T \big|\hat{f}(\rho_s^n,\alpha_s^n)-\hat{f}(\rho_s,\alpha_s)\big|ds + \big|\hat{g}(\rho_T^n)-\hat{g}(\rho_T)\big|\Big], \end{aligned} \tag{4.3.14}$$

and noting that $\hat{f}$ and $\hat{g}$ are continuous on $\mathcal{P}_2(\mathbb{R}^d)\times\mathbf{A}$, resp. on $\mathcal{P}_2(\mathbb{R}^d)$, under the continuity assumption in **(H2)**(ii), we conclude by the same arguments as in [Kry08] using (4.3.13) (see Chapter 3, Sec. 2, or also Lemma 4.1 in [FP15]) that the r.h.s. of (4.3.14) tends to zero as n goes to infinity, which proves the continuity of $J(t,\mu,.)$ on $\mathcal{A}$.

(4) Finally, the global continuity of the cost functional J on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{A}$ is a direct consequence of the continuity of $J(.,.,\alpha)$ on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ uniformly w.r.t. $\alpha\in\mathcal{A}$, and the continuity of $J(t,\mu,.)$ on $\mathcal{A}$, while the continuity of the value function v on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ follows immediately from the fact that

$$|v(t,\mu)-v(s,\pi)| \quad \leq \quad \sup_{\alpha\in\mathcal{A}}|J(t,\mu,\alpha)-J(s,\pi,\alpha)|, \quad t,s\in[0,T],\ \mu,\pi\in\mathcal{P}_2(\mathbb{R}^d),$$

and again from the continuity of $J(.,.,\alpha)$ on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ uniformly w.r.t. $\alpha\in\mathcal{A}$. □

Remark 4.3.1. Notice that the supremum defining the value function $v(t,\mu)$ can be taken over the subset $\mathcal{A}_t$ of elements in $\mathcal{A}$ which are independent of $\mathcal{F}_t^0$ under $\mathbb{P}^0$, i.e.

$$v(t,\mu) \quad = \quad \inf_{\alpha\in\mathcal{A}_t}\mathbb{E}^0\Big[\int_t^T \hat{f}(\rho_s^{t,\mu},\alpha_s)ds+\hat{g}(\rho_T^{t,\mu})\Big]. \tag{4.3.15}$$

Indeed, denoting by $\tilde{v}(t,\mu)$ the r.h.s. of (4.3.15), and since $\mathcal{A}_t\subset\mathcal{A}$, it is clear that $v(t,\mu)\leq\tilde{v}(t,\mu)$. To prove the reverse inequality, we apply the conditioning relation (4.3.6) for $\theta=t$, and get in particular for all $\alpha\in\mathcal{A}$:

$$\int_{\Omega^0} J(t,\mu,\alpha^{t,\omega^0})\mathbb{P}^0(d\omega^0) \quad = \quad J(t,\mu,\alpha). \tag{4.3.16}$$

Now, recalling that for any fixed $\omega^0\in\Omega^0$, α^{t,ω^0} lies in $\mathcal{A}_t$, we have $J(t,\mu,\alpha^{t,\omega^0})\geq\tilde{v}(t,\mu)$, which proves the required result since α is arbitrary in (4.3.16). □

We can now state the dynamic programming principle (DPP) for the value function to the stochastic McKean-Vlasov control problem.

Proposition 4.3.1. *(Dynamic Programming Principle)*

We have for all $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} v(t,\mu) \quad &= \quad \inf_{\alpha\in\mathcal{A}}\inf_{\theta\in\mathcal{T}_{t,T}^0}\mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big] \\ &= \quad \inf_{\alpha\in\mathcal{A}}\sup_{\theta\in\mathcal{T}_{t,T}^0}\mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big], \end{aligned}$$

which means equivalently that

(i) for all $\alpha\in\mathcal{A}$, $\theta\in\mathcal{T}_{t,T}^0$,

$$v(t,\mu) \quad \leq \quad \mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big], \tag{4.3.17}$$

(ii) for all $\varepsilon>0$, *there exists* $\alpha\in\mathcal{A}$, *such that for all* $\theta\in\mathcal{T}_{t,T}^0$,

$$v(t,\mu)+\varepsilon \quad \geq \quad \mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big]. \tag{4.3.18}$$

Remark 4.3.2. The above formulation of the DPP implies in particular that for all $\theta \in \mathcal{T}^0_{t,T}$,

$$v(t,\mu) \quad = \quad \inf_{\alpha\in\mathcal{A}} \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big],$$

which is the usual formulation of the DPP. The formulation in Proposition 4.3.1 is stronger, and the difference relies on the fact that in the inequality (4.3.18), the ε-optimal control $\alpha = \alpha^\varepsilon$ does not depend on θ. This condition will be useful to show later the viscosity supersolution property of the value function. □

Proof. 1. Fix $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$. From the conditioning relation (4.3.6), we have for all $\theta \in \mathcal{T}^0_{t,T}$, $\alpha \in \mathcal{A}$,

$$J(t,\mu,\alpha) \quad = \quad \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds + J(\theta,\rho_\theta^{t,\mu,\alpha},\alpha^\theta)\Big]. \tag{4.3.19}$$

Since $J(.,.,\alpha^\theta) \geq v(.,.)$, and θ is arbitrary in $\mathcal{T}^0_{t,T}$, we have

$$J(t,\mu,\alpha) \quad \geq \quad \sup_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds + v(\theta,\rho_\theta^{t,\mu,\alpha})\Big],$$

and since α is arbitrary in $\mathcal{A}$, it follows that

$$v(t,\mu) \quad \geq \quad \inf_{\alpha\in\mathcal{A}} \sup_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds \ + \ v(\theta,\rho_\theta^{t,\mu,\alpha})\Big] \tag{4.3.20}$$

2. Fix $(t,\mu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{T}^0_{t,T}$. For any $\varepsilon > 0$, $\omega^0 \in \Omega^0$, one can find from (4.3.15) some $\alpha^{(\varepsilon,\omega^0)} \in \mathcal{A}_{\theta(\omega^0)}$ s.t.

$$v(\theta(\omega^0),\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0)) + \varepsilon \quad \geq \quad J(\theta(\omega^0),\rho^{t,\mu,\alpha}_{\theta(\omega^0)}(\omega^0),\alpha^{(\varepsilon,\omega^0)}). \tag{4.3.21}$$

Since J and v are continuous (by Lemma 4.3.3), one can invoke measurable selection arguments (see e.g. [Wag80]), to claim that the map $\omega^0 \in (\Omega^0,\mathcal{F}^0) \mapsto \alpha^{(\varepsilon,\omega^0)} \in (\mathcal{A},\mathcal{B}_\mathcal{A})$ can be chosen measurable. Let us now define the process $\bar\alpha$ on $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0)$ obtained by concatenation at θ of the processes α and $\alpha^{(\varepsilon,\omega^0)}$ in $\mathcal{A}$, namely:

$$\bar\alpha_s(\omega^0) \quad := \quad \alpha_s(\omega^0)1_{s<\theta(\omega^0)} + \alpha^{(\varepsilon,\omega^0)}(\omega^0)1_{s\geq\theta(\omega^0)}, \quad 0 \leq s \leq T.$$

By Lemma 2.1 in [ST02], and since $\mathcal{A}$ is a separable metric space, the process $\bar\alpha$ is $\mathbb{F}^0$-progressively measurable, and thus $\bar\alpha \in \mathcal{A}$. Notice with our notations of shifted control process that $\bar\alpha^{\theta(\omega^0),\omega^0} = \alpha^{(\varepsilon,\omega^0)}$ for all ω^0 in Ω^0, and then (4.3.21) reads as

$$v(\theta,\rho_\theta^{t,\mu,\alpha}) + \varepsilon \quad \geq \quad J(\theta,\rho_\theta^{t,\mu,\alpha},\bar\alpha^\theta), \quad \mathbb{P}^0-\text{a.s.}$$

Therefore, by using again (4.3.19) to $\bar\alpha$, and since $\rho_s^{t,\mu,\bar\alpha} = \rho_s^{t,\mu,\alpha}$ for $s \leq \theta$ (recall that $\bar\alpha_s = \alpha_s$ for $s < \theta$, and $\rho^{t,\mu,\alpha}$ has continuous trajectories), we get

$$\begin{aligned} v(t,\mu) \leq \ J(t,\mu,\bar\alpha) \quad &= \quad \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds + J(\theta,\rho_\theta^{t,\mu,\alpha},\bar\alpha^\theta)\Big] \\ &\leq \quad \mathbb{E}^0\Big[\int_t^\theta \hat f(\rho_s^{t,\mu,\alpha},\alpha_s)ds + v(\theta,\rho_\theta^{t,\mu,\alpha})\Big] + \varepsilon \end{aligned}$$

Since α, θ and ε are arbitrary, this gives the inequality

$$v(t,\mu) \quad \leq \quad \inf_{\alpha\in\mathcal{A}} \inf_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds \; + \; v(\theta,\rho_\theta^{t,\mu,\alpha})\Big],$$

which, combined with the first inequality (4.3.20), proves the DPP result. □

4.4 Bellman equation and viscosity solutions

4.4.1 Dynamic programming equation

Based on the DPP in Proposition 4.3.1 and Itô's formula for functions of measure-valued processes in Proposition 1.3.2, the dynamic programming Bellman equation associated to the value function of the stochastic McKean-Vlasov control problem takes the following form:

$$\begin{cases} -\partial_t v - \inf\limits_{a\in\mathbf{A}}\Big[\hat{f}(\mu,a) + \mu\big(\text{Ł}^a v(t,\mu)\big) + \mu\otimes\mu\big(\mathbb{M}^a v(t,\mu)\big)\Big] &= 0, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ v(T,\mu) &= \hat{g}(\mu), \quad \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{4.4.1}$$

where for $\phi\in\mathcal{C}_c^2(\mathcal{P}_2(\mathbb{R}^d))$, $a\in\mathbf{A}$, and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $\text{Ł}^a\phi(\mu)\in L^2_\mu(\mathbb{R})$ is the function $\mathbb{R}^d\to\mathbb{R}$ defined by

$$\text{Ł}^a\phi(\mu)(x) \quad := \quad \partial_\mu\phi(\mu)(x).b(x,\mu,a) + \frac{1}{2}\text{tr}\big(\partial_x\partial_\mu\phi(\mu)(x)(\sigma\sigma^\intercal + \sigma_0\sigma_0^\intercal)(x,\mu,a)\big), \tag{4.4.2}$$

and $\mathbb{M}^a\phi(\mu)\in L^2_{\mu\otimes\mu}(\mathbb{R})$ is the function $\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined by

$$\mathbb{M}^a\phi(\mu)(x,x') \quad := \quad \frac{1}{2}\text{tr}\big(\partial^2_\mu\phi(\mu)(x,x')\sigma_0(x,\mu,a)\sigma_0^\intercal(x',\mu,a)\big). \tag{4.4.3}$$

Alternatively, by viewing the value function as a function on $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ via the lifting identification, and keeping the same notation $v(t,\xi)=v(t,\mathcal{L}(\xi))$ (recall that v depends on ξ only via its distribution), we see from the connection (1.2.1)-(1.3.8) between derivatives in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ and in the Hilbert space $L^2(\mathcal{G};\mathbb{R}^d)$ that the Bellman equation (4.4.1) is written also in $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ as

$$\begin{cases} -\partial_t v - H\big(\xi,Dv(t,\xi),D^2v(t,\xi)\big) &= 0, \quad (t,\xi)\in[0,T)\times L^2(\mathcal{G};\mathbb{R}^d), \\ v(T,\xi) &= \tilde{\mathbb{E}}^1\big[g(\xi,\mathcal{L}(\xi))\big], \quad \xi\in L^2(\mathcal{G};\mathbb{R}^d), \end{cases} \tag{4.4.4}$$

where $H: L^2(\mathcal{G};\mathbb{R}^d)\times L^2(\mathcal{G};\mathbb{R}^d)\times S(L^2(\mathcal{G};\mathbb{R}^d))\to\mathbb{R}$ is defined by

$$\begin{aligned} H(\xi,P,Q) \quad = \quad & \inf_{a\in\mathbf{A}}\tilde{\mathbb{E}}^1\Big[f(\xi,\mathcal{L}(\xi),a) + P.b(\xi,\mathcal{L}(\xi),a) \\ & + \frac{1}{2}Q(\sigma_0(\xi,\mathcal{L}(\xi),a)).\sigma_0(\xi,\mathcal{L}(\xi),a) + \frac{1}{2}Q(\sigma(\xi,\mathcal{L}(\xi),a)N).\sigma(\xi,\mathcal{L}(\xi),a)N\Big], \end{aligned} \tag{4.4.5}$$

with $N\in L^2(\mathcal{G};\mathbb{R}^n)$ of zero mean, and unit variance, and independent of ξ.

The purpose of this section is to prove an analytic characterization of the value function in terms of the dynamic programming Bellman equation. We shall adopt a notion of viscosity solutions following the approach in [Lio12], which consists via the lifting identification in working in the Hilbert space $L^2(\mathcal{G};\mathbb{R}^d)$ instead of working in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. Indeed, comparison principles for viscosity solutions in the Wasserstein space, or more generally in metric spaces, are difficult to obtain as we have to deal with locally non compact spaces (see e.g. [AGS08], [GNT08], [FK09]), and instead by working in separable Hilbert spaces, one can essentially reduce to the case of Euclidian spaces by projection, and then take advantage of the results developed for viscosity solutions, in particular here, for second order Hamilton-Jacobi-Bellman equations, see [Lio89b], [FGS15]. We shall assume that the σ-algebra $\mathcal{G}$ is countably generated upto null sets, which ensures that the Hilbert space $L^2(\mathcal{G};\mathbb{R}^d)$ is separable, see [Doo94], p. 92. This is satisfied for example when $\mathcal{G}$ is the Borel σ-algebra of a canonical space $\tilde\Omega^1$ of continuous functions on $\mathbb{R}_+$ (see Exercise 4.21 in Chapter 1 of [RY99]).

Definition 4.4.1. *We say that a continuous function $u : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is a viscosity (sub, super) solution to* (4.4.1) *if its lifted version $\tilde u$ on $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ is a viscosity (sub, super) solution to* (4.4.4)*, that is:*

(i) $\tilde u(T,\xi)\le\tilde{\mathbb{E}}^1\big[g(\xi,\mathcal{L}(\xi))\big]$, and for any test function $\varphi\in\mathcal{C}^2([0,T]\times L^2(\mathcal{G};\mathbb{R}^d))$ (the set of real-valued continuous functions on $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ which are continuously differentiable in $t\in[0,T)$, and twice continuously Fréchet differentiable on $L^2(\mathcal{G};\mathbb{R}^d)$) s.t. $\tilde u-\varphi$ has a maximum at $(t,\xi)\in[0,T)\times L^2(\mathcal{G};\mathbb{R}^d)$, one has

$$-\partial_t\varphi(t,\xi) - H\big(\xi, D\varphi(t,\xi), D^2\varphi(t,\xi)\big) \quad\le\quad 0.$$

(ii) $\tilde u(T,\xi)\ge\tilde{\mathbb{E}}^1\big[g(\xi,\mathcal{L}(\xi))\big]$, and for any test function $\varphi\in\mathcal{C}^2([0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ s.t. $\tilde u-\varphi$ has a minimum at $(t,\xi)\in[0,T)\times L^2(\mathcal{G};\mathbb{R}^d)$, one has

$$-\partial_t\varphi(t,\xi) - H\big(\xi, D\varphi(t,\xi), D^2\varphi(t,\xi)\big) \quad\ge\quad 0.$$

Remark 4.4.1. Since the lifted function $\tilde u$ of a smooth solution $u\in\mathcal{C}^2([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ to (4.4.1), may not be smooth in $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$, it says that u cannot be viewed in general as a viscosity solution to (4.4.1) in the sense of Definition 4.4.1 unless we add the extra-assumption that its lifted function is indeed twice continuously Fréchet differentiable on $L^2(\mathcal{G};\mathbb{R}^d)$. Hence, a more natural and intrinsic definition of viscosity solutions would use test functions on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$: in this case, it would be possible to get the viscosity property from the dynamic programming principle and Itô's formula (1.3.7), but as pointed out above, the uniqueness result (and so the characterization) in the Wasserstein space is a challenging issue, beyond the scope of this paper. We have then chosen here to work with test functions on $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$, not necessarily of the lifted form. □

The main result of this section is the viscosity characterization of the value function for the stochastic McKean-Vlasov control problem (4.2.4) to the dynamic programming Bellman equation (4.4.1) (or (4.4.4)).

Theorem 4.4.1. *The value function v is the unique continuous viscosity solution to* (4.4.1) *satisfying a quadratic growth condition* (4.3.5)*.*

Proof. *(1) Viscosity property.* Let us first reformulate the dynamic programming principle (DPP) of Proposition 4.3.1 for the value function viewed now as a function on $[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$. For this, we take a copy $\tilde B$ of B on the probability space $(\tilde\Omega^1,\mathcal{G},\tilde{\mathbb{P}}^1)$, and given $(t,\xi)\in[0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$, $\alpha\in\mathcal{A}$, we consider on $(\tilde\Omega=\Omega^0\times\tilde\Omega^1,\tilde{\mathcal{F}}=\mathcal{F}^0\otimes\mathcal{G},\tilde{\mathbb{P}}=\mathbb{P}^0\otimes\tilde{\mathbb{P}}^1)$ the solution $\tilde X^{t,\xi,\alpha}$, $t\le s\le T$, to the McKean-Vlasov equation

$$\begin{aligned}\tilde X_s^{t,\xi,\alpha} &= \xi + \int_t^s b(\tilde X_r^{t,\xi,\alpha},\tilde{\mathbb{P}}^{W^0}_{\tilde X_r^{t,\xi,\alpha}},\alpha_r)dr + \int_t^s \sigma(\tilde X_r^{t,\xi,\alpha},\tilde{\mathbb{P}}^{W^0}_{\tilde X_r^{t,\xi,\alpha}},\alpha_r)d\tilde B_r \\ &\quad + \int_t^s \sigma_0(\tilde X_r^{t,\xi,\alpha},\tilde{\mathbb{P}}^{W^0}_{\tilde X_r^{t,\xi,\alpha}},\alpha_r)dW_r^0, \quad t\le s\le T,\end{aligned}$$

where $\tilde{\mathbb{P}}^{W^0}_{\tilde{X}^{t,\xi,\alpha}_s}$ denotes the regular conditional distribution of $\tilde{X}^{t,\xi,\alpha}_s$ given $\mathcal{F}^0$. In other words, $\tilde{X}^{t,\xi,\alpha}$ is a copy of $X^{t,\xi,\alpha}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and denoting by $\check{X}^{t,\xi,\alpha}_s(\omega^0) = \tilde{X}^{t,\xi,\alpha}_s(\omega^0, .)$, $t \leq s \leq T$, we see that the process $\{\check{X}^{t,\xi,\alpha}_s, t \leq s \leq T\}$ is $\mathbb{F}^0$-progressive, valued in $L^2(\mathcal{G};\mathbb{R}^d)$, and $\tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,\alpha}_s} = \rho^{t,\mu,\alpha}_s$ for $\mu = \mathcal{L}(\xi)$. Therefore, the lifted value function on $[0,T] \times L^2(\mathcal{G};\mathbb{R}^d)$ identified with the value function on $[0,T] \times \mathcal{P}_2(\mathbb{R}^d)$ satisfies $v(s, \check{X}^{t,\xi,\alpha}_s) = v(s, \rho^{t,\mu,\alpha}_s)$, $t \leq s \leq T$. By noting that $\hat{f}(\rho^{t,\mu,\alpha}_s, \alpha_s) = \tilde{\mathbb{E}}^1\big[f(\tilde{X}^{t,\xi,\alpha}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,\alpha}_s}, \alpha_s)\big]$, we obtain from Proposition 4.3.1 the lifted DPP: for all $(t,\xi) \in [0,T] \times L^2(\mathcal{G};\mathbb{R}^d)$,

$$v(t,\xi) = \inf_{\alpha\in\mathcal{A}} \inf_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \tilde{\mathbb{E}}^1\big[f(\tilde{X}^{t,\xi,\alpha}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,\alpha}_s}, \alpha_s)\big]ds + v(\theta, \check{X}^{t,\xi,\alpha}_\theta)\Big] \tag{4.4.6}$$

$$= \inf_{\alpha\in\mathcal{A}} \sup_{\theta\in\mathcal{T}^0_{t,T}} \mathbb{E}^0\Big[\int_t^\theta \tilde{\mathbb{E}}^1\big[f(\tilde{X}^{t,\xi,\alpha}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,\alpha}_s}, \alpha_s)\big]ds + v(\theta, \check{X}^{t,\xi,\alpha}_\theta)\Big]. \tag{4.4.7}$$

We already know that v is continuous on $[0,T] \times L^2(\mathcal{G};\mathbb{R}^d)$, hence in particular at T, so that $v(T,\xi) = \tilde{\mathbb{E}}^1[g(\xi, \mathcal{L}(\xi))]$, and it remains to derive the viscosity property for the value function in $[0,T) \times L^2(\mathcal{G};\mathbb{R}^d)$ by following standard arguments that we adapt in our context.

(i) Subsolution property. Fix $(t,\xi) \in [0,T) \times L^2(\mathcal{G};\mathbb{R}^d)$, and consider some test function $\varphi \in \mathcal{C}^2([0,T] \times L^2(\mathcal{G};\mathbb{R}^d))$ s.t. $v - \varphi$ has a maximum at (t,ξ), and w.l.o.g. $v(t,\xi) = \varphi(t,\xi)$, so that $v \leq \varphi$. Let a be an arbitrary element in $\mathbf{A}$, $\alpha \equiv a$ the constant control in $\mathcal{A}$ equal to a, and consider the stopping time in $\mathcal{T}^0_{t,T}$: $\theta_h = \inf\{s \geq t : \tilde{\mathbb{E}}^1[|\check{X}^{t,\xi,a}_s - \xi|^2] \geq \delta^2\} \wedge (t+h)$, with $h \in (0, T-t)$, and δ some positive constant small enough (depending on ξ), so that φ and its continuous derivatives $\partial_t\varphi$, $D\varphi$, $D^2\varphi$ are bounded on the ball in $L^2(\mathcal{G};\mathbb{R}^d)$ of center ξ and radius δ. From the first part (4.4.6) of the DPP, we get

$$\varphi(t,\xi) \leq \mathbb{E}^0\Big[\int_t^{\theta_h} \tilde{\mathbb{E}}^1\big[f(\tilde{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a)\big]ds + \varphi(\theta_h, \check{X}^{t,\xi,a}_{\theta_h})\Big].$$

Applying Itô's formula (1.3.9) to $\varphi(s, \check{X}^{t,\xi,a}_s)$, and noting that the stochastic integral w.r.t. W^0 vanishes under expectation $\mathbb{E}^0$ by the localization with the stopping time θ_h, we then have

$$\begin{aligned}
0 \leq\ & \mathbb{E}^0\Big[\frac{1}{h}\int_t^{\theta_h} \partial_t\varphi(s, \check{X}^{t,\xi,a}_s) + \tilde{\mathbb{E}}^1\big[f(\tilde{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a) + D\varphi(s, \check{X}^{t,\xi,a}_s).b(\check{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a) \\
& + \frac{1}{2}D^2\varphi(s, \check{X}^{t,\xi,a}_s)(\sigma(\check{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a)N).\sigma(\check{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a)N \\
& + \frac{1}{2}D^2\varphi(s, \check{X}^{t,\xi,a}_s)(\sigma_0(\check{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a)).\sigma_0(\check{X}^{t,\xi,a}_s, \tilde{\mathbb{P}}^1_{\check{X}^{t,\xi,a}_s}, a)\big]ds\Big] \\
=:\ & \mathbb{E}^0\Big[\frac{1}{h}\int_t^{\theta_h} F_s(t,\xi,a)ds\Big],
\end{aligned} \tag{4.4.8}$$

with $N \in L^2(\mathcal{G};\mathbb{R}^n)$ of zero mean, and unit variance, and independent of $(\tilde{B}, \xi)$. Since the map $s \in [t,T] \mapsto \tilde{\mathbb{E}}^1[\psi(\tilde{X}^{t,\xi,a}_s)] = \mathbb{E}[\psi(X^{t,\xi,a}_s)|\mathcal{F}^0] = \rho^{t,\mu,a}_s(\psi)$ (for $\mu = \mathcal{L}(\xi)$) is continuous $\mathbb{P}^0$-a.s. (recall that $\rho^{t,\mu,\alpha}_s$ is continuous in s), for any bounded continuous function ψ on $\mathbb{R}^d$, we see that the process $\{F_s(t,\xi,a), t \leq s \leq \theta_h\}$ has continuous paths $\mathbb{P}^0$ almost surely. Moreover, by (standard) Itô's formula, we have for all $t \leq s \leq T$,

$$\begin{aligned}
\tilde{\mathbb{E}}^1\big[|\check{X}^{t,\xi,a}_s - \xi|^2\big] = \mathbb{E}\big[|X^{t,\xi,a}_s - \xi|^2|\mathcal{F}^0\big] &= \int_t^s \mathbb{E}\big[2(X^{t,\xi,a}_r - \xi).b_r + \sigma_r\sigma_r^\intercal + \sigma^0_r(\sigma^0_r)^\intercal|\mathcal{F}^0\big]dr \\
&+ \int_t^s \mathbb{E}\big[2(X^{t,\xi,a}_r - \xi)^\intercal\sigma^0_r|\mathcal{F}^0\big]dW^0_r,
\end{aligned}$$

where we set $b_s = b(X_s^{t,\xi,a}, \mathbb{P}^{W^0}_{X_s^{t,\xi,a}}, a)$, $\sigma_s = \sigma(X_s^{t,\xi,a}, \mathbb{P}^{W^0}_{X_s^{t,\xi,a}}, a)$, $\sigma_s^0 = \sigma_0(X_s^{t,\xi,a}, \mathbb{P}^{W^0}_{X_s^{t,\xi,a}}, a)$. This shows that the map $s \in [t,T] \mapsto \tilde{\mathbb{E}}^1[|\check{X}_s^{t,\xi,a} - \xi|^2]$ is continuous $\mathbb{P}^0$-a.s., and thus $\theta_h(\omega^0) = t+h$ for h small enough ($\leq \bar{h}(\omega^0)$), $\mathbb{P}^0(d\omega^0)$-a.s. By the mean-value theorem, we then get $\mathbb{P}^0$ almost surely, $\frac{1}{h}\int_t^{\theta_h} F_s(t,\xi,a)ds \rightarrow F_t(t,\xi,a)$, as h goes to zero, and so from the dominated convergence theorem in (4.4.8):

$$
\begin{aligned}
0 \;\leq\; F_t(t,\xi,a) \;\;=\;\; & \partial_t\varphi(t,\xi) + \tilde{\mathbb{E}}^1\big[f(\xi,\mathcal{L}(\xi),a) + D\varphi(t,\xi).b(\xi,\mathcal{L}(\xi),a) \\
& + \frac{1}{2}D^2\varphi(t,\xi)(\sigma(\xi,\mathcal{L}(\xi),a)N).\sigma(\xi,\mathcal{L}(\xi),a)N \\
& + \frac{1}{2}D^2\varphi(t,\xi)(\sigma_0(\xi,\mathcal{L}(\xi),a)).\sigma_0(\xi,\mathcal{L}(\xi),a)\Big].
\end{aligned}
$$

Since a is arbitrary in $\mathbf{A}$, this shows the required viscosity subsolution property.

(ii) Supersolution property. Fix $(t,\xi) \in [0,T) \times L^2(\mathcal{G};\mathbb{R}^d)$, and consider some test function $\varphi \in \mathcal{C}^2([0,T] \times L^2(\mathcal{G};\mathbb{R}^d))$ s.t. $v - \varphi$ has a minimum at (t,ξ), and w.l.o.g. $v(t,\xi) = \varphi(t,\xi)$, so that $v \geq \varphi$. From the continuity assumptions in **(H1)-(H2)**, we observe that the function $\mathcal{H}$ defined on $[0,T] \times L^2(\mathcal{G};\mathbb{R}^d)$ by

$$
\mathcal{H}(s,\zeta) \;\;:=\;\; H(\zeta, D\varphi(s,\zeta), D^2\varphi(s,\zeta)),
$$

is continuous. Then, given an arbitrary $\varepsilon > 0$, there exists $\bar{h} \in (0, T-t)$, $\delta > 0$ s.t. for all $s \in [t, t+\bar{h}]$, and $\zeta \in L^2(\mathcal{G};\mathbb{R}^d)$ with $\tilde{\mathbb{E}}^1[|\zeta - \xi|^2] \leq \delta$,

$$
\Big|\big(\partial_t\varphi + \mathcal{H}\big)(s,\zeta) - \big(\partial_t\varphi + \mathcal{H}\big)(t,\xi)\Big| \;\;\leq\;\; \varepsilon.
$$

From the second part (4.4.7) of the DPP, for any $h \in (0,\bar{h})$, there exists $\alpha \in \mathcal{A}$ s.t.

$$
\varphi(t,\xi) + \varepsilon h \;\;\geq\;\; \mathbb{E}^0\Big[\int_t^{\theta_h} \tilde{\mathbb{E}}^1\big[f(\check{X}_s^{t,\xi,\alpha}, \tilde{\mathbb{P}}^1_{\check{X}_s^{t,\xi,\alpha}}, \alpha_s)\big]ds \;+\; \varphi(\theta_h, \check{X}_{\theta_h}^{t,\xi,\alpha})\Big],
$$

where we take $\theta_h = \inf\{s \geq t : \tilde{\mathbb{E}}^1[|\check{X}_s^{t,\xi,\alpha} - \xi|^2] \geq \delta^2\} \wedge (t+h)$ (assuming w.l.o.g. that δ is small enough (depending on ξ), so that φ and its continuous derivatives $\partial_t\varphi$, $D\varphi$, $D^2\varphi$ are bounded on the ball in $L^2(\mathcal{G};\mathbb{R}^d)$ of center ξ and radius δ). Applying again Itô's formula (1.3.9) to $\varphi(s, \check{X}_s^{t,\xi,\alpha})$, and by definition of $\mathcal{H}$, we get

$$
\begin{aligned}
\varepsilon \;\;\geq\;\; & \mathbb{E}^0\Big[\frac{1}{h}\int_t^{\theta_h} \big(\partial_t\varphi + \mathcal{H}\big)(s, \check{X}_s^{t,\xi,\alpha})ds\Big] \\
\geq\;\; & \Big[\big(\partial_t\varphi + \mathcal{H}\big)(t,\xi) - \varepsilon\Big]\frac{\mathbb{E}^0[\theta_h] - t}{h},
\end{aligned}
\tag{4.4.9}
$$

by the choice of h, δ, and θ_h. Now, by noting from Chebyshev's inequality that

$$
\begin{aligned}
\mathbb{P}^0[\theta_h < t+h] \;\;\leq\;\; & \mathbb{P}^0\Big[\sup_{t \leq s \leq t+h} \tilde{\mathbb{E}}^1[|\check{X}_s^{t,\xi,\alpha} - \xi|^2] \geq \delta\Big] \\
\leq\;\; & \frac{\mathbb{E}^0\Big[\sup_{t \leq s \leq t+h} \tilde{\mathbb{E}}^1[|\check{X}_s^{t,\xi,\alpha} - \xi|^2]\Big]}{\delta} \;\;\leq\;\; \frac{C(1 + \tilde{\mathbb{E}}^1[|\xi|^2])h}{\delta}
\end{aligned}
$$

and using the obvious inequality: $1 - \mathbb{P}^0[\theta_h < t+h] = \mathbb{P}[\theta_h = t+h] \leq \frac{\mathbb{E}^0[\theta_h]-t}{h} \leq 1$, we see that $\frac{\mathbb{E}^0[\theta_h]-t}{h}$ converges to 1 when h goes to zero, and deduce from (4.4.9) that

$$
2\varepsilon \;\;\geq\;\; \big(\partial_t\varphi + \mathcal{H}\big)(t,\xi).
$$

We obtain the required viscosity supersolution property by sending ε to zero.

(2) Uniqueness property. In view of our definition of viscosity solution, we have to show a comparison principle for viscosity solutions to the lifted Bellman equation (4.4.4). We use the comparison principle proved in Theorem 3.50 in [FGS15] and only need to check that the hypotheses of this theorem are satisfied in our context for the lifted Hamiltonian H defined in (4.4.5). Notice that the Bellman equation (4.4.4) is a bounded equation in the terminology of [FGS15] (see their section 3.3.1) meaning that there is no linear dissipative operator on $L^2(\mathcal{G};\mathbb{R}^d)$ in the equation. Therefore, the notion of B-continuity reduces to the standard notion of continuity in $L^2(\mathcal{G};\mathbb{R}^d)$ since one can take for B the identity operator. Their Hypothesis 3.44 follows from the uniform continuity of b, σ, σ_0 and f in **(H1)**-**(H2)**. Hypothesis 3.45 is immediately satisfied since there is no discount factor in our equation, i.e. H does not depend on v but only on its derivatives. The monotonicity condition in $Q \in S(L^2(\mathcal{G};\mathbb{R}^d))$ of H in Hypothesis 3.46 is clearly satisfied. Hypothesis 3.47 holds directly when dealing with bounded equations. Hypothesis 3.48 is obtained from the Lipschitz condition of b, σ, σ_0 in **(H1)**, and the uniform continuity condition on f in **(H2)**, while Hypothesis 3.49 follows from the growth condition of σ, σ_0 in **(H1)**. One can then apply Theorem 3.50 in [FGS15] and conclude that comparison principle holds for the Bellman equation (4.4.4). □

We conclude this section with a verification theorem, which gives an analytic feedback form of the optimal control when there is a smooth solution to the Bellman equation (4.4.1) in the Wasserstein space. We refer to the recent paper [GŚ15a] for existence result of smooth solution to the Bellman equation on small time horizon.

Theorem 4.4.2. *(Verification theorem)*

Let $w : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a function in $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$, i.e. w is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, $w(t,.) \in \mathcal{C}_c^2(\mathcal{P}_2(\mathbb{R}^d))$, and $w(.,\mu) \in C^1([0,T))$, and satisfying a quadratic growth condition as in (4.3.5), together with a linear growth condition for its derivative:

$$|\partial_\mu w(t,\mu)(x)| \quad \leq \quad C(1+|x|+\|\mu\|_2), \quad \forall (t,x,\mu) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \tag{4.4.10}$$

for some positive constant C. Suppose that w is solution to the Bellman equation (4.4.1), and there exists for all $(t,\mu) \in [0,T)\times\mathcal{P}_2(\mathbb{R}^d)$ an element $\hat{a}(t,\mu) \in \mathbf{A}$ attaining the infimum in (4.4.1) s.t. the map $(t,\mu) \mapsto \hat{a}(t,\mu)$ is measurable, and the stochastic McKean-Vlasov equation

$$\begin{aligned} d\hat{X}_s \quad &= \quad b(\hat{X}_s, \mathbb{P}^{W^0}_{\hat{X}_s}, \hat{a}(s,\mathbb{P}^{W^0}_{\hat{X}_s}))ds + \sigma(\hat{X}_s, \mathbb{P}^{W^0}_{\hat{X}_s}, \hat{a}(s,\mathbb{P}^{W^0}_{\hat{X}_s}))dB_s \\ &\quad + \sigma_0(\hat{X}_s, \mathbb{P}^{W^0}_{\hat{X}_s}, \hat{a}(s,\mathbb{P}^{W^0}_{\hat{X}_s}))dW^0_s, \quad t \leq s \leq T, \ \hat{X}_t = \xi, \end{aligned}$$

admits a unique solution denoted $(\hat{X}_s^{t,\xi})_{t\leq s\leq T}$, for any $(t,\xi) \in [0,T]\times L^2(\mathcal{G};\mathbb{R}^d)$ (This is satisfied e.g. when $\mu \mapsto \hat{a}(t,\mu)$ is Lipschitz on $\mathcal{P}_2(\mathbb{R}^d)$). Then, $w = v$, and the feedback control $\alpha^* \in \mathcal{A}$ defined by

$$\alpha^*_s \quad = \quad \hat{a}(s, \mathbb{P}^{W^0}_{\hat{X}_s^{t,\xi}}), \quad t \leq s < T, \tag{4.4.11}$$

is an optimal control for $v(t,\mu)$, i.e. $v(t,\mu) = J(t,\mu,\alpha^*)$, with $\mu = \mathcal{L}(\xi)$.

Proof. Fix $(t,\mu = \mathcal{L}(\xi)) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, and consider some arbitrary control $\alpha \in \mathcal{A}$ associated to $\rho_s^{t,\mu,\alpha} = \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}$, $t \leq s \leq T$. Denote by $X_s^{\prime t,\xi,\alpha}$ a copy of $X_s^{t,\xi,\alpha}$ on another probability space $(\Omega' = \Omega^0 \times \Omega'^1\,\mathcal{F}^0 \otimes \mathcal{F}'^1, \mathbb{P}^0 \times \mathbb{P}'^1)$, with $(\Omega'^1, \mathcal{F}'^1, \mathbb{P}'^1)$ supporting B' a copy of B. Applying Itô's formula

(1.3.7) to $w(s,\rho_s^{t,\mu,\alpha})$ between t and the $\mathbb{F}^0$-stopping time $\theta_T^n = \inf\{s \geq t : \|\rho_s^{t,\mu,\alpha}\|_2 \geq n\} \wedge T$, we obtain

$$\begin{aligned}
& w(\theta_T^n, \rho_{\theta_T^n}^{t,\mu,\alpha}) \\
= \;& w(t,\mu) + \int_t^{\theta_T^n} \Big\{ \frac{\partial w}{\partial t}(s,\rho_s^{t,\mu,\alpha}) + \mathbb{E}_{W^0}\Big[\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha}).b(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s) \\
& + \frac{1}{2}\mathrm{tr}\big[\partial_x\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha})(\sigma\sigma^\intercal(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s) + \sigma_0\sigma_0^\intercal(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s))\big]\Big] \\
& + \mathbb{E}_{W^0}\Big[\mathbb{E}'_{W^0}\big[\frac{1}{2}\mathrm{tr}\big(\partial_\mu^2 w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha},X_s'^{t,\xi,\alpha})\sigma_0(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s)\sigma_0^\intercal(X_s'^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s)\big)\big]\Big]\Big\}ds \\
& + \int_t^{\theta_T^n} \mathbb{E}_{W^0}\big[\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha})^\intercal\sigma_0(X_s^{t,\xi,\alpha},\rho_s^{t,\xi,\alpha},\alpha_s)\big]dW_s^0 \\
= \;& w(t,\mu) + \int_t^{\theta_T^n} \Big[\frac{\partial w}{\partial t}(s,\rho_s^{t,\mu,\alpha}) + \rho_s^{t,\mu,\alpha}\big(\text{Ł}^{\alpha_s} w(s,\rho_s^{t,\mu,\alpha})\big) + \rho_s^{t,\mu,\alpha}\otimes\rho_s^{t,\mu,\alpha}\big(\mathbb{M}^{\alpha_s} w(s,\rho_s^{t,\mu,\alpha})\big)\Big]ds \\
& + \int_t^{\theta_T^n} \mathbb{E}_{W^0}\big[\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha})^\intercal\sigma_0(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s)\big]dW_s^0, \qquad (4.4.12)
\end{aligned}$$

by definition of Ł^a and $\mathbb{M}^a$ in (4.4.2)-(4.4.3), and recalling again that $\rho_s^{t,\mu,\alpha} = \mathbb{P}^{W^0}_{X_s^{t,\xi,\alpha}}$. Now, the integrand of the stochastic integral w.r.t. W^0 in (4.4.12) satisfies:

$$\begin{aligned}
& \Big|\mathbb{E}_{W^0}\big[\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(X_s^{t,\xi,\alpha})^\intercal\sigma_0(X_s^{t,\xi,\alpha},\rho_s^{t,\mu,\alpha},\alpha_s)\big]\Big|^2 \\
\leq \;& \Big(\int_{\mathbb{R}^d}\big|\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(x)^\intercal\sigma_0(x,\rho_s^{t,\mu,\alpha},\alpha_s)\big|\rho_s^{t,\mu,\alpha}(dx)\Big)^2 \\
\leq \;& \int_{\mathbb{R}^d}\big|\partial_\mu w(s,\rho_s^{t,\mu,\alpha})(x)\big|^2\rho_s^{t,\mu,\alpha}(dx)\int_{\mathbb{R}^d}\big|\sigma_0(x,\rho_s^{t,\mu,\alpha},\alpha_s)\big|^2\rho_s^{t,\mu,\alpha}(dx) \\
\leq \;& C(1+n^2)^2 \;<\; \infty, \qquad t \leq s \leq \theta_T^n,
\end{aligned}$$

from Cauchy-Schwarz inequality, the linear growth condition of σ_0 in **(H1)**, the choice of θ_T^n, and condition (4.4.10). Therefore, the stochastic integral in (4.4.12) vanishes in $\mathbb{E}^0$-expectation, and we get

$$\begin{aligned}
\mathbb{E}^0\big[w(\theta_T^n,\rho_{\theta_T^n}^{t,\mu,\alpha})\big] \;&=\; w(t,\mu) \;+\; \mathbb{E}^0\Big[\int_t^{\theta_T^n}\frac{\partial w}{\partial t}(s,\rho_s^{t,\mu,\alpha}) + \rho_s^{t,\mu,\alpha}\big(\text{Ł}^{\alpha_s}w(s,\rho_s^{t,\mu,\alpha})\big) \\
& \qquad + \rho_s^{t,\mu,\alpha}\otimes\rho_s^{t,\mu,\alpha}\big(\mathbb{M}^{\alpha_s}w(s,\rho_s^{t,\mu,\alpha})\big)ds\Big] \\
&\geq\; w(t,\mu) \;-\; \mathbb{E}^0\Big[\int_t^{\theta_T^n}\hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds\Big], \qquad (4.4.13)
\end{aligned}$$

since w satisfies the Bellman equation (4.4.1). By sending n to infinity into (4.4.13), and from the dominated convergence theorem (under the condition that w, f satisfy a quadratic growth condition and recalling the estimation (4.3.10)), we obtain:

$$w(t,\mu) \;\leq\; J(t,\mu,\alpha) \;=\; \mathbb{E}^0\Big[\int_t^T \hat{f}(\rho_s^{t,\mu,\alpha},\alpha_s)ds + \hat{g}(\rho_T^{t,\mu,\alpha})\Big].$$

Since α is arbitrary in $\mathcal{A}$, this shows that $w \leq v$.

Finally, by applying the same Itô's argument with the feedback control $\alpha^* \in \mathcal{A}$ in (4.4.11), and noting that $\hat{X}_s^{t,\xi} = X_s^{t,\xi,\alpha^*}$, $\mathbb{P}^{W^0}_{\hat{X}_s^{t,\xi}} = \rho_s^{t,\mu,\alpha^*}$, we have now equality in (4.4.13), hence $w(t,\mu) = J(t,\mu,\alpha^*)$ ($\geq v(t,\mu)$), and thus finally the required equality: $w(t,\mu) = v(t,\mu) = J(t,\mu,\alpha^*)$. □

4.5 Linear quadratic stochastic McKean-Vlasov control

We consider the linear-quadratic (LQ) stochastic McKean-Vlasov control problem where the control set $\mathbf{A}$ is a functional space, which corresponds to the McKean-Vlasov problem with common noise as presented in the introduction.

The control set $\mathbf{A}$ is the set $L(\mathbb{R}^d;\mathbb{R}^m)$ of Lipschitz functions from $\mathbb{R}^d$ into $A=\mathbb{R}^m$, and we consider a multivariate linear McKean-Vlasov controlled dynamics with coefficients given by

$$\begin{array}{rcl} b(x,\mu,a) & = & b_0 + Bx + \bar B\bar\mu + Ca(x), \\ \sigma(x,\mu,a) & = & \vartheta + Dx + \bar D\bar\mu + Fa(x), \\ \sigma_0(x,\mu,a) & = & \vartheta_0 + D_0x + \bar D_0\bar\mu + F_0a(x), \end{array} \tag{4.5.1}$$

for $(x,\mu,a)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times L(\mathbb{R}^d;\mathbb{R}^m)$, where we set

$$\bar\mu \;:=\; \int_{\mathbb{R}^d} x\mu(dx).$$

Here B, $\bar B$, D, $\bar D$, D_0, $\bar D_0$, are constant matrices in $\mathbb{R}^{d\times d}$, C, F, F_0 are constant matrices in $\mathbb{R}^{d\times m}$, and b_0, ϑ, ϑ_0 are constant vectors in $\mathbb{R}^d$. The quadratic cost functions are given by

$$\begin{array}{rcl} f(x,\mu,a) & = & x^\intercal Q_2x + \bar\mu^\intercal\bar Q_2\bar\mu + a(x)^\intercal R_2a(x) \\ g(x,\mu) & = & x^\intercal P_2x + \bar\mu^\intercal\bar P_2\bar\mu, \end{array} \tag{4.5.2}$$

where Q_2, $\bar Q_2$, P_2, $\bar P_2$ are constant matrices in $\mathbb{R}^{d\times d}$, R_2 is a constant matrix in $\mathbb{R}^{m\times m}$. Since f and g are real-valued, we may assume w.l.o.g. that all the matrices Q_2, $\bar Q_2$, R_2, P_2, $\bar P_2$ are symmetric. We denote by $\mathbb{S}^d$ the set of symmetric matrices in $\mathbb{R}^{d\times d}$, by $\mathbb{S}^d_+$ the subset of nonnegative symmetric matrices, by $\mathbb{S}^d_{>+}$ the subset of symmetric positive definite matrices, and similarly for $\mathbb{S}^m$, $\mathbb{S}^m_+$, $\mathbb{S}^m_{>+}$.

The functions $\hat f$ and $\hat g$ defined in (4.3.4) are then given by

$$\left\{\begin{array}{rcl} \hat f(t,\mu,a) & = & \mathrm{Var}(\mu)(Q_2) + \bar\mu^\intercal(Q_2+\bar Q_2)\bar\mu + \overline{a\star\mu}_2(R_2) \\ \hat g(\mu) & = & \mathrm{Var}(\mu)(P_2) + \bar\mu^\intercal(P_2+\bar P_2)\bar\mu \end{array}\right. \tag{4.5.3}$$

for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $a\in\mathbf{A}=L(\mathbb{R}^d;\mathbb{R}^m)$, where we set for any Λ in $\mathbb{S}^d$ (resp. in $\mathbb{S}^m$), and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ (resp. $\mathcal{P}_2(\mathbb{R}^m)$):

$$\bar\mu_2(\Lambda) \;:=\; \int x^\intercal\Lambda x\mu(dx), \qquad \mathrm{Var}(\mu)(\Lambda) \;:=\; \bar\mu_2(\Lambda) - \bar\mu^\intercal\Lambda\bar\mu,$$

and $a\star\mu\in\mathcal{P}_2(\mathbb{R}^m)$ is the image by $a\in L(\mathbb{R}^d;\mathbb{R}^m)$ of the measure $\mu\in\mathbb{R}^m$, so that

$$\overline{a\star\mu} \;=\; \int_{\mathbb{R}^d} a(x)\mu(dx), \qquad \overline{a\star\mu}_2(\Lambda) \;:=\; \int a(x)^\intercal\Lambda a(x)\mu(dx).$$

We look for a value function solution to the Bellman equation (4.4.1) in the form

$$w(t,\mu) \;=\; \mathrm{Var}(\mu)(\Lambda(t)) + \bar\mu^\intercal\Gamma(t)\bar\mu + \bar\mu^\intercal\gamma(t) + \chi(t), \tag{4.5.4}$$

for some functions $\Lambda,\Gamma\in C^1([0,T];\mathbb{S}^d)$, $\gamma\in C^1([0,T];\mathbb{R}^d)$, and $\chi\in C^1([0,T];\mathbb{R})$. One easily checks that w lies in $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ with

$$\begin{array}{rcl} \partial_tw(t,\mu) & = & \mathrm{Var}(\mu)(\Lambda'(t)) + \bar\mu^\intercal\Gamma'(t)\bar\mu + \gamma'(t)\bar\mu + \chi'(t), \\ \partial_\mu w(t,\mu)(x) & = & 2\Lambda(t)(x-\bar\mu) + 2\Gamma(t)\bar\mu + \gamma(t), \\ \partial_x\partial_\mu w(t,\mu)(x) & = & 2\Lambda(t), \\ \partial_\mu^2 w(t,\mu)(x,x') & = & 2(\Gamma(t)-\Lambda(t)). \end{array}$$

Together with the quadratic expression (4.5.3) of $\hat{f}$, $\hat{g}$, we then see after some tedious but direct calculations that w satisfies the Bellman equation (4.4.1) iff

$$\begin{aligned} & \mathrm{Var}(\mu)(\Lambda(T)) + \bar{\mu}^\intercal \Gamma(T)\bar{\mu} + \bar{\mu}^\intercal \gamma(T) + \chi(T) \\ = \quad & \mathrm{Var}(\mu)(P_2) + \bar{\mu}^\intercal (P_2 + \bar{P}_2)\bar{\mu}, \end{aligned} \tag{4.5.5}$$

holds for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and

$$\begin{aligned} & \mathrm{Var}(\mu)\big(\Lambda'(t) + Q_2 + D^\intercal \Lambda(t) D + D_0^\intercal \Lambda(t) D_0 + \Lambda(t) B + B^\intercal \Lambda(t)\big) + \inf_{a \in L(\mathbb{R}^d;\mathbb{R}^m)} G_t^\mu(a) \\ & + \bar{\mu}^\intercal \Big(\Gamma'(t) + Q_2 + \bar{Q}_2 + (D+\bar{D})^\intercal \Lambda(t)(D+\bar{D}) \\ & \quad + (D_0+\bar{D}_0)^\intercal \Gamma(t)(D_0+\bar{D}_0) + \Gamma(t)(B+\bar{B}) + (B + \bar{B})^\intercal \Gamma(t)\Big)\bar{\mu} \\ & + \bar{\mu}^\intercal \big(\gamma'(t) + (B+\bar{B})^\intercal \gamma(t) + 2(D+\bar{D})^\intercal \Lambda(t)\vartheta + 2(D_0+\bar{D}_0)^\intercal \Gamma(t)\vartheta_0 + 2\Gamma(t) b_0\big) \\ & \quad + \chi'(t) + \gamma(t)^\intercal b_0 + \vartheta^\intercal \Lambda(t)\vartheta + \vartheta_0{}^\intercal \Gamma(t)\vartheta_0 \\ = \quad & 0, \end{aligned} \tag{4.5.6}$$

holds for all $t \in [0,T)$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, where the function $G_t^\mu : L(\mathbb{R}^d;\mathbb{R}^m) \to \mathbb{R}$ is defined by

$$\begin{aligned} G_t^\mu(a) \quad = \quad & \mathrm{Var}(a \star \mu)(U_t) + \overline{a \star \mu}^\intercal V_t \overline{a \star \mu} + 2\int_{\mathbb{R}^d} (x-\bar{\mu})^\intercal S_t a(x)\mu(dx) \\ & + 2\bar{\mu}^\intercal Z_t \overline{a \star \mu} + Y_t.\overline{a \star \mu}, \end{aligned}$$

and we set $U_t = U(t,\Lambda(t))$, $V_t = V(t,\Lambda(t),\Gamma(t))$, $S_t = S(t,\Lambda(t))$, $Z_t = Z(t,\Lambda(t),\Gamma(t))$, $Y_t = Y(t,\Gamma(t),\gamma(t))$ with

$$\left\{ \begin{aligned} U(t,\Lambda(t)) &= F^\intercal \Lambda(t) F + F_0^\intercal \Lambda(t) F_0 + R_2, \\ V(t,\Lambda(t),\Gamma(t)) &= F^\intercal \Lambda(t) F + F_0^\intercal \Gamma(t) F_0 + R_2 \\ S(t,\Lambda(t)) &= D^\intercal \Lambda(t) F + D_0^\intercal \Lambda(t) F_0 + \Lambda(t) C + M_2, \\ Z(t,\Lambda(t),\Gamma(t)) &= (D+\bar{D})^\intercal \Lambda(t) F + (D_0+\bar{D}_0)^\intercal \Gamma(t) F + \Gamma(t) C + M_2 \\ Y(t,\Gamma(t),\gamma(t)) &= C^\intercal \gamma(t) + 2F^\intercal \Lambda(t)\vartheta + 2F_0^\intercal \Gamma(t)\vartheta_0. \end{aligned} \right. \tag{4.5.7}$$

Then, under the condition that the symmetric matrices U_t and V_t in (4.5.7) are positive, hence invertible (this will be discussed later on), we get after square completion:

$$\begin{aligned} G_t^\mu(a) \quad = \quad & \mathrm{Var}((a - a^*(t,.,\mu)) \star \mu)(U_t) + \overline{(a - a^*(t,.,\mu)) \star \mu}^\intercal V_t \overline{(a - a^*(t,.,\mu)) \star \mu} \\ & - \mathrm{Var}(\mu)\big(S_t U_t^{-1} S_t^\intercal\big) - \bar{\mu}^\intercal \big(Z_t V_t^{-1} Z_t^\intercal\big)\bar{\mu} - Y_t^\intercal V_t^{-1} Z_t^\intercal \bar{\mu} - \frac{1}{4} Y_t^\intercal V_t^{-1} Y_t. \end{aligned}$$

where $a(t,.,\mu) \in L(\mathbb{R}^d;\mathbb{R}^m)$ is given by

$$a^*(t,x,\mu) \quad = \quad -U_t^{-1} S_t^\intercal (x - \bar{\mu}) - V_t^{-1} Z_t^\intercal \bar{\mu} - \frac{1}{2} V_t^{-1} Y_t. \tag{4.5.8}$$

This means that G_t^μ attains its infimum at $a^*(t,.,\mu)$, and plugging the above expression of $G_t^\mu(a^*(t,.,\mu))$ in (4.5.6), we observe that the relation (4.5.5)-(4.5.6), hence the Bellman equation, is satisfied by identifying the terms in Var(.), $\bar{\mu}^\intercal(.)\bar{\mu}$, $\bar{\mu}$, which leads to the system of ordinary differential equations (ODEs) for $(\Lambda,\Gamma,\gamma,\chi)$:

$$\left\{ \begin{aligned} & \Lambda'(t) + Q_2 + D^\intercal \Lambda(t) D + D_0^\intercal \Lambda(t) D_0 + \Lambda(t) B + B^\intercal \Lambda(t) \\ & \qquad - S(t,\Lambda(t)) U(t,\Lambda(t))^{-1} S(t,\Lambda(t))^\intercal = 0, \\ & \qquad \Lambda(T) = P_2, \end{aligned} \right. \tag{4.5.9}$$

$$\left\{\begin{array}{rcl}\Gamma'(t)+Q_2+\bar{Q}_2+(D+\bar{D})^{\intercal}\Lambda(t)(D+\bar{D}) & & \\ +(D_0+\bar{D}_0)^{\intercal}\Gamma(t)(D_0+\bar{D}_0)+\Gamma(t)^{\intercal}(B+\bar{B}) & & \\ +(B+\bar{B})^{\intercal}\Gamma(t)-Z(t,\Lambda(t),\Gamma(t))V(t,\Lambda(t),\Gamma(t))^{-1}Z(t,\Lambda(t),\Gamma(t))^{\intercal} & = & 0, \\ \Gamma(T) & = & P_2+\bar{P}_2,\end{array}\right. \tag{4.5.10}$$

$$\left\{\begin{array}{rcl}\gamma'(t)+\big(B+\bar{B}\big)^{\intercal}\gamma(t)-Z(t,\Lambda(t),\Gamma(t))V(t,\Lambda(t),\Gamma(t))^{-1}Y(t,\Gamma(t),\gamma(t)) & & \\ +2\big(D+\bar{D}\big)^{\intercal}\Lambda(t)\vartheta+2\big(D_0+\bar{D}_0\big)^{\intercal}\Gamma(t)\vartheta_0+2\Gamma(t)b_0 & = & 0, \\ \gamma(T) & = & 0\end{array}\right. \tag{4.5.11}$$

$$\left\{\begin{array}{rcl}\chi'(t)-\frac{1}{4}Y(t,\Gamma(t),\gamma(t))^{\intercal}V(t,\Lambda(t),\Gamma(t))^{-1}Y(t,\Gamma(t),\gamma(t)) & & \\ +\gamma(t)^{\intercal}b_0+\vartheta^{\intercal}\Lambda(t)\vartheta+\vartheta_0^{\intercal}\Gamma(t)\vartheta_0 & = & 0, \\ \chi(T) & = & 0.\end{array}\right. \tag{4.5.12}$$

Therefore, the resolution of the Bellman equation in the LQ framework is reduced to the resolution of the Riccati equations (4.5.9) and (4.5.10) for Λ and Γ, and then given (Λ,Γ), to the resolution of the linear ODEs (4.5.11) and (4.5.12) for γ and χ. Suppose that there exists a solution $(\Lambda,\Gamma)\in C^1([0,T];\mathbb{S}^d)\times C^1([0,T];\mathbb{S}^d)$ to (4.5.9)-(4.5.10) s.t. (U_t,V_t) in (4.5.7) lies in $\mathbb{S}^m_{>+}\times\mathbb{S}^m_{>+}$ for all $t\in[0,T]$ (see Remark 4.5.1). Then, the above calculations are justified a posteriori, and by noting also that the mapping $(x,\mu)\mapsto a^*(t,x,\mu)$ is Lipschitz on $\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, we deduce by the verification theorem that the value function v is equal to w in (4.5.4) with $(\Lambda,\Gamma,\gamma,\chi)$ solution to (4.5.9)-(4.5.10)-(4.5.11)-(4.5.12). Moreover, the optimal control is given in feedback form from (4.5.8) by

$$\begin{aligned}\alpha_t^*(X_t^*) &= a^*(t,X_t^*,\mathbb{P}^{W^0}_{X_t^*}) \\ &= -U_t^{-1}S_t^{\intercal}\big(X_t^*-\mathbb{E}[X_t^*|\mathcal{F}_t^0]\big)-V_t^{-1}Z_t^{\intercal}\mathbb{E}[X_t^*|\mathcal{F}_t^0]-\frac{1}{2}V_t^{-1}Y_t,\end{aligned}\tag{4.5.13}$$

where X^* is the state process controlled by α^*.

Remark 4.5.1. It is known from [Won68] that under the condition

$$P_2\ \geq\ 0,\ P_2+\bar{P}_2\ \geq\ 0,\qquad Q_2\ \geq\ 0,\ Q_2+\bar{Q}_2\ \geq\ 0,\quad R_2\ \geq\delta I_m, \tag{4.5.14}$$

for some $\delta>0$, the matrix Riccati equations (4.5.9)-(4.5.10) admit unique solutions $(\Lambda,\Gamma)\in\mathcal{C}^1([0,T];\mathbb{S}^d_+)\times\mathcal{C}^1([0,T];\mathbb{S}^d_+)$, and then U_t,V_t in (4.5.7) are symmetric positive definite matrices, i.e. lie in $\mathbb{S}^m_{>+}$ for all $t\in[0,T]$. The expression in (4.5.13) of the optimal control extends then to the case of stochastic LQ McKean-Vlasov control problem the feedback form obtained in [Yon13] for LQ McKean-Vlasov without common noise, i.e. $\sigma_0=0$. □

Example: Interbank systemic risk model

We consider a model of interbank borrowing and lending studied in [CFS15] where the log-monetary reserve of each bank in the asymptotics when the number of banks tend to infinity, is governed by the McKean-Vlasov equation:

$$\begin{aligned}dX_t &= \big[\kappa(\mathbb{E}[X_t|W^0]-X_t)+\alpha_t(X_t)\big]dt \\ &\quad +(\sigma_0+\sigma_1X_t)(\sqrt{1-\rho^2}dB_t+\rho dW_t^0),\ X_0\ =\ x_0\in\mathbb{R}.\end{aligned}\tag{4.5.15}$$

Here, $\kappa\geq 0$ is the rate of mean-reversion in the interaction from borrowing and lending between the banks, $\sigma_0>0$, $\sigma_1\in\mathbb{R}$ are the affine coefficients of the volatility of the bank reserve, and there is a common noise W^0 for all the banks. This is a slight extension of the model considered in [CFS15] where

$\sigma_1 = 0$. Moreover, all banks can control their rate of borrowing/lending to a central bank with the same feedback policy α in order to minimize a cost functional of the form

$$\begin{aligned} J(\alpha) &= \mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}\alpha_t(X_t)^2 - q\alpha_t(X_t)(\mathbb{E}[X_t|W^0] - X_t) + \frac{\eta}{2}(\mathbb{E}[X_t|W^0] - X_t)^2\Big)dt \\ &\quad + \frac{c}{2}(\mathbb{E}[X_T|W^0] - X_T)^2\Big], \end{aligned}$$

where $q > 0$ is a positive parameter for the incentive to borrowing ($\alpha_t > 0$) or lending ($\alpha_t < 0$), and $\eta > 0$, $c > 0$ are positive parameters for penalizing departure from the average. After square completion, we can rewrite the cost functional as

$$J(\alpha) = \mathbb{E}\Big[\int_0^T \Big(\frac{1}{2}\tilde{\alpha}_t(X_t)^2 + \frac{\eta - q^2}{2}(\mathbb{E}[X_t|W^0] - X_t)^2\Big)dt + \frac{c}{2}(\mathbb{E}[X_T|W^0] - X_T)^2\Big],$$

with $\tilde{\alpha}_t(X_t) = \alpha_t(X_t) - q(\mathbb{E}[X_t|W^0] - X_t)$. This model fits into the framework of (4.5.1)-(4.5.2) of the LQ stochastic McKean-Vlasov problem with

$$\begin{aligned} b_0 = 0,\ B = -(\kappa + q),\ \bar{B} = \kappa + q,\ C = 1, \\ D = \sigma_1\sqrt{1-\rho^2}, D_0 = \sigma_1\rho,\ \bar{D} = F = \bar{D}^0 = F^0 = 0,\ \vartheta = \sigma_0\sqrt{1-\rho^2},\ \vartheta_0 = \sigma_0\rho, \\ Q_2 = \frac{\eta - q^2}{2},\ \bar{Q}_2 = -\frac{\eta - q^2}{2},\ R_2 = \frac{1}{2}, P_2 = \frac{c}{2},\ \bar{P}_2 = -\frac{c}{2}. \end{aligned}$$

The Riccati system (4.5.9)-(4.5.10)-(4.5.11)-(4.5.12) for $(\Lambda(t), \Gamma(t), \gamma(t), \chi(t))$ is written in this case as

$$\begin{cases} \Lambda'(t) - 2(\kappa + q - \frac{\sigma_1^2}{2})\Lambda(t) - 2\Lambda^2(t) + \frac{1}{2}(\eta - q^2) = 0, & \Lambda(T) = \frac{c}{2}, \\ \Gamma'(t) - 2\Gamma^2(t) + \sigma_1^2\rho^2\Gamma(t) + \sigma_1^2(1-\rho^2)\Lambda(t) = 0, & \Gamma(T) = 0, \\ \gamma'(t) - 2\Gamma(t)\gamma(t) + 2\sigma_0\sigma_1\rho^2\Gamma(t) + 2\sigma_0\sigma_1(1-\rho^2)\Lambda(t) = 0, & \gamma(T) = 0, \\ \chi'(t) - \frac{1}{2}\gamma^2(t) + \sigma_0^2\rho^2\Gamma(t) + \sigma_0^2(1-\rho^2)\Lambda(t) = 0, & \chi(T) = 0. \end{cases} \tag{4.5.16}$$

Assuming that $q^2 \le \eta$, the explicit solution to the Riccati equation for Λ is given by

$$\Lambda(t) = \frac{1}{2}\frac{(\eta - q^2)\big(e^{(\delta^+ - \delta^-)(T-t)} - 1\big) + c\big(\delta^+ e^{(\delta^+ - \delta^-)(T-t)} - \delta^-\big)}{c\big(e^{(\delta^+ - \delta^-)(T-t)} - 1\big) + \delta^+ - \delta^- e^{(\delta^+ - \delta^-)(T-t)}} > 0,$$

where we set

$$\delta^{\pm} = -\big(\kappa + q - \frac{\sigma_1^2}{2}\big) \pm \sqrt{\big(\kappa + q - \frac{\sigma_1^2}{2}\big)^2 + \eta - q^2}.$$

Since $\Lambda \ge 0$, there exists a unique solution to the Riccati equation for Γ, and then γ, and finally χ are determined the linear ordinary differential equations in (4.5.16). Moreover, the functions (U_t, V_t, Z_t, Y_t) in (4.5.7) are explicitly given by: $U_t = V_t = \frac{1}{2}$ (hence > 0), $S_t = \Lambda(t) + \frac{q}{2}$, $Z_t = \Gamma(t)$, $Y_t = \gamma(t)$. Therefore, the optimal control is given in feedback form from (4.5.13) by

$$\begin{aligned} \alpha_t^*(X_t^*) &= a^*(t, X_t^*, \mathbb{P}_{X_t^*}) \\ &= -(2\Lambda(t) + q)(X_t^* - \mathbb{E}[X_t^*|W^0]) - 2\Gamma(t)\mathbb{E}[X_t^*|W^0] - \gamma(t), \end{aligned} \tag{4.5.17}$$

where X^* is the optimal log-monetary reserve controlled by the rate of borrowing/lending α^*. Moreover, denoting by $\bar{X}_t^* = \mathbb{E}[X_t^*|W^0]$ the conditional mean of the optimal log monetary reserve, we see that $\mathbb{E}[\alpha_t^*(X_t^*)|W^0] = -2\Gamma(t)\bar{X}_t^* - \gamma(t)$, and thus $\bar{X}^*$ is given from (4.5.15) by

$$d\bar{X}_t^* = -\big(2\Gamma(t)\bar{X}_t^* + \gamma(t)\big)dt + (\sigma_1\bar{X}_t^* + \sigma_0)\rho dW_t^0, \quad \bar{X}_0^* = x_0.$$

When $\sigma_1 = 0$, we have $\Gamma(t) = \gamma(t) = 0$, hence $\bar{X}_t^* = x_0 + \sigma_0 \rho W_t^0$, and we retrieve the expression found in [CFS15] by sending the number of banks N to infinity in their formula for the optimal control of the borrowing/lending rate:

$$\alpha_t^*(X_t^*) \quad = \quad -(2\Lambda(t) + q)(X_t^* - x_0 - \sigma \rho W_t^0), \quad 0 \leq t \leq T.$$

Part II

Robust mean-variance problem under model uncertainty

Chapter 5

Portfolio diversification and model uncertainty: a robust dynamic mean-variance approach [a]

Abstract: This paper is concerned with a multi-asset mean-variance portfolio selection problem under model uncertainty. We develop a continuous time framework for taking into account ambiguity aversion about both expected rate of return and correlation matrix of stocks, and for studying the effects on portfolio diversification. We prove a separation principle for the associated robust control problem, which allows to reduce the determination of the optimal dynamic strategy to the parametric computation of the minimal risk premium function. Our results provide a justification for under-diversification, as documented in empirical studies, and that we explicitly quantify in terms of correlation and Sharpe ratio ambiguity parameters. In particular, we show that an investor with a poor confidence in the expected return estimation does not hold any risky asset, and on the other hand, trades only one risky asset when the level of ambiguity on correlation matrix is large. This extends to the continuous-time setting the results obtained by Garlappi, Uppal and Wang [GUW06], and Liu and Zeng [LZ17] in a one-period model.

Key words: Continuous-time Markowitz problem, model uncertainty, ambiguous drift and correlation, separation principle, portfolio diversification.

a. This chapter is based on joint work with Pham Huyên and Zhou Chao [PWZ18]. The preprint is submitted and available at arXiv: arXiv1809.01464, 2018.

5.1 Introduction

In the Finance and Economics literature, there are many studies on under-diversification, i.e. when investors hold only a small part of risky assets among a large number of available risky assets. In the extreme case the anti-diversification means that investor holds only a single asset(or even do not hold any risky asset and exclude many others). Empirical studies reported in numerous papers, [FP91], [CK94], [MV07], [CCS07], [GL16], have shown the evidence of the under-diversification in practice. For example, in [FP91], [CK94], it's observed that there exists concentration on(bias towards) the domestic assets compared to foreign assets in investors' international equity portfolios. These results are in contrast to the portfolio well-diversification suggested by the classical mean-variance portfolio theory initialed in single period [Mar52], later in a continuous time model [LZ00].

A possible explanation to under-diversification is provided in the Finance and Economics literature by model uncertainty, often also called ambiguity or Knightian uncertainty. In the classical portfolio theory, the model and the parameters are assumed to be perfectly known. However, in reality, due to statistical estimation errors. there is always ambiguity about the model or the parameters, see e.g. [Tal09], It's well known that robust approach is very notable to address the model uncertainty, where the investor takes portfolio decisions under the worst-case scenario that corresponds to the least favorable distribution implied by the set of ambiguous parameters.

Abundant research has been conducted to tackle different model uncertainty. Related works include [GUW06], [Sch07], [BGUW12], [JZ15] among others for uncertainty about solely drift with a family of dominated probability measure, [MPZ15] for ambiguity about volatility or equivalent covariance matrix with a family of nondominated probability measures in a probabilistic setup, and [TTU13], [LR14], [BK17], [NN18] for combined uncertainty about both drift and volatility, and also [GX13] for uncertainty about probability law generating market data. It's usually assumed that the covariance matrix falls into the region $[\underline{\Sigma}, \bar{\Sigma}]$ in the matrix sense. The worst-case scenario for ambiguity on the covariance matrix is upper bound of covariance matrix $\bar{\Sigma}$. Alternatively, the ambiguity on covariance matrix can be characterized in terms of correlation with marginal volatility known. It is known that the estimation of correlation between assets may be extremely inaccurate, due to the asynchronous data and lead-lag effect, especially when the number of assets is large, see [JM03], [LW04]. We are aware of only a few results on correlation ambiguity [FPW16], [LZ17], [IP17], in which they have shown that the worst-case scenario for ambiguity on the correlation depends upon the correlation parameter. In the above cited papers, only a few work connects the model uncertainty with portfolio diversification. We mention the work [BGUW12], [UW03], [LZ17]. The authors in [BGUW12], [UW03] considered the ambiguity about the assets' rates of return. Their framework allows both for uncertainty about the joint distribution of returns for all assets and for different levels of uncertainty for the marginal distribution of returns for any subsets of these assets. They showed that the different levels of uncertainty on different asset subclass could result in significant under-diversification. They also applied their theoretical results to real data and found consistent results with the empirical studies by [CK94] and [FP91] among others, showing that international equity portfolios are strongly biased toward domestic stocks, and in [Hub01] and [Sch96], where a similar lack of diversification is revealed on domestic portfolios. The model in [BGUW12], [UW03] offer a partial explanation for the observed under-diversification and bias toward familiar securities. More recently, in [LZ17], the authors considered the ambiguity about the correlation of the assets. With a static mean-variance investment, they found that the robust optimal portfolio is of under-diversification depends on the level of correlation ambiguity. They also provided results with market data and showed that using their ambiguous correlation model, the investor only holds less than

20 (17 stocks in average) among 100 stocks randomly selected from about 100 stocks in S&P500.

A further possible explanation for under-diversification is that investors can reduce the uncertainty on the model or the parameters through learning. In [VNV10], the authors built a framework to solve jointly for investment and information choices, with general preferences and information cost functions. They showed that, for some special preferences and information acquisition technologies, investors tend to learn more about the assets they are familiar among many available assets (typically, the domestic assets rather than foreign ones)and become even more familiar with those assets after learning. As a result of this learning procedure, the investors select those assets they have learnt at the expense of others for which they have less information. Their results are consistent with the empirical studies on the portfolios of international investors.

In the existing literature on model uncertainty, investor seeks to maximize the expected utility criterion. However, mean-variance criterion has received little attention, especially in the continuous-time framework, see [IP17]. Inspired by [IP17], we develop a robust model that take into account uncertainty about both drift and correlation of multi risky assets for $d \geq 2$, in a dynamic mean-variance portfolio setting. In view of drift uncertainty, our framework allows for both polyhedral set in [TTU13], [LR14] and ellipsoidal set in [BP17]. The ellipsoidal representation for the drift uncertainty allows to take into account drift structure of the assets in the correlation ambiguity modelling. Our purpose is to explore the joint effects of ambiguity about drift and correlation on portfolio selection and diversification with dynamic mean-variance criterion.

The paper's first contribution is to derive a separation principle for solving the associated robust control problem formulated as a McKean-Vlasov differential game, which allows us to reduce the original min-max problem to the parametric computation of minimal risk premium. The main methodology for the separation principle is based on a weak version of the martingale optimality principle. While the robust dynamics mean-variance problem under covariance matrix uncertainty, in particular, correlation ambiguity, has been analyzed in [IP17], the methods and techniques applied therein cannot tackle the drift uncertainty. One key assumption in [IP17] is that one can aggregate a family of processes, however in the case of drift uncertainty, this condition does not hold anymore. As a byproduct, to the best of our knowledge, ours is the first paper to tackle the robust dynamic mean-variance portfolio selection under drift uncertainty.

Our second contribution is to provide a possible explanation for under-diversification. The existing literature mainly concentrate on two asset case see [FPW16], [IP17], [LZ17]. We distinguish the case of ellipsoidal set and rectangular set in terms of drift uncertainty and quantify explicitly the diversification effects on the optimal robust portfolio in terms of the ambiguity level. We provide notably a complete picture of the diversification for the optimal robust portfolio strategy in the three risky assets case, which is new to the best of our knowledge. In particular, our findings consist in no trading in assets with large expected return ambiguity, and trading only one risky asset with high level of ambiguity about correlation. A similar finding on phenomenon of trading only one risky asset with high level of ambiguity about correlation is derived in [LZ17] with static mean-variance investment. By incorporating drift uncertainty into our framework, the anti-diversification in our paper is in a more general sense that the investor does not hold risky asset or holds one single risky asset. For our future studies, we may incorporate the different uncertainty levels for return as in [UW03] or introduce the information acquisition procedure as in [VNV10] in our framework.

The rest of paper is organized as follows. Section 2 presents the formulation of the model uncertainty setting and the robust multi-asset mean-variance problem in continuous time. In section 3, we derive the separation principle for the associated robust control problem. Section 4 provides several examples arising from the separation principle, and the implications for the optimal robust portfolio strategy and the portfolio diversification.

5.2 Problem formulation

5.2.1 Model uncertainty setting

We consider a financial market with one risk-free asset, assumed to be constant equal to one, and d risky assets on a finite investment horizon $[0,T]$. Model uncertainty is formulated by using a probabilistic setup as in [NN18]. We define the canonical state space by $\Omega = \{\omega = (\omega(t))_{t\in[0,T]} \in C([0,T],\mathbb{R}^d) : \omega(0) = 0\}$ representing the continuous paths driving the risky assets. We equip Ω with the uniform norm and the corresponding Borel σ-field $\mathcal{F}$. We denote by $B = (B_t)_{t\in[0,T]}$ the canonical process, i.e., $B_t(\omega) = \omega(t)$, and by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration, i.e. the natural (raw) filtration generated by B.

We assume that the investor knows the marginal volatilities $\sigma_i > 0$ of each asset $i = 1,\dots,d$, typically through a quadratic variation estimation of the assets, and we denote by $\mathfrak{S}$ the known constant diagonal matrix with i-th diagonal term equal to σ_i, $i = 1,\dots,d$. However, there is uncertainty about the drift (expected rate of return) and the correlation between the multi-assets, which are parameters notoriously difficult to estimate in practice.

The ambiguity about drift and correlation matrix is parametrized by a nonempty convex set

$$\Theta \quad \subset \quad \mathbb{R}^d \times \mathbb{C}^d_{>+},$$

where $\mathbb{C}^d_{>+}$ is the subset of all elements $\rho = (\rho_{ij})_{1\le i<j\le d} \in [-1,1]^{d(d-1)/2}$ s.t. the symmetric matrix $C(\rho)$ with diagonal terms 1 and anti-diagonal terms ρ_{ij}:

$$C(\rho) \quad = \quad \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1d} \\ \rho_{12} & 1 & \dots & . \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1d} & . & \dots & 1 \end{pmatrix}$$

lies in $\mathbb{S}^d_{>+}$, the set of positive definite symmetric matrices in $\mathbb{R}^{d\times d}$. Notice that $\mathbb{C}^d_{>+}$ is an open convex set of $[-1,1]^{d(d-1)/2}$. The first component set of Θ represent the prior values taken by the (possibly random) drift of the assets, while the matrices $C(\rho)$, when ρ runs in the second component set of Θ, represent the prior correlation matrices of the multi-assets. The prior covariance matrices of the assets are given by

$$\Sigma(\rho) \quad = \quad \mathfrak{S}C(\rho)\mathfrak{S} \;=\; \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \dots & \sigma_1\sigma_d\rho_{1d} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \dots & . \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_d\rho_{1d} & . & \dots & \sigma_d^2 \end{pmatrix},$$

and we denote by $\sigma(\rho) = \Sigma^{\frac{1}{2}}(\rho)$ the square-root matrix, called volatility matrix. Let us also introduce the prior (square) risk premium

$$R(\theta) \quad = \quad b^{\intercal}\Sigma^{-1}(\rho)b \;=\; \|\sigma(\rho)^{-1}b\|_2^2 \quad \text{for} \;\; \theta = (b,\rho) \in \Theta. \tag{5.2.1}$$

Hereafter, $^\intercal$ denotes the transpose of matrix and $\|\cdot\|_2$ denotes the L_2-norm in $\mathbb{R}^d$.

Remark 5.2.1. There exists different conditions for characterizing the positive definiteness of the correlation matrix $C(\rho)$. For example, Sylvester's criterion states that $C(\rho)$ is positive definite if and only if all the leading principal minors are positive, e.g., in dimension $d=2$, $\rho\in(-1,1)$; in dimension $d=3$, $\rho_{ij}\in(-1,1)$ $1\le i<j\le 3$ and $\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-1-2\rho_{12}\rho_{13}\rho_{23}<0$. Alternatively, one can characterize the positive definiteness of $C(\rho)$ using angular coordinates as in [RBM07].

In the sequel, we shall focus on the two following cases for the parametrization of the ambiguity set Θ, which are relevant for practical applications:

(HΘ)

(i) *Product set*: $\Theta=\Delta\times\Gamma$, where Δ is a compact convex set of $\mathbb{R}^d$, e.g., in rectangular form $\Delta=\prod_{i=1}^d[\underline{b}_i,\bar{b}_i]$, for some constants $\underline{b}_i\le\bar{b}_i$, $i=1,\dots,d$, and Γ is a convex set of $\mathbb{C}^d_{>+}$. In this product formulation, one considers that the uncertainty on drift is independent of the uncertainty on the correlation.

(ii) *Ellipsoidal set*: $\Theta=\{(b,\rho)\in\mathbb{R}^d\times\Gamma:\|\sigma(\rho)^{-1}(b-\hat{b})\|_2\le\delta\}$, for some convex set Γ of $\mathbb{C}^d_{>+}$, where $\hat{b}$ is a known vector, representing *a priori* expected rates of return, and $\delta>0$ represents a level of ambiguity around $\hat{b}$ due to estimation error. It is known from Lemma 2.2 in [CDH18] that Θ is a convex set. This ellipsoidal set in which varies the uncertain drift, for fixed correlation, is used in [BTMN00], and allows to take into account the correlation structure of the assets in the drift uncertainty modelling.

We denote by $\mathcal{V}_\Theta$ the set of $\mathbb{F}$-progressively measurable processes $\theta=(\theta_t)=(b_t,\rho_t)_t=(b,\rho)$ valued in Θ, and introduce the set of prior probability measures $\mathcal{P}^\Theta$:

$$\mathcal{P}^\Theta \quad = \quad \{\mathbb{P}^\theta:\theta\in\mathcal{V}_\Theta\},$$

where $\mathbb{P}^\theta$ is the probability measure on $(\Omega,\mathcal{F})$ s.t. B is a semimartingale on $(\Omega,\mathcal{F},\mathbb{P}^\theta)$ with absolutely continuous characteristics (w.r.t. the Lebesgue measure dt) $(b,\Sigma(\rho))$. The *prior* probabilities $\mathbb{P}^\theta$ are in general non-equivalent, and actually mutually singular, and we say that a property holds $\mathcal{P}^\Theta$-quasi surely ($\mathcal{P}^\Theta$-q.s. in short) if it holds $\mathbb{P}^\theta$-a.s. for all $\theta\in\mathcal{V}_\Theta$.

The (positive) price process of the d risky assets is given by the dynamics

$$\begin{aligned} dS_t &= \operatorname{diag}(S_t)dB_t, \quad 0\le t\le T,\ \mathcal{P}^\Theta-q.s.\\ &= \operatorname{diag}(S_t)\big(b_tdt+\sigma(\rho_t)dW_t^\theta\big), \quad \mathbb{P}^\theta-a.s., \quad \text{for } \theta = (b,\rho)\in\mathcal{V}_\Theta, \end{aligned}$$

where W^θ is a d-dimensional Brownian motion under $\mathbb{P}^\theta$. Notice that in this uncertainty modeling, we allow the unknown drift and correlation to be *a priori* random processes, valued in Θ.

5.2.2 Robust mean-variance problem

An admissible portfolio strategy $\alpha=(\alpha_t)_{0\le t\le T}$ representing the amount invested in the d risky assets, is an $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable process, satisfying the integrability condition

$$\sup_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}_\theta\Big[\int_0^T|\alpha_t^\intercal b_t|dt\ +\ \int_0^T\alpha_t^\intercal\Sigma(\rho_t)\alpha_tdt\Big]\ <\ \infty, \tag{5.2.2}$$

and denoted by $\alpha \in \mathcal{A}$. Hereafter, $\mathbb{E}_\theta$ denotes the expectation under $\mathbb{P}^\theta$. This integrability condition (5.2.2) ensures that $\text{diag}(S)^{-1}\alpha$ is S-integrable under any $\mathbb{P} \in \mathcal{P}^\Theta$. For a portfolio strategy $\alpha \in \mathcal{A}$, and an initial capital $x_0 \in \mathbb{R}$, the dynamics of the self-financed wealth process are driven by

$$\begin{aligned} dX_t^\alpha &= \alpha_t^\intercal \text{diag}(S_t)^{-1} dS_t \;=\; \alpha_t^\intercal dB_t, \quad 0 \le t \le T, \; X_0^\alpha = x_0, \; \mathcal{P}^\Theta - q.s. \\ &= \alpha_t^\intercal \big(b_t dt + \sigma(\rho_t) dW_t^\theta\big), \quad 0 \le t \le T, \; X_0^\alpha = x_0 \in \mathbb{R}, \; \mathbb{P}^\theta - a.s. \end{aligned} \tag{5.2.3}$$

for all $\theta = (b, \rho) \in \mathcal{V}_\Theta$.

Given a risk aversion parameter $\lambda > 0$, the worst-case mean-variance functional under ambiguous drift and correlation is

$$J_{wc}(\alpha) \;=\; \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} \Big(\mathbb{E}_\theta[X_T^\alpha] - \lambda \text{Var}_\theta(X_T^\alpha)\Big) \;<\; \infty, \; \alpha \in \mathcal{A},$$

where $\text{Var}_\theta(.)$ denotes the variance under $\mathbb{P}^\theta$, and the robust mean-variance portfolio selection is formulated as

$$\begin{cases} V_0 &:= \sup_{\alpha \in \mathcal{A}} J_{wc}(\alpha) \;=\; \sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta) \\ J(\alpha, \theta) &:= \mathbb{E}_\theta[X_T^\alpha] - \lambda \text{Var}_\theta(X_T^\alpha), \quad \alpha \in \mathcal{A}, \theta \in \mathcal{V}_\Theta. \end{cases} \tag{5.2.4}$$

Notice that problem (5.2.4) is a non standard stochastic differential game due to the presence of the variance term in the criterion, which prevents the use of classical control method by dynamic programming or maximum principle. We end this section by recalling the solution to the mean-variance problem when there is no ambiguity on the model parameters, and which will serve later as benchmark for comparison when studying the uncertainty case.

Remark 5.2.2 (Case of no uncertainty model)**.** When $\Theta = \{\theta^o = (b^o, \rho^o)\}$ is a singleton, we are reduced to the Black-Scholes model with drift b^o, covariance matrix $\Sigma^o = \Sigma(\rho^o)$, volatility $\sigma = \sigma(\rho^o)$, and risk premium $R^o = R(\theta^o)$. In this case, it is known, see e.g. [LZ00], that the optimal mean-variance strategy is given by

$$\alpha_t^* \;=\; \Big[x_0 + \frac{e^{R^o T}}{2\lambda} - X_t^*\Big](\Sigma^o)^{-1} b^o \;=:\; \Lambda^o(X_t^*)(\Sigma^o)^{-1} b^o, \quad 0 \le t \le T,$$

where X^* is wealth process associated to α^*, while the optimal performance value is

$$V_0 \;=\; x_0 + \frac{1}{4\lambda}\big[e^{R^o T} - 1\big].$$

The vector $(\Sigma^o)^{-1} b^o$, which depends only on the model parameters of the stock price, determines the allocation in the multi-assets. The above expression of α^* shows that, once we know the exact values of the rate of return and covariance matrix, one diversifies her portfolio among all the assets according to the components of the vector $(\Sigma^o)^{-1} b^o$, and this is weighted by the scalar term $\Lambda^o(X_t^*)$, which depends on the risk aversion of the investor via the parameter λ, on the current wealth but also on the initial capital x_0 (which is sometimes refereed to as the pre-committment of the mean-variance criterion). Notice that $\Lambda^o(X_t^*)$ is positive. Indeed, observe that

$$\begin{aligned} d\Lambda^o(X_t^*) \;=\; -dX_t^* &= -(\alpha_t^*)^\intercal (b^o dt + \sigma^o dW_t^o) \\ &= -\Lambda^o(X_t^*)(R^o dt + (\sigma^o)^{-1} b^o . dW_t^o), \quad 0 \le t \le T, \end{aligned}$$

with $\Lambda^o(X_0^*) = \frac{1}{2\lambda}e^{R^oT} > 0$, which shows clearly that $\Lambda^o(X_t^*) > 0$, $0 \leq t \leq T$, and decreases with λ.

Let us discuss in particular the allocation in the two-asset case. Notice that the vector $(\Sigma^o)^{-1}b^o$ of allocation is then given by

$$(\Sigma^o)^{-1}b^o \quad = \quad \frac{1}{1-|\rho^o|^2}\begin{pmatrix} \frac{\beta_1^o - \rho^o\beta_2^o}{\sigma_1^o} \\ \frac{\beta_2^o - \rho^o\beta_1^o}{\sigma_2^o} \end{pmatrix} =: \begin{pmatrix} \kappa_1^o \\ \kappa_2^o \end{pmatrix},$$

where $\beta_i^o = b_i^o/\sigma_i^o$ is the sharpe ratio of the i-th asset, $i = 1, 2$. To fix the idea, assume that $\beta_1^o > \beta_2^o > 0$. We then see that $\kappa_1^o > 0$, while $\kappa_2^o \geq 0$ if and only if $\frac{\beta_2^o}{\beta_1^o} \geq \rho^o$. The interpretation is the following: the ratio $\frac{\beta_2^o}{\beta_1^o} \in (0,1)$ measures the "proximity" in terms of Sharpe ratio between the two assets, and has to be compared with the correlation ρ^o between these assets in order to determine whether it is optimal to invest according to a directional trading, i.e., $\kappa_1^o\kappa_2^o > 0$ (thus here long in both assets) or according to a spread trading, i.e., $\kappa_1^o\kappa_2^o < 0$ (long in the first asset and short in the second one) or according to under-diversification, i.e. $\kappa_1^o\kappa_2^o = 0$(only long in the first asset). For example, when both assets have close Sharpe ratio, and their correlation is not too high, then one optimally invests in both assets with a directional trading. In contrast, when one asset has a much larger Sharpe ratio than the other one, or when the correlation between the assets is high, then one optimally invests in both assets with a spread trading. ◊

In the sequel, we study the quantitative impact of the uncertainty model and ambiguity on the drift and correlation, on the optimal robust mean-variance strategy, in particular regarding the portfolio diversification.

5.3 Separation principle and robust solution

The main result of this section is to state a separation principle for solving the robust dynamic mean-variance problem.

Theorem 5.3.1 (Separation Principle)**.** *Let us consider a parametric set Θ for model uncertainty as in* **(HΘ)**. *Suppose that there exists a (constant) pair $\theta^* = (b^*, \rho^*) \in \Theta$ solution to $\arg\min_{\theta\in\Theta} R(\theta)$. Then the robust mean-variance problem* (5.2.4) *admits an optimal portfolio strategy given by*

$$\alpha_t^* \quad = \quad \Lambda_{\theta^*}(X_t^*)\Sigma(\rho^*)^{-1}b^*, \quad 0 \leq t \leq T, \;\; \mathcal{P}^\Theta - q.s., \tag{5.3.1}$$

where X^* is the state process associated to α_t^*, and $\Lambda_{\theta^*}(X_t^*) > 0$ with

$$\Lambda_{\theta^*}(x) \; := \; x_0 + \frac{e^{R(\theta^*)T}}{2\lambda} - x, \;\; x \in \mathbb{R} \tag{5.3.2}$$

Moreover, the corresponding initial value function is

$$V_0 \quad = \quad x_0 + \frac{1}{4\lambda}\big[e^{R(\theta^*)T} - 1\big].$$

Interpretation. Theorem 5.3.1 means that the robust mean-variance problem (5.2.4) can be solved in two steps according to a separation principle: (i) First, we search for the infimum of the risk premium function $\theta \in \Theta \mapsto R(\theta)$ as defined in (5.2.1), which depends only on the inputs of the uncertainty model. Existence and explicit determination of an element $\theta^* = (b^*, \rho^*) \in \Theta$ attaining this infimum will be discussed and illustrated all along the paper through several examples. (ii) The solution to (5.2.4) is then given by the solution to the mean-variance problem in the Black-Scholes model with drift b^* and correlation ρ^*, see Remark 5.2.2, and the worst-case scenario of the robust dynamic mean-variance problem is simply given by the constant parameter $\theta^* = (b^*, \rho^*)$. Some interesting features show up, especially regarding portfolio diversification, as detailed in the next section. ♢

The rest of this section is devoted to the proof of Theorem 5.3.1, and the methodology is based on the following weak version of the martingale optimality principle.

Lemma 5.3.1 (Weak optimality principle)**.** *Let* $\{V_t^{\alpha,\theta}, t \in [0,T], \alpha \in \mathcal{A}, \theta \in \mathcal{V}_\Theta\}$ *be a family of real-valued processes in the form*

$$V_t^{\alpha,\theta} : \;=\; v_t(X_t^\alpha, \mathbb{E}_\theta[X_t^\alpha]),$$

for some measurable functions v_t *on* $\mathbb{R} \times \mathbb{R}$, $t \in [0,T]$*, such that :*

(i) $v_T(x, \bar{x}) = x - \lambda(x - \bar{x})^2$*, for all* $x, \bar{x} \in \mathbb{R}$*,*

(ii) the function $t \in [0,T] \mapsto \mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}]$ *is nonincreasing for all* $\alpha \in \mathcal{A}$ *and some* $\theta^* \in \mathcal{V}_\Theta$*,*

(iii) $\mathbb{E}_\theta[V_T^{\alpha^*,\theta} - V_0^{\alpha^*,\theta}] \geq 0$*, for some* $\alpha^* \in \mathcal{A}$ *and all* $\theta \in \mathcal{V}_\Theta$*.*

Then, α^* *is an optimal portfolio strategy for the robust mean-variance problem* (5.2.4) *with a worst-case scenario* θ^**, and*

$$\begin{aligned} V_0 \;&=\; J_{wc}(\alpha^*) \;=\; \sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta) \;=\; \inf_{\theta \in \mathcal{V}_\Theta} \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta) \;=\; v_0(x_0, x_0) \\ &=\; J(\alpha^*, \theta^*). \end{aligned} \tag{5.3.3}$$

Proof. First, observe that $V_0^{\alpha,\theta} = v_0(x_0, x_0)$ is a constant that does not depend on α, θ and from condition (i) that $\mathbb{E}_\theta[V_T^{\alpha,\theta}] = J(\alpha, \theta)$ for all $\alpha \in \mathcal{A}$, $\theta \in \mathcal{V}_\Theta$. Then, from condition (ii), we see that

$$v_0(x_0, x_0) \;=\; \mathbb{E}_{\theta^*}[V_0^{\alpha,\theta^*}] \;\geq\; \mathbb{E}_{\theta^*}[V_T^{\alpha,\theta^*}] \;=\; J(\alpha, \theta^*),$$

for all $\alpha \in \mathcal{A}$, and thus: $v_0(x_0, x_0) \geq \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta^*) \geq \inf_{\theta \in \mathcal{V}_\Theta} \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta)$. Similarly, from condition (iii), we have: $v_0(x_0, x_0) \leq J(\alpha^*, \theta)$ for all $\theta \in \mathcal{V}_\Theta$, and thus: $v_0(x_0, x_0) \leq \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha^*, \theta) = J_{wc}(\alpha^*) \leq \sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta)$. Recalling that we always have $\sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta) \leq \inf_{\theta \in \mathcal{V}_\Theta} \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta)$, we obtained the required equality in (5.3.3). Then, finally, from (ii) with α^* and (iii) with θ^*, we obtain that $v_0(x_0, x_0)$ $= J(\alpha^*, \theta^*)$. □

Remark 5.3.1. The usual martingale optimality principle for stochastic differential games as in robust portfolio selection problem, and with classical expected utility criterion for some nondecreasing and concave utility function U on $\mathbb{R}$, e.g. $U(x) = -e^{-\eta x}$, $\eta > 0$:

$$\sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} \mathbb{E}_\theta[U(X_T^\alpha)],$$

would consist in finding a family of processes $V_t^{\alpha,\theta}$ in the form $v_t(X_t^\alpha)$ for some measurable functions v_t on $\mathbb{R}$ s.t. (i) $v_T(x) = U(x)$, (ii') the process $(V_t^{\alpha,\theta^*})_t$ is a supermartingale under $\mathbb{P}_{\theta^*}$ for all α, and some

θ^*, and (iii') the process $(V_t^{\alpha^*,\theta})_t$ is a submartingale under $\mathbb{P}_\theta$ for some α^* and all θ. Due to the nonlinear dependence on the law of the state wealth process via the variance term in the mean-variance criterion, making the problem a *priori* time inconsistent, we have to adopt a weaker version of the optimality principle: first, the functions v_t depend not only on the state process X_t^α but also on its mean $\mathbb{E}_\theta[X_t^\alpha]$. Second, we replace condition (ii') by the weaker condition (ii) on the mean in Lemma 5.3.1, and third, condition (iii') is substituted by the even weaker condition (iii) than (iii") $t \mapsto \mathbb{E}_\theta[V_t^{\alpha^*,\theta}]$ is nondecreasing for some α^* and all θ. This asymmetry of condition between (ii) and (iii) is explained in more detail in Remark 5.3.3. ♢

We shall also use the following saddle-point property on the infimum of the *prior* risk premium function.

Lemma 5.3.2 (Saddle point property). *Given Θ as in* **(HΘ)**, *and assuming that there exists $\theta^* = (b^*, \rho^*) \in \arg\min_{\theta\in\Theta} R(\theta)$, let us define the function H on Θ by*

$$H(\theta) \quad := \quad b^\intercal \Sigma(\rho^*)^{-1}\Sigma(\rho)\Sigma(\rho^*)^{-1}b^*, \quad \text{for } \theta = (b,\rho) \in \Theta. \tag{5.3.4}$$

Then, we have for all $\theta = (b, \rho) \in \Theta$:

$$H(b^*, \rho) \;\le\; H(\theta^*) \;=\; R(\theta^*) \;\le\; H(b, \rho^*). \tag{5.3.5}$$

Proof. See Section 5.5.2 in Appendix. □

Proof of Theorem 5.3.1. We aim to construct a family of processes $\{V_t^{\alpha,\theta}, t \in [0,T], \alpha \in \mathcal{A}, \theta \in \mathcal{V}_\Theta\}$ as in Lemma 5.3.1, and given the linear-quadratic structure of our optimization problem, we look for measurable functions v_t in the form:

$$v_t(x,\bar{x}) \quad = \quad K_t(x-\bar{x})^2 + Y_t x + \chi_t, \quad t \in [0,T], (x,\bar{x}) \in \mathbb{R}^2, \tag{5.3.6}$$

for some deterministic processes $(K_t, Y_t, \chi_t)_t$ to be determined. Condition (i) in Lemma 5.3.1 fixes the terminal condition

$$K_T \;=\; -\lambda, \;\; Y_T \;=\; 1, \;\; \chi_T \;=\; 0. \tag{5.3.7}$$

We now consider $\theta^* \in \Theta$ as in Theorem 5.3.1, hence defining in particular a (constant) process $\theta^* \in \mathcal{V}_\Theta$, and α^* given by (5.3.1). Let us first check that $\alpha^* \in \mathcal{A}$. The corresponding wealth process X^* satisfies under any $\mathbb{P}^\theta$, $\theta = (b,\rho) \in \mathcal{V}_\Theta$, a linear stochastic differential equation with bounded random coefficients (notice that b and $\sigma(\rho)$ are bounded process), and thus by standard estimates: $\mathbb{E}_\theta\big[\sup_{0\le t\le T}|X_t^*|^2\big] \le C(1+|x_0|^2)$ for some constant C independent of $\theta \in \mathcal{V}_\Theta$. It follows immediately that α^* satisfies the integrability condition in (5.2.2), i.e., $\alpha^* \in \mathcal{A}$.

The main issue is now to show that such a pair (α^*, θ^*) satisfies conditions (ii)-(iii) of Lemma 5.3.1.

- *Step 1: condition (ii) of Lemma 5.3.1.*

For any $\alpha \in \mathcal{A}$, with associated wealth process $X = X^\alpha$, let us compute the derivative of the deterministic function $t \mapsto \mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}] = \mathbb{E}_{\theta^*}[v_t(X_t, \mathbb{E}_{\theta^*}[X_t])]$ with v_t as in (5.3.6). From the dynamics of $X = X_t^\alpha$ in (5.2.3) under $\mathbb{P}^{\theta^*}$ and by applying Itô's formula, we obtain

$$\begin{aligned}
\frac{d\mathbb{E}_{\theta^*}[X_t]}{dt} &= \mathbb{E}_{\theta^*}[\alpha_t^\intercal b^*] \\
\frac{d\mathrm{Var}_{\theta^*}(X_t)}{dt} &= 2\mathrm{Cov}_{\theta^*}(X_t, \alpha_t^\intercal b^*) + \mathbb{E}_{\theta^*}[\alpha_t^\intercal \Sigma(\rho^*)\alpha_t].
\end{aligned}$$

From the quadratic form of v_t in (5.3.6), with (K, Y, χ) differentiable in time, we then have

$$\begin{aligned}
\frac{d\mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}]}{dt} &= \frac{d\mathbb{E}_{\theta^*}[v_t(X_t, \mathbb{E}_{\theta^*}[X_t])]}{dt} \\
&= \dot{K}_t \mathrm{Var}_{\theta^*}(X_t) + K_t \frac{d\mathrm{Var}_{\theta^*}(X_t)}{dt} + \dot{Y}_t \mathbb{E}_{\theta^*}[X_t] + Y_t \frac{d\mathbb{E}_{\theta^*}[X_t]}{dt} + \dot{\chi}_t \\
&= \dot{K}_t \mathrm{Var}_{\theta^*}(X_t) + \dot{Y}_t \mathbb{E}_{\theta^*}[X_t] + \dot{\chi}_t + \mathbb{E}_{\theta^*}[G_t(\alpha)] \qquad (5.3.8)
\end{aligned}$$

where

$$G_t(\alpha) = \alpha_t^\intercal Q_t \alpha_t + \alpha_t^\intercal \big[2U_t(X_t - \mathbb{E}_{\theta^*}[X_t]) + O_t\big],$$

with the deterministic coefficients

$$Q_t = K_t \Sigma(\rho^*), \quad U_t = K_t b^*, \quad O_t = Y_t b^*.$$

By square completion, we rewrite $G_t(\alpha)$ as

$$G_t(\alpha) = \big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big)^\intercal Q_t \big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big) - \zeta_t,$$

where for $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^2$,

$$\hat{a}_t(x, \bar{x}) := -Q_t^{-1} U_t (x - \bar{x}) - \frac{1}{2} Q_t^{-1} O_t$$

and

$$\zeta_t = U_t^\intercal Q_t^{-1} U_t \mathrm{Var}_{\theta^*}(X_t) + \frac{1}{4} O_t^\intercal Q_t^{-1} O_t = K_t R(\theta^*) \mathrm{Var}_{\theta^*}(X_t) + \frac{Y_t^2}{4K_t} R(\theta^*).$$

The expression in (5.3.8) is then rewritten as

$$\begin{aligned}
\frac{d\mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}]}{dt} &= (\dot{K}_t - K_t R(\theta^*)) \mathrm{Var}_{\theta^*}(X_t) + \dot{Y}_t \mathbb{E}_{\theta^*}[X_t] + \dot{\chi}_t - \frac{Y_t^2}{4K_t} R(\theta^*) \qquad (5.3.9) \\
&\quad + K_t \mathbb{E}_{\theta^*}\Big[\big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big)^\intercal \Sigma(\rho^*) \big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big)\Big].
\end{aligned}$$

Therefore, whenever

$$\begin{cases} \dot{K}_t - K_t R(\theta^*) = 0, \\ \dot{Y}_t = 0, \\ \dot{\chi}_t - \frac{Y_t^2}{4K_t} R(\theta^*) = 0, \end{cases} \qquad (5.3.10)$$

holds for all $t \in [0, T]$, which yields, together with the terminal condition (5.3.7), the explicit forms:

$$K_t = -\lambda e^{R(\theta^*)(t-T)} < 0, \ Y_t = 1, \ \chi_t = \frac{1}{4\lambda}\big[e^{R(\theta^*)(T-t)} - 1\big], \qquad (5.3.11)$$

we have

$$\frac{d\mathbb{E}_{\theta^*}[V_t^{\alpha,\theta^*}]}{dt} = K_t \mathbb{E}_{\theta^*}\Big[\big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big)^\intercal \Sigma(\rho^*) \big(\alpha_t - \hat{a}_t(X_t, \mathbb{E}_{\theta^*}[X_t])\big)\Big],$$

which is nonpositive for all $\alpha \in \mathcal{A}$, i.e., the process V_t^{α,θ^*} satisfies the condition (ii) of Lemma 5.3.1. Moreover, notice that in this case,

$$V_0^{\alpha,\theta^*} = v_0(x_0, x_0) = x_0 + \frac{1}{4\lambda}\big[e^{R(\theta^*)T} - 1\big], \qquad (5.3.12)$$

and

$$\hat{a}_t(x,\bar{x}) \;=\; -\Sigma(\rho^*)^{-1}b^*\big(x-\bar{x}-\frac{1}{2\lambda}e^{R(\theta^*)(T-t)}\big). \tag{5.3.13}$$

Notice that in this step, we have not yet used the property that θ^* attains the infimum of the *prior* risk premium function. This will be used in the next step.

• *Step 2: Condition (iii) of Lemma 5.3.1.*

Let us now prove that $V_0^{\alpha^*,\theta} \le \mathbb{E}_\theta[V_T^{\alpha^*,\theta}]$, for all $\theta \in \mathcal{V}_\Theta$. A sufficient condition is the nondecreasing monotonicity of the function $t \mapsto \mathbb{E}_\theta[V_t^{\alpha^*,\theta}]$, by proving that $\frac{d\mathbb{E}_\theta[V_t^{\alpha^*,\theta}]}{dt}$ is nonnegative, for all $\theta \in \mathcal{V}_\Theta$. However, while this nondecreasing property is valid when there is no uncertainty on the drift, this does not hold true in the general uncertainty case as shown in Remark 5.3.3. We then proceed by computing directly the difference: $\mathbb{E}_\theta[V_T^{\alpha^*,\theta}] - V_0^{\alpha^*,\theta}$. Notice from (5.3.1), (5.2.3), that the dynamics of $\Lambda_{\theta^*}(X^*)$, with $\Lambda_{\theta^*}(x)$ defined in (5.3.2), under $\mathbb{P}^\theta$, $\theta \in \mathcal{V}_\Theta$, are given by

$$d\Lambda_{\theta^*}(X_t^*) \;=\; -\Lambda_{\theta^*}(X_t^*)(b^*)^\intercal\Sigma(\rho^*)^{-1}\big[b_t dt + \sigma(\rho_t)dW_t^\theta\big],$$

with $\Lambda_{\theta^*}(x_0) = \frac{e^{R(\theta^*)T}}{2\lambda}$. By setting $N_t^* := \frac{2\lambda}{e^{R(\theta^*)T}}\Lambda_{\theta^*}(X_t^*)$, we deduce that

$$\begin{aligned}
N_t^* \;&=\; \exp\Big(-\int_0^t \big(b_s^\intercal\Sigma(\rho^*)^{-1}b^* + \frac{1}{2}(b^*)^\intercal\Sigma(\rho^*)^{-1}\Sigma(\rho)\Sigma(\rho^*)^{-1}b^*\big)ds \\
&\qquad - \int_0^t (b^*)^\intercal\Sigma(\rho^*)^{-1}\sigma(\rho_s)dW_s^\theta\Big), \quad 0 \le t \le T,\ \mathbb{P}^\theta - a.s. \\
X_t^* \;&=\; x_0 + \frac{e^{R(\theta^*)T}}{2\lambda}(1-N_t^*), \quad 0 \le t \le T,\ \mathcal{P}^\Theta - q.s.
\end{aligned}$$

and thus

$$\mathbb{E}_\theta[X_t^*] \;=\; x_0 + \frac{e^{R(\theta^*)T}}{2\lambda}(1-\mathbb{E}_\theta[N_t^*]), \qquad \mathrm{Var}_\theta(X_t^*) \;=\; \frac{e^{2R(\theta^*)T}}{4\lambda^2}\mathrm{Var}_\theta(N_t^*). \tag{5.3.14}$$

By using the quadratic form (5.3.6) of v_t, together with the terminal condition (5.3.7), (5.3.12), and (5.3.14), we then obtain for all $\theta \in \mathcal{V}_\Theta$:

$$\begin{aligned}
\mathbb{E}_\theta[V_T^{\alpha^*,\theta}] - V_0^{\alpha^*,\theta} \;&=\; \mathbb{E}_\theta\big[v_T(X_T^*,\mathbb{E}_\theta[X_T^*])\big] - v_0(x_0,x_0) \\
&=\; -\lambda\mathrm{Var}_\theta(X_T^*) + \mathbb{E}_\theta[X_T^*] - x_0 - \frac{1}{4\lambda}(e^{R(\theta^*)T}-1) \\
&=\; -\frac{e^{2R(\theta^*)T}}{4\lambda}\mathrm{Var}_\theta(N_T^*) + \frac{e^{R(\theta^*)T}}{2\lambda}(1-\mathbb{E}_\theta[N_T^*]) - \frac{1}{4\lambda}(e^{R(\theta^*)T}-1) \\
&=\; \frac{e^{R(\theta^*)T}}{4\lambda}\Big(1-e^{R(\theta^*)T}\mathbb{E}_\theta[|N_T^*|^2]\Big) + \frac{1}{4\lambda}\Big(e^{R(\theta^*)T}\mathbb{E}_\theta[N_T^*]-1\Big)^2 \\
&\ge\; \frac{e^{R(\theta^*)T}}{4\lambda}\Big(1-e^{R(\theta^*)T}\mathbb{E}_\theta[|N_T^*|^2]\Big) \;=:\; \frac{e^{R(\theta^*)T}}{4\lambda}\Delta_T^*(\theta).
\end{aligned} \tag{5.3.15}$$

Noting that N^* is rewritten in terms of H introduced in Lemma 5.3.2 as

$$N_t^* \;=\; \exp\Big(-\int_0^t \big(H(b_s,\rho^*) + \frac{1}{2}H(b^*,\rho_s)\big)ds - \int_0^t (b^*)^\intercal\Sigma(\rho^*)^{-1}\sigma(\rho_s)dW_s^\theta\Big), 0 \le t \le T,\ \mathbb{P}^\theta - a.s.$$

and observing that $|(b^*)^\intercal\Sigma(\rho^*)^{-1}\sigma(\rho_s)|^2 = H(b^*,\rho_s)$, we see that

$$|N_t^*|^2 \;=\; \exp\Big(-\int_0^t \big(2H(b_s,\rho^*) - H(b^*,\rho_s)\big)ds\Big)M_t^*,$$

where

$$M_t^* = \exp\Big(-2\int_0^t |(b^*)^\intercal \Sigma(\rho^*)^{-1}\sigma(\rho_s)|^2 ds - 2\int_0^t (b^*)^\intercal \Sigma(\rho^*)^{-1}\sigma(\rho_s) dW_s^\theta\Big),\ 0 \leq t \leq T,\ \ \mathbb{P}^\theta - a.s.$$

is an exponential Doléans-Dade local martingale under any $\mathbb{P}^\theta$, $\theta \in \mathcal{V}_\Theta$. Actually, the Novikov criterion

$$\begin{aligned}\mathbb{E}_\theta\Big[\exp\Big(\frac{1}{2}\int_0^T |2(b^*)^\intercal\Sigma(\rho^*)^{-1}\sigma(\rho_t)|^2 dt\Big)\Big] &= \mathbb{E}_\theta\Big[\exp\Big(2\int_0^T H(b^*,\rho_t)dt\Big)\Big] \\ &\leq \exp\big(2R(\theta^*)T\big) < \infty,\end{aligned}$$

is satisfied by (5.3.5), and then $(M_t^*)_{0\leq t\leq T}$ is a martingale under any $\mathbb{P}^\theta$, $\theta \in \mathcal{V}_\Theta$. Consequently, we have

$$\begin{aligned}\Delta_T^*(\theta) &= 1 - \mathbb{E}_\theta\Big[\exp\Big(\int_0^t \big(R(\theta^*) - 2H(b_s,\rho^*) + H(b^*,\rho_s)\big)ds\Big) M_T^*\Big] \\ &\geq 1 - \mathbb{E}_\theta[M_T^*] = 1 - M_0^* = 0,\end{aligned}$$

where we used (5.3.5) in the above inequality. From (5.3.15), this proves condition (iii) of Lemma (5.3.1), and finally concludes the proof of Theorem 5.3.1. □

Remark 5.3.2. The optimal strategy α^* given in (5.3.1) can be expressed in feedback form as

$$\alpha_t^* = \hat{a}_t(X_t^*, \mathbb{E}_{\theta^*}[X_t^*]),\quad 0 \leq t \leq T,\ \mathcal{P}^\Theta - q.s. \tag{5.3.16}$$

where $\hat{a}_t$ is defined in (5.3.13). Indeed, denoting by $\hat{\alpha} \in \mathcal{A}$ the process defined by $\hat{\alpha}_t = \hat{a}_t(\hat{X}_t, \mathbb{E}_{\theta^*}[\hat{X}_t])$, $0 \leq t \leq T$, $\mathcal{P}^\Theta - q.s$, where $\hat{X}$ is the wealth process associated to $\hat{\alpha}$, we see from (5.2.3) that $\hat{X}$ satisfies the dynamics under $\mathbb{P}^{\theta^*}$:

$$d\hat{X}_t = -\Big[\hat{X}_t - \mathbb{E}_{\theta^*}[\hat{X}_t] - \frac{1}{2\lambda}e^{R(\theta^*)(T-t)}\Big](b^*)^\intercal\Sigma(\rho^*)^{-1}\big[b^* dt + \sigma(\rho^*)dW_t^{\theta^*}\big].$$

By taking expectation under $\mathbb{P}^{\theta^*}$, we get: $d\mathbb{E}_{\theta^*}[\hat{X}_t] = \frac{1}{2\lambda}e^{R(\theta^*)(T-t)}R(\theta^*)dt$, and thus

$$\begin{aligned}\mathbb{E}_{\theta^*}[\hat{X}_t] &= x_0 + \frac{e^{R(\theta^*)T}}{2\lambda}\big[1 - e^{-R(\theta^*)t}\big], \\ \hat{\alpha}_t &= \Lambda_{\theta^*}(\hat{X}_t)\Sigma(\rho^*)^{-1}b^*,\quad 0 \leq t \leq T,\ \mathcal{P}^\Theta - q.s.\end{aligned}$$

This implies that $\hat{X}$ and X^* satisfy the same linear SDE under $\mathbb{P}^\theta$, for any $\theta \in \mathcal{V}_\Theta$, and so $\hat{X}_t = X_t^*$, $0 \leq t \leq T$, $\mathcal{P}^\Theta$-q.s. This proves that $\alpha^* = \hat{\alpha}$, equal to (5.3.16). ◊

Remark 5.3.3. By similar derivation as in (5.3.9), and using (5.3.10), (5.3.16), we have that for all $\theta = (\theta_t)_t = (b_t, \rho_t)_t \in \mathcal{V}_\Theta$, $t \in [0,T]$,

$$\begin{aligned}\frac{d\mathbb{E}_\theta[V_t^{\alpha^*,\theta}]}{dt} &= K_t\big(R(\theta^*) - R(\theta_t)\big)\mathrm{Var}_\theta(X_t^*) + \frac{1}{4K_t}\big(R(\theta^*) - R(\theta_t)\big) \qquad (5.3.17)\\ &\quad + K_t\mathbb{E}_\theta\Big[\big(\hat{a}_t(X_t^*, \mathbb{E}_{\theta^*}[X_t^*]) - \hat{a}_t(X_t^*, \mathbb{E}_\theta[X_t^*])\big)^\intercal\Sigma(\rho_t) \\ &\quad\quad \big(\hat{a}_t(X_t^*, \mathbb{E}_{\theta^*}[X_t^*]) - \hat{a}_t(X_t^*, \mathbb{E}_\theta[X_t^*])\big)\Big] \\ &\geq K_t\mathbb{E}_\theta\Big[\big(\hat{a}_t(X_t^*, \mathbb{E}_{\theta^*}[X_t^*]) - \hat{a}_t(X_t^*, \mathbb{E}_\theta[X_t^*])\big)^\intercal\Sigma(\rho_t) \\ &\quad\quad \big(\hat{a}_t(X_t^*, \mathbb{E}_{\theta^*}[X_t^*]) - \hat{a}_t(X_t^*, \mathbb{E}_\theta[X_t^*])\big)\Big] \qquad (5.3.18)\end{aligned}$$

by definition of $\theta^* \in \arg\min_{\theta\in\Theta} R(\theta)$, and as $K_t < 0$. In the case when there is no uncertainty on the drift, i.e., for any $\theta = (b,\rho) \in \mathcal{V}_\Theta$, b is a constant equal to b^o, the dynamics of X^* under any $\mathbb{P}^\theta$, $\theta \in \mathcal{V}_\Theta$, is given by

$$dX_t^* \quad = \quad \big[x_0 + \frac{e^{R(\theta^*)T}}{2\lambda} - X_t^*\big](b^o)^\intercal \Sigma(\rho^*)^{-1}\big[b^o dt + \sigma(\rho_t)dW_t^\theta\big],$$

from which, we deduce by taking expectation under $\mathbb{P}^\theta$:

$$\mathbb{E}_\theta[X_t^*] \quad = \quad x_0 + \frac{e^{R(\theta^*)T}}{2\lambda}\big[1 - e^{-R(\theta^*)t}\big].$$

This means that the expectation under $\mathbb{P}^\theta$ of the optimal wealth process X^* does not depend on $\theta \in \mathcal{V}_\Theta$, and the r.h.s. of (5.3.18) is then equal to zero. Therefore, the function $t \mapsto \mathbb{E}_\theta[V_t^{\alpha^*,\theta}]$ is nondecreasing for all $\theta \in \mathcal{V}_\Theta$, which implies in particular condition (iii) of Lemma 5.3.1.

However, in the case of drift uncertainty, we cannot conclude as above, and actually this nondecreasing property does not always hold true. Indeed, consider for example the case where there is only drift uncertainty in a single asset model $d = 1$, with $\Theta = \{\theta \in [\underline{b}, \bar{b}]\}$, $0 \le \underline{b} < \bar{b}$, and known variance Σ^o normalized to one. Notice that $R(\theta) = \theta^2$, and $\theta^* = \arg\min_{\theta\in\Theta} R(\theta) = \underline{b}$. For any $\theta \in \Theta$, we can compute explicitly from (5.3.14) the expectation and variance of X^* under $\mathbb{P}^\theta$:

$$\begin{aligned}
\mathbb{E}_\theta[X_t^*] \quad &= \quad \frac{1}{2\lambda}e^{R(\theta^*)T}\big[1 - e^{-\theta\theta^* t}\big], \\
\mathrm{Var}_\theta(X_t^*) \quad &= \quad \frac{1}{4\lambda^2}e^{2R(\theta^*)T}\big[e^{(R(\theta^*)-2\theta\theta^*)t} - e^{-2\theta\theta^* t}\big].
\end{aligned}$$

Plugging into (5.3.17), and using also the expression of K, $\hat{a}$ in (5.3.11), (5.3.13), we have for all $\theta \in \Theta$, $t \in [0,T]$, after some straightforward rearrangement:

$$\begin{aligned}
\frac{d\mathbb{E}_\theta[V_t^{\alpha^*,\theta}]}{dt} \quad &= \quad \frac{1}{2\lambda}e^{R(\theta^*)T}\Big[ce^{-2ct} - e^{-R(\theta^*)t}(1 - e^{-ct})\Big(\frac{R(\theta^*)}{2} - \big(\frac{R(\theta^*)}{2} + c\big)e^{-ct}\Big)\Big] \\
&=: \quad f(t,c),
\end{aligned}$$

where we set $c = (\theta - \theta^*)\theta^* \ge 0$. Now, we easily see that for all $t \in [0,T]$, $f(t,c)$ converges to $-\frac{R(\theta^*)}{4\lambda}e^{R(\theta^*)(T-t)} < 0$, as c goes to infinity. Then, by continuity of f with respect to c, we deduce that for θ large enough (hence for c large enough), $\frac{d\mathbb{E}_\theta[V_t^{\alpha^*,\theta}]}{dt}$ is negative, which means that the function $t \mapsto \mathbb{E}_\theta[V_t^{\alpha^*,\theta}]$ is not nondecreasing for all $\theta \in \Theta$. Actually, we have proved in Theorem 5.3.1 the weaker condition (iii) of Lemma 5.3.1 that $V_0^{\alpha^*,\theta} \le \mathbb{E}_\theta[V_T^{\alpha^*,\theta}]$, for all $\theta \in \mathcal{V}_\Theta$. $\Diamond$

5.4 Applications and examples

We provide in this section several examples for the determination of the minimal risk premium arising from the separation principle in Theorem 5.3.1, and the implications for the optimal robust portfolio strategy and the portfolio diversification.

5.4.1 Minimal risk premium and worst-case scenario

We compute the infimum of the prior risk premium function $\theta \in \Theta \mapsto R(\theta)$ as defined in (5.2.1), and (when it exists) the element $\theta^* \in \Theta$ which achieves this minimum, i.e., the worst-case scenario for uncertain parameters. According to condition **(HΘ)**, we distinguish the case of ellipsoidal set and rectangular sets for the ambiguity parameter Θ.

5.4.1.1 Case of ellipsoidal ambiguity parameter set

In this paragraph, we consider Θ in the form:

$$\Theta \;=\; \{(b,\rho) \in \mathbb{R}^d \times \Gamma : \|\sigma(\rho)^{-1}(b-\hat{b})\|_2 \leq \delta\}, \tag{5.4.1}$$

as in **(H Θ)**, and denote by $\hat{\beta}_i := \frac{\hat{b}_i}{\sigma_i}$ the Sharpe ratio of the i-th asset associated with a *priori* expected rate of return $\hat{b}_i$, and marginal volatility $\sigma_i > 0$, $i = 1, \ldots, d$. We assume w.l.o.g that $|\hat{\beta}_i|$ is in descending order: $|\hat{\beta}_1| \geq |\hat{\beta}_2| \geq \ldots |\hat{\beta}_d|$, and define the Sharpe ratio "proximity" between asset i and asset j by

$$\hat{\varrho}_{ij} \;=\; \frac{\hat{\beta}_j}{\hat{\beta}_i} \in [-1,1], \quad 1 \leq i < j \leq d. \tag{5.4.2}$$

with the convention that $\hat{\varrho}_{ij} = 0$ when $\hat{\beta}_i = 0$.

Lemma 5.4.1. *Let Θ be an ellipsoidal set , and assume that there exists $\rho^* \in \arg\min_{\rho \in \Gamma} R(\hat{b},\rho)$ $= \arg\min_{\rho\in\Gamma} \left\|\sigma(\rho)^{-1}\hat{b}\right\|_2$. Then $\theta^* = (b^*, \rho^*)$ with*

$$\begin{aligned} b^* &= \Big(1 - \frac{\delta}{\|\sigma(\rho^*)^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}}\, \hat{b}, \\ \textit{and} \quad R(\theta^*) &= \big(\|\sigma(\rho^*)^{-1}\hat{b}\|_2 - \delta\big)^2 1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}}. \end{aligned}$$

Proof. See proof of Lemma 5.5.2 in Appendix. □

Remark 5.4.1. The existence of ρ^* is guaranteed when Γ is a compact set of $\mathbb{C}^d_{>+}$ by continuity of the function $\rho \mapsto \left\|\sigma(\rho)^{-1}\hat{b}\right\|_2$. We also show in Proposition 5.4.1 its existence when $\Gamma = \mathbb{C}^d_{>+}$, and under the condition that there exists a highest *priori* Sharpe ratio. ♢

In the particular case when there is full ambiguity about the correlation, i.e. $\Gamma = \mathbb{C}^d_{>+}$, and there is an asset with a *priori* highest (absolute value) Sharpe ratio, one can compute explicitly the worst-case scenario $\rho^* \in \Theta$ for correlation.

Proposition 5.4.1 (Full ambiguity correlation)**.** *Let Θ as in* (5.4.1)*, with $\Gamma = \mathbb{C}^d_{>+}$, and assume that $|\hat{\beta}_1| > \max_{j\neq 1} |\hat{\beta}_j|$. Then, we have $\arg\min_{\Theta} R(b,\rho) \neq \varnothing$ and b^*, $\rho^* = (\rho^*_{ij})_{1\leq i<j\leq d} \in \mathbb{C}^d_{>+}$ with*

$$\rho^*_{1j} \;=\; \hat{\varrho}_{1j}, \qquad b^* \;=\; (1 - \frac{\delta}{|\hat{\beta}_1|}) 1_{\{|\hat{\beta}_1| > \delta\}} \hat{b}, \quad 1 < j \leq d, \tag{5.4.3}$$

and

$$R(\theta^*) \;=\; (|\hat{\beta}_1| - \delta)^2 1_{\{|\hat{\beta}_1| > \delta\}}.$$

In particular, when $|\hat{\beta}_1| > |\hat{\beta}_2| > \ldots > |\hat{\beta}_d|$, then $\rho^*_{ij} = \hat{\varrho}_{ij}$, $1 \leq i < j \leq d$, and the associated correlation matrix $C(\rho^*)$ is positive definite.

Proof. See 5.5.3 in Appendix. □

Remark 5.4.2. Proof of proposition 5.4.1 states that $R(\hat{b},\rho)$ has minimum value over $\mathbb{C}^d_{>+}$ iff $|\hat{\beta}_1| > \max_{j\neq 1}|\hat{\beta}_j|$. If there are more than one greatest (absolute) Sharpe ratio, $R(\hat{b},\rho)$ doesn't have minimum value. For example, if $d=2$, $\hat{\beta}_1=\hat{\beta}_2$, then $\mathbb{C}^2_{>+}=(-1,1)$, $R(\hat{b},\rho)=\frac{2|\hat{\beta}_1|^2}{1+\rho}$. We obtain $\inf_{\mathbb{C}^2_{>+}} R(\hat{b},\rho) = |\hat{\beta}_1|^2$ but it can't be attained. Therefore, the case with more than one greatest(absolute) Sharpe ratio under full ambiguity correlation can not be tackled with our approach. ♢

We now consider a model for two-risky assets, i.e. with $d=2$, mixing partial ambiguity about correlation and drift uncertainty. In this case, the following result provides the explicit expression of the worst-case scenario achieving the minimal risk premium.

Proposition 5.4.2 (Ambiguous drift and correlation in the two-assets case)**.** *Let* $\Theta=\{(b,\rho)\in\mathbb{R}^2\times[\underline{\rho},\bar{\rho}] : \|\sigma(\rho)^{-1}(b-\hat{b})\|_2\leq\delta\}$, *with* $-1<\underline{\rho}\leq\bar{\rho}<1$, *and assume w.l.o.g that* $|\hat{\beta}_1|\geq|\hat{\beta}_2|$, $(\hat{\beta}_1,\hat{\beta}_2)\neq(0,0)$. *Recall that* $\hat{\rho}_{12} := \frac{\hat{\beta}_2}{\hat{\beta}_1}$. *Then,*

1. *If* $\hat{\varrho}_{12}\in[\underline{\rho},\bar{\rho}]$, *then* $\rho^*=\hat{\varrho}_{12}$, *and* $b^*=\hat{b}(1-\frac{\delta}{|\hat{\beta}_1|})1_{\{|\hat{\beta}_1|>\delta\}}$.
2. *If* $\bar{\rho}<\hat{\varrho}_{12}$, *then* $\rho^*=\bar{\rho}$, *and* $b^*=\hat{b}(1-\frac{\delta}{\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2>\delta\}}$.
3. *If* $\underline{\rho}>\hat{\varrho}_{12}$, *then* $\rho^*=\underline{\rho}$, *and* $b^*=\hat{b}(1-\frac{\delta}{\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2>\delta\}}$.

Proof. See 5.5.4 in Appendix. □

Remark 5.4.3. The computation of the worst-case correlation ρ^* is determined according to three cases depending on the relation between the Sharpe ratio proximity $\hat{\rho}_{12}$ and the two correlation bounds $\underline{\rho}$ and $\bar{\rho}$.

In the first case when the range of correlation ambiguity is large enough so that $\hat{\rho}_{12}\in[\underline{\rho},\bar{\rho}]$, or in other words, no stock is clearly dominating the other one in terms of Sharpe ratio, then the worst-case correlation is attained at the point $\hat{\rho}_{12}$ inside the interval $[\underline{\rho},\bar{\rho}]$.

In the second case, when $\bar{\rho}<\hat{\varrho}_{12}$, meaning that both assets have close Sharpe ratios with a correlation upper bound not too large, then the worst-case correlation is attained at the prior highest correlation $\bar{\rho}$.

In the third case, when $\underline{\rho}>\hat{\varrho}_{12}$, meaning that Sharpe ratios of the two assets are rather distinctive with respect to the correlation lower bound, then the worst-case correlation is given by the prior lowest correlation $\underline{\rho}$. ♢

We finally consider a model for three-risky assets ($d=3$) mixing partial ambiguity about correlation and drift uncertainty, hence with Θ in the form $\Theta=\{(b,\rho)\in\mathbb{R}^3\times\Gamma : \|\sigma(\rho)^{-1}(b-\hat{b})\|_2\leq\delta\}$ with $\Gamma=[\underline{\rho}_{12},\bar{\rho}_{12}]\times[\underline{\rho}_{13},\bar{\rho}_{13}]\times[\underline{\rho}_{23},\bar{\rho}_{23}]$, a subset of $\mathbb{C}^3_{>+}$.

Recall that

$$\hat{\varrho}_{12} := \frac{\hat{\beta}_2}{\hat{\beta}_1}\in[-1,1],\quad \hat{\varrho}_{13} := \frac{\hat{\beta}_3}{\hat{\beta}_1}\in[-1,1],\quad \hat{\varrho}_{23} := \frac{\hat{\beta}_3}{\hat{\beta}_2}\in[-1,1]. \tag{5.4.4}$$

We introduce the so-called variance risk ratio $\hat{\kappa}(\rho)$,

$$\Sigma(\rho)^{-1}\hat{b} =: \hat{\kappa}(\rho) = (\hat{\kappa}^1(\rho),\hat{\kappa}^2(\rho),\hat{\kappa}^3(\rho))^{\intercal}, \tag{5.4.5}$$

which represents (up to a scalar term) the vector of allocation in the assets when the drift is $\hat{b}$ and the correlation is ρ.

We denote by $\hat{b}_{-i}$ the prior expected rate of return $\hat{b}$ with the i-th component $\hat{b}_i$ removed, and by $\Sigma_{-i}(\rho)$ the covariance matrix $\Sigma(\rho)$ with i-th row and i-th column removed, and $\sigma_{-i}(\rho) = \Sigma_{-i}(\rho)^{\frac{1}{2}}$. Notice that $\Sigma_{-1}(\rho)$ depends only on ρ_{23}. We will write $\Sigma_{-1}(\rho)$ as $\Sigma_{-1}(\rho_{23})$, similarly, $\Sigma_{-2}(\rho_{13})$, $\Sigma_{-3}(\rho_{12})$.

In this case, the following result provides the explicit expression of the worst-case scenario achieving the minimal risk premium.

Proposition 5.4.3 (Ambiguous drift and correlation in the three-asset case)**.** *Let* $\Theta = \{(b, \rho) \in \mathbb{R}^3 \times \Gamma : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \le \delta\}$ *with* $\Gamma = [\underline{\rho}_{12}, \bar{\rho}_{12}] \times [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}] \subset \mathbb{C}^3_{>+}$. *Then, we have the following possible exclusive cases:*

1. *(High-level correlation ambiguity for the second and third assets)*

If $\hat{\varrho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$, $\hat{\varrho}_{13} \in [\underline{\rho}_{13}, \bar{\rho}_{13}]$, *then* $\rho^* = (\hat{\varrho}_{12}, \hat{\varrho}_{13}, \rho^*_{23})$ *for any* $\rho^*_{23} \in [\underline{\rho}_{23}, \bar{\rho}_{23}]$, *and* $b^* = \hat{b}(1 - \frac{\delta}{|\hat{\beta}_1|})1_{\{|\hat{\beta}_1| > \delta\}}$.

2. *(High-level correlation ambiguity for the third asset)*

(i) If $\bar{\rho}_{12} < \hat{\varrho}_{12}$, $\hat{\kappa}^3(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^3(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \le 0$, *then* $\rho^* = (\bar{\rho}_{12}, \rho^*_{13}, \rho^*_{23})$ *satisfying* $\hat{\kappa}^3(\bar{\rho}_{12}, \rho^*_{13}, \rho^*_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2})1_{\{\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 > \delta\}}$.

(ii) If $\underline{\rho}_{12} > \hat{\varrho}_{12}$, $\hat{\kappa}^3(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^3(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \le 0$, *then* $\rho^* = (\underline{\rho}_{12}, \rho^*_{13}, \rho^*_{23})$ *satisfying* $\hat{\kappa}^3(\underline{\rho}_{12}, \rho^*_{13}, \rho^*_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2})1_{\{\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 > \delta\}}$.

3. *(High-level correlation ambiguity for the second asset)*

(i) If $\bar{\rho}_{13} < \hat{\varrho}_{13}$, $\hat{\kappa}^2(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^2(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \le 0$, *then* $\rho^* = (\rho^*_{12}, \bar{\rho}_{13}, \rho^*_{23})$ *satisfying* $\hat{\kappa}^2(\rho^*_{12}, \bar{\rho}_{13}, \rho^*_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2})1_{\{\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2 > \delta\}}$.

(ii) If $\underline{\rho}_{13} > \hat{\varrho}_{13}$, $\hat{\kappa}^2(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^2(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \le 0$, *then* $\rho^* = (\rho^*_{12}, \underline{\rho}_{13}, \rho^*_{23})$ *satisfying* $\hat{\kappa}^2(\rho^*_{12}, \underline{\rho}_{13}, \rho^*_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2})1_{\{\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2 > \delta\}}$.

4. *(High-level correlation ambiguity for the first asset)*

(i) If $\bar{\rho}_{23} < \hat{\varrho}_{23}$, $\hat{\kappa}^1(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^1(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \le 0$, *then* $\rho^* = (\rho^*_{12}, \rho^*_{13}, \bar{\rho}_{23})$ *satisfying* $\hat{\kappa}^1(\rho^*_{12}, \rho^*_{13}, \bar{\rho}_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2})1_{\{\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2 > \delta\}}$.

(ii) If $\underline{\rho}_{23} > \hat{\varrho}_{23}$, $\hat{\kappa}^1(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})\hat{\kappa}^1(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \le 0$, *then* $\rho^* = (\rho^*_{12}, \rho^*_{13}, \underline{\rho}_{23})$ *satisfying* $\hat{\kappa}^1(\rho^*_{12}, \rho^*_{13}, \underline{\rho}_{23}) = 0$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2})1_{\{\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2 > \delta\}}$.

5. *(Small ambiguity about correlation)*

(i) If $\hat{\kappa}^1\hat{\kappa}^2(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, $\hat{\kappa}^1\hat{\kappa}^3(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, *then* $\rho^* = (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})^{-1}\hat{b}\|_2 > \delta\}}$.

(ii) If $\hat{\kappa}^1\hat{\kappa}^2(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) < 0$, $\hat{\kappa}^1\hat{\kappa}^3(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) < 0$, *then* $\rho^* = (\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})^{-1}\hat{b}\|_2 > \delta\}}$.

(iii) If $\hat{\kappa}^1\hat{\kappa}^2(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) > 0$, $\hat{\kappa}^1\hat{\kappa}^3(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) < 0$, *then* $\rho^* = (\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})^{-1}\hat{b}\|_2 > \delta\}}$.

(iv) If $\hat{\kappa}^1\hat{\kappa}^2(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) < 0$, $\hat{\kappa}^1\hat{\kappa}^3(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) > 0$, *then* $\rho^* = (\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$, *and* $b^* = \hat{b}(1 - \frac{\delta}{\|\sigma(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})^{-1}\hat{b}\|_2})1_{\{\|\sigma(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})^{-1}\hat{b}\|_2 > \delta\}}$.

Proof. See 5.5.5 in Appendix. □

Remark 5.4.4. The different cases in the above proposition depend on the relation between the Sharpe ratio proximities and the correlation intervals bounds, and can be roughly divided into 5 cases with subcases with the following interpretation:

In case **1.** where the range of correlation ambiguity for the second and third asset is large enough, in the sense that the intervals $[\underline{\rho}_{12}, \bar{\rho}_{12}]$ and $[\underline{\rho}_{13}, \bar{\rho}_{13}]$ contain respectively $\hat{\rho}_{12}$ and $\hat{\rho}_{13}$, then the worst-case correlation is attained at the Sharpe ratio proximity value $\rho^* = (\hat{\varrho}_{12}, \hat{\varrho}_{13}, \rho^*_{23})$.

Let us now discuss case **2.**, and more specifically (i). This corresponds to the situation where the assets 1 and 2 have close Sharpe ratios with a correlation upper bound between these assets not too large, while the correlation ambiguity for the third asset is high, which is quantified by the fact that the function $(\rho_{13}, \rho_{23}) \mapsto \hat{\kappa}(\bar{\rho}_{12}, \rho_{13}, \rho_{23})$ evaluated at the prior lower bounds $(\underline{\rho}_{13}, \underline{\rho}_{23})$ and the prior upper bounds $(\bar{\rho}_{13}, \bar{\rho}_{23})$ have opposite signs. In this case, the worst-case correlation is achieved at the prior highest correlation $\bar{\rho}_{12}$ for ρ_{12}, and at the point $(\rho^*_{13}, \rho^*_{23})$ cancelling the term $\hat{\kappa}(\bar{\rho}_{12}, \rho^*_{13}, \rho^*_{23})$. Similar interpretations hold for cases **3.** and **4.**

Let us finally discuss case **5.**, which involves explicitly the signs of $\hat{\kappa}^1\hat{\kappa}^2$ and $\hat{\kappa}^1\hat{\kappa}^3$ at the prior correlation bounds. Assuming that these functions $\hat{\kappa}^1\hat{\kappa}^2$ and $\hat{\kappa}^1\hat{\kappa}^3$ do not vanish at some point $\rho \in [\underline{\rho}_{12}, \bar{\rho}_{12}] \times [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}]$, then by continuity, and provided that the range of these correlation bounds are small enough, we see that one should fall into one of the 4 subcases **5.**(i), (ii), (iii), (iv), and for which the worst-case correlation is obtained on the prior upper or lower correlation bounds. ◊

5.4.1.2 Case of rectangular ambiguity parameter set

In this paragraph, we consider Θ in the form

$$\Theta \;=\; \Delta \times \Gamma \;:=\; \prod_{i=1}^{d} [\underline{b}_i, \bar{b}_i] \times \Gamma, \tag{5.4.6}$$

for some constants $0 \leq \underline{b}_i \leq \bar{b}_i$, $i = 1, \dots, d$, and Γ is a convex set of $\mathbb{C}^d_{>+}$, as in $(\mathbf{H\Theta})$(i). For any $b = (b_i)_{i=1,\dots,d} \in \Delta$, we denote by $\beta = (\beta_i)_{i=1,\dots,d}$ with $\beta_i = \frac{b_i}{\sigma_i}$, and also set $\underline{\beta}_i = \frac{\underline{b}_i}{\sigma_i}$, minimum Sharpe ratio of asset i, $\bar{\beta}_i = \frac{\bar{b}_i}{\sigma_i}$, maximum Sharpe ratio of asset i, for $1 \leq i \leq d$.

We focus first on the particular case of full ambiguity correlation, i.e., $\Gamma = \mathbb{C}^d_{>+}$, and compute explicitly the worst-case scenario.

Proposition 5.4.4 (Full ambiguity correlation)**.** *Let Θ be a rectangular set $\Theta = \prod_{i=1}^d [\underline{b}_i, \bar{b}_i] \times \mathbb{C}^d_{>+}$, with $0 \leq \underline{b}_i \leq \bar{b}_i$, $i = 1, \dots, d$, and assume that there exists $\underline{\beta}_{i_1} > \max_{j \neq i_1} \underline{\beta}_j$ for some $i_1 \in \{1, \dots, d\}$, hence w.l.o.g., $i_1 = 1$. Then*

$$b^*_1 \;=\; \underline{b}_1, \;\; b^*_j \;=\; \frac{\underline{b}_1 \sigma_j}{\sigma_1} \rho^*_{1j} \;\;, \quad \text{for any} \quad \rho^*_{1j} \in \Big[\frac{\underline{\beta}_j}{\underline{\beta}_1}, \min(1, \frac{\bar{\beta}_j}{\underline{\beta}_1})\Big), \; j \neq 1, \tag{5.4.7}$$

and

$$R(\theta^*) = \underline{\beta}_1^2. \tag{5.4.8}$$

Proof. See 5.5.6 in Appendix. □

Remark 5.4.5. When $\Gamma = \mathbb{C}^d_{>+}$, this means the investor has no confidence on the correlation estimates. In this case, the worst-case scenario refers to the lower bound of the rate of return of the asset with the highest instantaneous minimum Sharpe ratio. $\Diamond$

We next consider a model for two-risky assets mixing ambiguous correlation and drift uncertainty, hence with $d = 2$. The following result provides the explicit expression of θ^* achieving the minimal risk premium.

Proposition 5.4.5 (Ambiguous drift and correlation in the two-assets case)**.** *Let* $\Theta = \prod_{i=1}^2 [\underline{b}_i, \bar{b}_i] \times [\underline{\rho}, \bar{\rho}]$*, and assume that* $\underline{\beta}_1 \geq \underline{\beta}_2 > 0$*. Then,*

1. *If* $\underline{\rho} \leq \min\big(\frac{\bar{\beta}_2}{\underline{\beta}_1}, 1\big)$ *and* $\bar{\rho} \geq \frac{\underline{\beta}_2}{\underline{\beta}_1}$*, then* $b^* = (\underline{b}_1, \frac{\underline{b}_1 \sigma_2}{\sigma_1}\rho^*)$*, for any* $\rho^* \in [\underline{\rho}, \bar{\rho}] \cap [\frac{\underline{\beta}_2}{\underline{\beta}_1}, \min(\frac{\bar{\beta}_2}{\underline{\beta}_1}, 1)]$.
2. *If* $\bar{\rho} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$*, then* $b^* = (\underline{b}_1, \underline{b}_2)$*,* $\rho^* = \bar{\rho}$.
3. *If* $\underline{\rho} > \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})$*, then* $b^* = (\underline{b}_1, \bar{b}_2)$*,* $\rho^* = \underline{\rho}$.

Proof. See 5.5.7 in Appendix. $\square$

Remark 5.4.6. As in Remark 5.4.3, the separation of the three cases depends on the relation between the Sharpe ratio proximities and the correlation interval bounds. Under the assumption $\underline{\beta}_1 \geq \underline{\beta}_2 > 0$, since the investor is looking for the worst case, she only takes the minimum Sharpe ratio for the first asset.

In the first case, the Sharpe ratios of the two assets don't have a proximate relation or a dominating relation, the investor is not sure of making directional trading or spread trading. In Section 4.2.2, we will see that in this case, the investor will only form a portfolio with the first asset, which has a larger positive Sharpe ratio. So the second asset is irrelevant, the optimal correlation can takes any value in an interval and the optimal drift b^* only depends on the minimum Sharpe ratio of the first asset.

In the second case, when $\bar{\rho} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$, the minimum Sharpe ratio of the two asset $\underline{\beta}_1$ and $\underline{\beta}_2$ are similar compared to the correlation upper bound $\bar{\rho}$. The investor will hold similar long positions in these assets (known as directional trading), thus the optimal drift $b^* = (\underline{b}_1, \underline{b}_2)$ and the optimal correlation is the upper bound.

In the third case, we have $\underline{\rho} > \frac{\bar{\beta}_2}{\underline{\beta}_1}$, which means that the maximum Sharpe ratio of the second asset is smaller than the minimum Sharpe ratio of the first asset compared to the correlation lower bound $\underline{\rho}$. The first asset dominates the second asset in term of Sharpe ratio. The investor will hold opposite positions in these assets (known as spread trading), i.e. long in the first asset and short in the second asset, thus the optimal drift $b^* = (\underline{b}_1, \bar{b}_2)$ and the optimal correlation is the lower bound. $\Diamond$

We finally consider a model for three-risky assets mixing ambiguous correlation and drift uncertainty, hence with $d = 3$, and Θ in the form $\Theta = \prod_{i=1}^3 [\underline{b}_i, \bar{b}_i] \times [\underline{\rho}_{12}, \bar{\rho}_{12}] \times [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}]$ with $[\underline{\rho}_{12}, \bar{\rho}_{12}] \times [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}] \subset \mathbb{C}^3_{>+}$.

We define $\kappa(b, \rho)$ as

$$\Sigma(\rho)^{-1} b =: \kappa(b, \rho) = (\kappa^1(b, \rho), \kappa^2(b, \rho), \kappa^3(b, \rho))^\intercal. \tag{5.4.9}$$

The following result provides the explicit expression of θ^* achieving the minimal risk premium.

Proposition 5.4.6 (Ambiguous drift and correlation in the 3-assets case)**.** *Let* $\Theta = \prod_{1\le i\le 3}[\underline{b}_i, \bar{b}_i] \times \prod_{1\le i<j\le 3}[\underline{\rho}_{ij}, \bar{\rho}_{ij}]$ *and assume that* $\underline{\beta}_1 \ge \underline{\beta}_2 \ge \underline{\beta}_3 > 0$. *Then,*

1. *(High-level ambiguity about drift and correlation for the second and third assets) If* $\underline{\rho}_{12} \le \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})$, $\bar{\rho}_{12} \ge \frac{\underline{\beta}_2}{\underline{\beta}_1}$, $\underline{\rho}_{13} \le \min(\frac{\bar{\beta}_3}{\underline{\beta}_1}, 1)$ *and* $\bar{\rho}_{13} \ge \frac{\underline{\beta}_3}{\underline{\beta}_1}$, *then* $b_1^* = \underline{b}_1$, $b_2^* = \frac{b_1\sigma_2}{\sigma_1}\rho_{12}^*$, $b_3^* = \frac{b_1\sigma_3}{\sigma_1}\rho_{13}^*$ *for any* $\rho_{12}^* \in [\underline{\rho}_{12}, \bar{\rho}_{12}] \cap [\frac{\underline{\beta}_2}{\underline{\beta}_1}, \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})]$, $\rho_{13}^* \in [\underline{\rho}_{13}, \bar{\rho}_{13}] \cap [\frac{\underline{\beta}_2}{\underline{\beta}_1}, \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})]$ *and* $\rho_{23}^* \in [\underline{\rho}_{23}, \bar{\rho}_{23}]$.

2. *(High-level ambiguity about drift and correlation for the third asset)*
   - *(i) If* $\bar{\rho}_{12} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$, $\kappa^3(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \ge 0 \ge \kappa^3(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *then* $b^* = (\underline{b}_1, \underline{b}_2, b_3^*)$, $\rho^* = (\bar{\rho}_{12}, \rho_{13}^*, \rho_{23}^*)$ *with* b_3^*, ρ_{13}^*, ρ_{23}^* *satisfying* $\kappa^3(\underline{b}_1, \underline{b}_2, b_3^*, \bar{\rho}_{12}, \rho_{13}^*, \rho_{23}^*) = 0$.
   - *(ii) If* $\underline{\rho}_{12} > \min(\frac{\bar{\beta}_2}{\underline{\beta}_1}, 1)$, $\kappa^3(\underline{b}_1, \bar{b}_2, \bar{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \ge 0 \ge \kappa^3(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$, *then* $b^* = (\underline{b}_1, \bar{b}_2, b_3^*)$, $\rho^* = (\underline{\rho}_{12}, \rho_{13}^*, \rho_{23}^*)$ *with* b_3^*, ρ_{13}^*, ρ_{23}^* *satisfying* $\kappa^3(\underline{b}_1, \bar{b}_2, b_3^*, \underline{\rho}_{12}, \rho_{13}^*, \rho_{23}^*) = 0$.

3. *(High-level ambiguity about drift and correlation for the second asset)*
   - *(i) If* $\bar{\rho}_{13} < \frac{\underline{\beta}_3}{\underline{\beta}_1}$, $\kappa^2(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \ge 0 \ge \kappa^2(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *then* $b^* = (\underline{b}_1, b_2^*, \underline{b}_3)$, $\rho^* = (\rho_{12}^*, \bar{\rho}_{13}, \rho_{23}^*)$ *with* b_2^*, ρ_{12}^*, ρ_{23}^* *satisfying* $\kappa^2(\underline{b}_1, b_2^*, \underline{b}_3, \rho_{12}^*, \bar{\rho}_{13}, \rho_{23}^*) = 0$.
   - *(ii) If* $\underline{\rho}_{13} > \min(\frac{\bar{\beta}_3}{\underline{\beta}_1}, 1)$, $\kappa^2(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \ge 0 \ge \kappa^2(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$, *then* $b^* = (\underline{b}_1, b_2^*, \bar{b}_3)$, $\rho^* = (\rho_{12}^*, \bar{\rho}_{13}, \rho_{23}^*)$ *with* b_2^*, ρ_{12}^*, ρ_{23}^* *satisfying* $\kappa^2(\underline{b}_1, b_2^*, \bar{b}_3, \rho_{12}^*, \underline{\rho}_{13}, \rho_{23}^*) = 0$.

4. *(High-level ambiguity about drift and correlation for the first asset)*
   - *(i) If* $\bar{\rho}_{23} < \frac{\underline{\beta}_3}{\underline{\beta}_2}$, $\kappa^1(\bar{b}_1, \underline{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \ge 0 \ge \kappa^1(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *then* $b^* = (b_1^*, \underline{b}_2, \underline{b}_3)$, $\rho^* = (\rho_{12}^*, \rho_{13}^*, \bar{\rho}_{23})$ *with* b_1^*, ρ_{12}^*, ρ_{13}^* *with* b_1^*, ρ_{12}^*, ρ_{13}^* *satisfying* $\kappa^1(b_1^*, \underline{b}_2, \underline{b}_3, \rho_{12}^*, \rho_{13}^*, \bar{\rho}_{23}) = 0$.
   - *(ii) If* $\underline{\rho}_{23} > \min(1, \frac{\bar{\beta}_3}{\underline{\beta}_2})$ $\kappa^1(\bar{b}_1, \underline{b}_2, \bar{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \ge 0 \ge \kappa^1(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$, *then* $b^* = (b_1^*, \underline{b}_2, \bar{b}_3)$, $\rho^* = (\rho_{12}^*, \rho_{13}^*, \underline{\rho}_{23})$ *with* b_1^*, ρ_{12}^*, ρ_{13}^* *satisfying* $\kappa^1(b_1^*, \underline{b}_2, \bar{b}_3, \rho_{12}^*, \rho_{13}^*, \underline{\rho}_{23}) = 0$.

5. *(Small ambiguity about drift and correlation)*
   - *(i) If* $\kappa^1(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, $\kappa^2(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, $\kappa^3(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, *then* $b^* = (\underline{b}_1, \underline{b}_2, \underline{b}_3)$ *and* $\rho^* = (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$.
   - *(ii) If* $\kappa^1(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) > 0$, $\kappa^2(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) < 0$, $\kappa^3(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) > 0$, *then* $b^* = (\underline{b}_1, \bar{b}_2, \underline{b}_3)$ *and* $\rho^* = (\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$.
   - *(iii) If* $\kappa^1(\bar{b}_1, \underline{b}_2, \bar{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) < 0$, $\kappa^2(\bar{b}_1, \underline{b}_2, \bar{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) > 0$, $\kappa^3(\bar{b}_1, \underline{b}_2, \bar{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) < 0$, *then* $b^* = (\bar{b}_1, \underline{b}_2, \bar{b}_3)$ *and* $\rho^* = (\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$.
   - *(iv) If* $\kappa^1(\underline{b}_1, \bar{b}_2, \bar{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) > 0$, $\kappa^2(\underline{b}_1, \bar{b}_2, \bar{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) < 0$, $\kappa^3(\underline{b}_1, \bar{b}_2, \bar{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) < 0$, *then* $b^* = (\underline{b}_1, \bar{b}_2, \bar{b}_3)$ *and* $\rho^* = (\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})$.

(v) *If* $\kappa^1(\bar{b}_1, \underline{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) < 0$, $\kappa^2(\bar{b}_1, \underline{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) > 0$, $\kappa^3(\bar{b}_1, \underline{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) > 0$, *then* $b^* = (\bar{b}_1, \underline{b}_2, \underline{b}_3)$ *and* $\rho^* = (\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})$.

(vi) *If* $\kappa^1(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) > 0$, $\kappa^2(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) > 0$, $\kappa^3(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) < 0$, *then* $b^* = (\underline{b}_1, \underline{b}_2, \bar{b}_3)$ *and* $\rho^* = (\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$.

(vii) *If* $\kappa^1(\bar{b}_1, \bar{b}_2, \underline{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) < 0$, $\kappa^2(\bar{b}_1, \bar{b}_2, \underline{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) < 0$, $\kappa^3(\bar{b}_1, \bar{b}_2, \underline{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) > 0$, *then* $b^* = (\bar{b}_1, \bar{b}_2, \underline{b}_3)$ *and* $\rho^* = (\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$.

Remark 5.4.7. For the ellipsoidal set in Remark 5.4.4, the level of drift uncertainty of each asset is the same characterized by the radius δ. If δ is not very large, the portfolio diversification is caused by the level of correlation ambiguity, see Remark 5.4.4, however, for the rectangular set, the level of drift uncertainty of each asset is different for each asset. For simplicity and clarity, we consider the only impact of drift uncertainty, or equivalently Sharpe ratio uncertainty, on the portfolio diversification, i.e. there is no ambiguity on correlation with $\rho_{ij}^o := \underline{\rho}_{ij} = \bar{\rho}_{ij}$, $1 \leq i < j \leq 3$. As in Remark 5.4.4, the different cases in the above proposition depend on the relation between Sharpe ratio proximities and correlation, and can be roughly divided into 5 cases with the following interpretation:

In case **1.** where the range of Sharpe ratio uncertainty for the second and third asset is large enough, in the sense that the intervals $[\frac{\underline{\beta}_2}{\underline{\beta}_1}, \frac{\bar{\beta}_2}{\underline{\beta}_1}]$ and $[\frac{\underline{\beta}_3}{\underline{\beta}_1}, \frac{\bar{\beta}_3}{\underline{\beta}_1}]$ contain respectively correlation ρ_{12}^o and ρ_{13}^o, then the worst-case Sharpe ratio (drift) is attained at the prior lower bound $\underline{\beta}_1$ for β_1, and at the point (β_2^*, β_3^*) $= (\underline{\beta}_1 \rho_{12}^o, \underline{\beta}_1 \rho_{13}^o)$.

Let us now discuss case **2.**, and more specifically (i). This corresponds to the situation where the assets 1 and 2 have close minimum Sharpe ratios with correlation between these assets not too large, while the Sharpe ratio uncertainty for the third asset is high, which is quantified by the fact that the function $b_3 \mapsto \kappa^3(\underline{b}_1, \underline{b}_2, b_3)$ (we omit ρ^o arguments in the function $\kappa^3(\underline{b}_1, \underline{b}_2, b_3, \rho_{12}^o, \rho_{13}^o, \rho_{23}^o)$ for simplicity) evaluated at the prior lower bounds $\underline{b}_3$ and the prior upper bounds $\bar{b}_3$ have opposite signs. In this case, the worst-case drift or Sharpe ratio is achieved at the prior lower bound $\underline{b}_1$ for b_1, prior lower bound $\underline{b}_2$ for b_2, and at the point b_3^* cancelling the term $\kappa(\underline{b}_1, \underline{b}_2, b_3^*)$. Similar interpretations hold for cases **3.** and **4.**

Let us finally discuss case **5.**, which involves explicitly the signs of κ^1, κ^2 and κ^3 at the prior drift or Sharpe ratio bounds. Assuming that these functions κ^1, κ^2 and κ^3 do not vanish at some point $b \in [\underline{b}_1, \bar{b}_1] \times [\underline{b}_2, \bar{b}_2] \times [\underline{b}_3, \bar{b}_3]$, then by continuity, and provided that the range of these correlation bounds are small enough, we see that one should fall into one of the 7 subcases **5.**(i), (ii), (iii), (iv), and for which the worst-case drift is obtained on the prior upper or lower drift (Sharpe ratio) bounds. ◊

5.4.2 Optimal robust strategy and portfolio diversification

We first provide the general explicit expression of the robust optimal strategy in the case of ellipsoidal ambiguity set. This follows directly from Theorem 5.3.1 and Lemma 5.4.1.

Proposition 5.4.7. *Let* Θ *be an ellipsoidal set as in* (5.4.1)*, and assume that there exists* $\rho^* \in \arg\min_{\rho \in \Gamma} \|\sigma(\rho)^{-1}\hat{b}\|_2$. *Then, an optimal portfolio strategy for* (5.2.4) *is given by*

$$\alpha_t^* = \Big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma(\rho^*)^{-1}\hat{b}\|_2 - \delta)^2 T} - X_t^*\Big]\Big(1 - \frac{\delta}{\|\sigma(\rho^*)^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}} \Sigma(\rho^*)^{-1}\hat{b}. \tag{5.4.10}$$

Remark 5.4.8. We have seen in the previous section that ρ^* exists when Γ is compact (in particular when it is a singleton, i.e., there is no ambiguity on correlation) or when $\Gamma = \mathbb{C}^d_{>+}$, i.e., there is full ambiguity on correlation. From (5.4.10), we observe notably that whenever $\delta \geq \|\sigma(\rho^*)^{-1}\hat{b}\|_2$, then $\alpha^* \equiv 0$. In other words, when the level of ambiguity about the expected rate of return is high (or when the investor is poorly confident about her estimation $\hat{b}$ on the expected rate of return), then she does not make risky investment at all. ♢

5.4.2.1 Full ambiguity correlation and anti-diversification

In this paragraph, we consider the case of full ambiguity on correlation, i.e., $\Gamma = \mathbb{C}^d_{>+}$, and investigate the impact on optimal robust portfolio strategy.

Theorem 5.4.1 (Full ambiguity correlation)**. I.** *Let Θ be an ellipsoidal set as in* (5.4.1)*, with $\Gamma = \mathbb{C}^d_{>+}$, and assume that $|\hat{\beta}_1| > \max_{j\neq 1} |\hat{\beta}_j|$. Then an optimal portfolio strategy for the robust mean-variance problem* (5.2.4) *is explicitly given by*

$$\alpha^*_t \quad = \quad \big[x_0 + \frac{1}{2\lambda} e^{(|\hat{\beta}_1|-\delta)^2 T} - X^*_t\big]\Big(1 - \frac{\delta}{|\hat{\beta}_1|}\Big) 1_{|\hat{\beta}_1|>\delta} \big(\frac{\hat{b}_1}{\sigma_1^2}, 0, \ldots, 0\big)^{\intercal}, \quad 0 \leq t \leq T,\ \mathcal{P}^{\Theta} - q.s.$$

II. *Let Θ be a rectangular set as in* (5.4.6)*, with $\Gamma = \mathbb{C}^d_{>+}$, and assume that $\underline{\beta}_1 > \max_{j\neq 1} \underline{\beta}_j$. Then, an optimal portfolio strategy for the robust mean-variance problem* (5.2.4) *is explicitly given by*

$$\alpha^*_t \quad = \quad \big[x_0 + \frac{1}{2\lambda} e^{\underline{\beta}_1^2 T} - X^*_t\big]\big(\frac{b_1}{\sigma_1^2}, 0, \ldots, 0\big)^{\intercal}, \ \ 0 \leq t \leq T,\ \mathcal{P}^{\Theta} - q.s..$$

Proof. I. From the formula (5.4.10) of the optimal portfolio strategy in Proposition 5.4.7, we only have to compute the vector $\hat{\kappa}(\rho^*) = \Sigma(\rho^*)^{-1}\hat{b}$, and $R(\hat{b}, \rho^*)$, which have been already given in (5.5.28), (5.5.29) in Section 5.5.3.
II. In view of the formula (5.3.1) of the optimal portfolio strategy in Theorem 5.3.1, we only have to compute the vector $\kappa(b^*, \rho^*) = \Sigma(\rho^*)^{-1}b^*$, and $R(b^*, \rho^*)$, which have been already given in (5.5.49), (5.5.48) in Section 5.5.6. □

Remark 5.4.9 (Financial interpretation: anti-diversification)**.** If the investor is poorly confident on the drift estimate, i.e., whenever δ is large enough, then she does not make risky investments at all, i.e. $\alpha^*_t \equiv 0$. When the investor has good knowledge of drift estimates but is poorly confident of correlation estimates, she only invests in one asset, namely the one with the highest estimated Sharpe ratio. This anti-diversification result under full ambiguity about correlation has been also observed in [LZ17] for a single-period mean-variance problem without drift uncertainty, and is extended here in a continuous time framework. ♢

5.4.2.2 Partial diversification

• **Two-asset model:** $d = 2$

We provide a complete picture of the optimal robust portfolio strategy in a two-asset model with ambiguous drift and correlation.

Theorem 5.4.2 (Ambiguous drift and correlation in the two-assets case)**. I.** *Let $\Theta = \{(b, \rho) \in \mathbb{R}^2 \times [\underline{\rho}, \bar{\rho}] : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \leq \delta\}$, with $-1 < \underline{\rho} \leq \bar{\rho} < 1$, and assume w.l.o.g that $|\hat{\beta}_1| \geq |\hat{\beta}_2|$, $(\hat{\beta}_1, \hat{\beta}_2) \neq (0, 0)$. We have the following possible cases:*

1. *If $\hat{\varrho}_{12} \in [\underline{\rho}, \bar{\rho}]$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{(|\hat{\beta}_1| - \delta)^2 T} - X_t^*\big]\big(1 - \frac{\delta}{|\hat{\beta}_1|}\big) 1_{\{|\hat{\beta}_1| > \delta\}} \begin{pmatrix} \frac{\hat{b}_1}{\sigma_1^2} \\ 0 \end{pmatrix}, \quad 0 \le t \le T, \ \mathcal{P}^\Theta - q.s.$$

2. *If $\bar{\rho} < \hat{\varrho}_{12}$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2 - \delta)^2 T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2 > \delta\}} \Sigma(\bar{\rho})^{-1}\hat{b},$$

and if $\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2 > \delta$, then $\alpha_t^{1,*}\alpha_t^{2,*} > 0$.

3. *If $\underline{\rho} > \hat{\varrho}_{12}$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2 - \delta)^2 T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2 > \delta\}} \Sigma(\underline{\rho})^{-1}\hat{b},$$

and if $\|\sigma(\underline{\rho})^{-1}\hat{b}\|_2 > \delta$, then $\alpha_t^{1,*}\alpha_t^{2,*} < 0$.

II. *Let $\Theta = \prod_{i=1}^2 [\underline{b}_i, \bar{b}_i] \times [\underline{\rho}, \bar{\rho}]$, and assume that $\underline{\beta}_1 \ge \underline{\beta}_2 > 0$. We have the following possible cases:*

1. *If $\underline{\rho} \le \min(\frac{\bar{\beta}_2}{\underline{\beta}_1}, 1)$ and $\bar{\rho} \ge \frac{\underline{\beta}_2}{\underline{\beta}_1}$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{\underline{\beta}_1^2 T} - X_t^*\big] \begin{pmatrix} \frac{\underline{b}_1}{\sigma_1^2} \\ 0 \end{pmatrix}, \quad 0 \le t \le T, \ \mathcal{P}^\Theta - q.s.,$$

2. *If $\bar{\rho} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{\|\sigma(\bar{\rho})^{-1}(\underline{b}_1, \underline{b}_2)^\intercal\|_2^2 T} - X_t^*\big]\Sigma(\bar{\rho})^{-1} \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \end{pmatrix}, \quad 0 \le t \le T, \ \mathcal{P}^\Theta - q.s.,$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{2,*} > 0$.

3. *If $\underline{\rho} > \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})$, then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda} e^{\|\sigma(\underline{\rho})^{-1}(\underline{b}_1, \bar{b}_2)^\intercal\|_2^2 T} - X_t^*\big]\Sigma(\underline{\rho})^{-1} \begin{pmatrix} \underline{b}_1 \\ \bar{b}_2 \end{pmatrix}, \quad 0 \le t \le T, \ \mathcal{P}^\Theta - q.s.,$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{2,*} < 0$.

Proof. I. In light of formula (5.4.10) of the optimal portfolio strategy in Proposition 5.4.7, we only need to compute $\|\sigma(\rho^*)^{-1}\hat{b}\|_2$, i.e., $R(\hat{b}, \rho^*)$, vector $\hat{\kappa}(\rho^*) = \Sigma(\rho^*)^{-1}\hat{b}$, explicitly given in the proof of Proposition 5.4.2 (see 5.5.4 in Appendix) when computing ρ^*, which leads to the three cases of Theorem 5.4.2 **I.**.
II. In light of formula (5.3.1) of the optimal portfolio strategy in Theorem 5.3.1, we only need to compute $\Sigma(\rho^*)^{-1}b^*$, $R(b^*, \rho^*)$, explicitly given in the proof of Proposition 5.4.5 (see 5.5.7 in Appendix) when computing (b^*, ρ^*), which lead to the three possible cases of Theorem 5.4.2 **II.**. □

Remark 5.4.10. When there is no ambiguity on the drift, which corresponds to $\delta = 0$ or $\underline{b}_i = \bar{b}_i$, $i = 1$, 2, we retrieve the results obtained in [IP17] for the correlation ambiguity between two assets (see their Theorem 4.2). Our Theorem includes in addition the case when there is uncertainty on the expected rate of return. $\Diamond$

Remark 5.4.11 (Financial interpretation)**.** We first look at the ellipsoidal set: In the first case when the range of correlation ambiguity is large enough so that $\hat{\rho}_{12} \in (\underline{\rho}, \bar{\rho})$, and thus no stock is clearly dominating the other one in terms of Sharpe ratio, it is optimal to invest only in one asset, namely the one with the highest estimated Sharpe ratio.

In the second case when $\bar{\rho} < \hat{\rho}_{12}$, this means that no stock is "dominating" the other one in terms of Sharpe ratio, and it is optimal to invest in both assets with a directional trading, that is buying or selling simultaneously, and the worst-case correlation refers to the highest prior correlation $\bar{\rho}$ (recall Remark 5.4.3) where the diversification effect is minimal.

Finally, when $\underline{\rho} > \hat{\rho}_{12}$, this means that one asset is clearly dominating the other one, and it is optimal to invest in both assets with a spread trading, that is buying one and selling another, and the worst-case correlation corresponds to the lowest prior correlation $\underline{\rho}$ where the profit from the spread trading is minimal.

For the rectangular set, the interpretation is similar, and the worst-case drift refers to lower bound of the drift if the asset is in long position and upper bound of the drift if the asset is in short position, see 5.4.6. $\Diamond$

• **Three-asset model:** $d = 3$

We finally provide an explicit description of the optimal robust strategy in a three-asset model under drift uncertainty and ambiguous correlation.

Theorem 5.4.3. I. *Let $\Theta = \{(b, \rho) \in \mathbb{R}^3 \times \Gamma : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \leq \delta\}$ with $\Gamma = [\underline{\rho}_{12}, \bar{\rho}_{12}] \times [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}] \subset \mathbb{C}^3_{>+}$, and assume w.l.o.g that $|\hat{\beta}_1| \geq |\hat{\beta}_2| \geq |\hat{\beta}_3|$, $(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3) \neq (0, 0, 0)$, Then, we have the following possible exclusive cases:*

1. *(Anti-diversification) If $\hat{\varrho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$, and $\hat{\varrho}_{13} \in [\underline{\rho}_{13}, \bar{\rho}_{13}]$, then an optimal portfolio strategy is explicitly given by*

$$\alpha^*_t = \Big[x_0 + \frac{1}{2\lambda} e^{(|\hat{\beta}_1| - \delta)^2 T} - X^*_t\Big]\Big(1 - \frac{\delta}{|\hat{\beta}_1|}\Big) 1_{\{|\hat{\beta}_1| > \delta\}} \begin{pmatrix} \frac{\hat{b}_1}{\sigma_1^2} \\ 0 \\ 0 \end{pmatrix}, \quad 0 \leq t \leq T, \ \mathcal{P}^\Theta - q.s.,$$

2. *(Under-diversification: no investment in the third asset)*

   (i) If $\bar{\rho}_{12} < \hat{\varrho}_{12}$, and $\hat{\kappa}^3(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \hat{\kappa}^3(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is

$$\begin{aligned} \begin{pmatrix} \alpha^{1,*}_t \\ \alpha^{2,*}_t \end{pmatrix} &= \Big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 - \delta)^2 T} - X^*_t\Big]\Big(1 - \frac{\delta}{\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2}\Big) \\ &\qquad 1_{\{\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 > \delta\}} \Sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}, \\ \alpha^{3,*}_t &\equiv 0, \end{aligned}$$

   and if $\|\sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 > \delta$, then $\alpha^{1,*}_t \alpha^{2,*}_t > 0$.

(ii) If $\underline{\rho}_{12} > \hat{\varrho}_{12}$, and $\hat{\kappa}^3(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^3(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is

$$\begin{aligned}\begin{pmatrix}\alpha_t^{1,*}\\ \alpha_t^{2,*}\end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2)-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2}\Big)\\ &\qquad 1_{\{\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2>\delta\}}\Sigma_{-3}(\bar{\rho}_{12})^{-1}\hat{b}_{-3}\\ \alpha_t^{3,*} &\equiv 0,\end{aligned}$$

and if $\|\sigma_{-3}(\underline{\rho}_{12})^{-1}\hat{b}_{-3}\|_2 > \delta$, then $\alpha_t^{1,*}\alpha_t^{2,*} < 0$.

3. *(Under-diversification: no investment in the second asset)*

(i) If $\bar{\rho}_{13} < \hat{\varrho}_{13}$, and $\hat{\kappa}^2(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^2(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is

$$\begin{aligned}\begin{pmatrix}\alpha_t^{1,*}\\ \alpha_t^{3,*}\end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2}\Big)\\ &\qquad 1_{\{\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2>\delta\}}\Sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\\ \alpha_t^{2,*} &\equiv 0,\end{aligned}$$

and if $\|\sigma_{-2}(\bar{\rho}_{13})^{-1}\hat{b}_{-2}\|_2 > \delta$, then $\alpha_t^{1,*}\alpha_t^{3,*} > 0$.

(ii) If $\underline{\rho}_{13} > \hat{\varrho}_{13}$, and $\hat{\kappa}^2(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^2(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is given by

$$\begin{aligned}\begin{pmatrix}\alpha_t^{1,*}\\ \alpha_t^{3,*}\end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2}\Big)\\ &\qquad 1_{\{\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2>\delta\}}\Sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2},\\ \alpha_t^{2,*} &\equiv 0,\end{aligned}$$

and if $\|\sigma_{-2}(\underline{\rho}_{13})^{-1}\hat{b}_{-2}\|_2 > \delta$, then $\alpha_t^{1,*}\alpha_t^{3,*} < 0$.

4. *(Under-diversification: no investment in the second asset)*

(i) If $\bar{\rho}_{23} < \hat{\varrho}_{23}$, and $\hat{\kappa}^1(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23})\hat{\kappa}^1(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is

$$\begin{aligned}\begin{pmatrix}\alpha_t^{2,*}\\ \alpha_t^{3,*}\end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2}\Big)\\ &\qquad 1_{\{\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2>\delta\}}\Sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1},\\ \alpha_t^{1,*} &\equiv 0,\end{aligned}$$

and if $\|\sigma_{-1}(\bar{\rho}_{23})^{-1}\hat{b}_{-1}\|_2 > \delta$, then $\alpha_t^{2,*}\alpha_t^{3,*} > 0$.

(ii) If $\underline{\rho}_{23} > \hat{\varrho}_{23}$, and $\hat{\kappa}^1(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})\hat{\kappa}^1(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \leq 0$, then an optimal portfolio strategy is

$$\begin{aligned}\begin{pmatrix}\alpha_t^{2,*}\\ \alpha_t^{3,*}\end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2}\Big)\\ &\qquad 1_{\{\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2>\delta\}}\Sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\\ \alpha_t^{1,*} &\equiv 0,\end{aligned}$$

and if $\|\sigma_{-1}(\underline{\rho}_{23})^{-1}\hat{b}_{-1}\|_2 > \delta$, then $\alpha_t^{2,*}\ \alpha_t^{3,*} < 0$.

5. *(Well-diversification)*

(i) If $\hat{\kappa}^1\hat{\kappa}^2(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23}) > 0$, *and* $\hat{\kappa}^1\hat{\kappa}^3(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23}) > 0$, *then an optimal portfolio strategy is given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2>\delta\}}\Sigma(\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}.$$

(ii) If $\hat{\kappa}^1\hat{\kappa}^2(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23}) < 0$, *and* $\hat{\kappa}^1\hat{\kappa}^3(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23}) < 0$, *then an optimal portfolio strategy is given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}\|_2>\delta\}}\Sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23})^{-1}\hat{b}.$$

(iii) If $\hat{\kappa}^1\hat{\kappa}^2(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23}) > 0$, *and* $\hat{\kappa}^1\hat{\kappa}^3(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23}) < 0$, *then an optimal portfolio strategy is given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda}e^{(\|\sigma(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2>\delta\}}\Sigma(\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}.$$

(iv) If $\hat{\kappa}^1\hat{\kappa}^2(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23}) < 0$, *and* $\hat{\kappa}^1\hat{\kappa}^3(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23}) > 0$, *then an optimal portfolio strategy is given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda}e^{\|(\sigma(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2-\delta)^2T} - X_t^*\big]\Big(1 - \frac{\delta}{\|\sigma(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2}\Big) 1_{\{\|\sigma(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}\|_2>\delta\}}\Sigma(\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23})^{-1}\hat{b}.$$

II. *Let* $\Theta = \prod_{i=1}^3[\underline{b}_i,\bar{b}_i] \times \prod_{1\le i<j\le 3}[\underline{\rho}_{ij},\bar{\rho}_{ij}]$ *and assume that* $\underline{\beta}_1 \ge \underline{\beta}_2 \ge \underline{\beta}_3 > 0$. *Then, we have the following possible cases:*

1. *(Anti-diversification) If* $\underline{\rho}_{12} \le \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})$, $\bar{\rho}_{12} \ge \frac{\underline{\beta}_2}{\underline{\beta}_1}$ *and* $\underline{\rho}_{13} \le \min(1, \frac{\bar{\beta}_3}{\underline{\beta}_1})$, $\bar{\rho}_{13} \ge \frac{\underline{\beta}_3}{\underline{\beta}_1}$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0 + \frac{1}{2\lambda}e^{\underline{\beta}_1^2T} - X_t^*\big]\begin{pmatrix}\frac{\underline{b}_1}{\sigma_1^2}\\ 0\\ 0\end{pmatrix}, \quad 0 \le t \le T,\ \mathcal{P}^\Theta - q.s.,$$

2. *(Under-diversification: no investment in the third asset)*

(i) If $\bar{\rho}_{12} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$, $\kappa^3(\underline{b}_1,\underline{b}_2,\bar{b}_3,\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23}) \ge 0 \ge \kappa^3(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{pmatrix}\alpha_t^{1,*}\\ \alpha_t^{2,*}\end{pmatrix} = \big[x_0 + \frac{1}{2\lambda}e^{\|\sigma_{-3}(\bar{\rho}_{12})^{-1}(\underline{b}_1,\underline{b}_2)^\top\|_2^2T} - X_t^*\big]\Sigma_{-3}(\bar{\rho}_{12})^{-1}\begin{pmatrix}\underline{b}_1\\ \underline{b}_2\end{pmatrix},$$

$$\alpha_t^{3,*} \equiv 0,$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{2,*} > 0$.

(ii) *If* $\underline{\rho}_{12} > \min(\frac{\bar{\beta}_2}{\underline{\beta}_1}, 1)$, $\kappa^3(\underline{b}_1, \bar{b}_2, \bar{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \geq 0 \geq \kappa^3(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{aligned} \begin{pmatrix} \alpha_t^{1,*} \\ \alpha_t^{2,*} \end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda} e^{\|\sigma_{-3}(\underline{\rho}_{12})^{-1}(\underline{b}_1, \bar{b}_2)^\top\|_2^2 T} - X_t^*\big] \Sigma_{-3}(\underline{\rho}_{12})^{-1} \begin{pmatrix} \underline{b}_1 \\ \bar{b}_2 \end{pmatrix}, \\ \alpha_t^{3,*} &\equiv 0, \end{aligned}$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{2,*} < 0$.

3. *(Under-diversification: no investment in the second asset)*

(i) *If* $\bar{\rho}_{13} < \frac{\underline{\beta}_3}{\underline{\beta}_1}$, $\kappa^2(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \geq 0$ $\kappa^2(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{aligned} \begin{pmatrix} \alpha_t^{1,*} \\ \alpha_t^{3,*} \end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma_{-2}(\bar{\rho}_{13})^{-1}(\underline{b}_1, \underline{b}_3)^\top\|_2^2 T} - X_t^*\big] \Sigma_{-2}(\bar{\rho}_{13})^{-1} \begin{pmatrix} \underline{b}_1 \\ \underline{b}_3 \end{pmatrix}, \\ \alpha_t^{2,*} &\equiv 0, \end{aligned}$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{3,*} > 0$.

(ii) *If* $\underline{\rho}_{13} > \min(\frac{\bar{\beta}_3}{\underline{\beta}_1}, 1)$, $\kappa^2(\underline{b}_1, \bar{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \geq 0 \geq \kappa^2(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{aligned} \begin{pmatrix} \alpha_t^{1,*} \\ \alpha_t^{3,*} \end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda} e^{(\|\sigma_{-2}(\underline{\rho}_{13})^{-1}(\underline{b}_1, \bar{b}_3)^\top\|_2^2 T} - X_t^*\big] \Sigma_{-2}(\underline{\rho}_{13})^{-1} \begin{pmatrix} \underline{b}_1 \\ \bar{b}_3 \end{pmatrix}, \\ \alpha_t^{2,*} &\equiv 0, \end{aligned}$$

and $\alpha_t^{1,*} > 0$, $\alpha_t^{3,*} < 0$.

4. *(Under-diversification: no investment in the third asset)*

(i) *If* $\bar{\rho}_{23} < \frac{\underline{\beta}_3}{\underline{\beta}_2}$, $\kappa^1(\bar{b}_1, \underline{b}_2, \underline{b}_3, \underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \geq 0 \geq \kappa^1(\underline{b}_1, \underline{b}_2, \underline{b}_3, \bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{aligned} \begin{pmatrix} \alpha_t^{2,*} \\ \alpha_t^{3,*} \end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda} e^{\|\sigma_{-1}(\bar{\rho}_{23})^{-1}(\underline{b}_2, \underline{b}_3)^\top\|_2^2 T} - X_t^*\big] \Sigma_{-1}(\bar{\rho}_{23})^{-1} \begin{pmatrix} \underline{b}_2 \\ \underline{b}_3 \end{pmatrix}, \\ \alpha_t^{1,*} &\equiv 0, \end{aligned}$$

and $\alpha_t^{2,*} > 0$, $\alpha_t^{3,*} > 0$.

(ii) *If* $\underline{\rho}_{23} > \min(1, \frac{\bar{\beta}_3}{\underline{\beta}_2})$ $\kappa^1(\bar{b}_1, \underline{b}_2, \bar{b}_3, \underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23}) \geq 0 \geq \kappa^1(\underline{b}_1, \underline{b}_2, \bar{b}_3, \bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$, *then an optimal portfolio strategy is explicitly given by*

$$\begin{aligned} \begin{pmatrix} \alpha_t^{2,*} \\ \alpha_t^{3,*} \end{pmatrix} &= \big[x_0 + \frac{1}{2\lambda} e^{\|\sigma_{-1}(\underline{\rho}_{23})^{-1}(\underline{b}_2, \bar{b}_3)^\top\|_2^2 T} - X_t^*\big] \Sigma_{-1}(\underline{\rho}_{23})^{-1} \begin{pmatrix} \underline{b}_2 \\ \bar{b}_3 \end{pmatrix}, \\ \alpha_t^{1,*} &\equiv 0, \end{aligned}$$

and $\alpha_t^{2,*} > 0$, $\alpha_t^{3,*} < 0$.

5. *(Well diversification)*

(i) If $\kappa^1(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar\rho_{12},\bar\rho_{13},\bar\rho_{23})>0$, $\kappa^2(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar\rho_{12},\bar\rho_{13},\bar\rho_{23})>0$, $\kappa^3(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar\rho_{12},\bar\rho_{13},\bar\rho_{23})>0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\bar\rho_{12},\bar\rho_{13},\bar\rho_{23})^{-1}(\underline{b}_1,\underline{b}_2,\underline{b}_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\bar\rho_{12},\bar\rho_{13},\bar\rho_{23})^{-1}(\underline{b}_1,\underline{b}_2,\underline{b}_3)^\top.$$

(ii) If $\kappa^1(\underline{b}_1,\bar b_2,\underline{b}_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})>0$, $\kappa^2(\underline{b}_1,\bar b_2,\underline{b}_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})<0$, $\kappa^3(\underline{b}_1,\bar b_2,\underline{b}_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})>0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})^{-1}(\underline{b}_1,\bar b_2,\underline{b}_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})^{-1}(\underline{b}_1,\bar b_2,\underline{b}_3)^\top.$$

(iii) If $\kappa^1(\bar b_1,\underline{b}_2,\bar b_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})<0$, $\kappa^2(\bar b_1,\underline{b}_2,\bar b_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})>0$, $\kappa^3(\bar b_1,\underline{b}_2,\bar b_3,\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})<0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})^{-1}(\bar b_1,\underline{b}_2,\bar b_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\underline{\rho}_{12},\bar\rho_{13},\underline{\rho}_{23})^{-1}(\bar b_1,\underline{b}_2,\bar b_3)^\top.$$

(iv) If $\kappa^1(\underline{b}_1,\bar b_2,\bar b_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})>0$, $\kappa^2(\underline{b}_1,\bar b_2,\bar b_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})<0$, $\kappa^3(\underline{b}_1,\bar b_2,\bar b_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})<0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})^{-1}(\underline{b}_1,\bar b_2,\bar b_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})^{-1}(\underline{b}_1,\bar b_2,\bar b_3)^\top.$$

(v) If $\kappa^1(\bar b_1,\underline{b}_2,\underline{b}_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})<0$, $\kappa^2(\bar b_1,\underline{b}_2,\underline{b}_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})>0$, $\kappa^3(\bar b_1,\underline{b}_2,\underline{b}_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})>0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})^{-1}(\bar b_1,\underline{b}_2,\underline{b}_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\underline{\rho}_{12},\underline{\rho}_{13},\bar\rho_{23})^{-1}(\bar b_1,\underline{b}_2,\underline{b}_3)^\top.$$

(vi) If $\kappa^1(\underline{b}_1,\underline{b}_2,\bar b_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})>0$, $\kappa^2(\underline{b}_1,\underline{b}_2,\bar b_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})>0$, $\kappa^3(\underline{b}_1,\underline{b}_2,\bar b_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})<0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}(\underline{b}_1,\underline{b}_2,\bar b_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}(\underline{b}_1,\underline{b}_2,\bar b_3)^\top.$$

(vii) If $\kappa^1(\bar b_1,\bar b_2,\underline{b}_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})<0$, $\kappa^2(\bar b_1,\bar b_2,\underline{b}_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})<0$, $\kappa^3(\bar b_1,\bar b_2,\underline{b}_3,\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})>0$, *then an optimal portfolio strategy is explicitly given by*

$$\alpha_t^* = \big[x_0+\frac{1}{2\lambda}e^{\|\sigma(\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}(\bar b_1,\bar b_2,\underline{b}_3)^\top\|_2^2T}-X_t^*\big]\Sigma(\bar\rho_{12},\underline{\rho}_{13},\underline{\rho}_{23})^{-1}(\bar b_1,\bar b_2,\underline{b}_3)^\top.$$

Proof. I. In view of formula (5.4.10) of the optimal portfolio strategy in Proposition 5.4.7, we only need to compute $\hat\kappa(\rho^*)=\Sigma(\rho^*)^{-1}\hat b$, and $\|\sigma(\rho^*)^{-1}\hat b\|_2$, i.e., $R(\hat b,\rho^*)$, which have been given explicitly in the proof of Proposition 5.4.3 (see 5.5.5 in Appendix) when computing ρ^*. In the case **1.**, we obtained (see (5.5.33) in Appendix) $\hat\kappa(\rho^*)=(\frac{\hat b_1}{\sigma_1^2},0,0)^\intercal$, and $R(\hat b,\rho^*)=\hat b^\intercal\hat\kappa(\rho^*)=\hat\beta_1^2$. In the case **2.**, let us focus on subcase (i) as the other subcase (ii) is dealt with similarly: we have $\rho_{12}^*=\bar\rho_{12}$, $(\hat\kappa^1(\bar\rho_{12}),\hat\kappa^2(\bar\rho_{12}))^\intercal=\Sigma_{-3}(\bar\rho_{12})\hat b_{-3}$, $\hat\kappa^3(\bar\rho_{12},\rho_{13}^*,\rho_{23}^*)=0$, and $R(\hat b,\rho^*)=\hat b_{-3}^\intercal\Sigma_{-3}(\bar\rho_{12})\hat b_{-3}$, by (5.5.40), (5.5.41). The other cases are computed in the same way and are omitted here.

II. From (5.3.1) of the optimal portfolio strategy in Theorem 5.3.1, we only need to compute $\kappa(b^*,\rho^*)=\Sigma(\rho^*)^{-1}b^*$, and $\|\sigma(\rho^*)^{-1}b^*\|_2$, i.e. $R(b^*,\rho^*)$, which have been given explicitly in the proof of Proposition 5.4.6 (see 5.5.8 in Appendix) when computing (b^*,ρ^*). ☐

Remark 5.4.12 (Financial interpretation)**.** We will see that in the above proposition, both **I.** for ellipsoidal set and **II.** for rectangular set lead to possible positions formed with a single asset, two assets and all three assets.

In case **1.** corresponding to large correlation ambiguity for the second and third asset (recall Remark 5.4.4), it is optimal to invest only in the first asset, namely the one with the highest estimated Sharpe ratio, which is consistent with the anti-diversification result obtained in Theorem 5.4.1 (see also Remark 5.4.9).

In case **2.**, corresponding to a large correlation ambiguity for the third asset (see Remark 5.4.4), the investor does not invest in the third asset, but only in the first and second assets. Moreover, depending whether the assets 1 and 2 have close Sharpe ratios with a correlation upper bound between these assets not too large (subcase (i)), or the asset 1 dominates the asset 2 in terms of Sharpe ratio (subcase (ii)), the investment in assets 1 or 2 follows a directional trading or a spread trading. This under-diversification result has been also observed in [LZ17] for a single-period mean-variance problem without drift uncertainty, and is extended here in a continuous time framework.

We have similar under-diversification effect in cases **3.** and **4.**, and notice that it may happen that one does not invest in the first asset even though it has the highest estimated Sharpe ratio.

Finally, in the case **5.**, corresponding to a small correlation ambiguity (see Remark 5.4.4), the investor has interest to well-diversify her portfolio among the three assets. ◊

5.5 Appendix

5.5.1 Differentiation and characterization of convex function

Let us introduce some notations and state some results which will be used frequently in the proof of Lemma 5.3.2 and also for the next propositions.

1. We introduce the so-called variance risk ratio

$$\hat{\kappa}(\rho) \;:=\; \Sigma(\rho)^{-1}\hat{b} \;=\; (\hat{\kappa}^1(\rho), \ldots, \hat{\kappa}^d(\rho))^\intercal, \tag{5.5.1}$$

$$\kappa(b,\rho) \;:=\; \Sigma(\rho)^{-1}b \;=\; (\kappa^1(b,\rho), \ldots, \kappa^d(b,\rho))^\intercal. \tag{5.5.2}$$

2. From some matrix calculations (see e.g. corollary 95 and corollary 105 in [Dhr78]), we obtain the explicit expressions of the first partial derivatives of $R(b,\rho)$ with respect to b_i, ρ_{ij} denoted by $\frac{\partial R(b,\rho)}{\partial b_i}$ and $\frac{\partial R(b,\rho)}{\partial \rho_{ij}}$, $1 \le i < j \le d$,

$$\frac{\partial R(b,\rho)}{\partial b_i} \;=\; 2\kappa^i(b,\rho), \qquad \frac{\partial R(b,\rho)}{\partial \rho_{ij}} \;=\; -\sigma_i\sigma_j\kappa^i(b,\rho)\kappa^j(b,\rho). \tag{5.5.3}$$

We also denote by $\nabla_b R(b,\rho)$ and $\nabla_\rho R(b,\rho)$ gradient of $R(b,\rho)$ with respect to b and ρ respectively,

$$\begin{cases} \nabla_b R(b,\rho) &=& (\frac{\partial R(b,\rho)}{\partial b_1}, \ldots, \frac{\partial R(b,\rho)}{\partial b_d})^\intercal \\ \nabla_\rho R(b,\rho) &=& (\frac{\partial R(b,\rho)}{\partial \rho_{12}}, \ldots, \frac{\partial R(b,\rho)}{\partial \rho_{1d}}, \ldots, \frac{\partial R(b,\rho)}{\partial \rho_{(d-1)d}})^\intercal \end{cases} \tag{5.5.4}$$

3. (*Sufficient and necessary optimality condition*). It is known (see e.g. Lemma 2.2 in [CDH18]) that $R(b,\rho)$ is jointly convex in b and ρ. Similarly, $R(\hat{b},\rho)$ is convex in ρ. Then

(1) ρ^* is a global minimum of $R(\hat{b},\rho)$ over Γ if and only if, for any $\rho \in \Gamma$ (see e.g. section 4.2.3 in [BV04]),

$$(\rho-\rho^*)^\intercal \nabla_\rho R(\hat{b},\rho^*) \;=\; \sum_{j=1}^{d}\sum_{i=1}^{j-1}\frac{\partial R(\hat{b},\rho^*)}{\partial \rho_{ij}}(\rho_{ij}-\rho^*_{ij}) \;\geq\; 0,$$

which is written from (5.5.3) as,

$$\sum_{j=1}^{d}\sum_{i=1}^{j-1}\sigma_i\sigma_j\hat{\kappa}^i\hat{\kappa}^j(\rho^*)(\rho_{ij}-\rho^*_{ij}) \;\leq\; 0. \tag{5.5.5}$$

(2) (b^*,ρ^*) is a global minimum of $R(b,\rho)$ over $\Delta\times\Gamma$ if and only if for any $(b,\rho)\in\Delta\times\Gamma$,

$$\sum_{i=1}^{d}\frac{\partial R(b^*,\rho^*)}{\partial b_i}(b_i-b_i^*)+\sum_{j=1}^{d}\sum_{i=1}^{j-1}\frac{\partial R(b^*,\rho^*)}{\partial \rho_{ij}}(\rho_{ij}-\rho^*_{ij}) \;\geq\; 0, \tag{5.5.6}$$

which is written together with (5.5.3) as,

$$\sum_{1\leq i\leq d}2\kappa^i(b^*,\rho^*)(b_i-b_i^*)-\sum_{1\leq i<j\leq d}\sigma_i\sigma_j\kappa^i\kappa^j(b^*,\rho^*)(\rho_{ij}-\rho^*_{ij}) \;\geq\; 0. \tag{5.5.7}$$

5.5.2 Proof of Lemma 5.3.2

The statement of Lemma 5.3.2 is minimax type theorem, as it implies obviously in the case where $\Theta = \Delta\times\Gamma$ is a rectangular set that the function H in (5.3.4) satisfies

$$\min_{b\in\Delta}\max_{\rho\in\Gamma}H(b,\rho) \;=\; \max_{\rho\in\Gamma}\min_{b\in\Delta}H(b,\rho).$$

However, its proof cannot be deduced directly from standard minimax theorem (see e.g. Theorem 45.8 in [Str85]), as it does not fulfill totally their conditions: the function H is linear (hence convex) in b, linear (hence concave) in ρ, but we do not assume that Γ is a compact set, and we also consider the case where Θ is an ellipsoidal set.

We distinguish the two cases in **(HΘ)** whether Θ is a rectangular or ellipsoidal set.

Lemma 5.5.1. *Suppose that $\Theta = \Delta\times\Gamma$ is in product set as in* **(HΘ)***(i), and assume that there exists $\theta^*\in\arg\min_{\Theta}R(\theta)$. Then, we have for all $\theta=(b,\rho)\in\Theta$:*

$$H(b^*,\rho) \;\leq\; H(\theta^*) \;=\; R(\theta^*) \;\leq\; H(b,\rho^*).$$

Moreover, (b^*,ρ^*) is a saddle point, namely,

$$\inf_{b\in\Delta}\sup_{\rho\in\Gamma}H(b,\rho) \;=\; \sup_{\rho\in\Gamma}\inf_{b\in\Delta}H(b,\rho) \;=\; H(b^*,\rho^*).$$

Proof. Note that if there exists $(b^*,\rho^*)\in\arg\min_{\Theta}R(\theta)$, the first-order condition implies that for any $(b,\rho)\in\Theta$,

$$(b-b^*)^\intercal\nabla_b R(\theta^*)+(\rho-\rho^*)^\intercal\nabla_\rho R(\theta^*) \;\geq\; 0, \tag{5.5.8}$$

where $\nabla_b R(\theta^*)$ and $\nabla_\rho R(\theta^*)$ are given in (5.5.4).
Recalling $H(b,\rho)$ in (5.3.4) and explicit expressions (5.5.4) of $\nabla_b R(\theta^*)$ and $\nabla_\rho R(\theta^*)$, and taking $b = b^*$ or $\rho = \rho^*$ in (5.5.8) respectively, we get

$$
\begin{aligned}
H(b^*,\rho) - H(b^*,\rho^*) &= \sum_{j=1}^{d}\sum_{i=1}^{j-1}\kappa^i(b^*,\rho^*)\kappa^j(b^*,\rho^*)\sigma_i\sigma_j(\rho_{ij}-\rho^*_{ij}) \\
&= (\rho^*-\rho)^\intercal \nabla_\rho R(\theta^*) \ \le\ 0, \quad \forall \rho \in \Gamma, \\
H(b,\rho^*) - H(b^*,\rho^*) &= \sum_{i=1}^{d}(b_i - b_i^*)\kappa^i(b^*,\rho^*) \\
&= \frac{1}{2}(b-b^*)^\intercal \nabla_b R(\theta^*) \ \ge\ 0, \quad \forall b \in \Delta.
\end{aligned}
$$

It follows that

$$
\inf_{b\in\Delta}\sup_{\rho\in\Gamma} H(b,\rho) \ \le\ \sup_{\rho\in\Gamma} H(b^*,\rho) \ =\ H(b^*,\rho^*) \ =\ \inf_{b\in\Delta} H(b,\rho^*) \ \le\ \sup_{\rho\in\Gamma}\inf_{b\in\Delta} H(b,\rho). \tag{5.5.9}
$$

Since we always have $\inf_{b\in\Delta}\sup_{\rho\in\Gamma} H(b,\rho) \ge \sup_{\rho\in\Gamma}\inf_{b\in\Delta} H(b,\rho)$, the above inequality is indeed an equality, and this proves the required result. □

Lemma 5.5.2. *Suppose that $\Theta = \{(b,\rho) \in \mathbb{R}^d \times \Gamma : \|\sigma(\rho)^{-1}(b-\hat{b})\|_2 \le \delta\}$ is an ellipsoidal set as in* **(HΘ)***(ii), and assume that $\rho^* = \arg\min_\Gamma \|\sigma(\rho)^{-1}\hat{b}\|_2$ exists. Then there exists $\theta^* \in \arg\min_\Theta R(\theta)$ with*

$$
\rho^* \ \in\ \arg\min \|\sigma(\rho)^{-1}\hat{b}\|_2, \qquad b^* \ =\ \hat{b}\Big(1 - \frac{\delta}{\|\sigma(\rho^*)^{-1}\hat{b}\|_2}\Big)1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}}, \tag{5.5.10}
$$

and

$$
R(\theta^*) = (\|\sigma(\rho^*)^{-1}\hat{b}\|_2 - \delta)^2 1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}}. \tag{5.5.11}
$$

Moreover, we have for all $\theta = (b,\rho) \in \Theta$:

$$
H(b^*,\rho) \ \le\ H(\theta^*) \ =\ R(\theta^*) \ \le\ H(b,\rho^*). \tag{5.5.12}
$$

Proof. Due to the dependence of b on ρ in the ellipsoidal set Θ written as $\Theta = \{(b,\rho) \in \mathbb{R}^d \times \Gamma : b \in \Delta_\rho\}$ where

$$
\Delta_\rho \ :=\ \{b \in \mathbb{R}^d : \|\sigma(\rho)^{-1}(b-\hat{b})\|_2 \le \delta\}, \tag{5.5.13}
$$

We use a Lagrangian approach.
Step 1. For fixed $\rho \in \Gamma$, let us first focus on the inner minimization

$$
\min_{b\in\Delta_\rho} R(b,\rho). \tag{5.5.14}
$$

The Lagrangian with nonnegative multiplier μ associated to this constrained minimization problem is

$$
L_1(b,\mu) \ =\ R(b,\rho) - \mu\Big(\delta^2 - (b-\hat{b})^\intercal \Sigma(\rho)^{-1}(b-\hat{b})\Big), \tag{5.5.15}
$$

and the first-order condition gives

$$\frac{\partial L_1(b,\mu)}{\partial b} \;=\; 2\Sigma(\rho)^{-1}b + 2\mu\Sigma(\rho)^{-1}(b-\hat{b}) \;=\; 0$$
$$\frac{\partial L_1(b,\mu)}{\partial \mu} \;=\; \delta^2 - (b-\hat{b})^\intercal\Sigma(\rho)^{-1}(b-\hat{b}) \;=\; 0.$$

Solving these two equations for fixed ρ, and recalling that the Lagrange multiplier is nonnegative, yield

$$\begin{cases} \mu^*(\rho) \;=\; (\frac{\|\sigma(\rho)^{-1}\hat{b}\|_2}{\delta} - 1)1_{\{\|\sigma(\rho)^{-1}\hat{b}\|_2>\delta\}}, \\ b^*(\rho) \;=\; \hat{b}(1-\frac{\delta}{\|\sigma(\rho)^{-1}\hat{b}\|_2})1_{\{\|\sigma(\rho)^{-1}\hat{b}\|_2>\delta\}}. \end{cases} \tag{5.5.16}$$

Substituting these expressions into the Lagrangian (5.5.15), we get

$$L_1(b^*(\rho),\rho) \;=\; R(b^*(\rho),\rho) \;=\; \big(\|\sigma(\rho)^{-1}\hat{b}\|_2 - \delta\big)^2 1_{\{\|\sigma(\rho)^{-1}\hat{b}\|_2>\delta\}}.$$

and thus, the original problem $\inf_\Theta R(\theta)$ is reduced to

$$\begin{aligned} \inf_{\theta=(b,\rho)\in\Theta} R(\theta) &= \inf_{\rho\in\Gamma}\inf_{b\in\Delta_\rho} R(b,\rho) \;=\; \inf_{\rho\in\Gamma} R(b^*(\rho),\rho) \\ &= \inf_{\rho\in\Gamma}\Big\{\big(\|\sigma(\rho)^{-1}\hat{b}\|_2-\delta\big)^2 1_{\{\|\sigma(\rho)^{-1}\hat{b}\|_2>\delta\}}\Big\} \\ &= \Big(\inf_{\rho\in\Gamma}\|\sigma(\rho)^{-1}\hat{b}\|_2 - \delta\Big)^2 1_{\{\inf_{\rho\in\Gamma}\|\sigma(\rho)^{-1}\hat{b}\|_2\,>\,\delta\}}. \end{aligned} \tag{5.5.17}$$

Therefore, whenever $\rho^* \in \arg\min_\Gamma \|\sigma(\rho)^{-1}\hat{b}\|_2$ exists, we see from (5.5.17) that R attains its infimum at $\theta^* = (b^*,\rho^*)$ with $b^* = b^*(\rho^*)$ as in (5.5.16) with $\rho = \rho^*$, which leads to the expressions as described in (5.5.10) and (5.5.11) of Lemma 5.5.2.

Step 2. Suppose that there exists $\rho^* \in \arg\min_\Gamma \|\sigma(\rho)^{-1}\hat{b}\|_2$. From *Step 1*, there exists $\theta^* = (b^*,\rho^*) \in \arg\min_\Theta R(\theta)$. Let us now prove that $H(b^*,\rho) \leq R(\theta^*)$ for any $\rho\in\Gamma$. Substituting the expression (5.5.10) of b^* in $H(b^*,\rho)$, we rewrite $H(b^*,\rho)$ as

$$H(b^*,\rho) \;=\; \Big(1-\frac{\delta}{\|\sigma(\rho^*)\hat{b}\|_2}\Big)^2\hat{b}^\intercal\Sigma(\rho^*)^{-1}\Sigma(\rho)\Sigma(\rho^*)^{-1}\hat{b}1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2>\delta\}}.$$

As $\rho^* \in \arg\min_\Gamma \hat{b}^\intercal\Sigma(\rho)^{-1}\hat{b}$, we use Lemma 5.5.1 by setting $\Delta = \{\hat{b}\}$, and immediately obtain

$$\sup_{\rho\in\Gamma}\hat{b}^\intercal\Sigma(\rho^*)^{-1}\Sigma(\rho)\Sigma(\rho^*)^{-1}\hat{b} \;=\; \hat{b}^\intercal\Sigma(\rho^*)^{-1}\hat{b}. \tag{5.5.18}$$

By multiplying both sides of the above equality with the constant $\Big(1-\frac{\delta}{\|\sigma(\rho^*)\hat{b}\|_2}\Big)^2 1_{\{\|\sigma(\rho^*)^{-1}\hat{b}\|_2>\delta\}}$, we get

$$\sup_{\rho\in\Gamma} H(b^*,\rho) \;=\; H(b^*,\rho^*) \;=\; R(\theta^*), \tag{5.5.19}$$

which shows that

$$H(b^*,\rho) \;\leq\; R(\theta^*), \quad \text{for all } \rho \,\in\, \Gamma.$$

Step 3. Let us finally prove that $H(b,\rho^*) \geq R(\theta^*)$ for any $b \in \Theta_b$. Again, we use a Lagrangian approach. For fixed $\rho \in \Gamma$, we focus on the inner minimization

$$\inf_{b\in\Delta_\rho} H(b,\rho^*),$$

and consider the associated Lagrangian function with nonnegative multiplier μ

$$L_2(b,\mu) \quad = \quad H(b,\rho^*) - \mu\big(\delta^2 - (b-\hat{b})^\intercal\Sigma(\rho)^{-1}(b-\hat{b})\big). \tag{5.5.20}$$

The first-order condition gives

$$\begin{aligned}
\frac{\partial L_2(b,\mu)}{\partial b} &= \Sigma(\rho^*)^{-1}b^* + 2\mu\Sigma(\rho)^{-1}(b-\hat{b}) \;=\; 0,\\
\frac{\partial L_2(b,\mu)}{\partial \mu} &= \delta^2 - (b-\hat{b})^\intercal\Sigma(\rho)^{-1}(b-\hat{b}) \;=\; 0,
\end{aligned}$$

and by solving these two equations for fixed ρ (recalling also that the Lagrangian multiplier is nonnegative), we get

$$\begin{cases}
\mu^{**}(\rho) &= \frac{\sqrt{H(b^*,\rho)}}{2\delta} \;\geq\; 0\\
b^{**}(\rho) &= \hat{b} - \frac{\delta}{\sqrt{H(b^*,\rho)}}\Sigma(\rho)\Sigma(\rho^*)^{-1}b^*.
\end{cases} \tag{5.5.21}$$

Substituting these expressions into the Lagrangian (5.5.20), we get

$$L_2(b^{**}(\rho),\rho) \quad = \quad H(b^{**}(\rho),\rho^*) \;=\; \hat{b}^\intercal\Sigma(\rho^*)^{-1}b^* - \delta\sqrt{H(b^*,\rho)}.$$

The outer minimization over Γ then yields

$$\begin{aligned}
\inf_{b\in\Theta_b} H(b,\rho^*) &= \inf_{\rho\in\Gamma}\inf_{b\in\Delta_\rho} H(b,\rho^*) \;=\; \inf_{\rho\in\Gamma} H(b^{**}(\rho),\rho^*)\\
&= \inf_{\rho\in\Gamma}\Big\{\hat{b}^\intercal\Sigma(\rho^*)^{-1}b^* - \delta\sqrt{H(b^*,\rho)}\Big\}\\
&= \hat{b}^\intercal\Sigma(\rho^*)^{-1}b^* - \delta\sup_{\rho\in\Gamma}\sqrt{H(b^*,\rho)}\\
&= \hat{b}^\intercal\Sigma(\rho^*)^{-1}b^* - \delta\sqrt{R(\theta^*)}\\
&= R(\theta^*),
\end{aligned}$$

where we used (5.5.19) in the last second equality, and last equality comes from (5.5.10). This shows that the infimum of $H(b,\rho^*)$ over $b\in\Theta_b$ is attained at $b^{**}(\rho^*) = b^*$ as in (5.5.21) with $\rho=\rho^*$. We conclude that for any $b\in\Theta_b$,

$$H(b,\rho^*) \quad \geq \quad R(\theta^*), \tag{5.5.22}$$

which completes the proof. □

5.5.3 Proof of Proposition 5.4.1

Let us prove that under the condition $|\hat{\beta}_1| > |\hat{\beta}_2| = \max_{i\neq 1}|\hat{\beta}_i|$, the function $\rho \mapsto R(\hat{b},\rho)$ attains its infimum over $\mathbb{C}^d_{>+} \subset (-1,1)^{d(d-1)/2}$, and this infimum ρ^* can be computed explicitly. By convexity and differentiability of $\rho \mapsto R(\hat{b}, \rho)$ over the convex open set $\Gamma = \mathbb{C}^d_{>+}$, the existence of such minimum is equivalent to the existence of critical points to $R(\hat{b}, .)$, i.e.,

$$\frac{\partial R(\hat{b}, \rho^*)}{\partial \rho_{ij}} = 0, \quad 1 \leq i < j \leq d. \tag{5.5.23}$$

Recalling that $\sigma_i > 0$ $i = 1, \ldots, d$, this is written from (5.5.3) as the system of $d(d-1)/2$ equations:

$$\hat{\kappa}^i(\rho^*)\hat{\kappa}^j(\rho^*) = 0, \quad 1 \leq i < j \leq d, \tag{5.5.24}$$

which indicates that at most one component of $\hat{\kappa}(\rho^*)$ is not zero. Notice that due to the assumption that $\hat{b} \neq 0$, $\hat{\kappa}(\rho^*) = \Sigma(\rho^*)^{-1}\hat{b}$ is never zero, i.e. at least one component of $\hat{\kappa}(\rho^*)$ is not zero. Therefore, exactly one component of $\hat{\kappa}(\rho^*)$ is not zero. Then (5.5.24) is equivalent to $\hat{\kappa}^{i_1}(\rho^*) \neq 0$, $\hat{\kappa}^j(\rho^*) = 0$, $j \neq i_1$, for some $i_1 = 1, \ldots, d$. In other words, we have

$$(0, \ldots, 0, \hat{\kappa}^{i_1}(\rho^*), 0, \ldots, 0)^\intercal = \Sigma(\rho^*)^{-1}\hat{b}, \quad \text{for some } i_1 = 1, \ldots, d. \tag{5.5.25}$$

Pre-multiplying $\Sigma(\rho^*)$ on both sides of (5.5.25) and then writing out l.h.s, we obtain

$$\begin{cases} \sigma^2_{i_1}\hat{\kappa}^{i_1}(\rho^*) = \hat{b}_{i_1} \\ \sigma_{i_1}\sigma_i\rho^*_{i_1 i}\hat{\kappa}^{i_1}(\rho^*) = \hat{b}_i \quad 1 \leq i \leq d, \ i \neq i_1, \end{cases} \tag{5.5.26}$$

which yields the explicit form:

$$\begin{cases} \hat{\kappa}^{i_1}(\rho^*) = \frac{\hat{b}_{i_1}}{\sigma^2_{i_1}} \\ \rho^*_{i_1 i} = \frac{\hat{\beta}_i}{\hat{\beta}_{i_1}} \quad i \neq i_1, \ 1 \leq i \leq d. \end{cases} \tag{5.5.27}$$

As $|\rho^*_{i_1 i}| < 1$ in (5.5.27), together with condition $|\hat{\beta}_1| > \max\limits_{i \neq 1} |\hat{\beta}_i|$, we thus have $i_1 = 1$ and

$$\Sigma(\rho^*)^{-1}\hat{b} = \hat{\kappa}(\rho^*) = (\frac{\hat{b}_1}{\sigma_1^2}, 0, \ldots, 0)^\intercal, \quad \rho^*_{1i} = \hat{\varrho}_{1i}, \quad 2 \leq i \leq d. \tag{5.5.28}$$

Once $\{\rho^*_{1i}\}_{2 \leq i \leq d}$ is given as in (5.5.28), we can complete the other values of $\rho^*_{ij} \in (-1, 1)$ such that ρ^* belongs to $\mathbb{C}^d_{>+}$. For instance, by choosing as in Corollary 2 in [LZ17], $\rho^*_{ij} = \rho^*_{1i}\rho^*_{1j} = \hat{\varrho}_{1i}\hat{\varrho}_{1j}$, $2 \leq i < j \leq d$, we check that $C(\rho^*) \in \mathbb{S}^d_{>+}$. Indeed, in this case we have

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ -\hat{\varrho}_{12} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\varrho}_{1d} & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 & \hat{\varrho}_{12} & \hat{\varrho}_{13} & \ldots & \hat{\varrho}_{1d} \\ \hat{\varrho}_{12} & 1 & \hat{\varrho}_{12}\hat{\varrho}_{13} & \ldots & \hat{\varrho}_{12}\hat{\varrho}_{1d} \\ \hat{\varrho}_{13} & \hat{\varrho}_{12}\hat{\varrho}_{13} & 1 & \ldots & \hat{\varrho}_{13}\hat{\varrho}_{1d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\varrho}_{1d} & \hat{\varrho}_{12}\hat{\varrho}_{1d} & \hat{\varrho}_{13}\hat{\varrho}_{1d} & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ -\hat{\varrho}_{12} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\varrho}_{1d} & 0 & \ldots & 1 \end{pmatrix}^\intercal$$
$$= \text{diag}\{1, 1 - |\hat{\varrho}_{12}|^2, 1 - |\hat{\varrho}_{13}|^2, \ldots, 1 - |\hat{\varrho}_{1d}|^2\},$$

which is positive definite since $1 - |\hat{\varrho}_{1i}|^2 > 0$, $i = 2, \ldots, d$.

In particular when $|\hat{\beta}_i|$ is in strictly descending order, i.e. $|\hat{\beta}_1| > |\hat{\beta}_2| > \ldots > |\hat{\beta}_d|$, $\rho^* = (\rho^*_{ij})_{1 \leq i < j \leq d}$ $= (\hat{\varrho}_{ij})_{1 \leq i \leq j \leq d} =: \hat{\varrho}$ also belongs to $\mathbb{C}^d_{>+}$. Indeed, in this case, observe that

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ -\hat{\varrho}_{12} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\varrho}_{1d} & 0 & \ldots & 1 \end{pmatrix} C(\hat{\varrho}) \begin{pmatrix} 1 & 0 & \ldots & 0 \\ -\hat{\varrho}_{12} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\varrho}_{1d} & 0 & \ldots & 1 \end{pmatrix}^\intercal = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 - \hat{\varrho}^2_{12} & \hat{\varrho}_{23} - \hat{\varrho}_{12}\hat{\varrho}_{13} & \ldots & \hat{\varrho}_{2d} - \hat{\varrho}_{12}\hat{\varrho}_{1d} \\ 0 & \hat{\varrho}_{23} - \hat{\varrho}_{12}\hat{\varrho}_{13} & 1 - \hat{\varrho}^2_{13} & \ldots & \hat{\varrho}_{3d} - \hat{\varrho}_{13}\hat{\varrho}_{1d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{\varrho}_{2d} - \hat{\varrho}_{12}\hat{\varrho}_{1d} & \hat{\varrho}_{3d} - \hat{\varrho}_{13}\hat{\varrho}_{1d} & \ldots & 1 - \hat{\varrho}^2_{1d} \end{pmatrix}.$$

Denote by $C_1(\hat{\varrho})$ the matrix in the r.h.s of the above equality and note that $\hat{\varrho}_{1i} = \hat{\varrho}_{12}\hat{\varrho}_{2i}$, $i = 3, \ldots, d$. Then we have

$$
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & -\hat{\varrho}_{23} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -\hat{\varrho}_{2d} & 0 & \ldots & 1
\end{pmatrix}
C_1(\hat{\varrho})
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & -\hat{\varrho}_{23} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -\hat{\varrho}_{2d} & 0 & \ldots & 1
\end{pmatrix}^{\intercal}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1-\hat{\varrho}_{12}^2 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1-\hat{\varrho}_{23}^2 & \hat{\rho}_{34}-\hat{\rho}_{23}\hat{\rho}_{24} & \ldots & \hat{\rho}_{3d}-\hat{\varrho}_{23}\hat{\varrho}_{2d} \\
0 & 0 & \hat{\varrho}_{34}-\hat{\varrho}_{23}\hat{\varrho}_{24} & 1-\hat{\varrho}_{24}^2 & \ldots & \hat{\varrho}_{4d}-\hat{\varrho}_{24}\hat{\varrho}_{2d} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \hat{\varrho}_{3d}-\hat{\varrho}_{23}\hat{\varrho}_{2d} & \hat{\varrho}_{4d}-\hat{\varrho}_{24}\hat{\varrho}_{2d} & \ldots & 1-\hat{\varrho}_{2d}^2
\end{pmatrix}.
$$

Denote by $C_2(\hat{\varrho})$ the matrix in the r.h.s of the above equality and again note that $\hat{\varrho}_{2i} = \hat{\varrho}_{23}\ \hat{\varrho}_{3i}$, $i = 4, \ldots, d$. Then we can do the similar matrix congruence with $C_2(\hat{\varrho})$ as with $C_1(\hat{\varrho})$. And so on. After $d-1$ steps of matrix congruence, we arrive at the diagonalization of the matrix $C(\hat{\varrho})$

$$
TC(\hat{\varrho})T^{\intercal} \;=\; \mathrm{diag}\{1, 1-|\hat{\varrho}_{12}|^2, 1-|\hat{\varrho}_{23}|^2, \ldots, 1-|\hat{\varrho}_{d-1d}|^2\},
$$

where $T = T_d \cdots T_1$ with T_i being invertible matrix with diagonal terms 1, (j,i)-th term $-\hat{\varrho}_{ij}$, $j > i$, and other terms 0.

We deduce that the system of equations (5.5.24) has solutions in $\mathbb{C}^d_{>+}$ given by (5.5.28). Moreover, we have from (5.5.28)

$$
\min_{\rho\in\mathbb{C}^d_{>+}} R(\hat{b},\rho) \;=\; R(\hat{b},\rho^*) \;=\; \hat{b}^{\intercal}\hat{\kappa}(\rho^*) \;=\; \hat{b}_1\hat{\kappa}^1(\rho^*) \;=\; \hat{\beta}_1^2. \tag{5.5.29}
$$

Combining this with Lemma 5.5.2, we obtain b^* described in 5.4.1. □

5.5.4 Proof of Proposition 5.4.2

As $\Gamma = [\underline{\rho}, \bar{\rho}]$ is compact, we already know that $\rho^* = \arg\min_{\rho\in\Gamma} R(\hat{b},\rho)$ exists, and from Lemma 5.5.2, we only need to compute the minimum of the function $\rho \mapsto R(\hat{b},\rho)$ over Γ. From (5.5.5) with $d = 2$, we obtain the sufficient and necessary condition of ρ^* for being global minima of $R(\hat{b},\rho)$ over Γ:

$$
\hat{\kappa}^1(\rho^*)\hat{\kappa}^2(\rho^*)(\rho-\rho^*) \le 0, \quad \forall\rho\in[\underline{\rho},\bar{\rho}], \tag{5.5.30}
$$

where $\hat{\kappa}(\rho)$ is explicitly written as

$$
\hat{\kappa}(\rho) \;=\; \frac{1}{1-\rho^2}\begin{pmatrix} \frac{\hat{b}_1}{\sigma_1^2} - \frac{\hat{b}_2}{\sigma_1\sigma_2}\rho \\ \frac{\hat{b}_2}{\sigma_2^2} - \frac{\hat{b}_1}{\sigma_1\sigma_2}\rho \end{pmatrix} \;=\; \frac{1}{1-\rho^2}\begin{pmatrix} \frac{\hat{\beta}_1-\hat{\beta}_2\rho}{\sigma_1} \\ \frac{\hat{\beta}_2-\hat{\beta}_1\rho}{\sigma_2} \end{pmatrix}. \tag{5.5.31}
$$

From (5.5.30), we have three possible cases:

1. $\hat{\kappa}^1(\rho^*)\hat{\kappa}^2(\rho^*) = 0$. From the explicit expression (5.5.31) of $\hat{\kappa}(\rho^*)$, and as ρ^* has to belong to $[\underline{\rho},\bar{\rho}] \subset (-1,1)$, we obtain $\hat{\kappa}^2(\rho^*) = 0$, i.e., $\rho^* = \hat{\varrho}_{12}$, and so $R(\hat{b},\rho^*) = \hat{\beta}_1^2$.

2. $\hat{\kappa}^1(\rho^*)\hat{\kappa}^2(\rho^*) > 0$. Then (5.5.30) is satisfied iff $\rho^* = \bar{\rho}$. Moreover, from the above explicit expression of $\hat{\kappa}(\rho^*)$, we obtain $\bar{\rho} < \hat{\varrho}_{12}$.

3. $\hat{\kappa}^1(\rho^*)\hat{\kappa}^2(\rho^*) < 0$. Then (5.5.30) is satisfied iff $\rho^* = \underline{\rho}$. Moreover, from the explicit expression of $\hat{\kappa}(\rho^*)$, we obtain $\underline{\rho} > \hat{\varrho}_{12}$.

By combining this with Lemma 5.5.2, we obtain b^* described as in Proposition 5.4.2. ☐

5.5.5 Proof of Proposition 5.4.3

As $\Gamma = \prod_{j=1}^{3}\prod_{i=1}^{j-1} [\underline{\rho}_{ij}, \bar{\rho}_{ij}]$ is compact, we already know that $\rho^* = \arg\min_{\rho\in\Gamma} R(\hat{b},\rho)$ exists. From Lemma 5.5.2, we only need to compute the minimum of the function $\rho \mapsto R(\hat{b},\rho)$ over Γ by applying the optimality principle (5.5.5) when $d = 3$,

$$\sum_{j=1}^{3}\sum_{i=1}^{j-1} \sigma_i\sigma_j\hat{\kappa}^i(\rho^*)\hat{\kappa}^j(\rho^*)(\rho_{ij} - \rho^*_{ij}) \le 0 \quad \text{for any } \rho \in \Gamma. \tag{5.5.32}$$

We observe from (5.5.32) that similar as Proposition 5.4.2, each ρ^*_{ij}, $1 \le i < j \le 3$ may be lower bound $\underline{\rho}_{ij}$, upper bound $\bar{\rho}_{ij}$, or an interior point in $(\underline{\rho}_{ij}, \bar{\rho}_{ij})$, which corresponds to $\kappa^i\kappa^j(\rho^*) > 0$, $\kappa^i\kappa^j(\rho^*) < 0$, or $\kappa^i\kappa^j(\rho^*) = 0$ respectively. Therefore, we consider the following possible exclusive cases depending on the number of zero components in $\hat{\kappa}(\rho^*)$:

1. $\hat{\kappa}^1\hat{\kappa}^2(\rho^*) = 0$, $\hat{\kappa}^1\hat{\kappa}^3(\rho^*) = 0$, $\hat{\kappa}^2\hat{\kappa}^3(\rho^*) = 0$.
   In this case, (5.5.32) is immediately satisfied. As we assume that $\hat{b} \ne 0$, $\hat{\kappa}(\rho^*)$ is not zero, i.e. at least one component of $\hat{\kappa}(\rho^*)$ is nonzero. Then, two components of $\hat{\kappa}(\rho^*)$ are zero. Under the assumption that $|\hat{\beta}_1| \ge |\hat{\beta}_2| \ge |\hat{\beta}_3|$, (5.5.28) and (5.5.29) in Section 5.5.3 yield the explicit expressions of ρ^*, $\hat{\kappa}(\rho^*)$ and $R(\hat{b},\rho^*)$

   $$\rho^*_{12} \;=\; \hat{\varrho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}],\ \rho^*_{13} \;=\; \hat{\varrho}_{13} \in [\underline{\rho}_{13}, \bar{\rho}_{13}],\ \text{any } \rho^*_{23} \in [\underline{\rho}_{23}, \bar{\rho}_{23}]$$

   and

   $$\hat{\kappa}(\rho^*) \;=\; (\frac{\hat{b}_1}{\sigma_1^2}, 0, 0)^\intercal, \quad R(\hat{b},\rho^*) \;=\; \hat{\beta}_1^2. \tag{5.5.33}$$

   Let us show that $\hat{\beta}_1^2$ in (5.5.33) is strict minimum value in the sense that $R(\hat{b},\rho^*) = \hat{\beta}_1^2$ if and only if $\rho^*_{12} = \hat{\varrho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$, $\rho^*_{13} = \hat{\varrho}_{13} \in [\underline{\rho}_{13}, \bar{\rho}_{13}]$ and any $\rho^*_{23} \in [\underline{\rho}_{23}, \bar{\rho}_{23}]$. We express $\Sigma(\rho)$ as the following block matrix

   $$\Sigma(\rho) = \begin{pmatrix} \sigma_1^2 & C_1^\intercal \\ C_1 & \Sigma_{-1}(\rho_{23}) \end{pmatrix},$$

   where the vector $C_1 = (\sigma_1\sigma_2\rho_{12}, \sigma_1\sigma_3\rho_{13})^\intercal$.

   $$\begin{pmatrix} 1 & \mathbf{0}_{1\times2} \\ -\frac{C_1}{\sigma_1^2} & I_{2\times2} \end{pmatrix}\begin{pmatrix} \sigma_1^2 & C_1^\intercal \\ C_1 & \Sigma_{-1}(\rho_{23}) \end{pmatrix}\begin{pmatrix} 1 & -\frac{C_1^\intercal}{\sigma_1^2} \\ \mathbf{0}_{2\times1} & I_{2\times2} \end{pmatrix} \;=\; \begin{pmatrix} \sigma_1^2 & \mathbf{0}_{1\times2} \\ \mathbf{0}_{2\times1} & A \end{pmatrix}, \tag{5.5.34}$$

   where $A = \Sigma_{-1}(\rho_{23}) - \frac{C_1C_1^\intercal}{\sigma_1^4}$ is 2×2 positive definite matrix.
   Inverting on both sides of (5.5.34), we get

   $$\Sigma^{-1}(\rho) \;=\; \begin{pmatrix} 1 & -\frac{C_1^\intercal}{\sigma_1^2} \\ \mathbf{0}_{2\times1} & I_{2\times2} \end{pmatrix}\begin{pmatrix} \sigma_1^{-2} & \mathbf{0}_{1\times2} \\ \mathbf{0}_{2\times1} & A^{-1} \end{pmatrix}\begin{pmatrix} 1 & \mathbf{0}_{1\times2} \\ -\frac{C_1}{\sigma_1^2} & I_{2\times2} \end{pmatrix}. \tag{5.5.35}$$

We express $\hat{b}$ as $(\hat{b}_1, \hat{b}_{-1}^\intercal)^\intercal$ and then write $R(\hat{b},\rho)$ as two nonnegative decompositions from (5.5.35),

$$\begin{aligned} R(\hat{b},\rho) &= \hat{\beta}_1^2 + (\hat{b}_{-1} - \frac{\hat{b}_1}{\sigma_1^2}C_1)^\intercal A^{-1}(\hat{b}_{-1} - \frac{\hat{b}_1}{\sigma_1^2}C_1) \\ &\geq \hat{\beta}_1^2, \end{aligned}$$

where in the last inequality, '=' holds if and only if $\hat{b}_{-1} - \frac{\hat{b}_1}{\sigma_1^2}C_1 = 0$, i.e. $\rho_{12}^* = \hat{\varrho}_{12}$, $\rho_{13}^* = \hat{\varrho}_{13}$. This corresponds to case **1.** of Proposition 5.4.3.

2. $\hat{\kappa}^1\hat{\kappa}^2(\rho^*) \neq 0$, $\hat{\kappa}^1\hat{\kappa}^3(\rho^*) = 0$, $\hat{\kappa}^2\hat{\kappa}^3(\rho^*) = 0$.
In this case, we express $\Sigma(\rho)$ as the following block-matrix form for convenience,

$$\Sigma(\rho) = \begin{pmatrix} \Sigma_{-3}(\rho_{12}) & C_3 \\ C_3^\intercal & \sigma_3^2 \end{pmatrix},$$

where the vector $C_3 = (\sigma_1\sigma_3\rho_{13}, \sigma_2\sigma_3\rho_{23})^\intercal$.
By first transforming $\Sigma(\rho)$ to block diagonal matrix as (5.5.35) and then taking inverse, we obtain

$$\begin{aligned} \Sigma(\rho)^{-1} &= \begin{pmatrix} I_{2\times2} & -\Sigma_{-3}(\rho_{12})^{-1}C_3 \\ \mathbf{0}_{1\times2} & 1 \end{pmatrix} \begin{pmatrix} \Sigma_{-3}(\rho_{12})^{-1} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{1\times2} & a(\rho)^{-1} \end{pmatrix} \\ &\quad \begin{pmatrix} I_{2\times2} & \mathbf{0}_{2\times1} \\ -C_3^\intercal\Sigma_{-3}(\rho_{12})^{-1} & 1 \end{pmatrix}, \end{aligned} \tag{5.5.36}$$

where $a(\rho) = \sigma_3^2 - C_3^\intercal\Sigma_{-3}(\rho_{12})^{-1}C_3$ is positive.

Recalling the definition of $\kappa(\hat{b},\rho)$ and $R(\hat{b},\rho)$, we obtain from (5.5.36)

$$\begin{cases} \begin{pmatrix} \hat{\kappa}^1(\rho) \\ \hat{\kappa}^2(\rho) \end{pmatrix} = \Sigma_{-3}(\rho_{12})^{-1}\hat{b}_{-3} - \hat{\kappa}^3(\rho)\Sigma_{-3}(\rho_{12})^{-1}C_3 \\ \hat{\kappa}^3(\rho) = \frac{1}{a(\rho)}(\hat{b}_3 - C_3^\intercal\Sigma_{-3}(\rho_{12})^{-1}\hat{b}_{-3}) \end{cases} \tag{5.5.37}$$

and

$$R(\hat{b},\rho) = \hat{b}_{-3}^\intercal\Sigma_{-3}(\rho_{12})^{-1}\hat{b}_{-3} + a(\rho)\big(\hat{\kappa}^3(\rho)\big)^2. \tag{5.5.38}$$

In the following, we write $\hat{b}_{-3}^\intercal\Sigma_{-3}(\rho_{12})^{-1}\hat{b}_{-3}$ as $R(\hat{b}_{-3},\rho_{12})$.
As $\hat{\kappa}^3(\rho^*) = 0$, we obtain from (5.5.32) that

$$\sigma_1\sigma_2\hat{\kappa}^1\hat{\kappa}^2(\rho_{12}^*)(\rho_{12} - \rho_{12}^*) \leq 0 \quad \text{for all } \rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}] \tag{5.5.39}$$

and from (5.5.37) and (5.5.38) that

$$\begin{cases} \begin{pmatrix} \hat{\kappa}^1(\rho_{12}^*) \\ \hat{\kappa}^2(\rho_{12}^*) \end{pmatrix} = \Sigma_{-3}(\rho_{12}^*)^{-1}\hat{b}_{-3} \\ R(\hat{b},\rho^*) = R(\hat{b}_{-3},\rho_{12}^*). \end{cases} \tag{5.5.40}$$

This is the case of ambiguous correlation in the two-risky assets: risky asset 1 and risky asset 2 with ambiguous correlation ρ_{12} in $[\underline{\rho}_{12}, \bar{\rho}_{12}]$. In this case, $\hat{\kappa}^1(\rho^*)$ and $\hat{\kappa}^2(\rho^*)$ are not zero, therefore we have that from Proposition 5.4.2

$$\rho_{12}^* = \bar{\rho}_{12}1_{\{\bar{\rho}_{12}<\hat{\varrho}_{12}\}} + \underline{\rho}_{12}1_{\{\underline{\rho}_{12}>\hat{\varrho}_{12}\}}. \tag{5.5.41}$$

By setting $g(\rho_{12}^*, \rho_{13}, \rho_{23}) := a(\rho_{12}^*, \rho_{13}, \rho_{23})\kappa^3(\rho_{12}^*, \rho_{13}, \rho_{23})$ for fixed ρ_{12}^* in (5.5.41), we deduce from (5.5.37) that the function

$$(\rho_{13}, \rho_{23}) \mapsto g(\rho_{12}^*, \rho_{13}, \rho_{23}) \quad = \quad \hat{b}_3 - \sigma_1\sigma_3\hat{\kappa}^1(\rho_{12}^*)\rho_{13} - \sigma_2\sigma_3\hat{\kappa}^2(\rho_{12}^*)\rho_{23},$$

is linear in $(\rho_{13}, \rho_{23}) \in [\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}]$, and has the same sign as $\hat{\kappa}^3(\rho_{12}^*, \rho_{13}, \rho_{23})$ due to the positiveness of $a(\rho_{12}^*, \rho_{13}, \rho_{23})$. To study the condition of $\kappa^3(\rho^*) = 0$, we discuss it in the following two cases:

(i) if $\bar{\rho}_{12} < \hat{\varrho}_{12}$, then $\hat{\kappa}^1\hat{\kappa}^2(\bar{\rho}_{12}) > 0$, the function $(\rho_{13}, \rho_{23}) \mapsto g(\bar{\rho}_{12}, \rho_{13}, \rho_{23})$ has the same monotonicity with respect to ρ_{13}, ρ_{23}. Therefore, to ensure that the function $g(\bar{\rho}_{12}, \rho_{13}, \rho_{23})$ has a root in $[\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}]$, we need $g(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$ $g(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \leq 0$, or equivalently $\hat{\kappa}^3(\bar{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23})$ $\hat{\kappa}^3(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \leq 0$.

(ii) if $\underline{\rho}_{12} > \hat{\varrho}_{12}$, then $\hat{\kappa}^1\hat{\kappa}^2(\underline{\rho}_{12}) < 0$, the function $(\rho_{13}, \rho_{23}) \mapsto g(\underline{\rho}_{12}, \rho_{13}, \rho_{23})$ has the opposite monotonicity with respect to ρ_{13}, ρ_{23}. Therefore, when $g(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})$ $g(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{23}) \leq 0$, or equivalently $\hat{\kappa}^3(\underline{\rho}_{12}, \bar{\rho}_{13}, \underline{\rho}_{23})\hat{\kappa}^3(\underline{\rho}_{12}, \underline{\rho}_{13}, \bar{\rho}_{13}) \leq 0$, the function $g(\underline{\rho}_{12}, \rho_{13}, \rho_{23})$ has a root in $[\underline{\rho}_{13}, \bar{\rho}_{13}] \times [\underline{\rho}_{23}, \bar{\rho}_{23}]$.

Therefore, we deduce that $R(\hat{b}, \rho) \geq R(\hat{b}_{-3}, \rho_{12}) \geq R(\hat{b}_{-3}, \bar{\rho}_{12}1_{\{\bar{\rho}_{12}<\hat{\varrho}_{12}\}} + \underline{\rho}_{12}1_{\{\underline{\rho}_{12}>\hat{\varrho}_{12}\}})$ and that '=' holds if and only if $\rho_{12}^* = \bar{\rho}_{12}1_{\{\bar{\rho}_{12}<\hat{\varrho}_{12}\}} + \underline{\rho}_{12}1_{\{\underline{\rho}_{12}>\hat{\varrho}_{12}\}}$ and ρ_{13}^*, ρ_{23}^* satisfies $\hat{\kappa}^3(\rho_{12}^*, \rho_{13}^*, \rho_{23}^*) = 0$.This corresponds to subcases **2.**(i) and **2.**(ii) of Proposition 5.4.3.

3. $\hat{\kappa}^1\hat{\kappa}^2(\rho^*) = 0$, $\hat{\kappa}^1\hat{\kappa}^3(\rho^*) \neq 0$, $\hat{\kappa}^2(\rho^*)\hat{\kappa}^3(\rho^*) = 0$.
In this case, we make permutations as follows,

$$\begin{pmatrix} \hat{\kappa}^{-2}(\rho) \\ \hat{\kappa}^2(\rho) \end{pmatrix} = \begin{pmatrix} \Sigma_{-2}(\rho_{13}) & C_2 \\ C_2^\intercal & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{b}_{-2} \\ \hat{b}_2 \end{pmatrix}, \tag{5.5.42}$$

where $\hat{\kappa}^{-2}(\rho) = (\hat{\kappa}^1(\rho), \hat{\kappa}^3(\rho))^\intercal$ and $C_2 = (\sigma_1\sigma_2\rho_{12}, \sigma_2\sigma_3\rho_{23})^\intercal$. Using (5.5.42) and proceeding with the same arguments as in the case **2.**, we obtain the result of $\hat{\kappa}^2(\rho^*) = 0$, $\hat{\kappa}^1(\rho^*)\hat{\kappa}^3(\rho^*) \neq 0$ as described in the subcases **3.**(i) and **3.**(ii) of Proposition 5.4.3.

4. $\hat{\kappa}^1\hat{\kappa}^2(\rho^*) = 0$, $\hat{\kappa}^1\hat{\kappa}^3(\rho^*) = 0$, $\hat{\kappa}^2(\rho^*)\hat{\kappa}^3(\rho^*) \neq 0$.
Note that

$$\begin{pmatrix} \hat{\kappa}^{-1}(\rho) \\ \hat{\kappa}^1(\rho) \end{pmatrix} = \begin{pmatrix} \Sigma_{-1}(\rho_{23}) & C_1 \\ C_1^\intercal & \sigma_1^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{b}_{-1} \\ \hat{b}_1 \end{pmatrix}, \tag{5.5.43}$$

where $\hat{\kappa}^{-1}(\rho) = (\hat{\kappa}^2(\rho), \hat{\kappa}^3(\rho))^\intercal$ and $C_1 = (\sigma_1\sigma_2\rho_{12}, \sigma_1\sigma_3\rho_{13})^\intercal$. Using (5.5.43) and proceeding with the same arguments as in the case **2.**, we obtain the result of $\hat{\kappa}^1(\rho^*) = 0$, $\hat{\kappa}^2(\rho^*)\hat{\kappa}^3(\rho^*) \neq 0$ as described in subcases **4.**(i) and **4.**(ii) of Proposition 5.4.3.

5. $\hat{\kappa}^1\hat{\kappa}^2(\rho^*) \neq 0$, $\hat{\kappa}^1\hat{\kappa}^3(\rho^*) \neq 0$, $\hat{\kappa}^2(\rho^*)\hat{\kappa}^3(\rho^*) \neq 0$.
In this case, we see from (5.5.32) that each ρ_{ij}^* takes value in $\{\underline{\rho}_{ij}, \bar{\rho}_{ij}\}$ relying on the sign of $\hat{\kappa}^i\hat{\kappa}^j(\rho^*)$. Note that once the signs of $\hat{\kappa}^1\hat{\kappa}^2(\rho^*)$ and $\hat{\kappa}^1\hat{\kappa}^3(\rho^*)$ are known, the sign of $\hat{\kappa}^2(\rho^*)\hat{\kappa}^3(\rho^*)$ is determined. Therefore, by combination, there are 4 possible sub-cases as described in the case **5.** of Proposition 5.4.3.
As $\hat{\kappa}^i(\rho^*)\hat{\kappa}^j(\rho^*) \neq 0$ in each sub case, l.h.s of (5.5.32) is strictly negative for any $\rho \in \Gamma \setminus \{\rho^*\}$. From the first-order characterization for convexity of $R(\hat{b}, \rho)$ (see e.g. Section 3.1.3 in [BV04]) and

(5.5.3), we obtain for any $\rho \in \Gamma \setminus \{\rho^*\}$,

$$\begin{aligned} R(\hat{b}, \rho) &\geq R(\hat{b}, \rho^*) + (\rho - \rho^*)^\intercal D_\rho R(\hat{b}, \rho^*) \\ &= R(\hat{b}, \rho^*) - \sum_{j=1}^{3} \sum_{i=1}^{j-1} \sigma_i \sigma_j \hat{\kappa}^i(\rho^*) \hat{\kappa}^j(\rho^*)(\rho_{ij} - \rho^*_{ij}) \\ &> R(\hat{b}, \rho^*), \end{aligned}$$

which indicates that ρ^* in each sub-case of case **5.** in Proposition 5.4.3 is a strict minimum of $R(\hat{b}, \rho)$.

As $R(\hat{b}, \rho^*)$ in this subcase is strict minimum value, we conclude that each subcase in Proposition 5.4.3 is exclusive.
By combining this with Lemma 5.5.2, we obtain b^* described as in Proposition 5.4.3. □

5.5.6 Proof of Proposition 5.4.4

We express $\Sigma(\rho)$ in block matrix form,

$$\Sigma(\rho) = \begin{pmatrix} \sigma_1^2 & C^\intercal \\ C & \Sigma_{-1}(\rho) \end{pmatrix}, \tag{5.5.44}$$

where the vector $C = (\sigma_1\sigma_2\rho_{12}, \ldots, \sigma_1\sigma_d\rho_{1d})^\intercal$.

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times(d-1)} \\ -\frac{C}{\sigma_1^2} & I_{(d-1)\times(d-1)} \end{pmatrix} \Sigma(\rho) \begin{pmatrix} 1 & -\frac{C^\intercal}{\sigma_1^2} \\ \mathbf{0}_{(d-1)\times 1} & I_{(d-1)\times(d-1)} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \mathbf{0}_{1\times(d-1)} \\ \mathbf{0}_{(d-1)\times 1} & A \end{pmatrix}, \tag{5.5.45}$$

where $I_{(d-1)\times(d-1)}$ is $d-1 \times d-1$ identity matrix and $A = \Sigma_{-1}(\rho) - \frac{CC^\intercal}{\sigma_1^4}$, $d-1 \times d-1 > \mathbf{0}$.
Inverting on both sides of (5.5.45), we get

$$\Sigma(\rho)^{-1} = \begin{pmatrix} 1 & -\frac{C^\intercal}{\sigma_1^2} \\ \mathbf{0}_{(d-1)\times 1} & I_{(d-1)\times(d-1)} \end{pmatrix} \begin{pmatrix} \sigma_1^{-2} & \mathbf{0}_{1\times(d-1)} \\ \mathbf{0}_{(d-1)\times 1} & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}_{1\times(d-1)} \\ -\frac{C}{\sigma_1^2} & I_{(d-1)\times(d-1)} \end{pmatrix}. \tag{5.5.46}$$

We write b as $(b_1, b_{-1}^\intercal)^\intercal$ where b_{-1} denotes b with the first component removed, and express from (5.5.46) $R(b, \rho)$ as two nonnegative parts

$$\begin{aligned} R(b, \rho) &= \beta_1^2 + (b_{-1} - \frac{b_1}{\sigma_1^2} C)^\intercal A^{-1} (b_{-1} - \frac{b_1}{\sigma_1^2} C) \\ &\geq \underline{\beta}_1^2. \end{aligned}$$

Note that $\underline{\beta}_1^2$ can be attained if and only if $b_1 = \underline{b}_1$ and $b_{-1} - \frac{b_1}{\sigma_1^2} C = 0$, which is equivalent to

$$b_1^* = \underline{b}_1, \quad \beta_i^* = \underline{\beta}_1 \rho_{1i}^* \quad 2 \leq i \leq d. \tag{5.5.47}$$

As long as $|\rho^*_{1i}|$ in (5.5.47) is less than 1, we can complete the other values of $\rho^*_{ij} \in (-1, 1)$ such that ρ^* belongs to $\mathbb{C}^d_{>+}$. For instance, by choosing $\rho^*_{ij} = \rho^*_{1i}\rho^*_{1j} = \frac{\beta_i^*\beta_j^*}{\underline{\beta}_{-1}^2}$, $2 \leq i < j \leq d$, we check that $C(\rho^*) \in$

$\mathbb{S}^d_{>+}$. A sufficient condition is $\underline{\beta}_1 > \max_{j\neq 1} \underline{\beta}_j$. Therefore, if $\underline{\beta}_1 > \max_{i\neq 1} \underline{\beta}_i$, then

$$\begin{aligned} \min_{\Delta\times\mathbb{C}^d_{>+}} R(b,\rho) &= \underline{\beta}_1^2, \\ b_1^* = \underline{b}_1, \ \ \rho^*_{1i} &= \frac{\beta_i^*}{\underline{\beta}_1} \quad \text{for any } \beta_i^* \in [\underline{\beta}_i, \min(\bar{\beta}_i, \underline{\beta}_1)). \end{aligned} \tag{5.5.48}$$

Moreover, in this case, we obtain from (5.5.46)

$$\kappa(b^*,\rho^*) \quad = \quad \Sigma(\rho^*)^{-1}b^* \;=\; \begin{pmatrix} \frac{b_1^*}{\sigma_1^2} - \frac{C^\intercal}{\sigma_1^2}A^{-1}(b^*_{-1} - \frac{C}{\sigma_1^2}) \\ A^{-1}(b^*_{-1} - \frac{C}{\sigma_1^2}) \end{pmatrix} \;=\; \begin{pmatrix} \frac{\underline{b}_1}{\sigma_1^2} \\ \mathbf{0}_{(d-1)\times 1} \end{pmatrix}. \tag{5.5.49}$$

□

5.5.7 Proof of Proposition 5.4.5

As Δ and Γ are compact, we know that $\min_{\Delta\times\Gamma} R(b,\rho)$ exists. In the following, we compute the minimum (b^*,ρ^*) of $R(b,\rho)$ by using (5.5.7) with $d=2$: for any $(b,\rho) \in \prod_{i=1}^2[\underline{b}_i,\bar{b}_i] \times [\underline{\rho},\bar{\rho}]$,

$$\sum_{1\le i\le 2} \kappa^i(b^*,\rho^*)(b_i - b_i^*) - \sigma_1\sigma_2\kappa^1(b^*,\rho^*)\kappa^2(b^*,\rho^*)(\rho-\rho^*) \ge 0, \tag{5.5.50}$$

where $\kappa(b,\rho)$ is explicitly written as

$$\kappa(b,\rho) = \Sigma(\rho)^{-1}b = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{b_1}{\sigma_1^2} - \frac{b_2}{\sigma_1\sigma_2}\rho \\ \frac{b_2}{\sigma_2^2} - \frac{b_1}{\sigma_1\sigma_2}\rho \end{pmatrix}. \tag{5.5.51}$$

From (5.5.50), we have the following cases:

1. If $\kappa^1(b^*,\rho^*) \neq 0$, $\kappa^2(b^*,\rho^*) = 0$, then solving (5.5.51) yields $\rho^* = \frac{\beta_2^*}{\beta_1^*}$ and $\kappa^1(b^*,\rho^*) = \frac{b_1^*}{\sigma_1^2} \ge \frac{\underline{b}_1}{\sigma_1^2} > 0$. Substituting these back in (5.5.50), we have $b_1^* = \underline{b}_1$ and $\rho^* = \frac{\beta_2^*}{\underline{\beta}_1} \in [\frac{\underline{\beta}_2}{\underline{\beta}_1}, \min(1, \frac{\bar{\beta}_2}{\underline{\beta}_1})]$. As ρ^* has to belong to $[\underline{\rho},\bar{\rho}]$, we have $[\frac{\underline{\beta}_2}{\underline{\beta}_1}, \min(1,\frac{\bar{\beta}_2}{\underline{\beta}_1})] \cap [\underline{\rho},\bar{\rho}] \neq \varnothing$, i.e. $\underline{\rho} \le \min(1,\frac{\bar{\beta}_2}{\underline{\beta}_1})$ and $\bar{\rho} \ge \frac{\underline{\beta}_2}{\underline{\beta}_1}$. This is the case as described in **1.** of Proposition 5.4.5.
2. If $\kappa^1(b^*,\rho^*) = 0$, $\kappa^2(b^*,\rho^*) \neq 0$, then solving (5.5.51) yields $\rho^* = \frac{\beta_1^*}{\beta_2^*}$ and $\kappa^2(b^*,\rho^*) = \frac{b_2^*}{\sigma_2^2} \ge \frac{\underline{b}_2}{\sigma_2^2} > 0$. Substituting these in (5.5.50), we have $b_2^* = \underline{b}_2$ and $\rho^* = \frac{\beta_1^*}{\underline{\beta}_2} \ge \frac{\underline{\beta}_1}{\underline{\beta}_2} \ge 1$, a contradiction with $|\rho^*| < 1$.
3. If $\kappa^1(b^*,\rho^*) > 0$, $\kappa^2(b^*,\rho^*) > 0$, then we have $b^* = (\underline{b}_1,\underline{b}_2)$, $\rho^* = \bar{\rho}$. Using (5.5.51), we obtain $\bar{\rho} < \frac{\underline{\beta}_2}{\underline{\beta}_1}$, which is described as in case **2.** of Proposition 5.4.5.
4. If $\kappa^1(b^*,\rho^*) > 0$, $\kappa^2(b^*,\rho^*) < 0$, then we have $b^* = (\underline{b}_1,\bar{b}_2)$, $\rho^* = \underline{\rho}$. Using (5.5.51), we obtain $\underline{\rho} > \min(1,\frac{\bar{\beta}_2}{\underline{\beta}_1})$, which is described in case **3.** of Proposition 5.4.5.
5. If $\kappa^1(b^*,\rho^*) < 0$, $\kappa^2(b^*,\rho^*) > 0$, then we have $b^* = (\bar{b}_1,\underline{b}_2)$, $\rho^* = \underline{\rho}$. Using (5.5.51), we obtain $\bar{\beta}_1 < \underline{\rho}\,\underline{\beta}_2$, a contradiction with $\underline{\beta}_1 \ge \underline{\beta}_2$.

6. If $\kappa^1(b^*,\rho^*) < 0$, $\kappa^2(b^*,\rho^*) < 0$, then we have $b^* = (\bar{b}_1, \bar{b}_2)$, $\rho^* = \bar{\rho}$. Using (5.5.51), we obtain $\bar{\beta}_1 < \bar{\rho}\bar{\beta}_2$ and $\bar{\beta}_2 < \bar{\rho}\bar{\beta}_1$, a contradiction with natural bound of $|\bar{\rho}| < 1$.

5.5.8 Proof of Proposition 5.4.6

As $\Delta = \prod_{i=1}^3 [\underline{b}_i, \bar{b}_i]$ and $\Gamma = \prod_{1\le i<j\le 3}[\underline{\rho}_{ij}, \bar{\rho}_{ij}]$ are compact, we already know that $(b^*,\rho^*) = \arg\min_{\Delta\times\Gamma} R(b,\rho)$ exists. we compute the minimum of the function $(b,\rho) \mapsto R(b,\rho)$ over $\Delta \times \Gamma$ by applying the optimality condition (5.5.7) when $d = 3$:

$$\sum_{i=1}^{3} \kappa^i(b^*,\rho^*)(b_i - b_i^*) - \sum_{j=1}^{3}\sum_{i=1}^{j-1} \sigma_i\sigma_j\kappa^i\kappa^j(\rho^*)(\rho_{ij} - \rho_{ij}^*) \quad \geq \quad 0. \tag{5.5.52}$$

We consider the following possible exclusive cases depending on the number of zero components in $\hat{\kappa}(b^*,\rho^*)$:

1. $\kappa^1\kappa^2(b^*,\rho^*) = 0$, $\kappa^1\kappa^3(b^*,\rho^*) = 0$, $\kappa^2\kappa^3(b^*,\rho^*) = 0$.
In this case, (5.5.52) immediately holds. As we assume that $\underline{\beta}_1 \geq \underline{\beta}_2 \geq \underline{\beta}_3 > 0$, (5.5.48) and (5.5.49) in Section 5.5.6 yields the explicit expression of (b^*,ρ^*), $\kappa(b^*,\rho^*)$ and $R(b^*,\rho^*)$

$$b_1^* \;=\; \underline{b}_1,\; \rho_{1j}^* \;=\; \frac{\beta_j^*}{\underline{\beta}_1}, \quad \text{for any } \rho_{1j}^* \in [\underline{\rho}_{1j}, \bar{\rho}_{1j}] \cap [\frac{\underline{\beta}_j}{\underline{\beta}_1}, \min(1, \frac{\bar{\beta}_j}{\underline{\beta}_1})), \quad j = 2,3, \tag{5.5.53}$$

and

$$\kappa(b^*,\rho^*) \;=\; (\frac{\underline{b}_1}{\sigma_1^2}, 0, 0)^\intercal, \quad R(\theta^*) \;=\; \underline{\beta}_1^2. \tag{5.5.54}$$

Let us show that $\underline{\beta}_1^2$ is strict minimum value in the sense that $R(\theta^*) = \underline{\beta}_1^2$ if and only if $b_1^* = \underline{b}_1$, $b_2^* = \underline{b}_1\rho_{12}^*$, $b_3^* = \underline{b}_1\rho_{13}^*$. From the block-matrix form of $\Sigma(\rho)^{-1}$ in (5.5.35), we write $R(b,\rho)$ as two nonnegative parts

$$\begin{aligned} R(b,\rho) &= \beta_1^2 + (b_{-1} - \frac{b_1}{\sigma_1^2}C_1)^\intercal A^{-1}(b_{-1} - \frac{b_1}{\sigma_1^2}C_1) \\ &\geq \underline{\beta}_1^2, \end{aligned}$$

where $C_1 = (\sigma_1\sigma_2\rho_{12}, \sigma_1\sigma_3\rho_{13})^\intercal$ and $A = \Sigma_{-1}(\rho_{23}) - \frac{C_1C_1^\intercal}{\sigma_1^4} > \mathbf{0}$. '=' holds true if and only if $b_1 = \underline{b}_1$ and $b_{-1} - \frac{\underline{b}_1}{\sigma_1^2}C_1 = 0$, i.e. $b_1^* = \underline{b}_1$, $\beta_2^* = \underline{\beta}_1\rho_{12}^*$, $\beta_3^* = \underline{\beta}_1\rho_{13}^*$.

2. $\kappa^1\kappa^2(b^*,\rho^*) \neq 0$, $\kappa^1\kappa^3(b^*,\rho^*) = 0$, $\kappa^2\kappa^3(b^*,\rho^*) = 0$.
In this case, we obtain from (5.5.36)

$$\left\{\begin{aligned} \begin{pmatrix}\kappa^1(b,\rho)\\ \kappa^2(b,\rho)\end{pmatrix} &= \Sigma_{-3}(\rho_{12})^{-1}b_{-3} - \kappa^3(\rho)\Sigma_{-3}(\rho_{12})^{-1}C_3 \\ \kappa^3(b,\rho) &= \tfrac{1}{a(\rho)}(b_3 - C_3^\intercal\Sigma_{-3}(\rho_{12})^{-1}b_{-3}) \end{aligned}\right. \tag{5.5.55}$$

and

$$R(b,\rho) \;=\; b_{-3}^\intercal\Sigma_{-3}(\rho_{12})^{-1}b_{-3} + a(\rho)\big(\kappa^3(b,\rho)\big)^2. \tag{5.5.56}$$

When there is no ambiguity, we write $b_{-3}\Sigma_{-3}(\rho_{12})b_{-3}$ as $R(b_{-3},\rho_{12})$.
As $\kappa^3(b^*,\rho^*)=0$, we get from (5.5.52) that

$$\kappa^1(b^*,\rho^*)(b_1-b_1^*)+\kappa^2(b^*,\rho^*)(b_2-b_2^*)-\sigma_1\sigma_2\kappa^1\kappa^2(b^*,\rho^*)(\rho_{12}-\rho_{12}^*) \quad \geq \quad 0. \tag{5.5.57}$$

and from (5.5.55) and (5.5.56) that

$$\begin{cases} \begin{pmatrix}\kappa^1(b^*_{-3},\rho^*_{12})\\ \kappa^2(b^*_{-3},\rho^*_{12})\end{pmatrix} & = \quad \Sigma_{-3}(\rho^*_{12})^{-1}b^*_{-3} \\ R(b^*,\rho^*) & = \quad R(b^*_{-3},\rho^*_{12}) \end{cases} \tag{5.5.58}$$

This is the case of mixing drift uncertainty (b_1,b_2) in $[\underline{b}_1,\bar{b}_1]\times[\underline{b}_2,\bar{b}_2]$ and correlation ambiguity $\rho_{12}\in[\underline{\rho}_{12},\bar{\rho}_{12}]$ in the two-risky asset described in Proposition 5.4.5. Therefore, we obtain the explicit expression of b_1^*, b_2^* and ρ_{12}^*

$$b_1^* \;=\; \underline{b}_1,\;\; b_2^* \;=\; \underline{b}_2 1_{\{\bar{\rho}_{12}<\frac{\underline{\beta}_2}{\underline{\beta}_1}\}}+\bar{b}_2 1_{\{\underline{\rho}_{12}>\frac{\bar{\beta}_2}{\underline{\beta}_1}\}},\;\; \rho^*_{12}=\underline{\rho}_{12}1_{\{\bar{\rho}_{12}<\frac{\underline{\beta}_2}{\underline{\beta}_1}\}}+\bar{\rho}_{12}1_{\{\underline{\rho}_{12}>\frac{\bar{\beta}_2}{\underline{\beta}_1}\}} \tag{5.5.59}$$

By setting $g(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23}) := a(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23})\ \kappa^3(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23})$ for fixed b_1^*, b_2^* and ρ_{12}^* in (5.5.59), we deduce from (5.5.55) that the function

$$(b_3,\rho_{13},\rho_{23})\mapsto g(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23}) \quad = \quad b_3-\sigma_1\sigma_3\rho_{13}\kappa^1(b^*_{-3},\rho^*_{12})-\sigma_2\sigma_3\rho_{23}\kappa^2(b^*_{-3},\rho^*_{12})$$

is linear in $(b_3,\rho_{13},\rho_{23})\in[\underline{b}_3,\bar{b}_3]\times[\underline{\rho}_{13},\bar{\rho}_{13}]\times[\underline{\rho}_{23},\bar{\rho}_{23}]$, and has the same sign as $\kappa^3(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23})$ due to the positiveness of $a(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23})=0$. To study the condition of $\kappa^3(b_1^*,b_2^*,b_3,\rho_{12}^*,\rho_{13},\rho_{23})$, we discuss it in the following two cases:

(i) If $\bar{\rho}_{12}<\frac{\underline{\beta}_2}{\underline{\beta}_1}$, then $\kappa^1(\underline{b}_1,\underline{b}_2,\bar{\rho}_{12})>0$, $\kappa^2(\underline{b}_1,\underline{b}_2,\bar{\rho}_{12})>0$, the linear function $(b_3,\rho_{13},\rho_{23})\mapsto g(\underline{b}_1,\underline{b}_2,b_3,\bar{\rho}_{12},\rho_{13},\rho_{23})$ is increasing in b_3, decreasing in ρ_{13} and ρ_{23}. To ensure that the function $g(\underline{b}_1,\underline{b}_2,b_3,\bar{\rho}_{12},\rho_{13},\rho_{23})$ has a root in $[\underline{b}_3,\bar{b}_3]\times[\underline{\rho}_{13},\bar{\rho}_{13}]\times[\underline{\rho}_{23},\bar{\rho}_{23}]$, we need

$$g(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23}) \;\leq\; 0 \;\leq\; g(\underline{b}_1,\underline{b}_2,\bar{b}_3,\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23}),$$

or equivalently,

$$\kappa^3(\underline{b}_1,\underline{b}_2,\underline{b}_3,\bar{\rho}_{12},\bar{\rho}_{13},\bar{\rho}_{23})\leq 0\leq \kappa^3(\underline{b}_1,\underline{b}_2,\bar{b}_3,\bar{\rho}_{12},\underline{\rho}_{13},\underline{\rho}_{23}).$$

(ii) If $\underline{\rho}_{12}>\min(1,\dfrac{\bar{\beta}_2}{\underline{\beta}_1})$, then $\kappa^1(\underline{b}_1,\bar{b}_2,\underline{\rho}_{12})>0$, $\kappa^2(\underline{b}_1,\bar{b}_2,\underline{\rho}_{12})<0$. The linear function $(b_3,\rho_{13},\rho_{23})\mapsto g(\underline{b}_1,\bar{b}_2,b_3,\underline{\rho}_{12},\rho_{13},\rho_{23})$ is increasing in b_3 and ρ_{23}, and decreasing in ρ_{13}. As $(b_3^*,\rho_{13}^*,\rho_{23}^*)$ satisfying $g(\underline{b}_1,\bar{b}_2,b_3^*,\underline{\rho}_{12},\rho_{13}^*,\rho_{23}^*)=0$ has to belong to $[\underline{b}_3,\bar{b}_3]\times[\underline{\rho}_{13},\bar{\rho}_{13}]\times[\underline{\rho}_{23},\bar{\rho}_{23}]$, we obtain

$$\kappa^3(\underline{b}_1,\bar{b}_2,\bar{b}_3,\underline{\rho}_{12},\underline{\rho}_{13},\bar{\rho}_{23})\geq 0\geq \kappa^3(\underline{b}_1,\bar{b}_2,\underline{b}_3,\underline{\rho}_{12},\bar{\rho}_{13},\underline{\rho}_{23}).$$

Therefore, we deduce that

$$R(b,\rho) \quad \geq \quad R(\underline{b}_1,\underline{b}_2,\bar{\rho}_{12})1_{\{\bar{\rho}_{12}<\frac{\underline{\beta}_2}{\underline{\beta}_1}\}}+R(\underline{b}_1,\bar{b}_2,\underline{\rho}_{12})1_{\{\underline{\rho}_{12}>\frac{\bar{\beta}_2}{\underline{\beta}_1}\}} \tag{5.5.60}$$

and that '=' holds if and only if $b_1^*=\underline{b}_1$, $b_2^*=\underline{b}_2 1_{\{\bar{\rho}_{12}<\frac{\underline{\beta}_2}{\underline{\beta}_1}\}}+\bar{b}_2 1_{\{\underline{\rho}_{12}>\frac{\bar{\beta}_2}{\underline{\beta}_1}\}}$, $\rho^*_{12}=\underline{\rho}_{12}1_{\{\underline{\rho}_{12}\geq\frac{\bar{\beta}_2}{\underline{\beta}_1}\}}+\bar{\rho}_{12}1_{\{\bar{\rho}_{12}\leq\frac{\underline{\beta}_2}{\underline{\beta}_1}\}}$. This corresponds to the subcases **2.**(i) and **2.**(ii) as described in Proposition 5.4.6.

3. $\kappa^1\kappa^2(b^*, \rho^*) = 0$, $\kappa^1\kappa^3(b^*, \rho^*) \neq 0$, $\kappa^2\kappa^3(b^*, \rho^*) = 0$.
In this case, we make permutations as follows,

$$\begin{pmatrix} \kappa^{-2}(b, \rho) \\ \kappa^2(b, \rho) \end{pmatrix} = \begin{pmatrix} \Sigma_{-2}(\rho_{13}) & C_2 \\ C_2^\intercal & \sigma_2^2)) \end{pmatrix}^{-1} \begin{pmatrix} b_{-2} \\ b_2 \end{pmatrix}, \tag{5.5.61}$$

where $\kappa^{-2}(b, \rho) = (\kappa^1(b, \rho), \kappa^3(b, \rho))^\intercal$ and $C_2 = (\sigma_1\sigma_2\rho_{12}, \sigma_2\sigma_3\rho_{23})^\intercal$. Using (5.5.61) and proceeding with the same arguments as in the case **2.**, we obtain the result of $\kappa^2(b^*, \rho^*) = 0$, $\kappa^1\kappa^3(b^*, \rho^*) \neq 0$ as described in case **3.**(i) and **3.**(ii) of Proposition 5.4.6. Proceeding with the same arguments as in the case of $\kappa^3(b^*, \rho^*) = 0$, we obtain the results of case $\kappa^2(b^*, \rho^*) = 0$ described in subcase **3.**(i) and **3.**(ii) of Proposition 5.4.6

4. $\kappa^1\kappa^2(b^*, \rho^*) = 0$, $\kappa^1\kappa^3(b^*, \rho^*) = 0$, $\kappa^2\kappa^3(b^*, \rho^*) \neq 0$.
Note that

$$\begin{pmatrix} \kappa^{-1}(b, \rho) \\ \kappa^1(b, \rho) \end{pmatrix} = \begin{pmatrix} \Sigma_{-1}(\rho_{23}) & C_1 \\ C_1^\intercal & \sigma_1^2 \end{pmatrix}^{-1} \begin{pmatrix} b_{-1} \\ b_1 \end{pmatrix}, \tag{5.5.62}$$

where $\kappa^{-1}(b, \rho) = (\kappa^2(b, \rho), \kappa^3(b, \rho))^\intercal$ and the vector $C_1 = (\sigma_1\sigma_2\rho_{12}, \sigma_1\sigma_3\rho_{13})^\intercal$. Using (5.5.62) and proceeding with the same arguments as in case **2.**, we obtain the result of $\kappa^1(b^*, \rho^*) = 0$, $\kappa^2\kappa^3(b^*, \rho^*) \neq 0$ as described in subcases **4.**(i) and **4.**(ii) of Proposition 5.4.6.

5. $\kappa^1\kappa^2(b^*, \rho^*) \neq 0$, $\kappa^1\kappa^3(b^*, \rho^*) \neq 0$, $\kappa^2\kappa^3(b^*, \rho^*) \neq 0$.
In this case, we see from (5.5.52) that each b_i^* takes value in $\{\underline{b}_i, \bar{b}_i\}$ depending on the sign of $\kappa^i(b^*, \rho^*)$ and that each ρ_{ij}^* takes value in $\{\underline{\rho}_{ij}, \bar{\rho}_{ij}\}$ depending on the sign of $\kappa^i\kappa^j(b^*, \rho^*)$. Moreover, due to the assumption $\underline{\beta}_1 > \underline{\beta}_2 > \underline{\beta}_3 > 0$, we obtain that $R(b^*, \rho^*) = \sum_{i=1}^{3} b_i^*\kappa^i(b^*, \rho^*) > 0$, hence at least one component of $\kappa(b^*, \rho^*)$ is positive. By combination, there are 7 possible subcases as described in case **5.** of Proposition 5.4.6.
As $\kappa^i\kappa^j(b^*, \rho^*) \neq 0$ in each possible subcase, l.h.s of (5.5.52) is strictly negative for any $(b, \rho) \in \Delta \times \Gamma \setminus \{(b^*, \rho^*)\}$. From first order condition of convexity of $R(b, \rho)$ at point (b^*, ρ^*), we have for any $(b, \rho) \in \Delta \times \Gamma \setminus \{(b^*, \rho^*)\}$

$$\begin{aligned} R(b, \rho) &\geq R(b^*, \rho^*) + \sum_{i=1}^{3} \kappa^i(b^*, \rho^*)(b_i - b_i^*) - \sum_{j=1}^{3}\sum_{i=1}^{j-1} \sigma_i\sigma_j\kappa^i\kappa^j(\rho^*)(\rho_{ij} - \rho_{ij}^*) \\ &> R(b^*, \rho^*), \end{aligned}$$

which indicates that (b^*, ρ^*) in each subcase of case **5.** in Proposition 5.4.6 is strict minimum of $R(b, \rho)$.

As $R(b, \rho)$ in each subcase is strict minimum value, we conclude that each subcase in Proposition 5.4.6 is exclusive. ☐

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